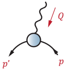
QCD contributions to the muon anomalous magnetic moment

Gernot Eichmann
University of Graz

Alps 2023
March 28, 2023

Muon g-2



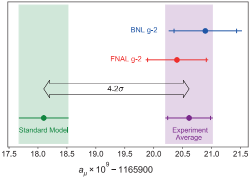
$$= ie\,\bar{u}(p')\left[F_1(Q^2)\,\gamma^\mu - F_2(Q^2)\,\frac{\sigma^{\mu\nu}Q^\nu}{2m}\right]u(p)$$

$$F_2(0) = a_\mu = \underbrace{\frac{\alpha_{\text{QED}}}{2\pi}} + \mathcal{O}(\alpha_{\text{QED}}^2)$$

0.001161...
Schwinger 1948

$a_\mu\ [10^{-10}]$

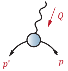
Exp:	11 659 206.1	(4.1)
QED:	11 658 471.9	(0.0)
EW:	15.4	(0.1)
Hadronic:		
• VP (LO+HO)	684.5	(4.0)
• LBL	9.2	(1.8)
SM:	11 659 181.0	(4.3)
Diff:	25.1	(5.9)



Muon g-2 Theory Initiative White Paper:
Aoyama et al., Phys. Rept. 887 (2020)

FNAL Run-1: Abi et al., PRL 126, 141801 (2021)

Muon g-2



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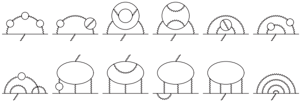
- **QED** contributions overwhelming,
  known up to 5 loops, precision good enough

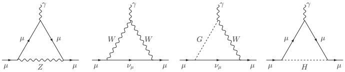


exemplary 4-loop diagrams
(892 in total)

Aoyama, Hayakawa, Kinoshita, Nio, PRL 109 (2012)

- Remainder small: 10^{-12} for electron, 10^{-8} for muon

- **EW** contributions known to 2 loops



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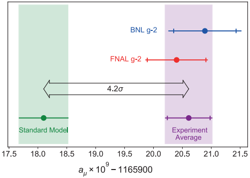

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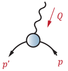

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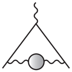
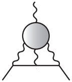
- **QCD** contributions dominate theory uncertainty:



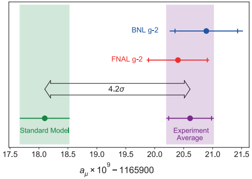
Hadronic vacuum polarization



Hadronic light-by-light scattering

Need to pin down as precisely as possible:

- **Dispersion theory**: data driven
- **Lattice QCD**: ab-initio
- **Functional methods**: in principle ab-initio, but no systematic error control yet for g-2
- **AdS/QCD, Hadronic models**, ...



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FNAL Run-1: Abi et al., PRL 126, 141801 (2021)

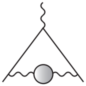
Hadronic vacuum polarization

- $\mathcal{O}(\alpha^2)$, largest QCD contribution, dominates uncertainty

- **Vector current correlator:**

$$\Pi^{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x}\, \langle j^\mu(x)\, j^\nu(0)\rangle$$
$$= \Pi(Q^2)\,(Q^2\,\delta^{\mu\nu} - Q^\mu Q^\nu)$$

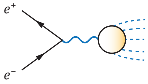
- Leading contribution to a_μ^{HVP}
dominated by small momenta $Q^2 \sim m_\mu^2$



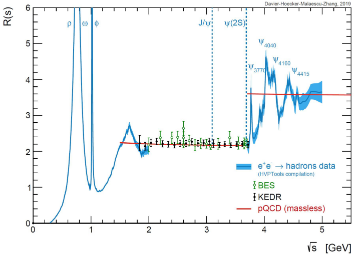
$$a_\mu^{\text{HVP, LO}} = \alpha^2 \int dQ^2\,(\dots)\,\Pi_{\text{ren}}(Q^2)$$

HVP - R ratio

- Direct relation to experiment: **R ratio**
  $e^+e^- \;\rightarrow\;$ hadrons

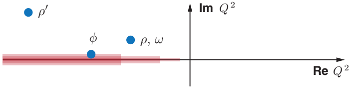


Data from CMD-2/3 & SND,
KLOE, BaBar, BES III, CLEO-c



$$s = -Q^2$$

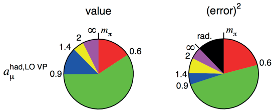
- Use optical theorem + dispersion relations
  to rewrite integral in terms of R ratio



$$a_\mu^{\text{HVP, LO}} = \frac{\alpha^2}{3\pi^2} \int ds \, \frac{K(s)}{s} \, R(s)$$

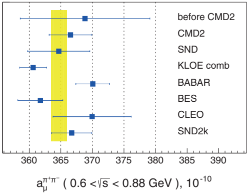
HVP - R ratio

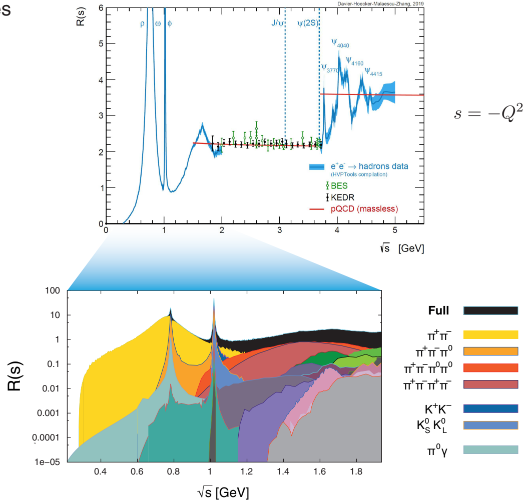
- Biggest contribution from low energies (0.6 ... 0.9 GeV) around ρ / ω poles



- Dominant channel is $\pi^+\pi^-$, followed by 3π, 4π, K^+K^-, ...

- $\pi\pi$ data not fully consistent





$s = -Q^2$

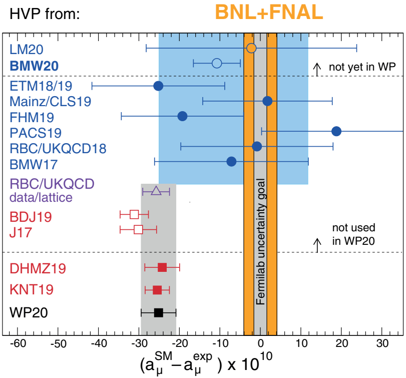
Keshavarzi, Nomura, Teubner, PRD 97 (2018), 1802.02995

HVP - Lattice QCD

$$\Pi^{\mu\nu}(x) = \langle\, j^\mu(x)\, j^\nu(0) \rangle \;= \int \mathcal{D}[\psi,\bar{\psi},A]\, e^{-S}\, j^\mu(x)\, j^\nu(0)$$

- Large efforts underway, results from several collaborations
- Difficult problem, spans scales from m_π to several GeV, finite-volume effects, disconnected contributions, isospin breaking & QED effects
- Biggest uncertainties in $\Pi(Q^2)$ at very low Q^2
- Tensions with dispersive results, but errors still comparatively large
- BMW: physical pion masses, isospin breaking & QED effects

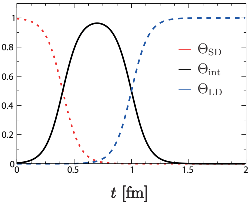
Borsanyi et al., Nature 593, 51 (2021)

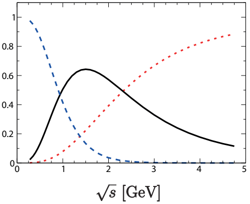


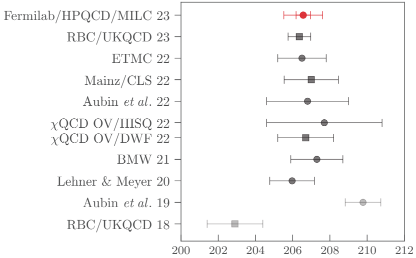
Colangelo et al., 2203.15810

HVP - Lattice QCD

- Cross-checks: “windows” in Euclidean time and $\sqrt{s}$
  Blum et al., PRL 121 (2018), Lehner & Meyer, PRD 101 (2020)
  Colangelo et al., PLB 833 (2022) 137313







Short distance: 10%

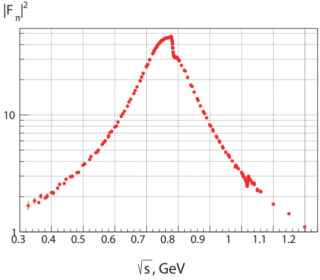
Intermediate window: 33%, precise lattice results

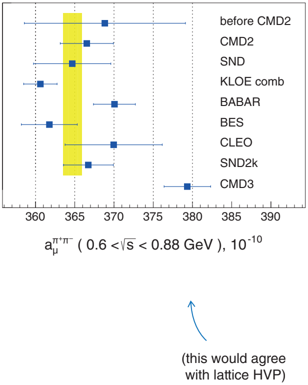
Long distance: 57%, finite-volume effects

Lattice calculations agree on light-quark connected contribution
Fermilab, HPQCD, MILC: 2301.08274

HVP - R ratio?

- New CMD-3 results for $e^+e^- \to \pi^+\pi^-$ are even more puzzling...
  CMD-3 Collaboration: Ignatov et al., 2302.08834

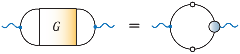




HVP - Functional methods

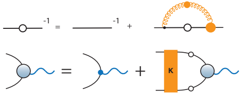
 $= \langle \underbrace{[\bar\psi\,\gamma^\mu\psi](x)}_{j^\mu(x)}\,\underbrace{[\bar\psi\,\gamma^\nu\psi](0)}_{j^\nu(0)} \rangle$

$= \gamma^\mu_{\alpha\beta}\,\gamma^\nu_{\rho\sigma}\,\langle \bar\psi_\alpha(x)\,\psi_\beta(x)\,\bar\psi_\rho(0)\,\psi_\sigma(0)\rangle$

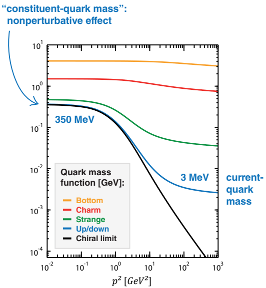


- Depends on **quark propagator** & **quark-photon vertex,** satisfy Dyson-Schwinger and Bethe-Salpeter equations

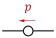
  GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016)



QCD ingredients rely on truncations, systematic improvements necessary (and underway)

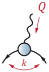


- Quark mass is not constant: **dynamical mass generation**



$$S_0(p) = \frac{-i\not p + m}{p^2+m^2} \longrightarrow S(p) = \frac{1}{A(p^2)}\,\frac{-i\not p + M(p^2)}{p^2+M^2(p^2)}$$

- Quark-photon vertex has 12 tensors:



$$i\gamma^\mu\,\Sigma_A + 2k^\mu(i\not k\,\Delta_A + \Delta_B)$$

Ball-Chiu vertex, determined by WTI, depends only on quark propagator

Ball, Chiu, PRD 22 (1980)

Transverse part, contains dynamics (VM poles, cuts, ...), 8 dressing functions

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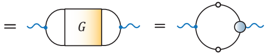

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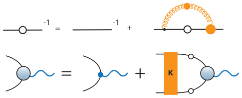
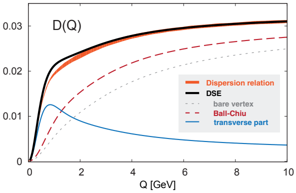


QCD ingredients rely on truncations, systematic improvements necessary (and underway)

Adler function: $D(Q^2) = -Q^2\,\dfrac{d\Pi(Q^2)}{dQ^2}$



$a_\mu^{\text{HVP}} = 710\,(34)\times 10^{-10}$

Goecke, Fischer, Williams, PLB 704 (2011)

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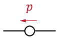

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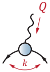

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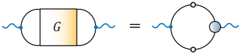

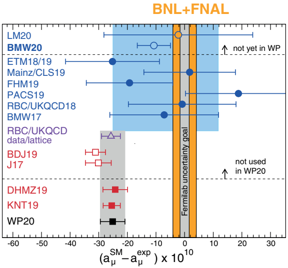

DSE

$m_\rho = m_\rho^{exp}$ $f_\pi = f_\pi^{exp}$

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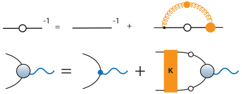
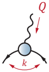
  GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016)



QCD ingredients rely on truncations, systematic improvements necessary (and underway)

contributes 80% to g-2,
resonance dynamics important

- Quark-photon vertex has 12 tensors:



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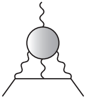
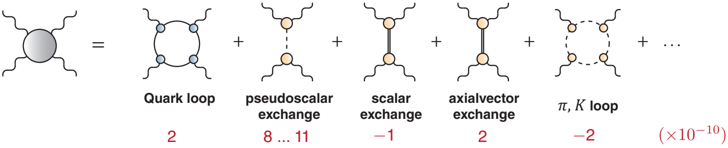
Hadronic light-by-light

- $\mathcal{O}(\alpha^3)$, known to ~20%, target: 10%
- more difficult, no direct data



Pre-WP: model results (”Glasgow consensus”)

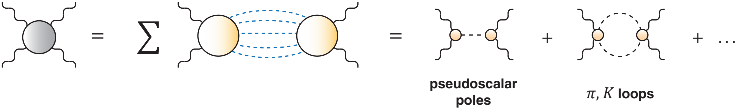
Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)

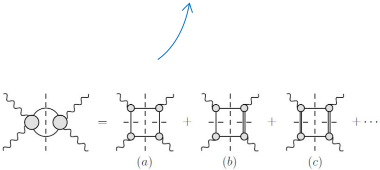


- Large uncertainties (and differences among calculations) in individual contributions
- Pion exchange dominant
- What about quark loop: double-counting?

HLbL

- Systematic treatment in dispersive approach, but significantly more complicated than HVP
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015), PRL 118 (2017), JHEP 04 (2017), Pauk & Vanderhaeghen, PRD 90 (2014), ...





WP20

HLbL

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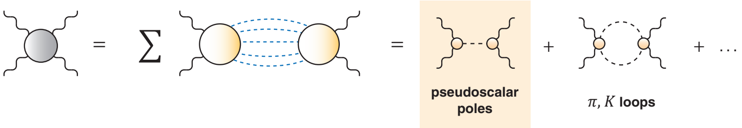
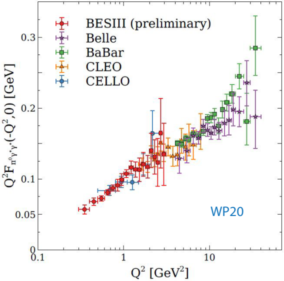
  but significantly more complicated than HVP
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015), PRL 118 (2017), JHEP 04 (2017),
  Pauk & Vanderhaeghen, PRD 90 (2014), . . .





Depend on $\pi^0 \to \gamma\gamma$, $\eta \to \gamma\gamma$ transition form factors

- Dispersion theory
  Hoferichter, Hoid, Kubis, Leupold, Schneider, JHEP 10 (2018)
- Padé and Canterbury approximants
  Masjuan, Sánchez-Puertas, PRD 95 (2017)

} **WP20**

- Functional methods (DSEs & BSEs)
  Raya et al., PRD 93 (2016), GE, Fischer, Weil, Williams, PLB 774 (2017), PRD 96 (2017), PLB 799 (2019)
- . . .

HLbL

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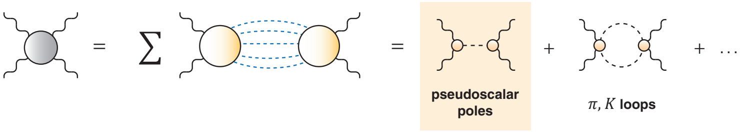
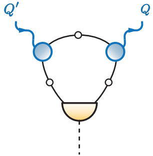
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015), PRL 118 (2017), JHEP 04 (2017),
  Pauk & Vanderhaeghen, PRD 90 (2014), . . .

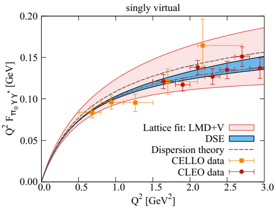


- DSEs and BSEs:

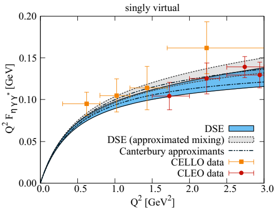


$\pi^0 \to \gamma\gamma$ transition form factor

GE, Fischer, Weil, Williams, PLB 799 (2019)

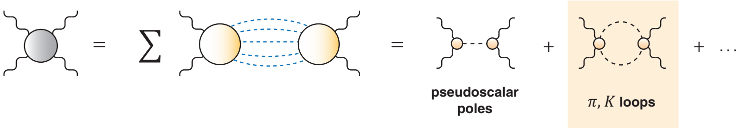


$\eta \to \gamma\gamma$ transition form factor



HLbL

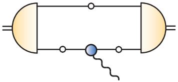
- Systematic treatment in dispersive approach, but significantly more complicated than HVP
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015), PRL 118 (2017), JHEP 04 (2017), Pauk & Vanderhaeghen, PRD 90 (2014), ...

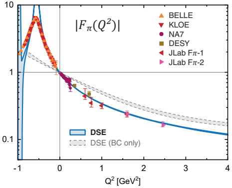


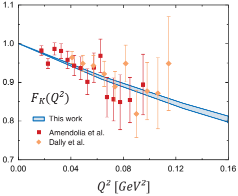
- DSEs and BSEs:

Depend on π, K electromagnetic FFs

GE, Fischer, Weil, Williams, PLB 799 (2019),
GE, Fischer, Williams, PRD 101 (2020)

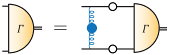


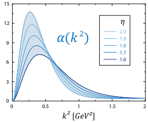




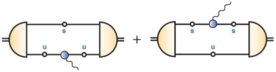
HLbL - Kaon box

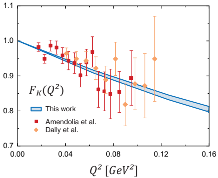
- Solve quark DSE & meson BSE in rainbow-ladder

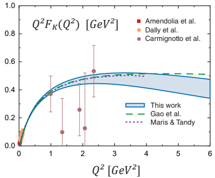




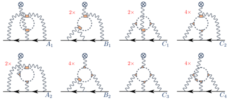
- Calculate quark-photon vertex in rainbow-ladder & kaon electromagnetic form factor







- Compute HLbL contribution, Meson box ↔ FsQED
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015)
  Kinoshita, Nizic, Okamoto, PRD 31 (1985)



- Kaon box is very small (~3% of pion box)
  GE, Fischer, Williams, PRD 101 (2020)

$$a_\mu^{\pi^\pm-\text{box}} = -15.7\,(2)(3)\times 10^{-11}$$

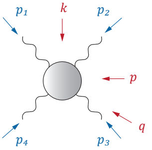
$$a_\mu^{K^\pm-\text{box}} = -0.48\,(2)(4)\times 10^{-11} \;\rightarrow\; -0.46(2)\times 10^{-11}$$

WP 20

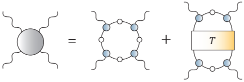
- Similar results from model calculations
  Bijnens, Pallante, Prades, Nucl. Phys. B 474 (1996),
  Hayakawa, Kinoshita, Sanda, PRL 75 (1995), PRD 54 (1996)

- Dispersive result
  Stamen, Hariharan, Hoferichter, Kubis, Stoffer, EPJC 82 (2022)

Structure of LbL amplitude



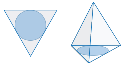
- Similar decomposition as for HVP
  GE, Fischer, PRD 85 (2012),
  Goecke, Fischer Williams, PRD 87 (2013)



\+ permutations
\+ 2 further topologies

- **136 tensors**, each dressing function depends on **6 variables**
  GE, Fischer, Heupel, PRD 92 (2015)

$$\Gamma^{\mu\nu\rho\sigma}(p,q,k) = \sum_{i=1}^{136} f_i(\mathcal{S}_0, \bigtriangledown, \Diamond)\, \tau_i^{\mu\nu\rho\sigma}(p,q,k)$$



relevant kinematic domain for g-2

- **Gauge invariance** ⇒ **41 tensors,** must find "minimal basis" (41 + 95), transverse projection of incomplete calculation can produce kinematic singularities
  Bardeen, Tung, Phys. Rev. 173 (1968), Tarrach, Nuovo Cim. A28 (1975), Drechsel et al., PRC 55 (1997),
  GE, Ramalho, PRD 98 (2018)

- Construct transverse basis from systematic power counting ⟹ 41 singlets
  GE, Fischer, Heupel, PRD 92 (2015)

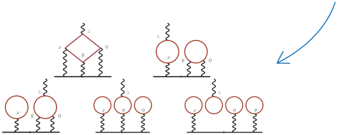
n	Seed element	#	Multiplets	$n=4$	$n=6$	$n=8$	$n=10$	$n=12$
4	$t_{12}^{\mu\nu}\, t_{34}^{\rho\sigma}$	3	$\mathcal{S}$, $\mathcal{D}_1$	1	1	1		
	$\varepsilon_{12}^{\mu\nu}\, \varepsilon_{34}^{\rho\sigma}$	3	$\mathcal{S}$, $\mathcal{D}_1$	1	1	1		
6	$\varepsilon_1^{\mu\lambda\alpha} t_{22}^{\alpha\nu}\, \varepsilon_3^{\rho\lambda\beta} t_{44}^{\beta\sigma}$	12	$\mathcal{S}$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{T}_2^+$, $\mathcal{T}_2^-$, $\mathcal{A}$		1	3	5	3
	$t_{12}^{\mu\nu}\, t_{33}^{\rho\lambda}\, t_{44}^{\lambda\sigma}$	6	$\mathcal{S}$, $\mathcal{D}_1$, $\mathcal{T}_1^+$		1	2	3	
	$t_{12}^{\mu\nu}\, t_{31}^{\rho\lambda}\, t_{24}^{\lambda\sigma}$	7	$\mathcal{S}$, $\mathcal{T}_1^+$, $\mathcal{T}_1^-$		1	1	3	2
	$\varepsilon_{12}^{\mu\nu}\, \varepsilon_{31}^{\rho\lambda}\, t_{24}^{\lambda\sigma}$	7	$\mathcal{D}_2$, $\mathcal{T}_2^+$, $\mathcal{T}_1^-$, $\mathcal{T}_2^-$			2	5	
8	$t_{12}^{\mu\nu}\, t_{31}^{\rho\alpha}\, t_{12}^{\alpha\beta}\, t_{24}^{\beta\sigma}$	3	$\mathcal{S}$, $\mathcal{D}_1$, $\mathcal{T}_1^+$			1	2	
	Total	41		2	5	11	18	5

7 equivalent seeds
in dispersive approach
Colangelo, Hoferichter, Procura,
Stoffer, JHEP 09 (2015)

HLbL - Lattice

- Difficult problem, QCD + QED, disconnected diagrams

  RBC/UKQCD: Blum, Christ, Hayakawa, Izubuchi, Jin, Jung, Lehner, PRL 124 (2020)
  Mainz: Chao, Hudspith, Gérardin, Green, Meyer, Ottnad, EPJ C 81 (2021)

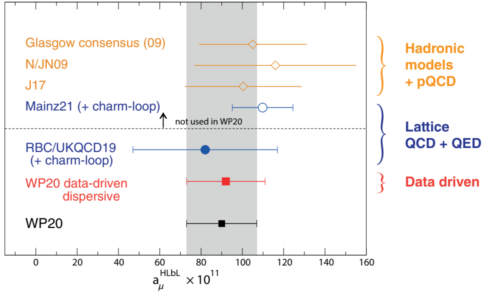


- Compatible with dispersive results

- Direct calculation of LbL amplitude in forward limit: pion pole dominance

  Gérardin, Green, Gryniuk, Hippel, Meyer, Pascalutsa, Wittig, PRD 98 (2018)

- HLbL likely may not explain discrepancy



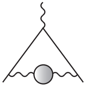
Colangelo et al., 2203.15810

Summary

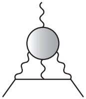
- Current g-2 status SM vs. experiment: 4.2 σ
  WP20: Aoyama et al., Phys. Rept. 887 (2020)
  FNAL Run-1: Abi et al., PRL 126, 141801 (2021)

- Experimental uncertainty to be improved:
  FermiLab (0.46 → 0.14 ppm), JPARC (0.45 ppm)

- SM uncertainty dominated by QCD



Hadronic vacuum polarization



Hadronic light-by-light scattering

- Current theory predictions mostly data driven, many lattice calculations underway

- HLbL seems increasingly unlikely to resolve discrepancy

- HVP: tension between lattice & R-ratio persists.
  New CMD-3 results → ?

$a_\mu \; [10^{-10}]$

Exp:	11 659 206.1	(4.1)
QED:	11 658 471.9	(0.0)
EW:	15.4	(0.1)
Hadronic:		
• VP (LO+HO)	684.5	(4.0)
• LBL	9.2	(1.8)
SM:	11 659 181.0	(4.3)
Diff:	25.1	(5.9)

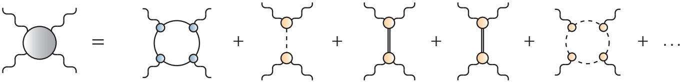
Thank you!

Backup slides

HLbL - Functional methods

LbL amplitude: model results

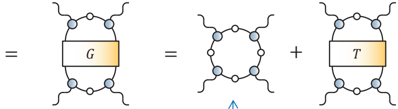
Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)



Quark loop	pseudoscalar exchange	scalar exchange	axialvector exchange	π, K loop	
2	8 ... 11	-1	2	-2	$(\times 10^{-10})$

Full expression:

GE, Fischer, PRD 85 (2012), Goecke, Fischer, Williams, PRD 87 (2013)



+ permutations

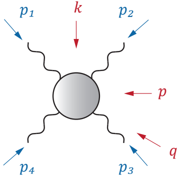
+ 2 further topologies (not in rainbow-ladder truncation)

nonperturbative **quark loop**

all **meson poles** (and more)

⇒ no double-counting! (same as for HVP)

Structure of LbL amplitude



3 independent momenta:

$$p = p_2 + p_3$$
$$q = p_3 + p_1$$
$$k = p_1 + p_2$$

6 Lorentz invariants:

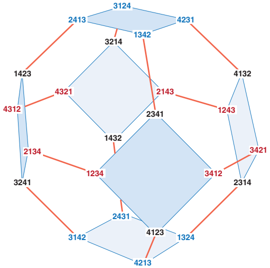
$$p^2, \quad q^2, \quad k^2, \quad p \cdot q, \quad p \cdot k, \quad q \cdot k$$

Bose symmetry:

$$\Gamma^{\mu\nu\rho\sigma}(p,q,k) = \sum_{i=1}^{136} f_i(\dots)\ \tau_i^{\mu\nu\rho\sigma}(p,q,k)$$

$\stackrel{!}{=}$ **symmetric**

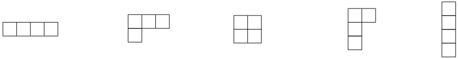
S4 multiplets



- Arrange in multiplets of **permutation group S4:**

GE, Fischer, Heupel, PRD 92 (2015)

Singlet	**Triplets**	**Doublets**	**Antitriplets**	**Antisinglet**
$\mathcal{S}$	$\mathcal{T}_i^+ = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}$	$\mathcal{D}_j = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}$	$\mathcal{T}_i^- = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}$	$\mathcal{A}$



- **6 Lorentz invariants** form singlet $\mathcal{S}_0$, doublet $\mathcal{D}$, triplet $\mathcal{T}^+$

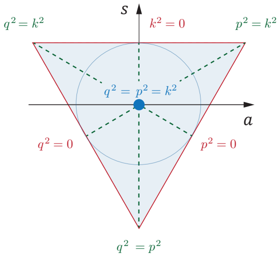
Structure of LbL amplitude

- **Singlet:** symmetric variable, carries overall scale:

$$\mathcal{S}_0 = \frac{p^2+q^2+k^2}{4} = \frac{p_1^2+p_2^2+p_3^2+p_4^2}{4}$$

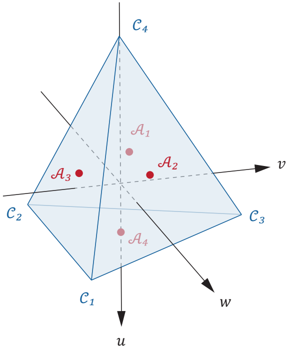
- **Doublet:** $\mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}$

  Mandelstam triangle,
  2-photon poles (pion, scalar, axialvector, ...)



- **Triplet:** $\mathcal{T} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

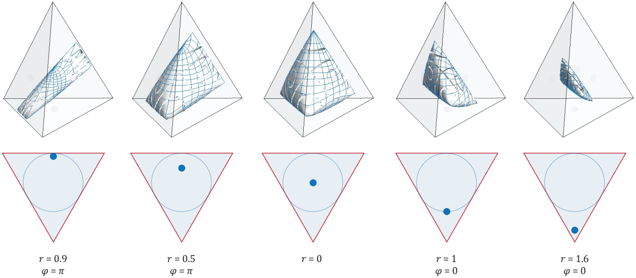
  tetrahedron bounded by $p_i^2 = 0$,
  vector-meson poles

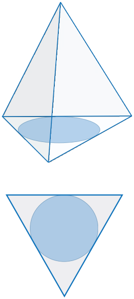


GE, Fischer, Heupel, PRD 92 (2015)

Structure of LbL amplitude

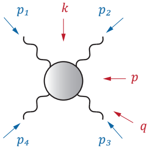
Fixed doublet variables ⟹ complicated geometric object inside tetrahedron:





relevant for g-2

Structure of LbL amplitude



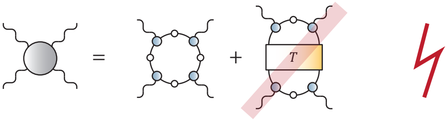
- **136 tensors,** each dressing function depends on **6 variables**
  GE, Fischer, Heupel, PRD 92 (2015)

$$\Gamma^{\mu\nu\rho\sigma}(p,q,k) = \sum_{i=1}^{136} f_i\,(\mathcal{S}_0, \triangledown, \triangle)\,\tau_i^{\mu\nu\rho\sigma}(p,q,k)$$

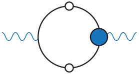
- Bose-symmetric ⇒ with symmetric tensors,
  dressings only depend on symmetric combinations

- **Gauge invariance** ⇒ transverse to $p_1{}^\mu,\ p_2{}^\nu, p_3{}^\rho, p_4{}^\sigma,$ **41 tensors**

Cannot do naive transverse projection:
incomplete calculation may break gauge invariance,
leads to **kinematic singularities**



Analogy: HVP



$= a(Q^2)\,\delta^{\mu\nu} + b(Q^2)\,Q^\mu Q^\nu$

- **Analyticity** ⇒ a, b cannot have poles at $Q^2 = 0$ (no intermediate massless particle)
- **Transversality** ⇒ Ward identity:
  $Q^\mu \Pi^{\mu\nu}(Q) = 0 \quad \Rightarrow \quad a = -b\,Q^2$

$$\Rightarrow \quad \Pi^{\mu\nu}(Q) = \underbrace{\Pi(Q^2)\,(Q^2\,\delta^{\mu\nu} - Q^\mu Q^\nu)}_{\text{transverse part}} + \underbrace{\widetilde{\Pi}(Q^2)\,\delta^{\mu\nu}}_{\text{„gauge part" = 0 by gauge invariance}}$$

transverse part → **"minimal basis"** - no kinematic dependencies at $Q^2 = 0$

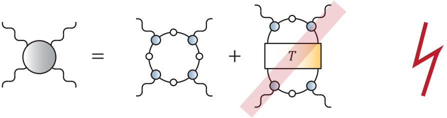
„gauge part" = 0 by gauge invariance

- Transverse projection:

$$\Rightarrow \left[\Pi(Q^2) + \frac{\widetilde{\Pi}(Q^2)}{Q^2}\right] (Q^2\,\delta^{\mu\nu} - Q^\mu Q^\nu)$$

we need value at $Q^2 = 0$, but **kinematic singularity**

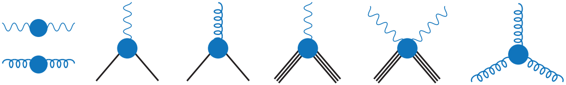
- No problem for HVP (gauge invariance preserved in truncation), but can be broken for HLbL due to incomplete calculation



How to do this for 136 tensors with multiple (potentially dangerous) kinematic limits?

Minimal bases

- Many known examples:



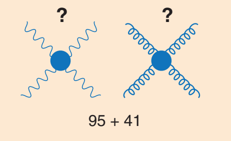
1 + 1 | 4 + 8 | 4 + 8 | (depends) | 14 + 18 | 10 + 4

Gauge + Transverse

Bardeen, Tung, Phys. Rev. 173 (1968)
Tarrach, Nuovo Cim. A28 (1975)
Ball, Chiu, PRD 22 (1980)
Drechsel et al., PRC 55 (1997)

- Systematic construction of minimal bases
  GE, Ramalho, PRD 98 (2018)

- If no **dynamical** longitudinal poles, minimal basis should exist (?)



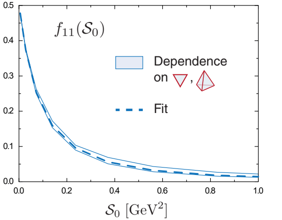
Systematic derivation extremely hard!

Minimal bases

- Alternative: construct transverse basis for LbL by systematic power counting $\Longrightarrow$ **41 singlets:**
  GE, Fischer, Heupel, PRD 92 (2015)

n	Seed element	#	Multiplets	$n=4$	$n=6$	$n=8$	$n=10$	$n=12$
4	$t_{12}^{\mu\nu}\, t_{34}^{\rho\sigma}$	3	$\mathcal{S},\, \mathcal{D}_1$	1	1	1		
	$\varepsilon_{12}^{\mu\nu}\, \varepsilon_{34}^{\rho\sigma}$	3	$\mathcal{S},\, \mathcal{D}_1$	1	1	1		
6	$\varepsilon_1^{\mu\lambda\alpha} t_{22}^{\alpha\nu}\, \varepsilon_3^{\rho\lambda\beta} t_{44}^{\beta\sigma}$	12	$\mathcal{S},\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{T}_2^+,\, \mathcal{T}_2^-,\, \mathcal{A}$		1	3	5	3
	$t_{12}^{\mu\nu}\, t_{33}^{\rho\lambda}\, t_{44}^{\lambda\sigma}$	6	$\mathcal{S},\, \mathcal{D}_1,\, \mathcal{T}_1^+$		1	2	3	
	$t_{12}^{\mu\nu}\, t_{31}^{\rho\lambda}\, t_{24}^{\lambda\sigma}$	7	$\mathcal{S},\, \mathcal{T}_1^+,\, \mathcal{T}_1^-$		1	1	3	2
	$\varepsilon_{12}^{\mu\nu}\, \varepsilon_{31}^{\rho\lambda}\, t_{24}^{\lambda\sigma}$	7	$\mathcal{D}_2,\, \mathcal{T}_2^+,\, \mathcal{T}_1^-,\, \mathcal{T}_2^-$			2	5	
8	$t_{12}^{\mu\nu}\, t_{31}^{\rho\alpha}\, t_{12}^{\alpha\beta}\, t_{24}^{\beta\sigma}$	3	$\mathcal{S},\, \mathcal{D}_1,\, \mathcal{T}_1^+$			1	2	
	Total	41		2	5	11	18	5

- With minimal bases, momentum dependencies become **maximally simple** (only physical poles & cuts): singlet dressings scale with $\mathcal{S}_0$



- 7 equivalent seeds in dispersive approach:
  Colangelo, Hoferichter, Procura, Stoffer, JHEP 09 (2015)

$$\varepsilon_{12}^{\mu\nu}\, \varepsilon_{34}^{\rho\sigma}, \quad t_{12}^{\mu\nu}\, t_{34}^{\rho\sigma},$$

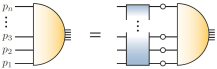
$$t_{12}^{\mu\nu}\, t_{31}^{\rho\lambda}\, t_{14}^{\lambda\sigma}, \quad t_{12}^{\mu\nu}\, t_{31}^{\rho\lambda}\, t_{24}^{\lambda\sigma}$$

$$t_{12}^{\mu\nu}\, t_{312}^{\rho}\, t_{412}^{\sigma}, \quad t_{134}^{\mu}\, t_2^{\nu\alpha\beta}\, t_3^{\rho\alpha\lambda}\, t_4^{\sigma\beta\lambda}, \quad (t_{14}^{\mu\alpha}\, t_{32}^{\beta\nu} - t_{13}^{\mu\beta}\, t_{42}^{\alpha\nu})\, t_3^{\rho\alpha\lambda}\, t_4^{\sigma\beta\lambda}$$

$$t_{ab}^{\mu\nu} = a\cdot b\, \delta^{\mu\nu} - b^\mu\, a^\nu$$
$$\varepsilon_{ab}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\, a^\alpha\, b^\beta$$

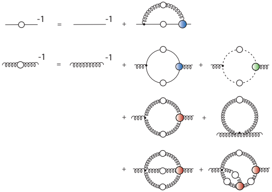
Functional methods

- Hadronic **bound-state equations** (BSEs, Faddeev eqs, ...)

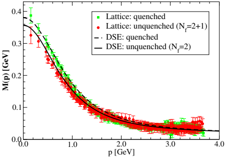


"QFT analogue of Schrödinger eq."
→ hadron masses & "wave functions"
→ **spectroscopy calculations**

- Ingredients: **QCD's n-point functions,** Satisfy Dyson-Schwinger equations (**DSEs**): QCD's quantum eqs. of motion

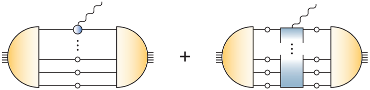


...



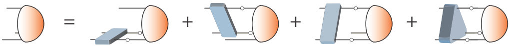
→ running **quark mass**

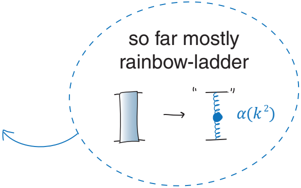
- Structure calculations: form factors, PDFs, GPDs, two-photon processes, ...

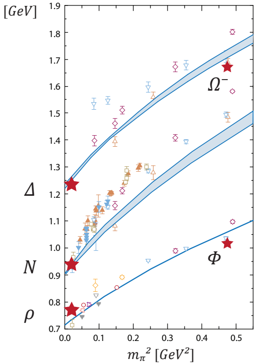


Baryons

- Covariant **3-quark Faddeev equation** for baryons
  GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)



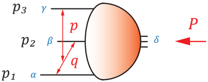




- Octet and decuplet spectra, elastic and transition form factors, ...
  GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016), 1606.09602

- Heavy baryons
  Qin, Roberts, Schmidt, PRD 97 (2018), FBS 60 (2019)

- Keep full structure of baryon's Faddeev amplitude:

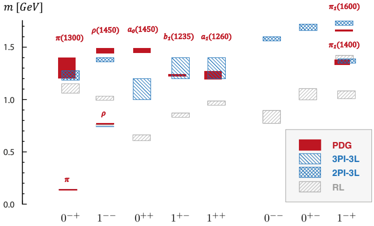


$$\Psi_{\alpha\beta\gamma\delta}(p,q,P) = \sum_i f_i(p^2, q^2, p\cdot q, p\cdot P, q\cdot P)\;\tau_i(p,q,P)_{\alpha\beta\gamma\delta}$$

Lorentz-invariant dressing functions

Dirac-Lorentz tensors: 64 (128) for spin 1/2 (3/2)

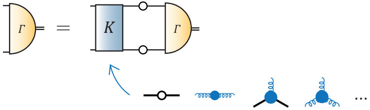
Towards ab-initio



Williams, Fischer, Heupel, PRD 93 (2016)

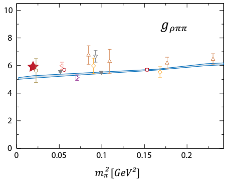
Beyond rainbow-ladder calculations improve meson spectrum, but π, K, ρ etc. stable

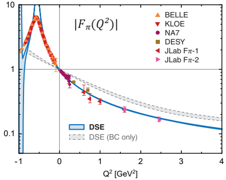
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016)

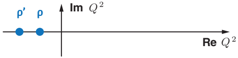


Residue at pole = $g_{\rho\pi\pi}$

Mader, GE, Blank, Krassnigg, PRD 84 (2011)



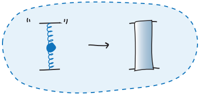




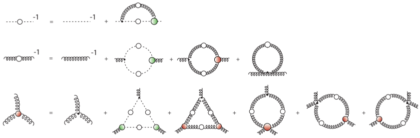
Pion form factor:
Absence of width has no visible effect on spacelike behavior

GE, Fischer, Weil, Williams, PLB 797 (2019)

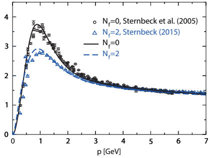
Towards ab-initio



- Compute higher n-point functions from **DSEs, FRG, lattice QCD**
  Binosi, Ibanez, Papavassiliou, JHEP 09 (2014),
  GE, Williams, Alkofer, Vujinovic, PRD 89 (2014),
  Williams, EPJA 51 (2015), Huber, PRD 101 (2020),
  Cyrol, Mitter, Pawlowski, Strodthoff, PRD 97 (2018),
  Oliveira, Silva, Skullerud, Sternbeck, PRD 99 (2019), ...



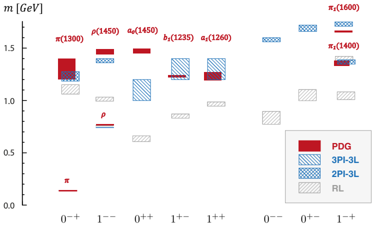
- Kernels from **chiral symmetry** constraints
  Chang, Roberts, PRL 103 (2009), Chang, Liu, Roberts, PRL 106 (2001),
  Qin, Roberts, 2009.13637



Gluon propagator:
DSE vs. lattice
Williams, Fischer, Heupel, PRD 93 (2016)

See also: Aguilar, De Soto, Ferreira, Papavassiliou, Rodriguez-Quintero, Zafeiropoulos, EPJ C 80 (2020)

Beyond rainbow-ladder calculations improve **light-meson spectrum**
Williams, Fischer, Heupel, PRD 93 (2016)



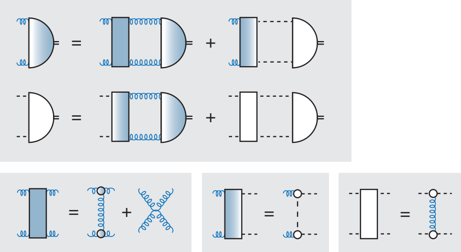
Towards ab-initio

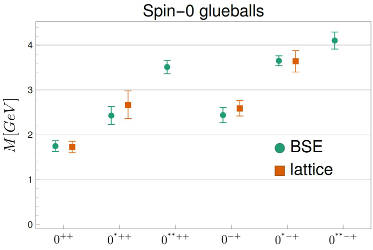
- **Glueballs** in pure Yang-Mills theory (no quarks)

  Meyers, Swanson, PRD 87 (2013)
  Sanchis-Alepuz, Fischer, Kellermann, Smekal, PRD 92 (2015)
  Souza, Ferreira, Aguilar, Papavassiliou, Roberts, Xu, EPJA 56 (2020)

Kernel derived from n-point functions,
no model input

Huber, Fischer, Sanchis-Alepuz, EPJ C 80 (2020)

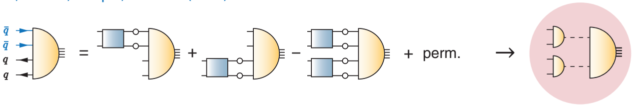




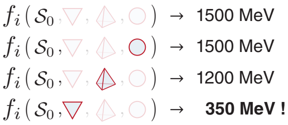
Lattice:
Morningstar, Peardon, PRD 60 (1999)
Chen, Alexandru, Dong, Draper, Horvath, PRD 73 (2006)
Athenodorou, Teper, JHEP 11 (2020)

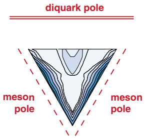
Four-quark states

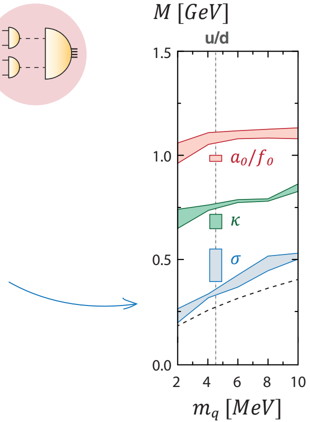
- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states:**
  GE, Fischer, Heupel, PLB 753 (2016)



- BSE dynamically generates **meson poles** in BS amplitude:







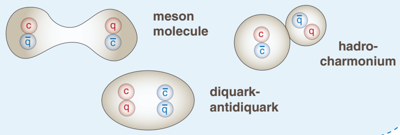
- "Light scalar mesons" look like **meson molecules,** diquark-antidiquark components almost negligible

- Lightness is inherited from pseudoscalar Goldstone bosons!

Four-quark states

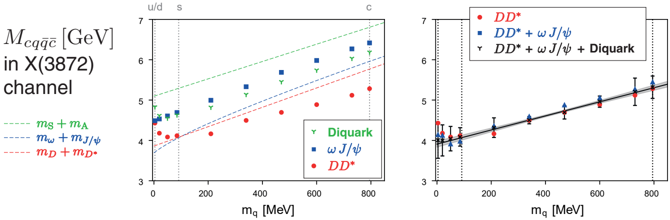
- Heavy-light **four-quark states:** what is their internal decomposition?

$cq\bar{q}\bar{c}$ →



- **Four-quark BSE:** all mix together

$M_{cq\bar{q}\bar{c}}$ [GeV] in X(3872) channel



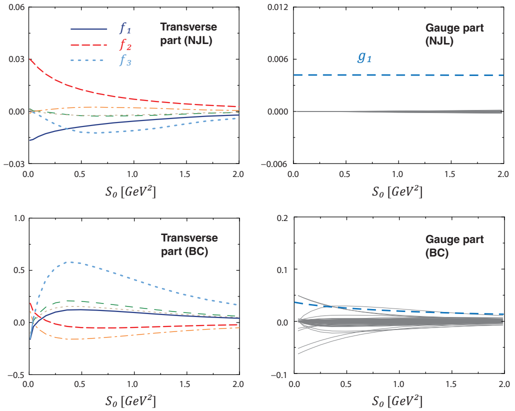
Wallbott, GE, Fischer, PRD 100 (2019), PRD 102 (2020)

$cq\bar{q}\bar{c}$ → strong meson-meson component: DD* for X(3872), Zc(3900):

$cc\bar{q}\bar{q}$ → diquarks also play role

Minimal bases

- **Quark loop for HLbL:** Project on transverse (derived) + gauge (conjecture)
  GE, Fischer, Heupel, Williams, AIP Conf. Proc. 1701 (2016)



NJL Model:
works well

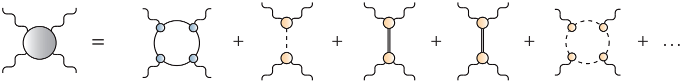
DSE (Ball-Chiu vertex):
gauge part likely
not yet fully consistent

⇒ still under construction

HLbL - Functional methods

LbL amplitude: model results

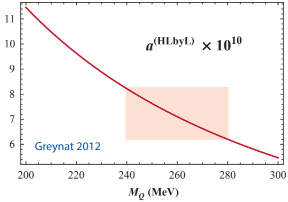
Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)



Quark loop	pseudoscalar exchange	scalar exchange	axialvector exchange	π, K loop	
2	8 ... 11	-1	2	-2	$(\times 10^{-10})$

How important is the **quark loop?**

- **Constituent quark loop** known analytically: 6 ... 8

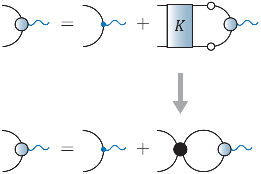


- **ENJL:** VM poles by summing up quark bubbles

Bijnens 1995

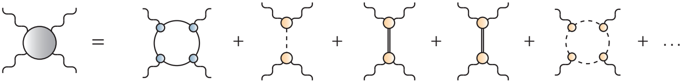
$$\gamma^\mu - \frac{1}{Q^2+m_V^2}\, t^{\mu\nu}_{QQ}\,\gamma^\nu$$

Large reduction: 2



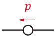
HLbL - Functional methods

LbL amplitude: model results
Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)



How important is the **quark loop?**

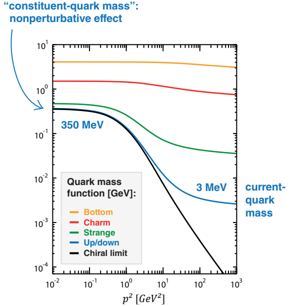
- Quark mass is not a constant:



$$S_0(p) = \frac{-i\not{p}+m}{p^2+m^2} \longrightarrow S(p) = \frac{1}{A(p^2)}\,\frac{-i\not{p}+M(p^2)}{p^2+M^2(p^2)}$$

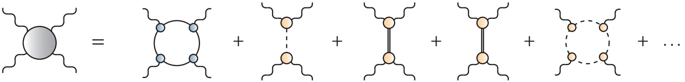
Dynamical mass generation
through spontaneous chiral symmetry breaking

Cloet, Roberts, Prog. Part. Nucl. Phys. 77 (2014)
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog. Part. Nucl. Phys. 91 (2016)
Cyrol, Mitter, Pawlowski, Strodthoff, PRD 97 (2018)



HLbL - Functional methods

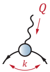
LbL amplitude: model results
Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)



Quark loop	pseudoscalar exchange	scalar exchange	axialvector exchange	π, K loop	
2	8 ... 11	−1	2	−2	$(\times 10^{-10})$

How important is the **quark loop?**

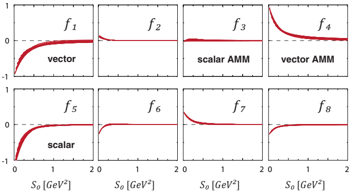
- Quark-photon vertex is not bare:



$$\Gamma^\mu(k,Q) = \left[i\gamma^\mu \Sigma_A + 2k^\mu (i\not{k}\, \Delta_A + \Delta_B) \right] + \left[i \sum_{j=1}^{8} f_j \, \tau_j^\mu(k,Q) \right]$$

Ball-Chiu vertex, determined by WTI, depends only on quark propagator
Ball, Chiu, PRD 22 (1980)

Transverse part, contains dynamics (VM poles, cuts, ...), 8 dressing functions

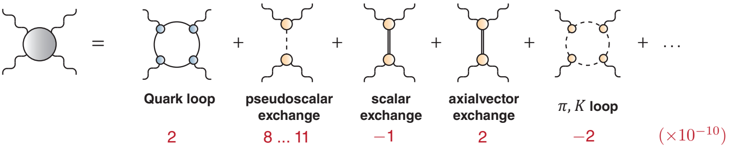


GE, Sternbeck, in preparation

HLbL - Functional methods

LbL amplitude: model results

Jegerlehner, Nyffeler, Phys. Rept. 477, 1 (2009)



How important is the **quark loop?**

- **DSE result** for quark loop:
  Goecke, Fischer, Williams, PRD 87 (2013)
  $a_\mu = 10.7 \times 10^{-10}$
- However, incomplete calculation (quark loop only) can induce artifacts from gauge invariance
- Can we calculate **full** four-point function?

$A(p^2)$	$M(p^2)$	γ^μ	Γ_T^μ	$a_\mu\ [10^{-10}]$
1	0.2 GeV	1	0	10
1	$M(p^2)$	1	0	10
$A(p^2)$	$M(p^2)$	1	0	5
$A(p^2)$	$M(p^2)$	Σ_A	0	10
$A(p^2)$	$M(p^2)$	Σ_A	$k=0$	4
$A(p^2)$	$M(p^2)$	Σ_A	Full	10

Electron vs. muon g-2

$a_e\ [10^{-10}]$

Exp:	11 596 521.81	(0.01)	
QED:	11 596 521.71	(0.09)	Cs
	11 596 521.81	(0.08)	Rb
EW:	0.00	(0.00)	
Hadronic:	0.02	(0.00)	
SM:	11 596 521.73	(0.09)	Cs
	.83	(0.08)	Rb

Theory error dominated by **QED**

$a_\mu\ [10^{-10}]$

Exp:	11 659 206.1	(4.1)
QED:	11 658 471.9	(0.0)
EW:	15.4	(0.1)
Hadronic:		
• VP (LO+HO)	684.5	(4.0)
• LBL	9.2	(1.8)
SM:	11 659 181.0	(4.3)
Diff:	25.1	(5.9)

Theory error dominated by **QCD**

FNAL Run-1: Abi et al., PRL 126, 141801 (2021)
WP20: Aoyama et al., Phys. Rept. 887 (2020)
Jegerlehner, Nyffeler, Phys. Rept. 477 (2009)