The central role of Gauss constraint across LQC and LQG

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Three points program

- Introduction to nondiagonal Bianchi models [Montani, MB ‘23]
  - Minisuperspace and Ashtekar variables
  - Flux quantization procedure

- Abelianization of the Gauss constraint [Montani, MB ‘23]
  - Gauge freedom and canonical transformation
  - Revised Gauss Constraint and Quantum-level implications

- Yang-Mills approach for the cosmological sector [MB ‘24] [MB ‘24]

Nondiagonal Bianchi models

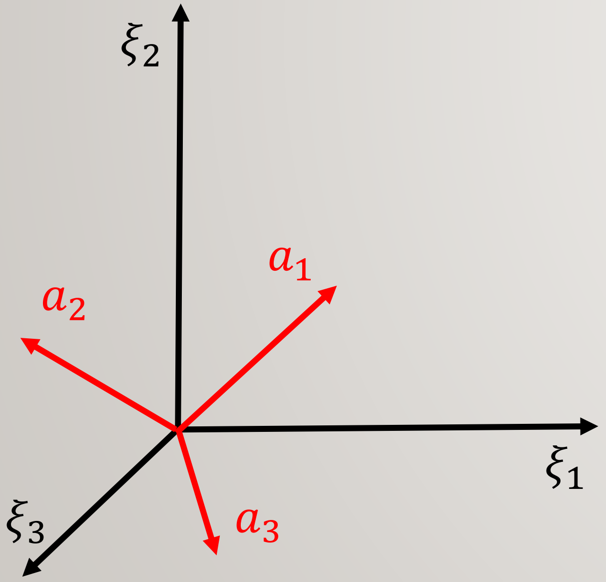
Minisuperspace

[Landau, Lifshits '74]
[Belinski '14]
[Montani, MB '23]

Globally hyperbolic spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$

Homogeneous space Σ prescription $q_{ij}(t,x) = \eta_{IJ}(t)\omega^I_i(x)\omega^J_j(x)$

Nondiagonal metric decomposition $\eta_{IJ} = \Gamma_{AB} R^A_I R^B_J$



Maurer-Cartan equation

$$d\omega^I + \frac{1}{2} f^I_{JK}\omega^J\omega^K = 0$$

Lie algebra generators

$$[\xi_I, \xi_J] = f^K_{IJ}\xi_K$$

Metric configuration variables $\{a_1, a_2, a_3, \theta, \psi, \phi\}$

Nondiagonal Bianchi models

Ashtekar variables

Lagrangian

$$L_{ADM} = N|\det(\omega_i^I)|\sqrt{\det(\Gamma_{AB})}\Big[\bar{R} + \frac{1}{4N^2}(\Gamma^{AC}\Gamma^{BD}\dot{\Gamma}_{AB}\dot{\Gamma}_{CD} + 2\Gamma^{AB}\Gamma_{CD}(R\dot{\Lambda})_A^D(R\dot{\Lambda})_B^C + 2(R\dot{\Lambda})_C^B(R\dot{\Lambda})_B^C$$
$$+ 2N^AN^B\left(f_{AJ}^I f_{BI}^J + \eta^{IJ}\eta_{KL}f_{AI}^K f_{BJ}^L\right) + 4N^K\eta^{IJ}\dot{\eta}_{JL}f_{KI}^L - \Gamma^{IJ}\dot{\Gamma}_{IJ}\Gamma^{KL}\dot{\Gamma}_{KL})\Big]$$

Ashtekar connection

$$A_i^a = \left[\frac{1}{2}\epsilon^{abc}\frac{a_c}{a_b}\Lambda_b^J R_K^c f_{IJ}^K - \frac{1}{4}\epsilon^{abc}\frac{1}{a_b a_c}\eta_{IJ}\Lambda_b^K\Lambda_c^L f_{LK}^J + \frac{\gamma}{2N}a_{(a)}R_L^a\left(\eta^{LJ}\dot{\eta}_{JI} + N^A\eta^{LK}\eta_{IJ}f_{AK}^J + N^A f_{AI}^L\right)\right]\omega_i^I$$

Electric field

$$E_a^i = |\det(\omega_i^I)|\,\mathrm{sgn}\left(a_{(a)}\right)|a_b a_c|\Lambda_a^I\xi_I^i$$

Nondiagonal Bianchi I model

Ashtekar variables

Lagrangian

$$
\begin{aligned}
L_{ADM} &= N|\det(\omega_i^I)|\sqrt{\det(\Gamma_{AB})}\Big[\cancelto{0}{\bar{R}} + \frac{1}{4N^2}\big(\Gamma^{AC}\Gamma^{BD}\dot{\Gamma}_{AB}\dot{\Gamma}_{CD} + 2\Gamma^{AB}\Gamma_{CD}(R\dot{\Lambda})_A^D(R\dot{\Lambda})_B^C + 2(R\dot{\Lambda})_C^B(R\dot{\Lambda})_B^C \\
&+ 2N^A N^B\cancelto{0}{\left(f_{AJ}^I f_{BI}^J + \eta^{IJ}\eta_{KL}f_{AI}^K f_{BJ}^L\right)} + \cancelto{0}{4N^K\eta^{IJ}\dot{\eta}_{JL}f_{KI}^L} - \Gamma^{IJ}\dot{\Gamma}_{IJ}\Gamma^{KL}\dot{\Gamma}_{KL}\big)\Big]
\end{aligned}
$$

Ashtekar connection

$$
A_i^a = \left[\frac{1}{2}\epsilon^{abc}\frac{a_c}{a_b}\Lambda_b^J R_K^c f_{IJ}^K - \cancelto{0}{\frac{1}{4}\epsilon^{abc}\frac{1}{a_b a_c}\eta_{IJ}\Lambda_b^K\Lambda_c^L f_{LK}^J} + \frac{\gamma}{2N}a_{(a)}R_L^a\left(\eta^{LJ}\dot{\eta}_{JI} + \cancelto{0}{N^A\eta^{LK}\eta_{IJ}f_{AK}^J} + \cancelto{0}{N^A f_{AI}^L}\right)\right]\omega_i^I
$$

Electric field

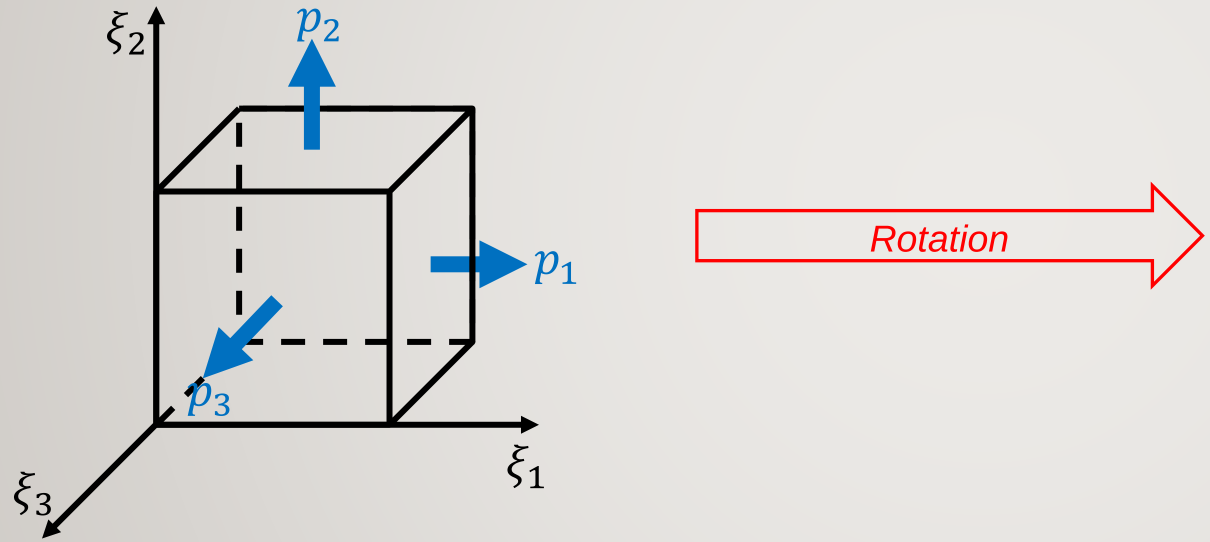
$$
E_a^i = |\det(\omega_i^I)|\,\mathrm{sgn}\big(a_{(a)}\big)\,|a_b a_c|\Lambda_a^I\xi_I^i
$$

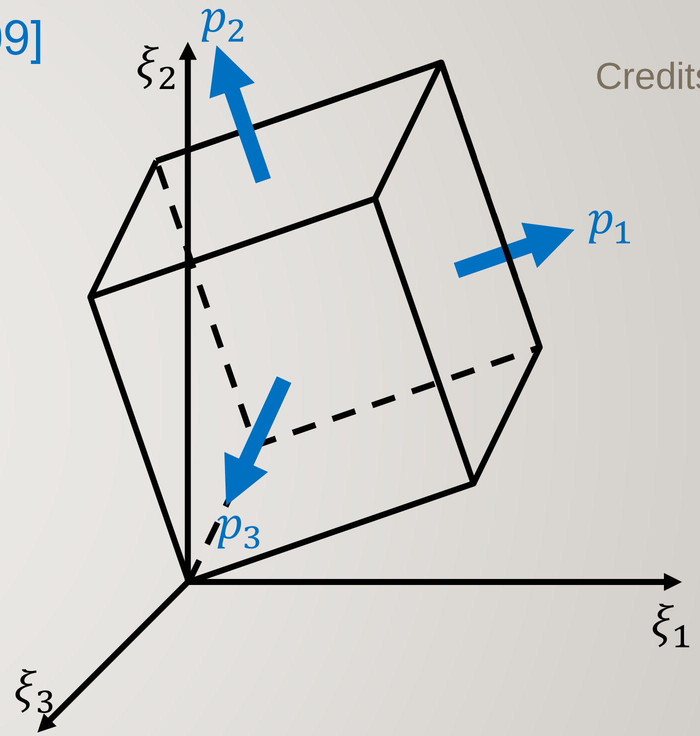
Nondiagonal Bianchi I model

Flux quantization

Quantization in the flux polarization as in [Ashtekar, Wilson-Ewing ‘09]

The fluxes computed on the faces of the fiducial cell





Credits: Beatrice Gorga

Geometric operators depend on diagonal fluxes only!

Basis states of the Hilbert space: $|p_1, p_2, p_3, \theta, \psi, \phi\rangle$

Abelianization of the Gauss constraint

Gauge freedom

$$G_a|_{\wp_{Mt}} = 0$$

$$\wp_{Mt} = \{a_1, a_2, a_3, \theta, \psi, \phi, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi\}$$

Mismatch in the number of degrees of freedom!

M. Bojowald's suggestion in [Bojowald '00, '13]

$$\left.\begin{aligned} A_i^a(t,x) &= \phi_I^a(t)\omega_i^I(x) \\ E_a^i(t,x) &= |\det(\omega(x))| p_a^I(t)\xi_I^i(x) \end{aligned}\right\} \quad G_a = \epsilon_{abc}\phi_I^b p_c^I$$

Recover the gauge freedom adding a rotation

$$\wp_{\overline{Mt}} = \{a_1, a_2, a_3, \theta, \psi, \phi, \alpha, \beta, \gamma, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi, \pi_\alpha, \pi_\beta, \pi_\gamma\}$$

Three abelian constraints $\begin{cases} \pi_\alpha = 0 \\ \pi_\beta = 0 \\ \pi_\gamma = 0 \end{cases}$

Abelianization of the Gauss constraint

Canonical transformation

Lie condition $\phi_I^a dp_a^I - \pi_n dq_n = 0$ provides, perturbative in configurational variables, a linear dependence between Gauss constraint and gauge momenta

Ansatz

Gauss constraint is linear in the gauge momenta $G_a = L_{ag}\pi_g$

System of 9 independent equations $\epsilon_{abc} = L_{ag}(O^t)_d^c \frac{\partial O_b^d}{\partial q_g}$

Admits a unique solution!

$$L_{ag} = \begin{pmatrix} -\csc\beta\cos\gamma & \sin\gamma & \cot\beta\cos\gamma \\ \csc\beta\sin\gamma & \cos\gamma & -\cot\beta\sin\gamma \\ 0 & 0 & 1 \end{pmatrix}$$

Abelianization of the Gauss constraint

The Abelian contraints

$$G_a = \begin{pmatrix} -\csc\beta\cos\gamma\,\pi_\alpha & \sin\gamma\,\pi_\beta & \cot\beta\cos\gamma\,\pi_\gamma \\ \csc\beta\sin\gamma\,\pi_\alpha & \cos\gamma\,\pi_\beta & -\cot\beta\sin\gamma\,\pi_\gamma \\ 0 & 0 & \pi_\gamma \end{pmatrix}$$

From a SU(2) symmetry, three U(1) appear!

The Gauss constraint is recast into three abelian constraints, namely the gauge momenta

This feature holds at the quantum level $\hat{G}_a|\Psi\rangle = 0 \;\Leftrightarrow\; \hat{\pi}_g|\Psi\rangle = 0$

The wavefunction factorizes $\Psi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3, \alpha, \beta, \gamma) = \varphi(\alpha, \beta, \gamma)\Phi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3)$

$$\hat{\pi}_g|\Psi\rangle = 0 \;\Rightarrow\; \varphi = const$$

The Hilbert space previously defined is the gauge-invariant one

Cosmological sector of Loop Quantum Gravity

A Yang-Mills approach

[Brodbeck '96]
[Bojowald, Kastrup '00]
[MB '24]
[MB '24]

$$P^{Spin}(\Sigma) \downarrow P^{SO}(\Sigma) \downarrow \Sigma$$

Yang-Mills variables

Connection ω is a 1-form on $P^{Spin}(\Sigma)$ with value in the Lie algebra of SU(2)

Dreibein e is a section in $P^{SO}(\Sigma)$

Ashtekar variables

Connection A is the local field $A = e^*\omega$

Electric field E is built from the dreibein $E = \sqrt{q}d^3x \otimes e$

The request of homogeneity for ω yields to a homogeneous geometry for Σ

Cosmological sector of Loop Quantum Gravity

Quantum states

Configurational space $\mathcal{A} = \{A \mid A = e^{*}\omega, \omega \text{ homogeneous}\}$

The set of constraints are the same of LQG

Spin-network states as cylindric functions on $\mathcal{A}$

Some properties analogous to the usual cosmological states naturally emerge:

- the spin networks are homogeneous, namely the curves of the graph are integral curves of linear combinations of ξ_I
- the invariant states bring pointwise holonomy

Conclusions

- The diagonal quantization in LQC is quite general within the minisuperspace approach
- The Abelianization of the quantum theory is a feature of the minisuperspace. The three U(1) simmetries arise from decomposing the Gauss constraint in three abelian ones
- We can identify a cosmological sector with the same constraints as LQG and perform a quantization that yields spin-network states exhibiting properties akin to those in LQC

Thank you for your attention

The central role of Gauss constraint across LQC and LQG

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