UV/IR Mixing, EFTs, Hidden Cancellations, and Origami:
Calculating the Higgs Mass and Running of Gauge Couplings in String Theory

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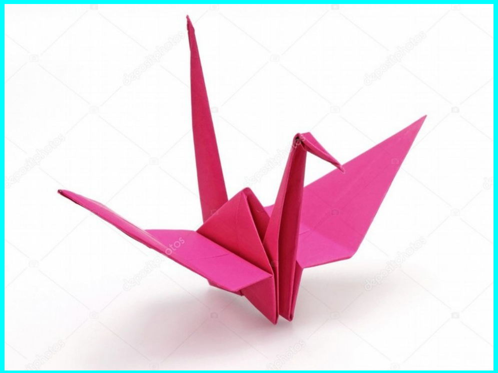
- S. Abel, KRD: `2106.04622`
- S. Abel, KRD, L. Nutricati: `2302.nnnnn`

Also partly based on a long line of earlier work:

- KRD, hep-th/9402006 (Nucl.Phys.B)
- KRD, M. Moshe, & R.C. Myers, hep-th/9503055 (Phys.Rev.Lett.)
- KRD, hep-ph/0104274 (Nucl.Phys.B)

CERN Exotic Naturalness Workshop
30 January 2023

This talk is dedicated to exploring a circle of ideas that are all connected to exotic approaches to naturalness...



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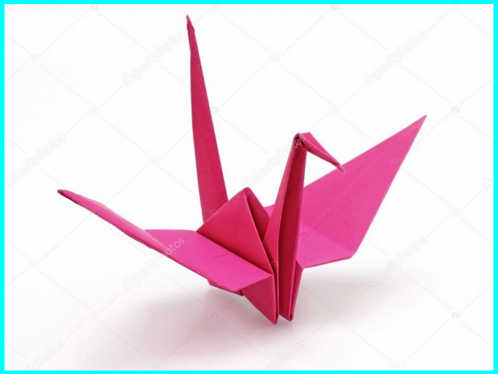

Naturalness concerns the existence and stability of widely separated energy scales with respect to quantum corrections

$\Rightarrow$ relations between UV and IR physics.

Therefore natural to explore ideas in which these interact and mix!

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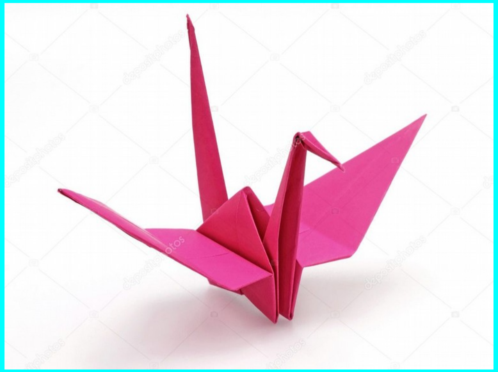

UV/IR mixing & associated symmetries

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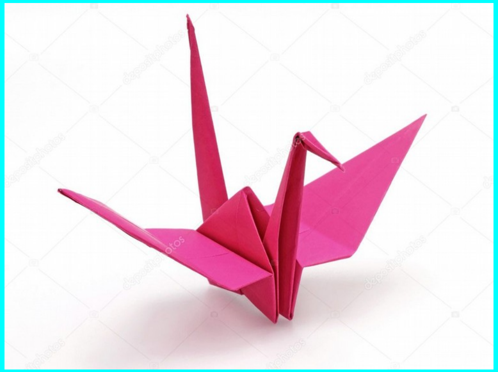

UV/IR mixing & associated symmetries

Because these UV/IR symmetries involve physics at all scales, they will not be readily apparent to low-energy observers.

In such cases, the resulting finiteness would appear to be the result of "hidden" cancellations! Coefficients of dangerous terms would "magically" be zero, akin to SUSY supertrace relations but *without* SUSY!

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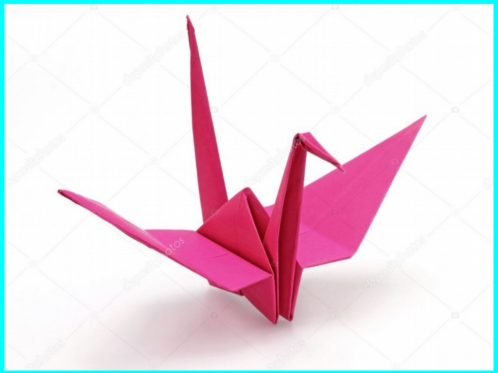

UV/IR mixing &
associated symmetries

Finiteness through
“hidden” cancellations

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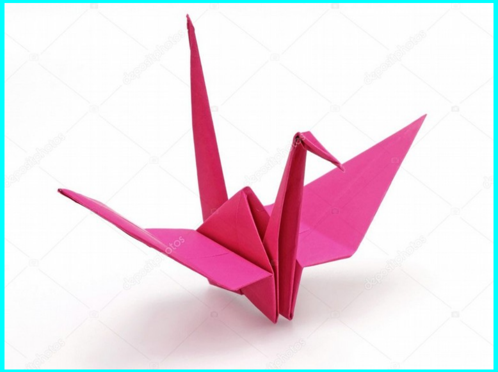


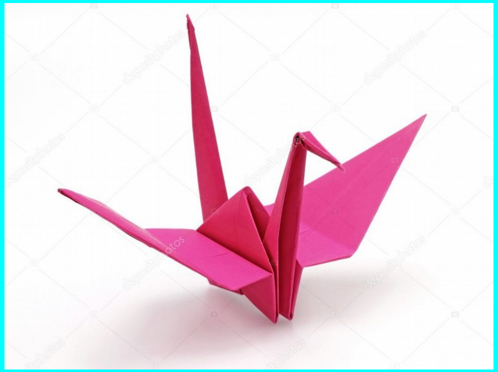
UV/IR mixing & associated symmetries

Finiteness through "hidden" cancellations

Because these cancellations are the result of (and protected by) UV/IR-mixed symmetries, they rely on conspiracies between physics *at all scales simultaneously*.

Such theories therefore have rich physics populating *all* scales, including infinite towers of states.

This talk is dedicated to exploring a circle of ideas that are all connected to exotic approaches to naturalness...



UV/IR mixing & associated symmetries

Finiteness through “hidden” cancellations

Infinite towers of states

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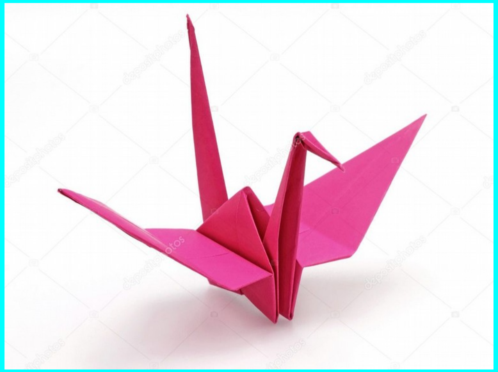

UV/IR mixing & associated symmetries

Finiteness through "hidden" cancellations

Infinite towers of states

Finally, if these theories have UV/IR mixing and achieve finiteness through conspiracies involving physics at all scales simultaneously, to what extent do they have low-energy EFT descriptions?

Is it possible to extract an EFT from such a theory?
What role would it play, and how could it be interpreted?

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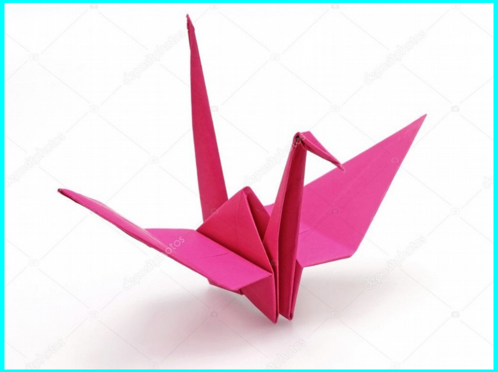

UV/IR mixing & associated symmetries

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Infinite towers of states

Extraction / role of EFTs

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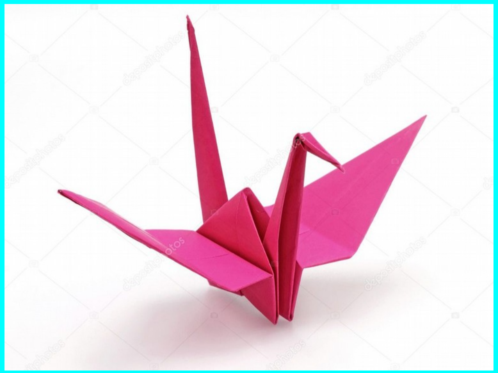

UV/IR mixing & associated symmetries

Finiteness through “hidden” cancellations

This is the circle of ideas we will be studying. But if we are talking about UV/IR-mixed theories, it is critical we study such ideas within frameworks that are UV-complete...

Infinite towers of states

Extraction / role of EFTs

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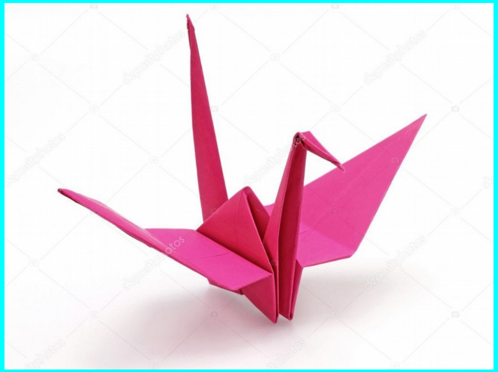

UV/IR mixing & associated symmetries

Finiteness through "hidden" cancellations

UV completion:
String theory

Infinite towers of states

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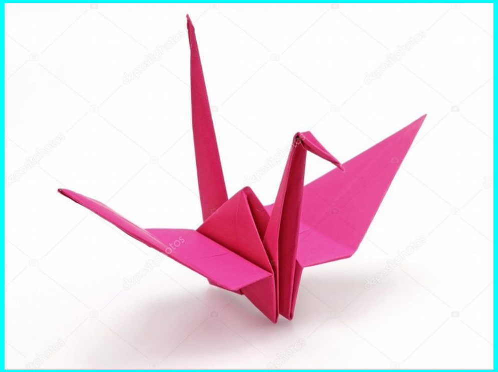

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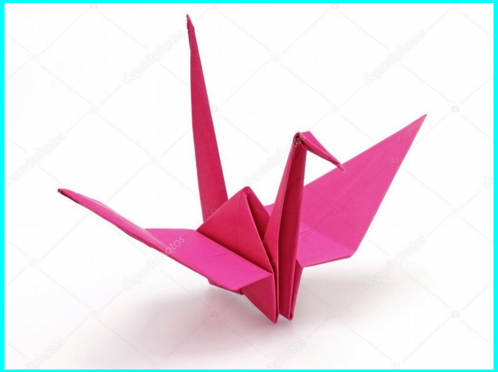

UV/IR mixing & associated symmetries

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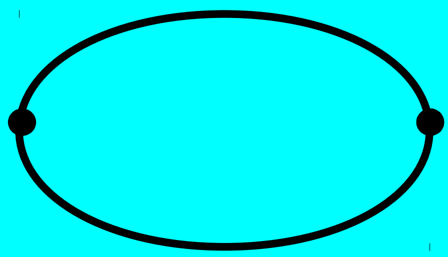
Extraction / role of EFTs

In this talk, we shall begin by discussing the main ideas in all generality. We will then illustrate their power by calculating the **Higgs mass** and **gauge-coupling running** in string theory --- all while keeping track of the full UV/IR string symmetries and complete towers of string states.

Let's start our story by examining the one-loop CW effective potential in field theory.



$$\Lambda = \frac{1}{2}\sum_n (-1)^{F_n}\, g_n \int \frac{d^D p}{(2\pi)^D}\, \log(p^2 + M_n^2)$$

Summation over spectrum

number of states
(# bosons - # fermions)
=0 for SUSY

It turns out that the best way to connect to these ideas is through the ***Schwinger worldline formalism***.

- Purely field-theoretic formalism
- Calculate *A(x,y,t)* = amplitude for particle to move from *x* to *y* within a fixed (proper) time *t*. "Schwinger proper time"
- In some sense, *t* is the total "length" of the worldline around the bubble.
- Total propagator Δ(x,y) is then the *integral* of this amplitude *A(t)* over all *t*.

Algebraically, this amounts to using the identity

$$\log\, x = \int_1^x \frac{dy}{y} = \int_1^x dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t} e^{-xt} + \ldots$$

and then dropping the *x*-independent term

We thus obtain

$$\Lambda = -\frac{1}{2}\sum_n (-1)^{F_n} g_n \int \frac{d^D p}{(2\pi)^D} \int \frac{dt}{t} e^{-(p^2+M_n^2)t}$$

perform p-integrations

$$= -\frac{1}{2}\frac{1}{(4\pi)^{D/2}}\sum_n (-1)^{F_n} g_n \int_0^\infty \frac{dt}{t^{1+D/2}} e^{-M_n^2 t}$$

But t has mass dimension -2.
Make dimensionless:

$$t \to \frac{\pi}{\mu^2}\hat{t}_2$$

(μ = arbitrary scale)

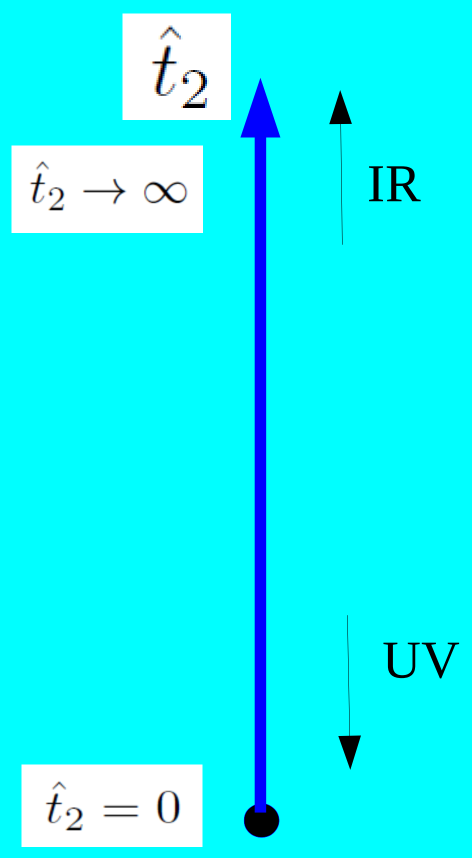
$$= -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_0^\infty \frac{d\hat{t}_2}{\hat{t}_2} \frac{1}{\hat{t}_2^{D/2}} \sum_n (-1)^{F_n} g_n e^{-\pi(M_n^2/\mu^2)\hat{t}_2}$$

Thus μ sets scale of Λ.

Identify as "partition function" with $(-1)^F$ statistics weighting

$$= -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_0^\infty \frac{d\hat{t}_2}{\hat{t}_2} Z(\hat{t}_2)$$

We thus have $$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_0^\infty \frac{d\hat{t}_2}{\hat{t}_2}\, Z(\hat{t}_2)$$



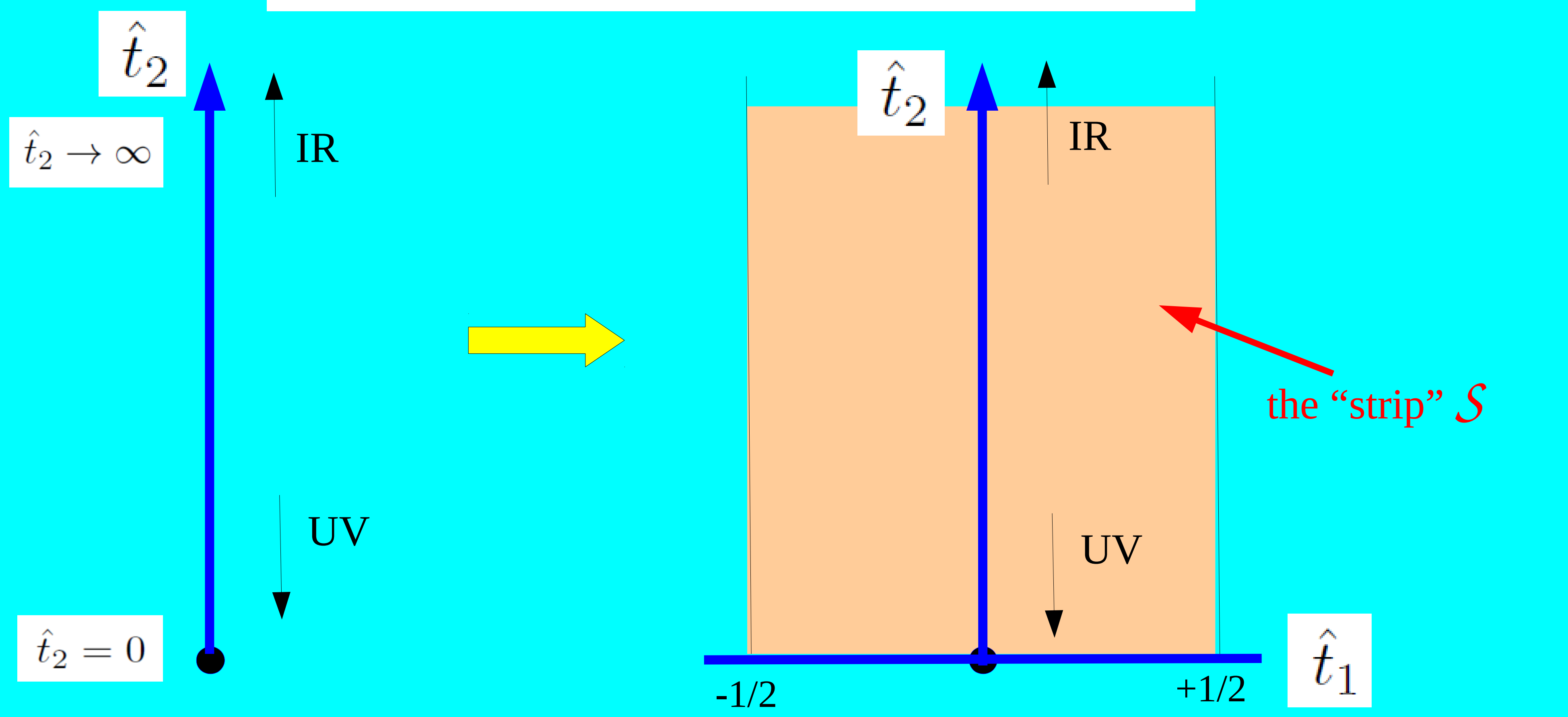
We thus have

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_0^\infty \frac{d\hat{t}_2}{\hat{t}_2}\, Z(\hat{t}_2)$$

__Final step__: With an eye towards an eventual connection to string theory, let's introduce a dummy variable and enlarge our region of integration.

$$= -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \boxed{\int_{-1/2}^{1/2} d\hat{t}_1} \int_0^\infty \frac{d\hat{t}_2}{\hat{t}_2}\, Z(\hat{t}_2)$$

$$= -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

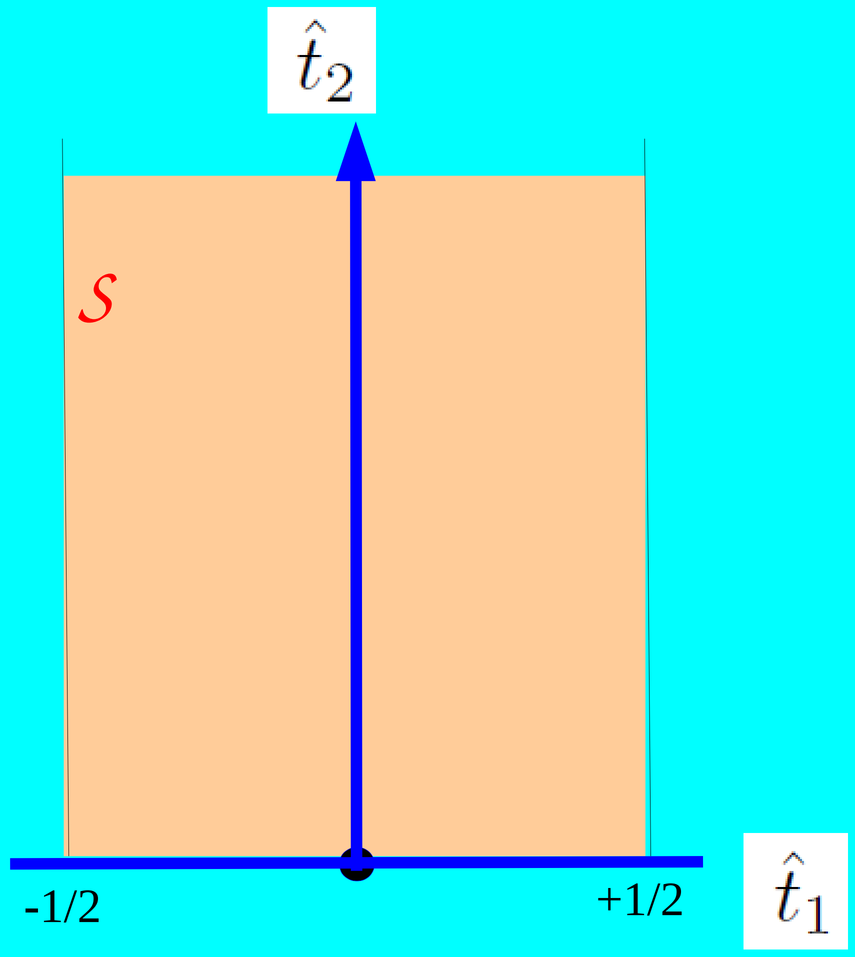


Thus, within ordinary QFT, we have

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

where

$$Z(\hat{t}_2) = \frac{1}{\hat{t}_2^{D/2}} \sum_{\text{states}} (-1)^F e^{-\pi M^2 \hat{t}_2/\mu^2}$$



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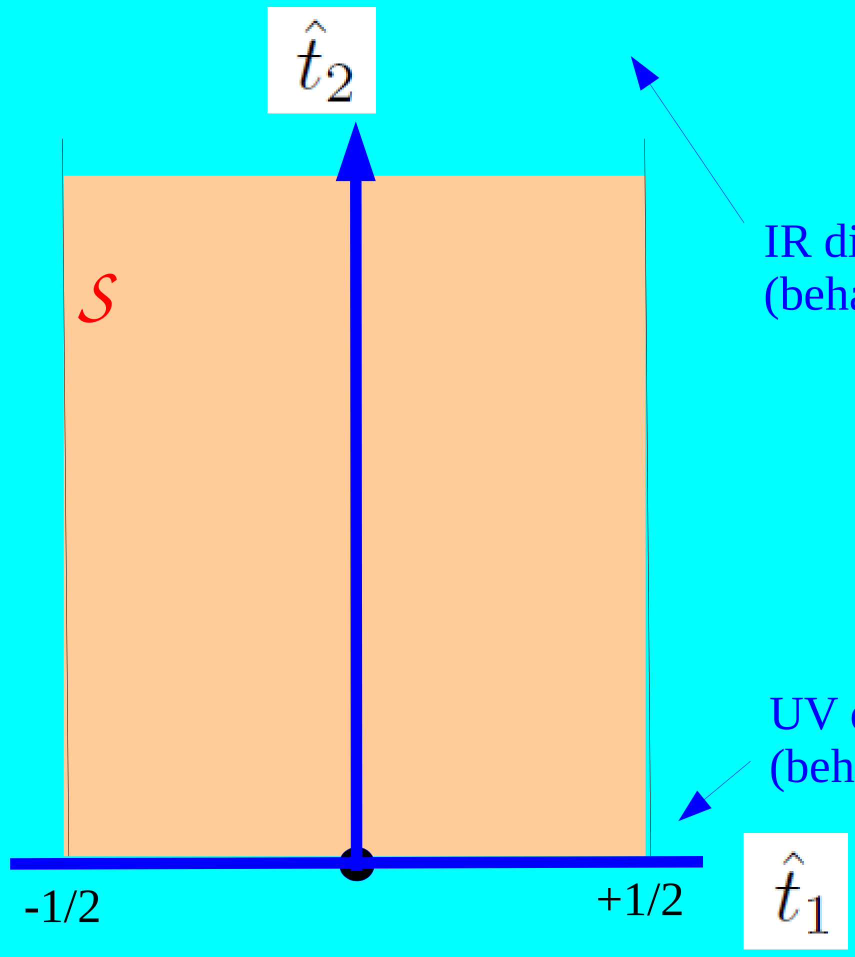
- IR divergences as $t_2 \to$ infinity
  (lightest states dominate)
- UV divergences as $t_2 \to 0$
  (opposite limit: all states contribute equally)



IR divergences arise here
(behavior of Z as $t_2 \to$ infinity)

UV divergences arise here
(behavior of Z as $t_2 \to 0$)

How to handle divergences?

Thus, within ordinary QFT, we have

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

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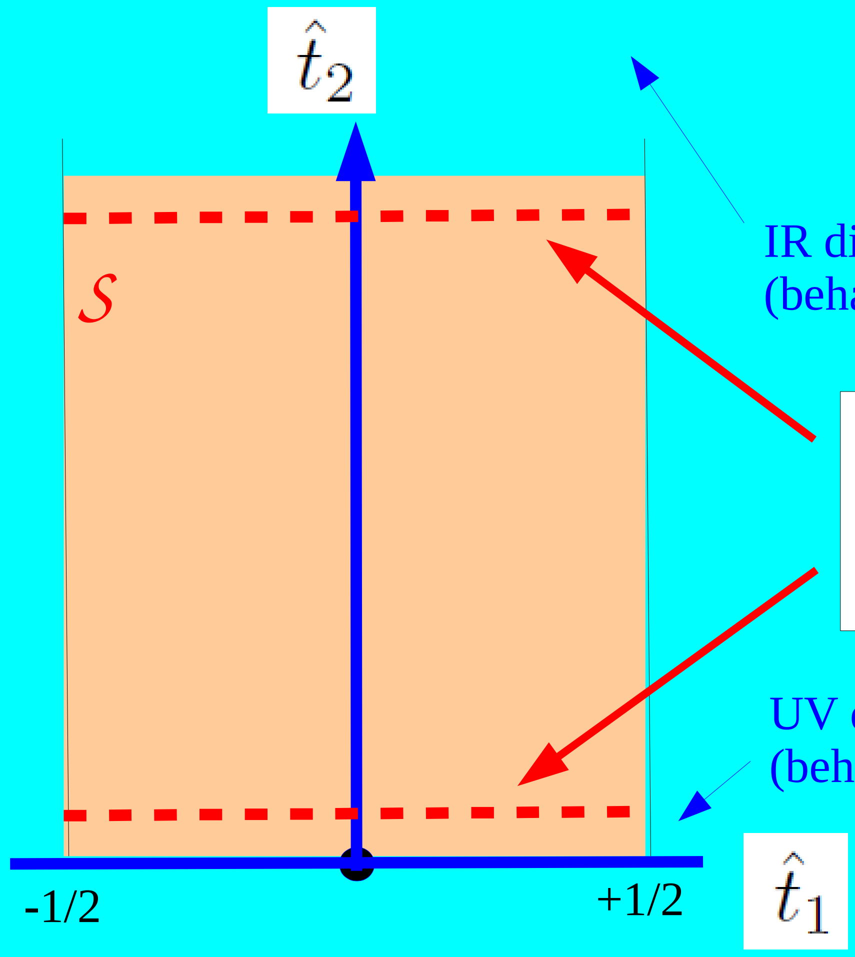


IR divergences arise here
(behavior of Z as $t_2 \to$ infinity)

IR cutoff: $t_2 \leq \Lambda_{\text{IR}}^{-2}$

UV cutoff: $t_2 \geq \Lambda_{\text{UV}}^{-2}$

introduce either cutoff as needed → new mass scales

UV divergences arise here
(behavior of Z as $t_2 \to 0$)

Thus far, we have stayed within traditional QFT.
But now let's ask a hypothetical question:

What if our theory had an exact symmetry under

$$\hat{t}_2 \;\to\; \frac{1}{\hat{t}_2}$$

?

Such a symmetry is clearly not field-theoretic!
But let's pursue this anyway.

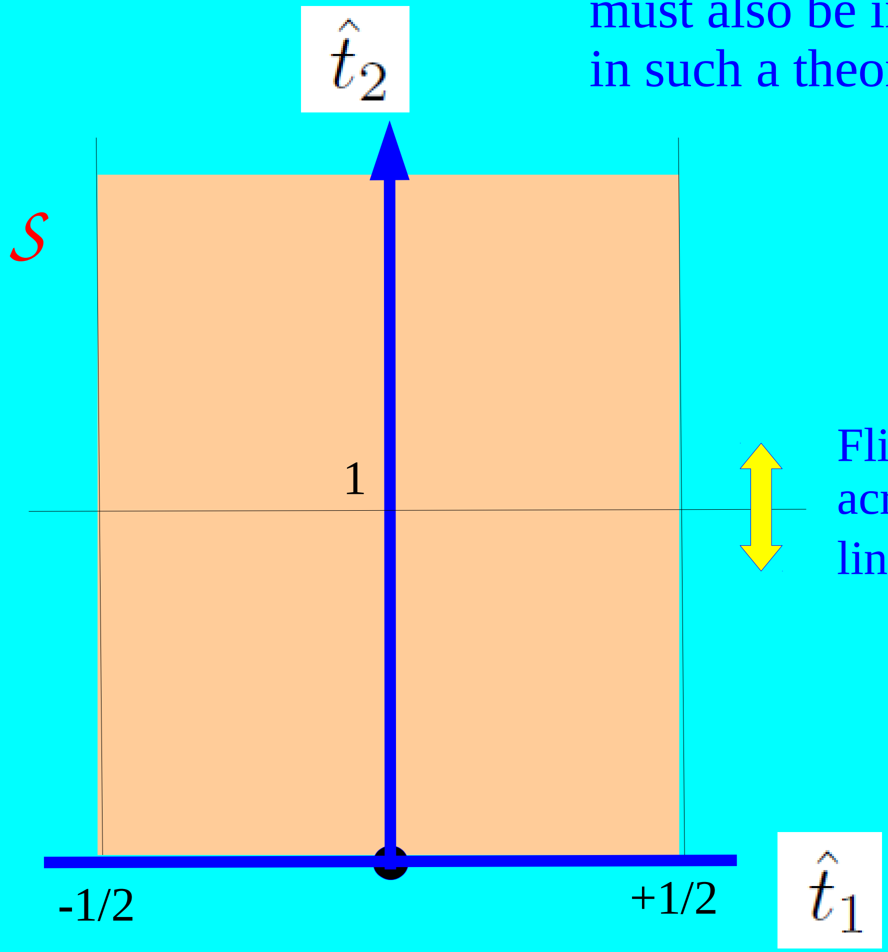
- What effects would this have?
- How could we interpret this?

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

where

$$Z(\hat{t}_2) = \frac{1}{\hat{t}_2^{D/2}} \sum_{\text{states}} (-1)^F e^{-\pi M^2 \hat{t}_2/\mu^2}$$

- Strip is invariant
- Measure is invariant
- Thus, partition function must also be invariant in such a theory!



Flip symmetry across t_2=1 line!

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

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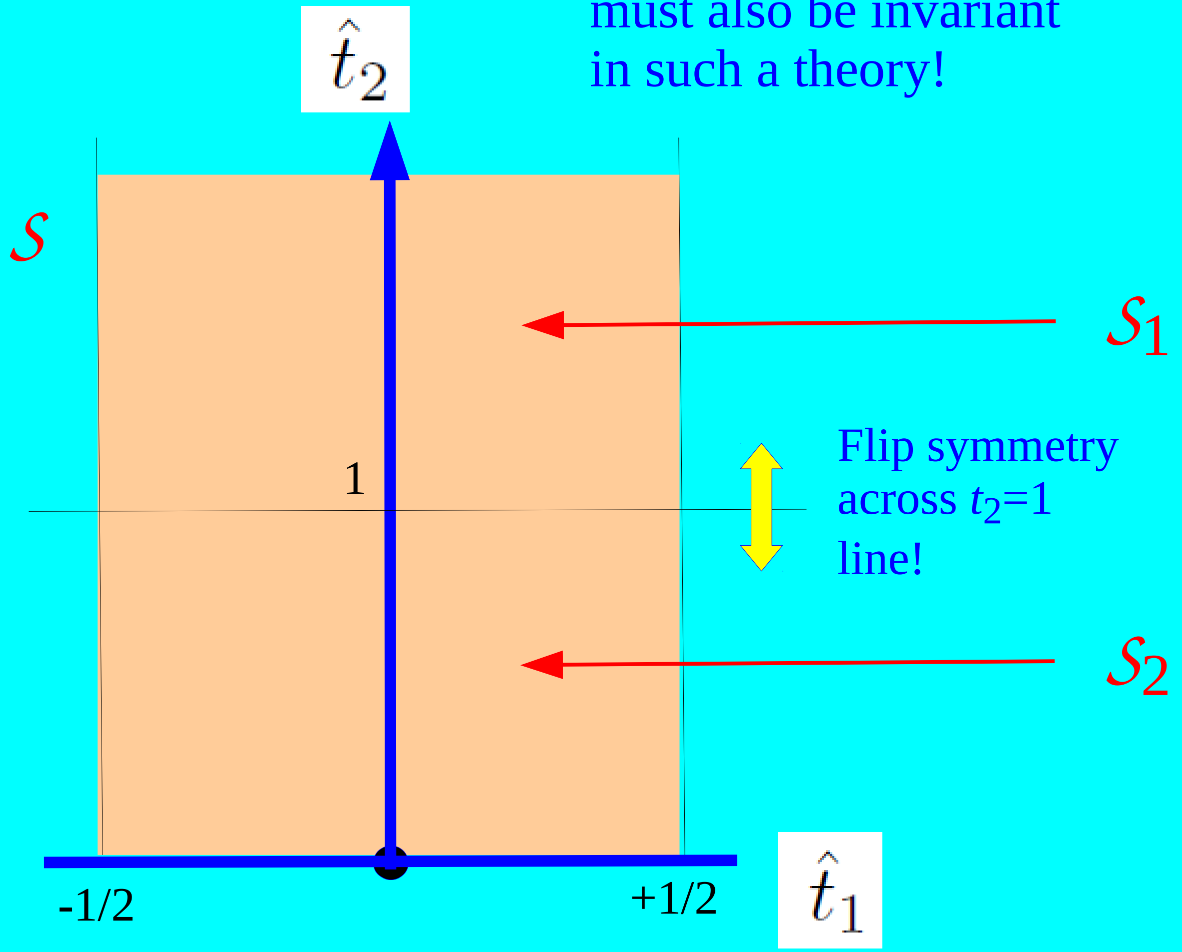

- Physics from $\mathcal{S}_1$ and $\mathcal{S}_2$ integration regions becomes identical!
- $\mathcal{S}_1$ and $\mathcal{S}_2$ provide redundant descriptions of the same physics!
- Thus, UV divergence must also be the same as IR divergence, likewise attributable to same underlying physics!

Sound familiar?

Two well-known analogues…

- Redundancy of description is like a *gauge* symmetry! Integrating over both $\mathcal{S}_1$ and $\mathcal{S}_2$ is like integrating over all of the gauge slices! Of course, there are only two gauge slices in this little example, and the overall factor of 2 is the "gauge volume". Still, the appropriate treatment is the same: **Effectively divide out by the gauge volume by choosing only *<u>one</u>* gauge slice!**

- Suppose we compactify a theory on a circle $y \sim y + 2\pi R$. Then we mod out by a Z_2 symmetry $y \to -y$ to construct an orbifold. What happens? The compactification volume of the orbifold is now only *half* that of the circle. We are effectively compactifying on the resulting line segment (= the orbifold). The point $y=0$ is self-dual and forms a new "edge".

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

where

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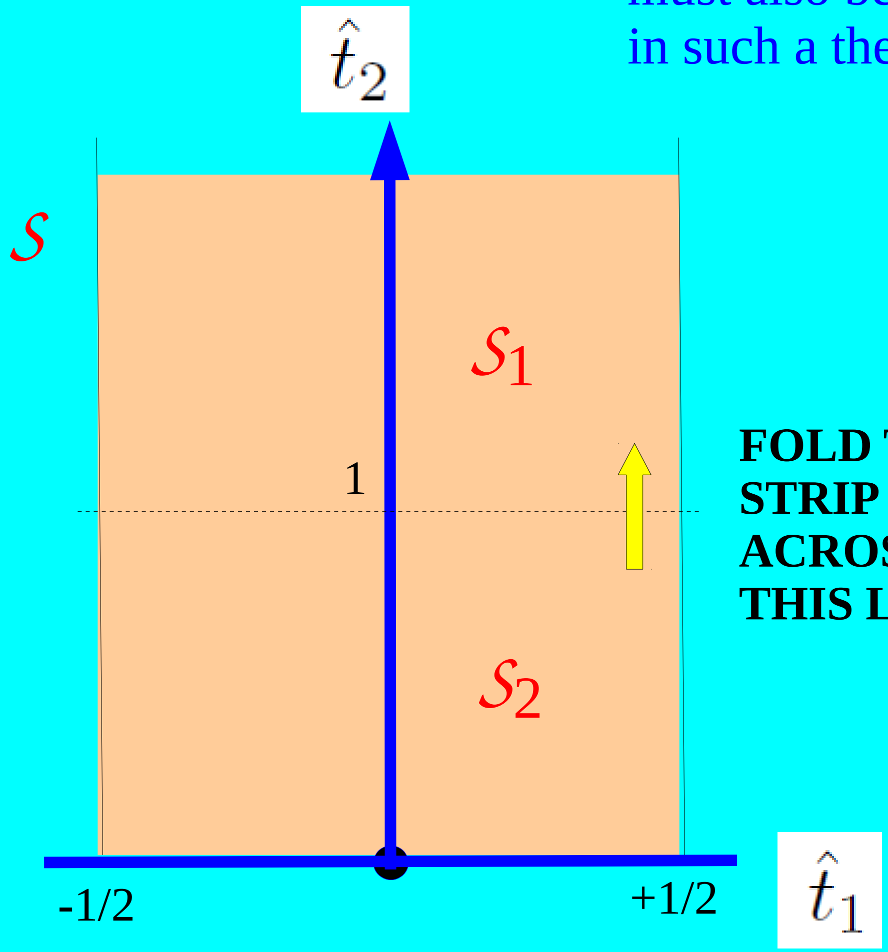

FOLD THE STRIP ACROSS THIS LINE

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

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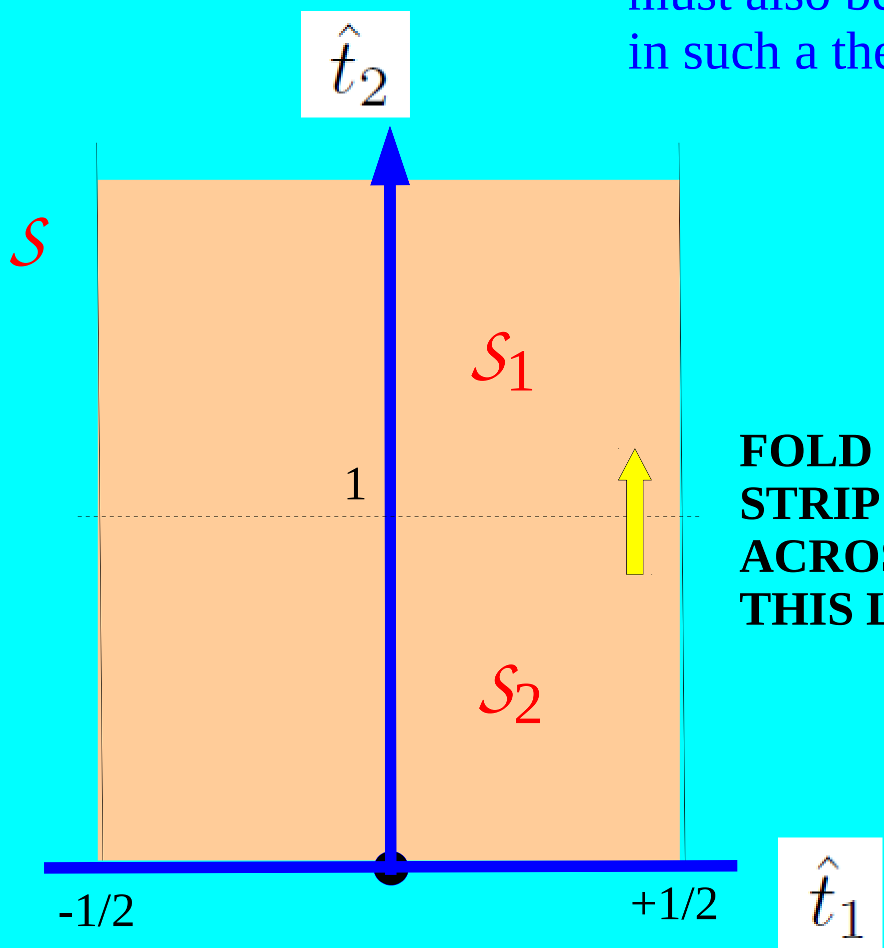

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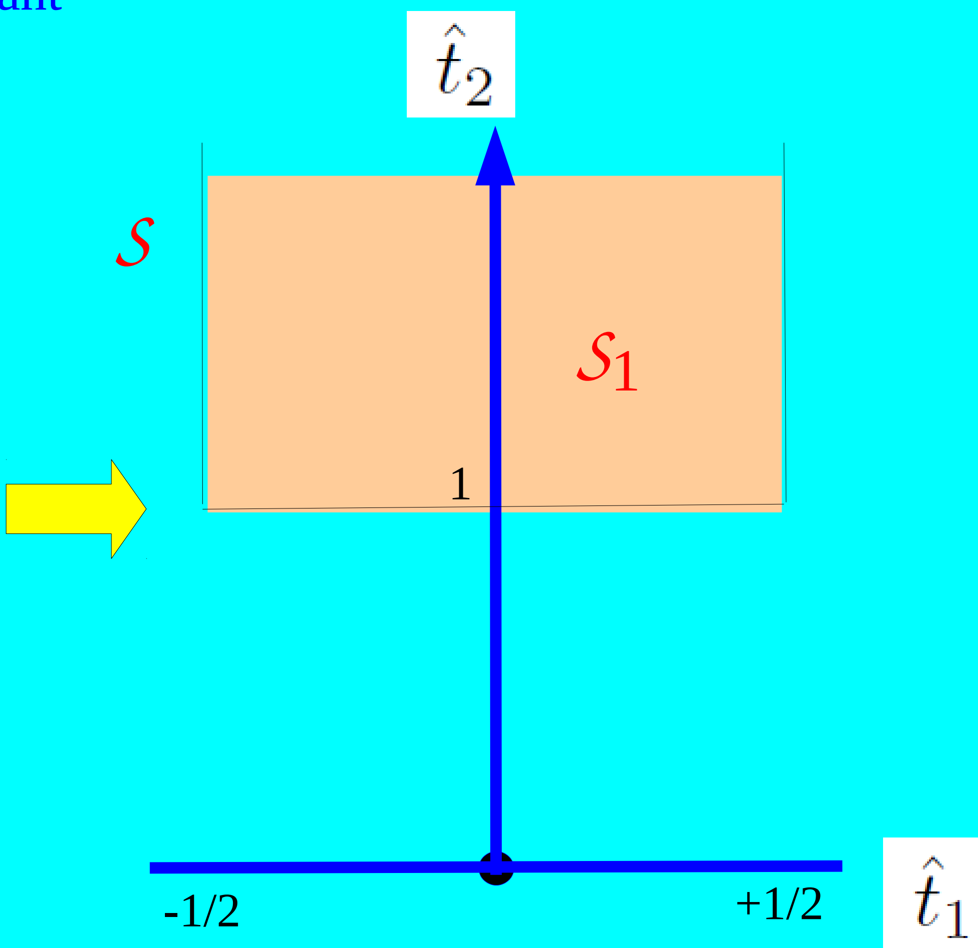

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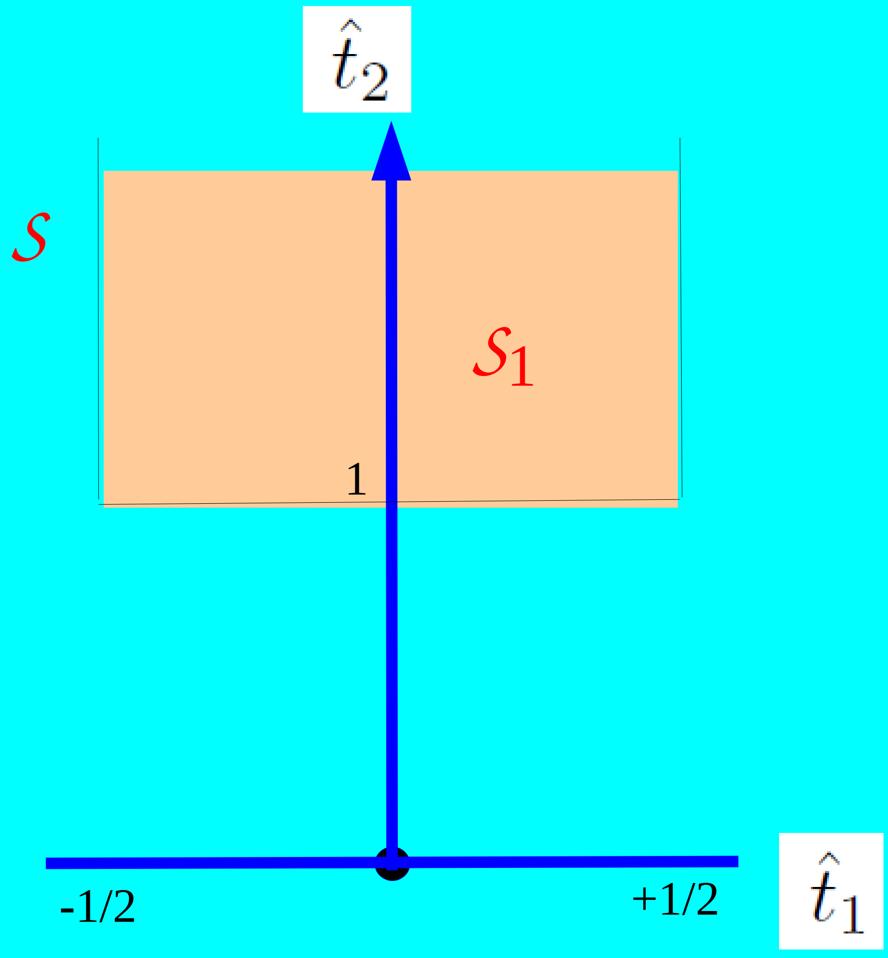

- Now integrate over $\mathcal{S}_1$, not $\mathcal{S}$.
- We have truncated the strip to just one slice.
- This eliminates the spurious factor of 2.
- Of course, could have chosen to fold the strip the other way, keeping $\mathcal{S}_2$ rather than $\mathcal{S}_1$.

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

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- Strip is invariant
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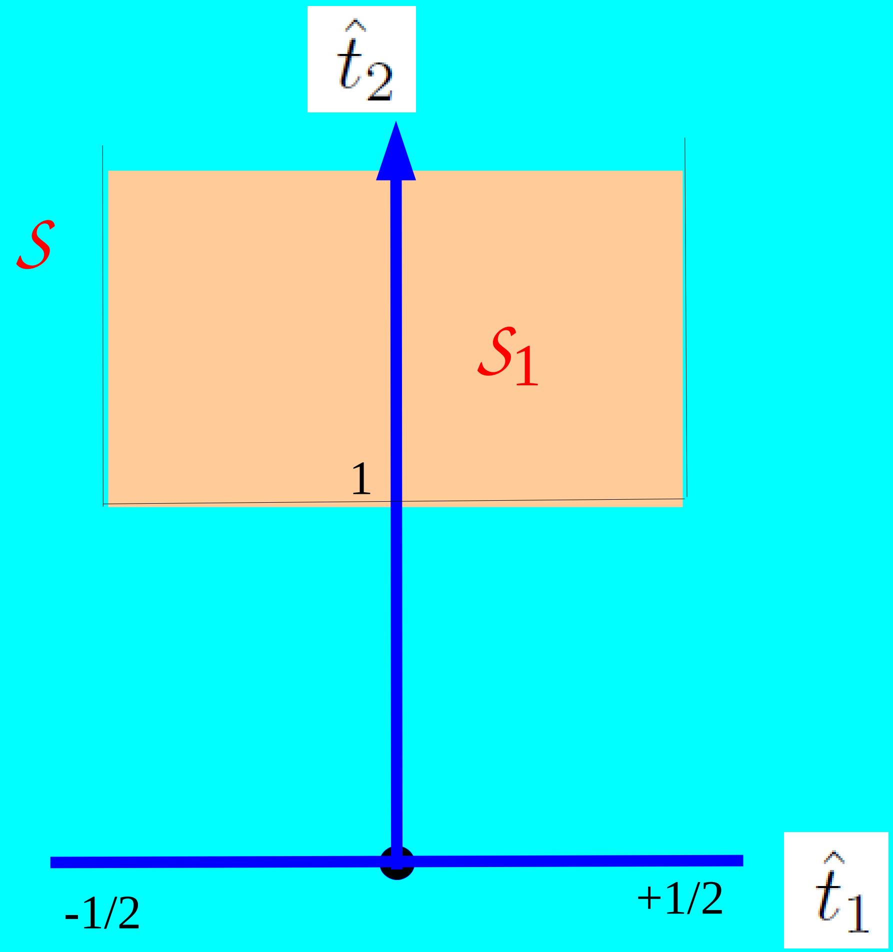
- Thus, partition function must also be invariant in such a theory!

But what does this folding imply for UV versus IR?

- The bottom part is folded *onto* the top part.
- There is no longer a unique up or down direction on the remaining segment! No notion of increasingly UV or IR "directions" → all directionality is lost. "Non-orientable"
- The two divergences (UV and IR) have been folded on top of each other!
- Thus, ***there is only one divergence.*** You can call it UV or IR according to your choice/convention → meaningless distinction!



Of course, this nightmare arises only if we have the $t_2 \rightarrow 1/t_2$ symmetry.

Can this ever really happen in field theory?

Not likely...

Recall

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

where

$$Z(\hat{t}_2) = \frac{1}{\hat{t}_2^{D/2}} \sum_{\text{states}} (-1)^F e^{-\pi M^2 \hat{t}_2/\mu^2}$$

- Strip is invariant … **ALREADY TRUE**
- Measure is invariant … **ALREADY TRUE**

Recall

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2}\, Z(\hat{t}_2)$$

where

$$Z(\hat{t}_2) = \frac{1}{\hat{t}_2^{D/2}} \sum_{\text{states}} (-1)^F e^{-\pi M^2 \hat{t}_2/\mu^2}$$

- Strip is invariant … **ALREADY TRUE**
- Measure is invariant … **ALREADY TRUE**
- Thus, partition function must also be invariant in such a theory! … **VERY HARD TO ARRANGE**

Each t_2 factor gets inverted in the exponential! *Is there some mathematical identity?*

Recall

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2} Z(\hat{t}_2)$$

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- Strip is invariant … **ALREADY TRUE**
- Measure is invariant … **ALREADY TRUE**
- Thus, partition function must also be invariant in such a theory! … **VERY HARD TO ARRANGE**

Each t_2 factor gets inverted in the exponential! *Is there some mathematical identity?*

Yes!

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \sqrt{t} \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t}$$

Just one of a whole series of similar identities involving infinite sums of exponentials

"Poisson resummation"

But this could only be useful if there were an *infinite* tower of states! Rather sick from a field-theoretic perspective…. (Must also have very tight balancing of masses and degeneracies at each level in order for such identities to apply.)

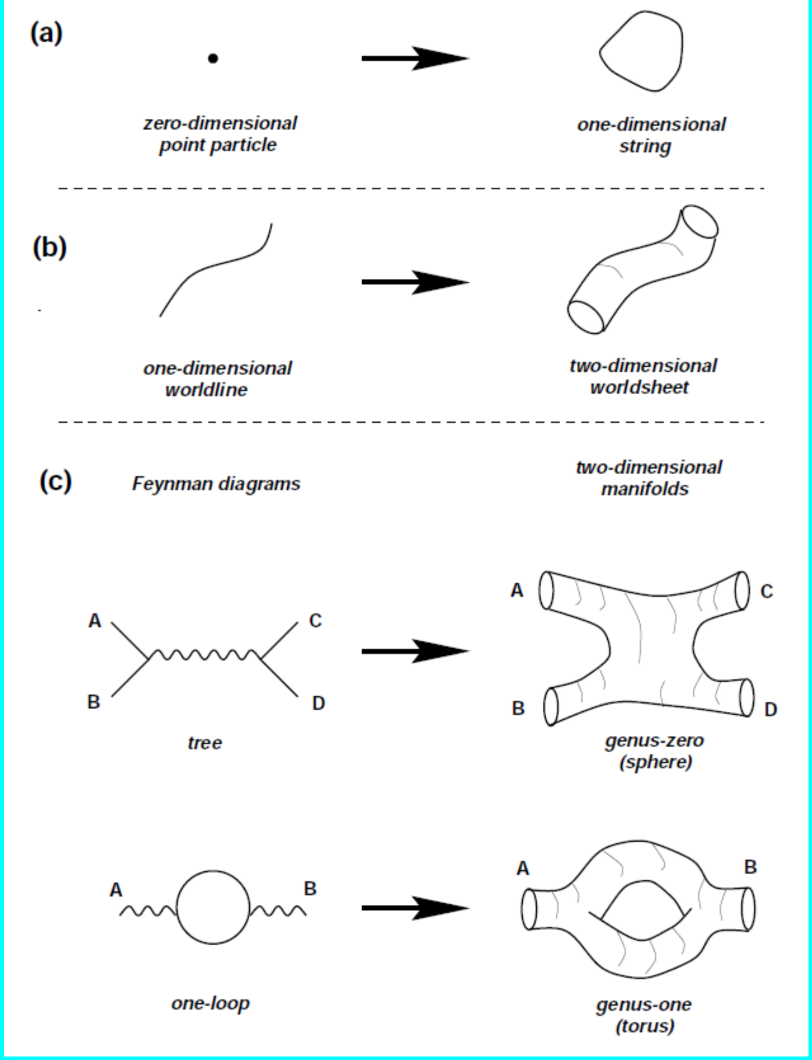
So now let's turn to string theory!

What actually happens in string theory?

Note: We shall focus on closed perturbative strings. This is a huge class, including Type II strings as well as heterotic strings formulated in any number of spacetime dimensions with any spacetime compactification manifold (or orbifold thereof), with or without spacetime SUSY. No restrictions on particle content, gauge symmetries, *etc*.

String Theory 101

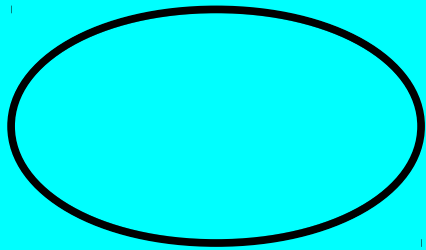
- Worldlines becomes *worldsheets*!
- Different configurations/excitations of worldsheet are different particles in spacetime.
- Excitations come in three varieties:
  - *oscillators*: quantum fluctuations of the string itself, masses depend on tension of the string (string scale)
  - *KK modes*: independent of string scale, masses depend on compactification radii
  - *winding modes*: strings wrapping around compactified directions, depend on both string tension and compactification radii.
- Excitations of string worldsheet are like waves on the worldsheet: can propagate clockwise or counterclockwise around worldsheet. “LM” versus “RM” modes.



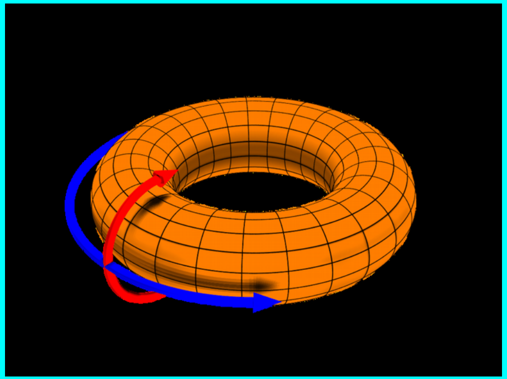
$$M^2 = \frac{1}{2}(M_L^2 + M_R^2)$$

- Because of oscillators, total number of string states of a given mass grows exponentially with mass. (Hagedorn)

How to calculate one-loop Λ in string theory?



Schwinger proper time t described the *length* of the circle (length around single cycle). We then integrated over all t.



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Note: We care about *shape*, not *volume*.

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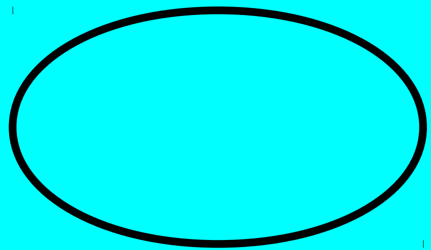

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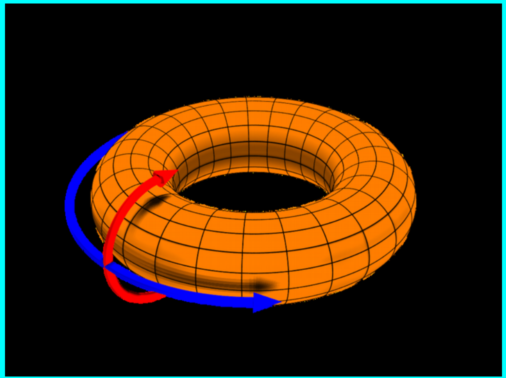

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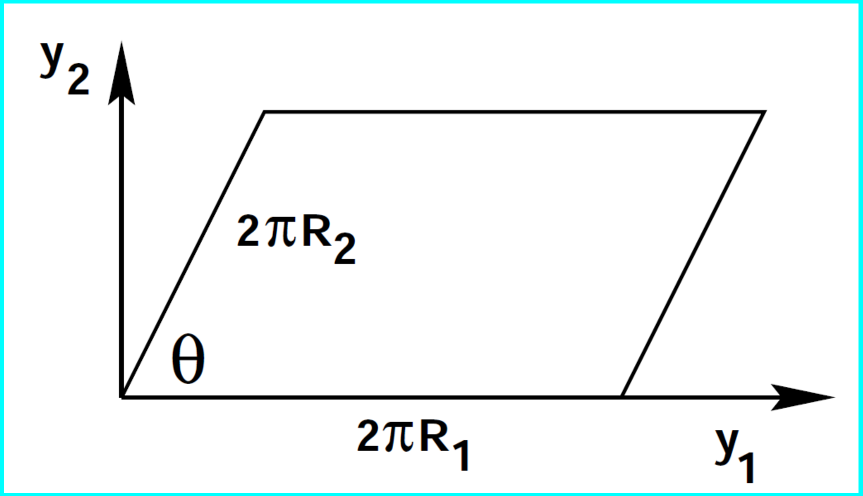
In general, three real quantities (R_1, R_2, θ) describe the two cycles of torus. Identify points related by...

$$\begin{cases} y_1 & \to & y_1 + 2\pi R_1 \\ y_2 & \to & y_2 \end{cases}$$

$$\begin{cases} y_1 & \to & y_1 + 2\pi R_2 \cos\,\theta \\ y_2 & \to & y_2 + 2\pi R_2 \sin\,\theta\ . \end{cases}$$



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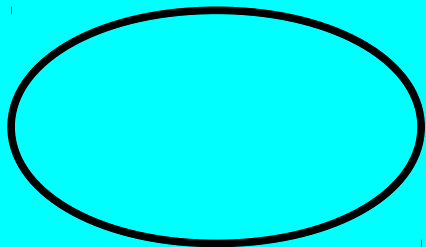

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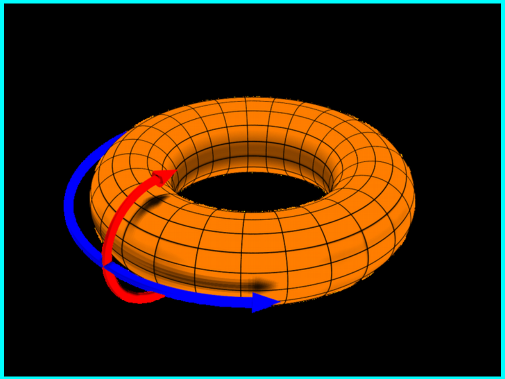

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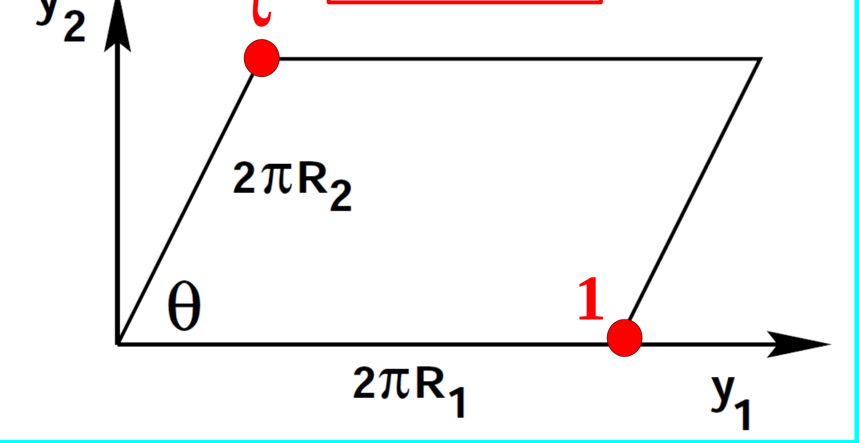
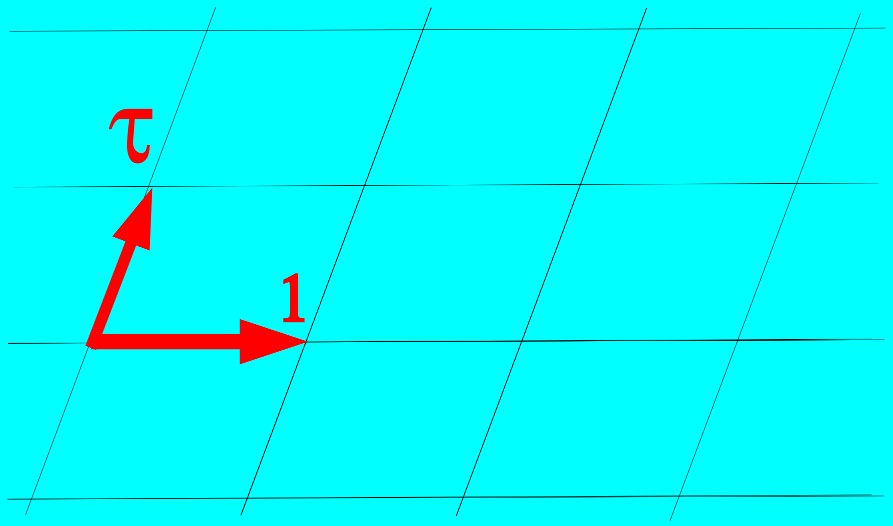
$$\tau \equiv \frac{R_2}{R_1} e^{i\theta}$$



Torus shape can be described through a complex number τ in the UHP. Recurring cycles of the torus map out a lattice in the plane.

- Is there any redundancy in this description?
- Are there different values of τ that give the same fundamental cell?



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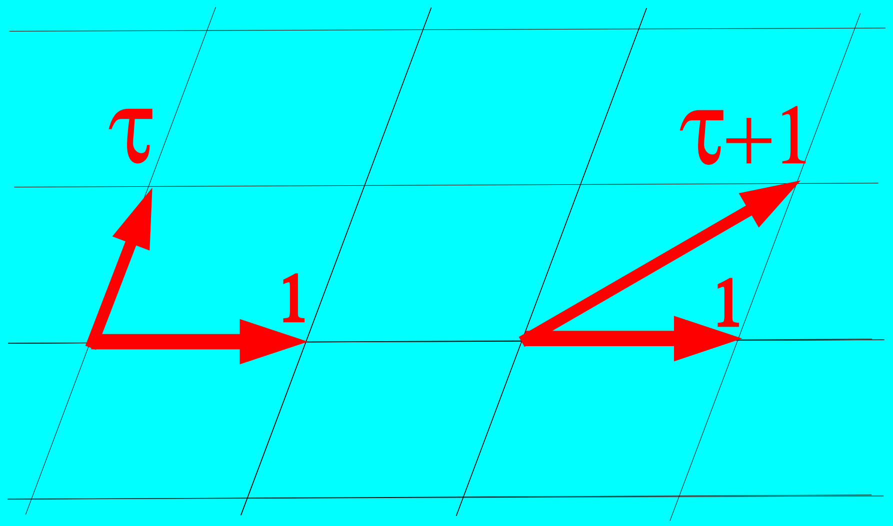

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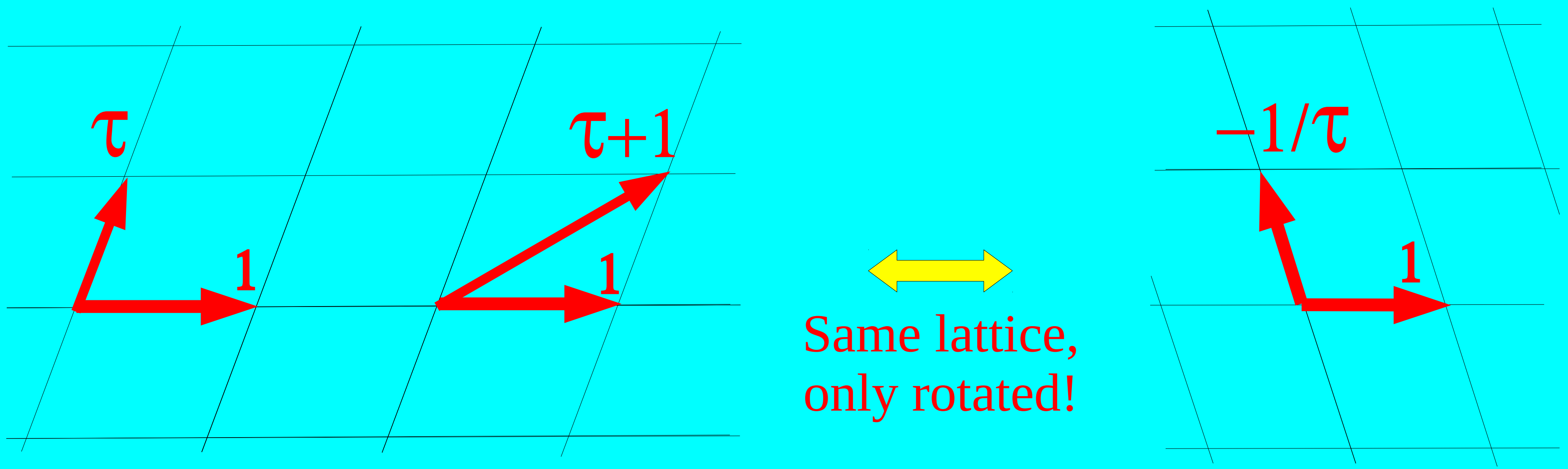

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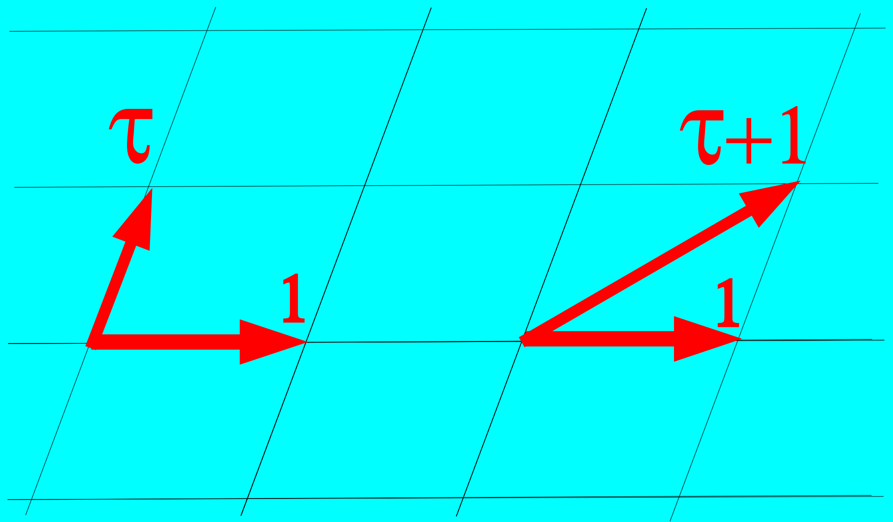
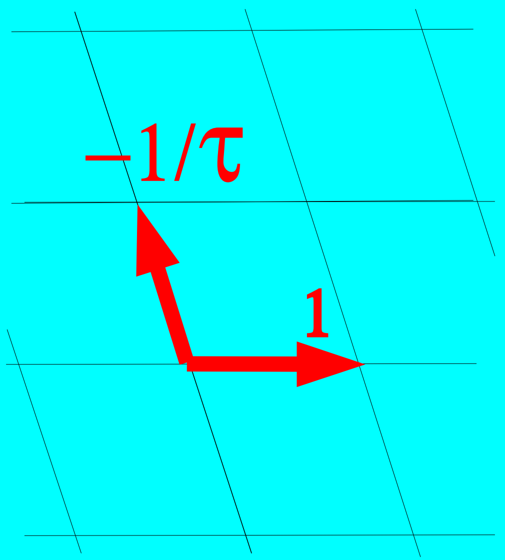


Same lattice, only rotated!



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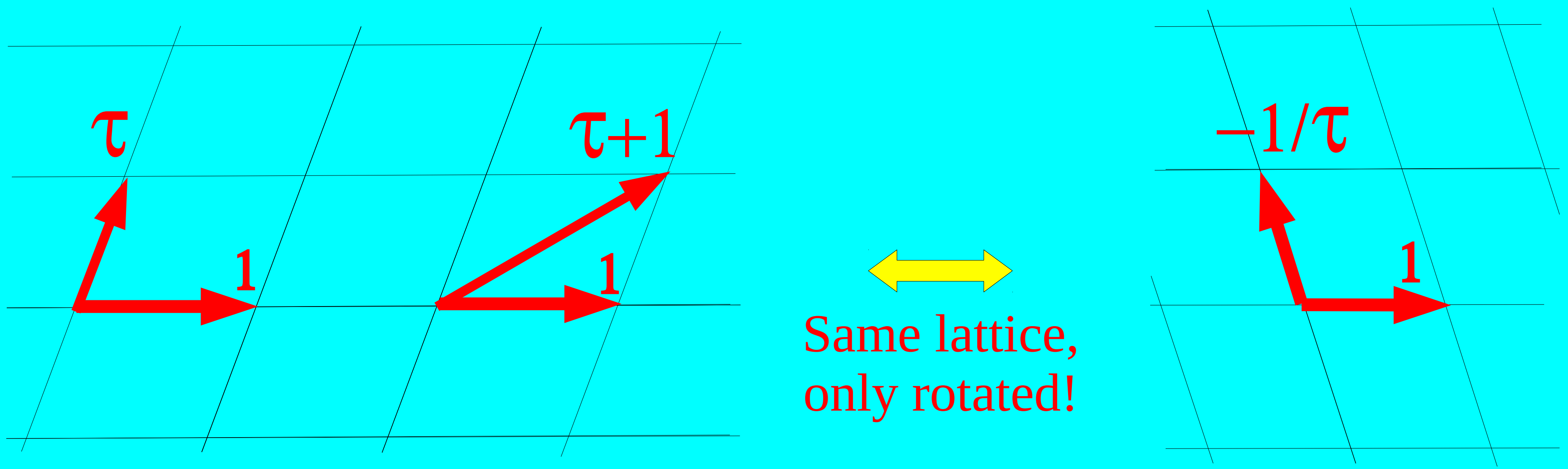

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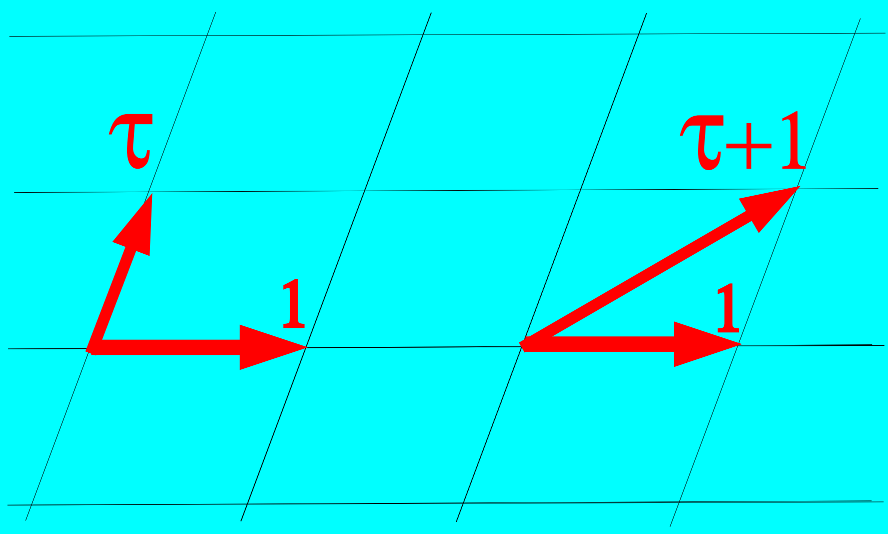

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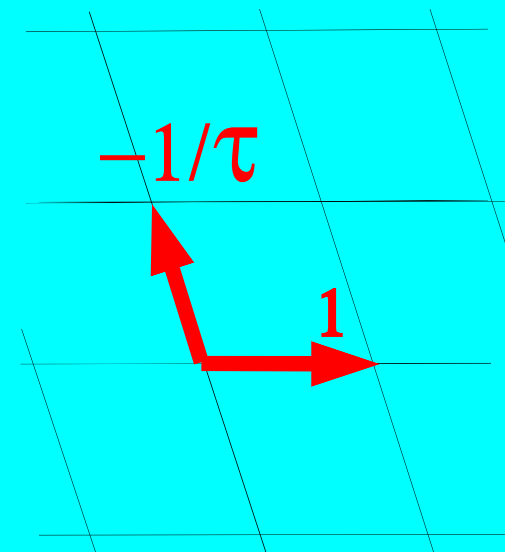

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Modular group!
For any τ, all of these describe same torus!

Thus the modular group describes the τ-redundancies inherent in describing tori. Tori are unchanged ("conformally equivalent" = same shape) under all transformations in the complex plane of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \in \mathrm{PSL}(2, \mathbb{Z})$$

Infinite-dimensional. Call this Γ.

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It turns out that all elements of Γ can be generated as *sequences* of two fundamental generators:

$$T: \quad \tau \to \tau + 1$$
$$S: \quad \tau \to -1/\tau$$

e.g.,

τ, $\tau+1$, $\tau+2$, …	$\Leftrightarrow$	1, T, T^2, …
$-1/\tau$, $-1/\tau + 1$, $-1/\tau + 2$, …		S, TS, T^2S, …
$-1/(\tau+1)$, $-1/(\tau+2)$, … $-1/(\tau+1)+1$, ...		ST, ST^2, …, TST, ...

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$\Longleftrightarrow$

1, T, T^2, …
S, TS, T^2S, …
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Important subgroup:

$$\Gamma' : \quad \text{generated by } \{1, T\} \text{ only}$$

- The complex upper half-plane of τ describes all possible tori, but redundancy group Γ describes the conformally equivalent tori.
- Again, just like a gauge theory redundancy!

What, then, is the subregion that captures just one copy of each torus without overcounting (one “gauge slice”)?

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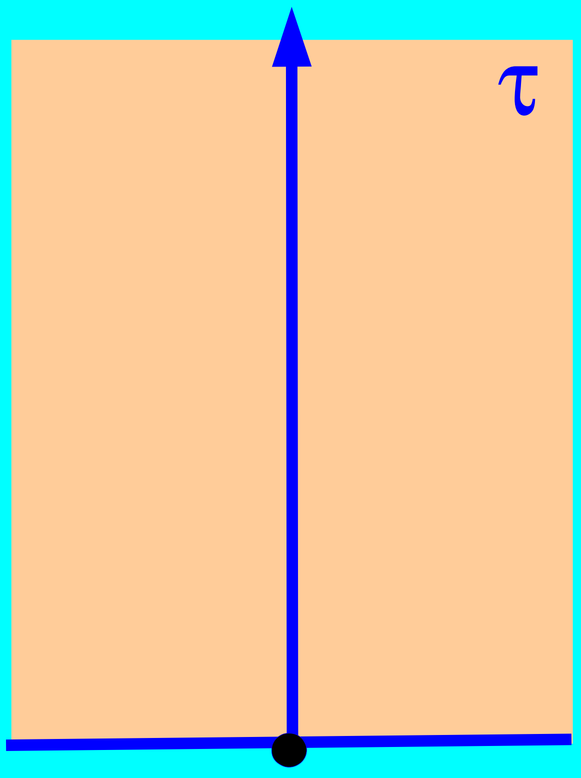

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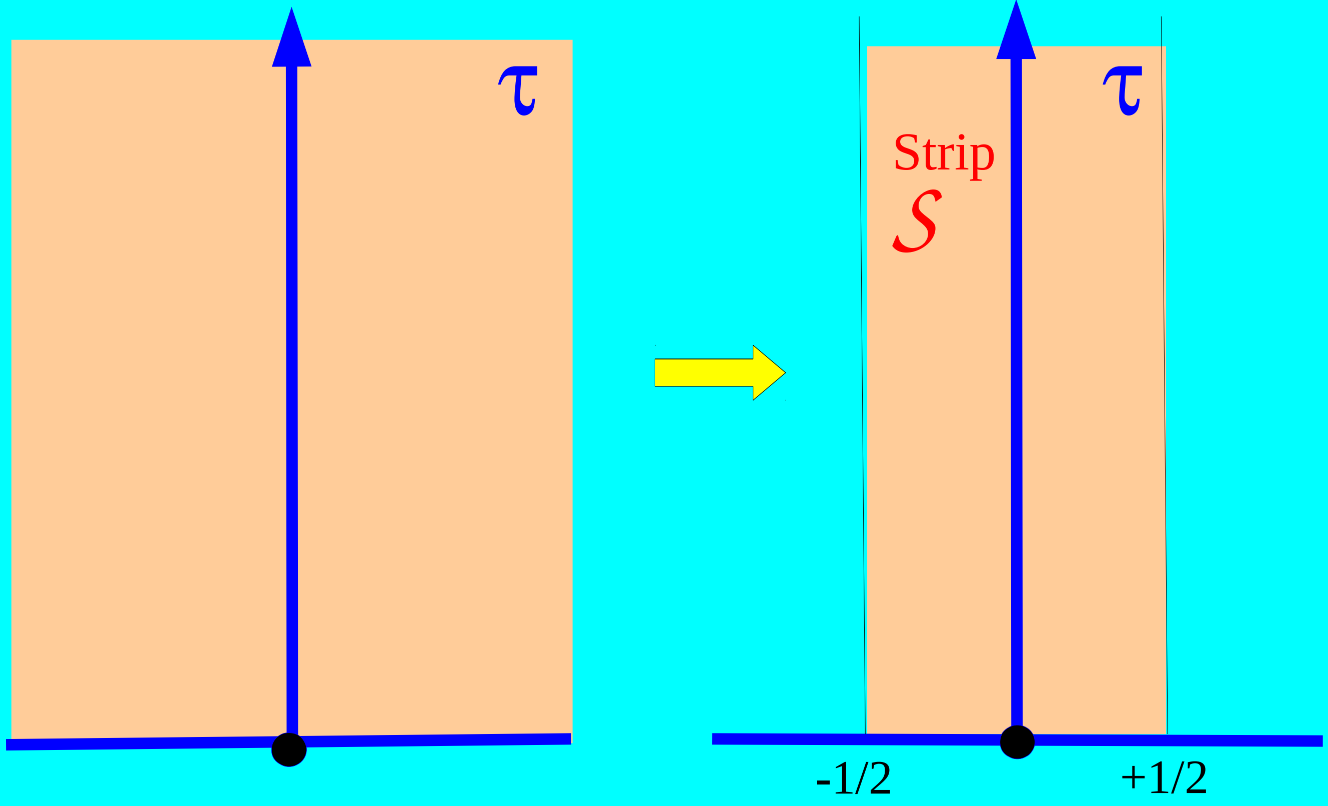

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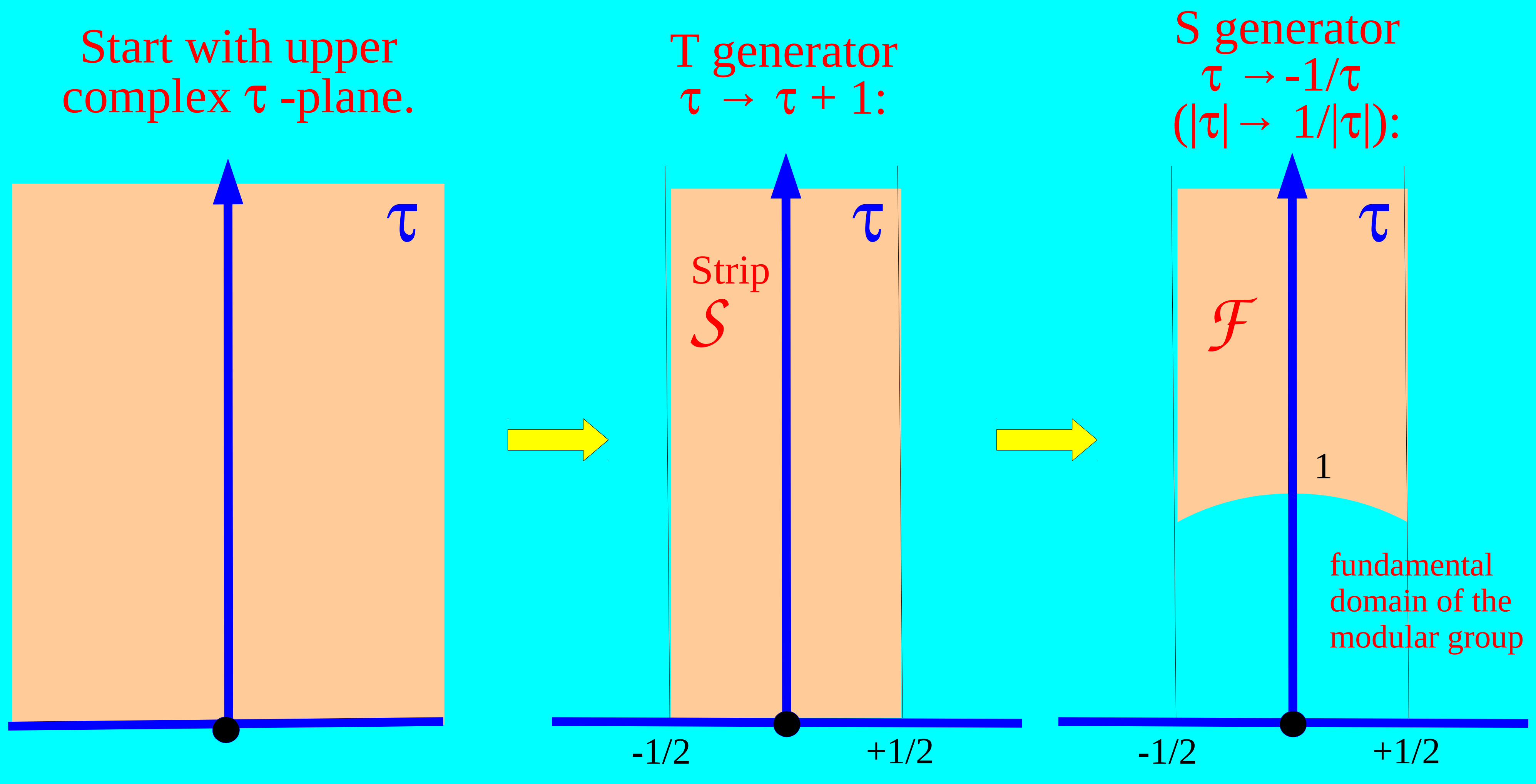
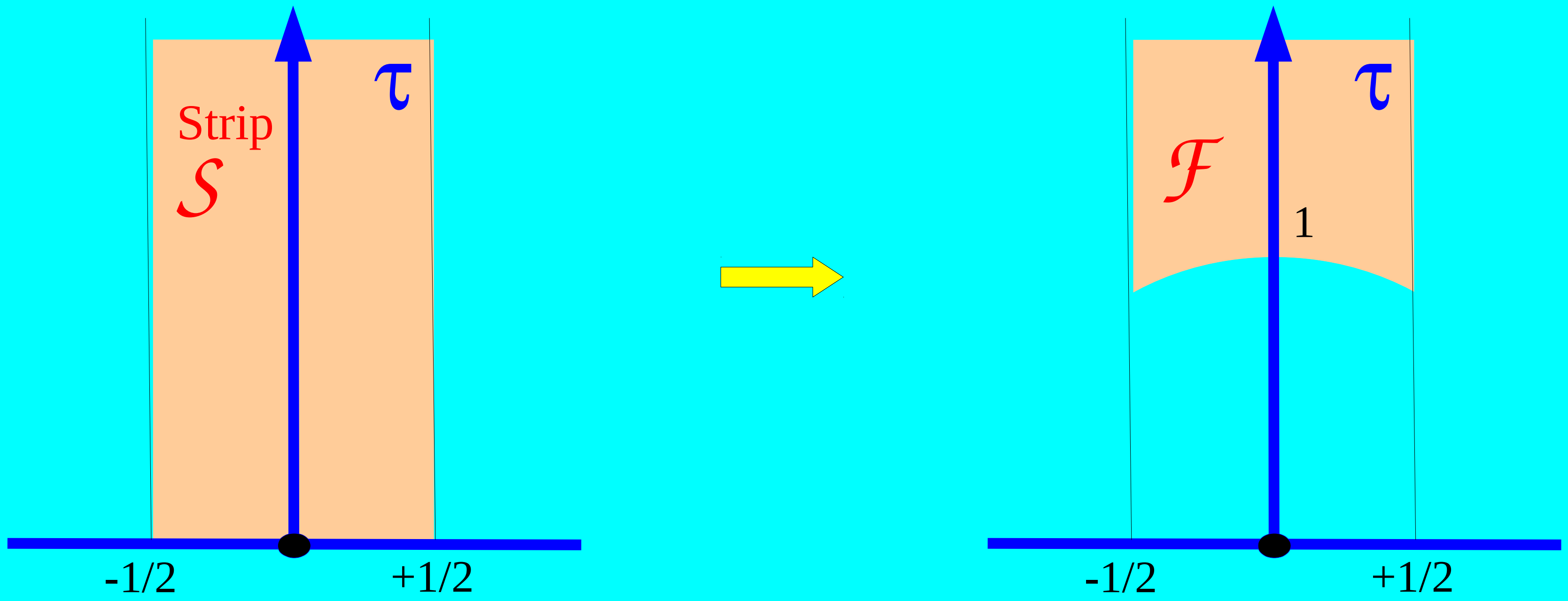


Let’s study this last step in more detail...



$\mathcal{S}$

- Looks like field theory!
- Fundamental domain of subgroup generated by T alone

$\mathcal{F}$

- String theory!
- Fundamental domain of full modular group Γ generated by both S and T

How many “gauge slices” --- *i.e.*, how many copies of $\mathcal{F}$ within $\mathcal{S}$?

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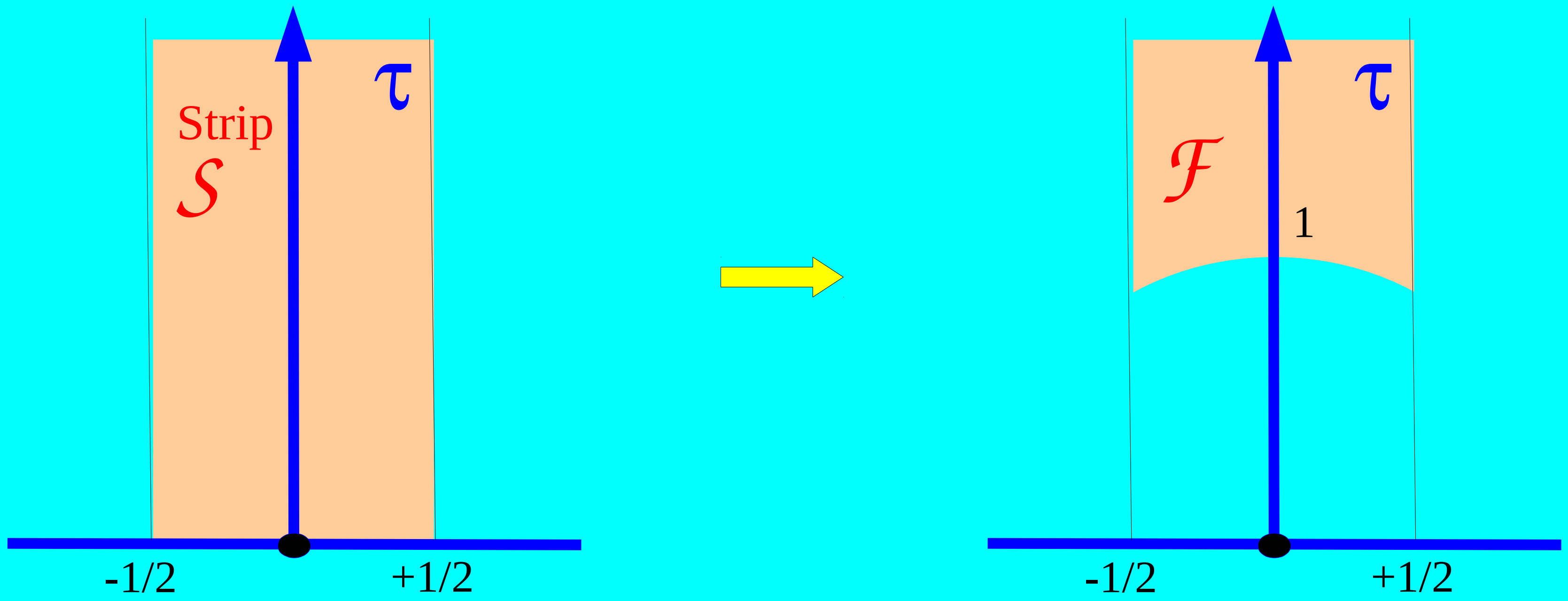

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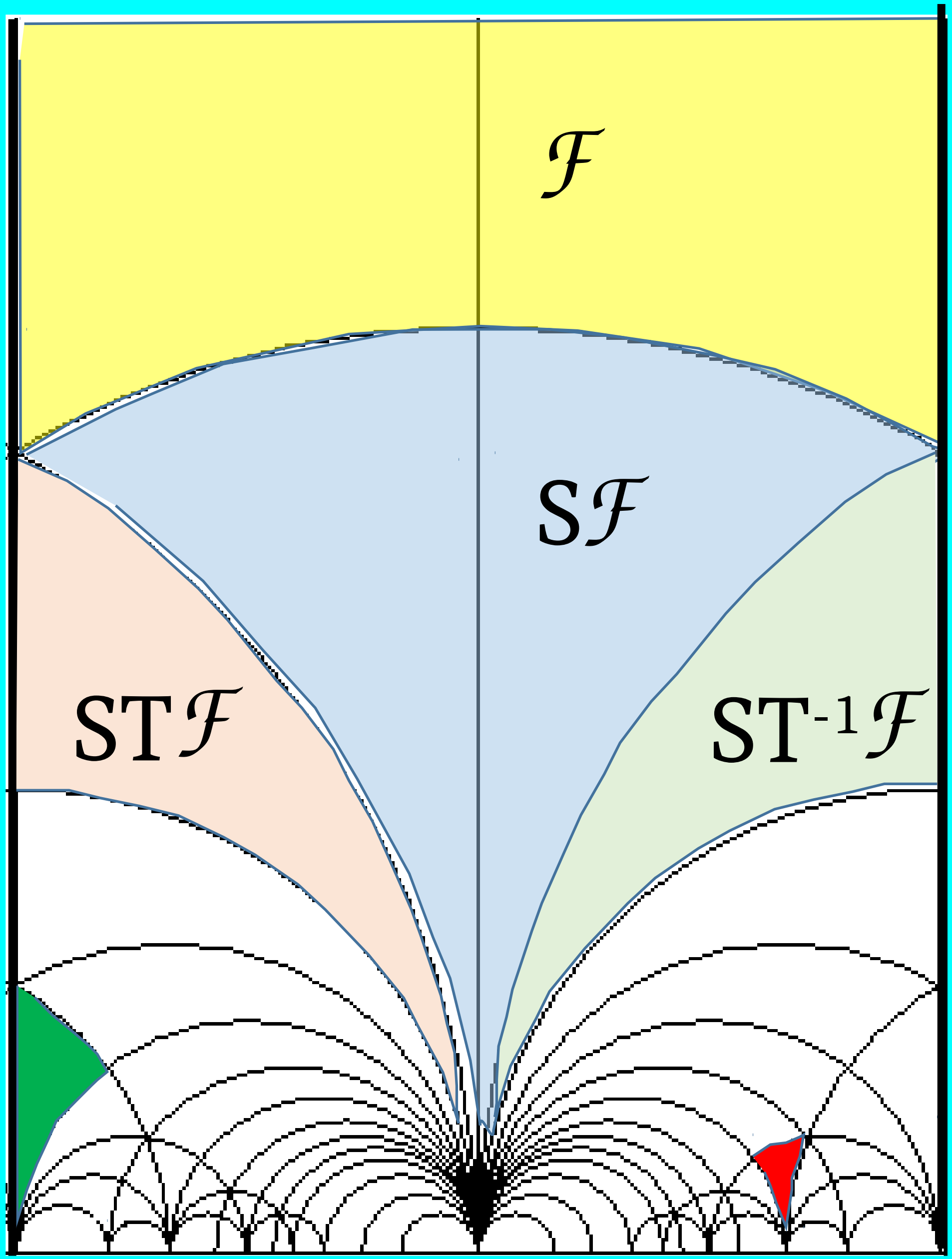
How many "gauge slices" --- *i.e.*, how many copies of $\mathcal{F}$ within $\mathcal{S}$?

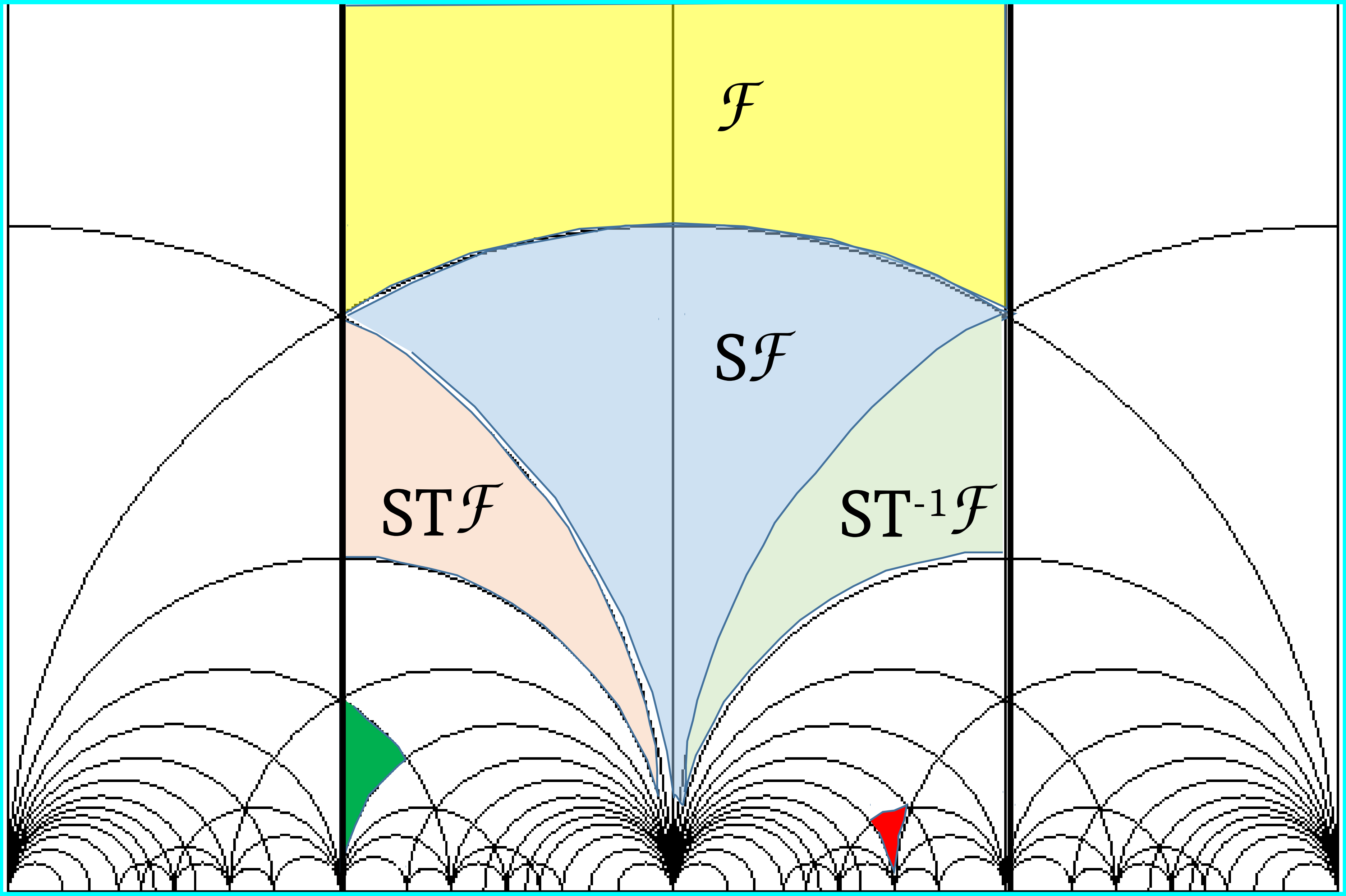
- In previous case with $t \to 1/t$ folding, answer was **2**.
- Answer now is the **dimensionality of the coset**

$$\boxed{\dim\left(\Gamma/\Gamma'\right) = \infty\ !}$$



- There are an *infinity* of domains!
- Each domain is equally valid
- Together these fundamental domains completely fill the strip
- Each domain has a unique UV/IR cusp.

Lines and circles!

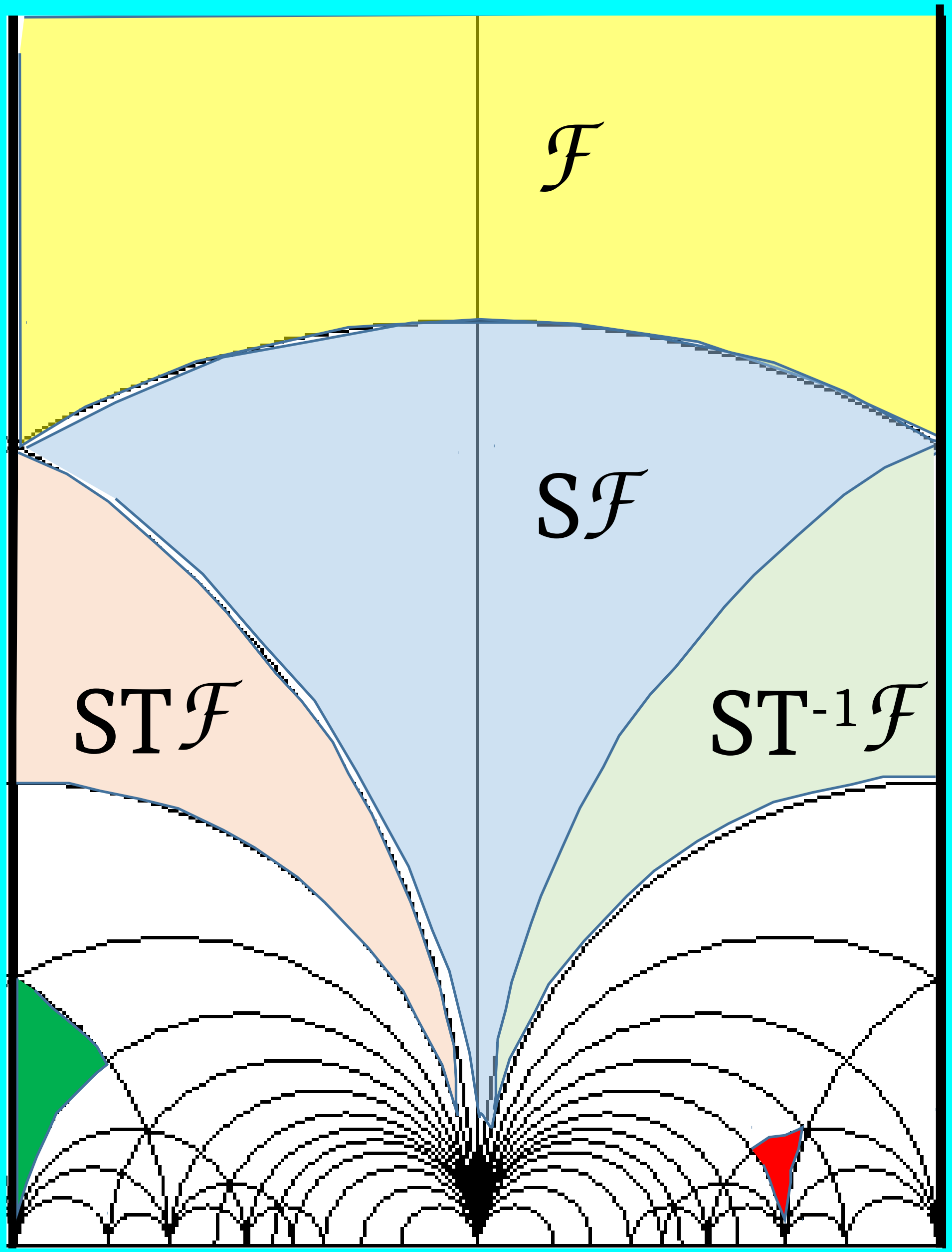


Linear fractional transformations *(az+b)/(cz+d)* map lines and circles to lines and circles.

A whole new meaning to the phrase “modular furniture”…

“The Modular Cabinet”

Richard Pink
https://people.math.ethz.ch/~pink/ModularCabinet/cabinet.html



- Thus in string theory we are instructed to "fold" the strip $\mathcal{S}$ into $\mathcal{F}$!
- Requires an *infinite* number of folds!
- Once folding is done, all "cusps" lie atop each other at the cusp at τ = *i**infinity!

We therefore have

$$\Lambda = -\frac{1}{2}\left(\frac{\mu}{2\pi}\right)^D \int_{\mathcal{S}} \frac{d^2\hat{t}}{\hat{t}_2} Z(\hat{t}_2)$$

where

$$Z(\hat{t}_2) = \frac{1}{\hat{t}_2^{D/2}} \sum_{\text{states}} (-1)^F e^{-\pi M^2 \hat{t}_2/\mu^2}$$

$$\Lambda = -\frac{1}{2}\left(\frac{M_s}{2\pi}\right)^D \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau)$$

$$Z(\tau) = \frac{1}{\tau_2^{D/2-1}} \sum_{\text{states}} (-1)^F e^{-\pi\tau_2 M^2/M_s^2} e^{-\pi i \tau_1 \Delta M^2/2M_s^2}$$

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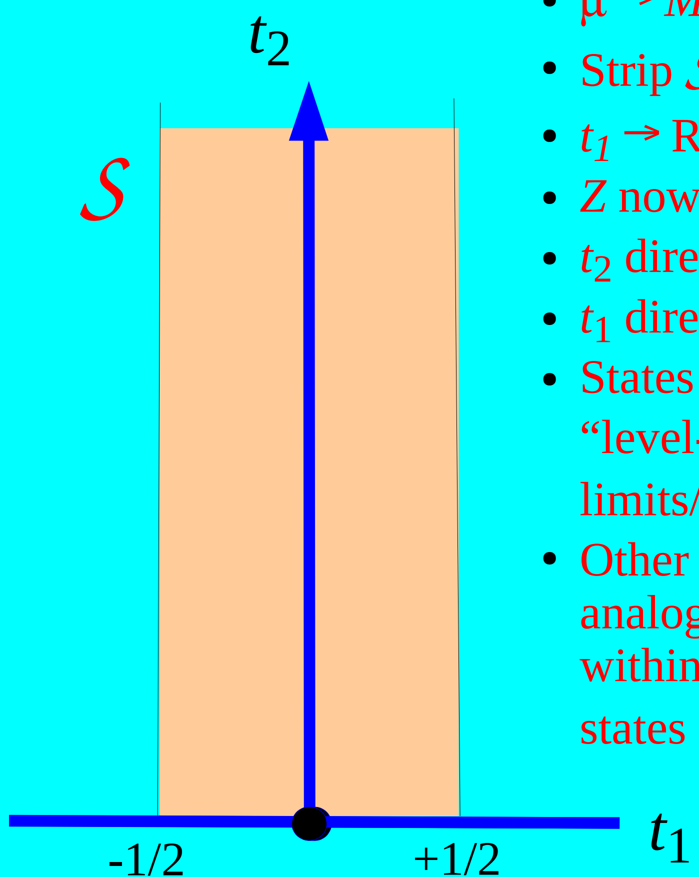
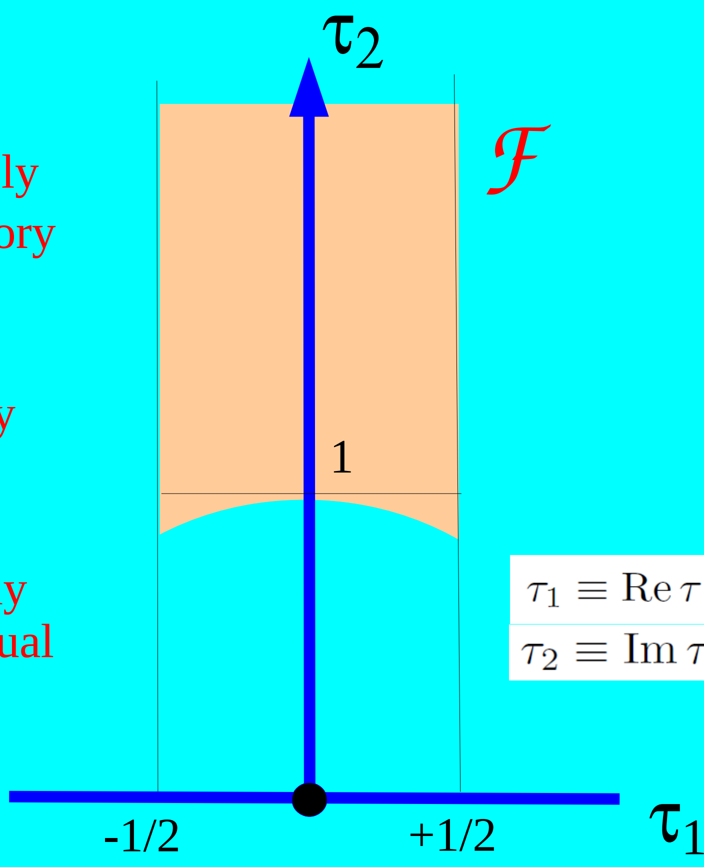


- $\mu \to M_s$
- Strip $\mathcal{S}$ → Fundamental domain $\mathcal{F}$.
- $t_1 \to \mathrm{Re}(\tau)$, $t_2 \to \mathrm{Im}(\tau)$
- Z now depends on (M_L, M_R) separately
- t_2 direction ⇔ total M, like field theory
- t_1 direction ⇔ M_L-M_R! Stringiness!
- States with $M_L = M_R$ are "physical" / "level-matched" → have field-theory limits/analogues, τ_1-independent
- Other states have no field-theory analogues but contribute significantly within amplitudes as "off-shell" virtual states from curved $\mathcal{F}$ region.



$\tau_1 \equiv \mathrm{Re}\,\tau$

$\tau_2 \equiv \mathrm{Im}\,\tau$

Modular invariance has compelled this structure!

- Modular invariance is unavoidable --- it's nothing more than the symmetry group reflecting the invariances of tori.

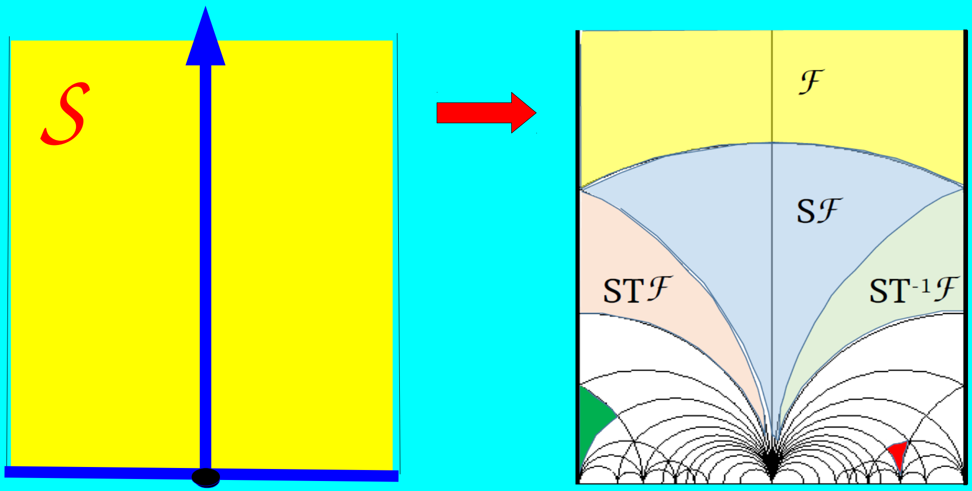
- As such, modular invariance is "baked into" string theory from the beginning and ensures that string theory is consistent, with consistent one-loop diagrams. It's part of reparametrization invariance: physics should not depend on WS coordinates.

- Modular invariance is sometimes called a "one-loop" symmetry because it ensures the consistency of one-loop string diagrams. However, this terminology is misleading, as this is not a symmetry that is violated at higher orders.

- Rather, this is an *exact* symmetry of the partition function and therefore of the entire string spectrum of states and their interactions. One cannot "break" modular invariance by small amounts or through interactions, just as one cannot break the Ward identities associated with gauge invariance through gauge-invariant interactions. Phenomena such as phase transitions may deform the theory and change its d.o.f.s, but only from one modular-invariant theory to another.

- So what about higher-loop diagrams in string theory? Do they introduce *additional* redundancies of this sort? Answer: **No.** Within certain closed string theories, amplitude factorization and physically-sensible state projections together ensure that one-loop modular invariance automatically implies multi-loop modular invariance. Thus one-loop modular invariance is sufficient.

But the consequences of "folding" $\mathcal{S}$ into $\mathcal{F}$ are profound!



- The lower portions of $\mathcal{S}$ are folded upwards into $\mathcal{F}$.
- Could have equivalently chosen to fold $\mathcal{S}$ into S$\mathcal{F}$.
- There is no longer a unique up or down direction on the remaining segment! No notion of increasingly UV or IR "directions".
- ***There is only one possible divergence.*** You can call it UV or IR according to your choice/ convention (e.g., $\mathcal{F}$ versus S$\mathcal{F}$) → meaningless distinction!

… just like previous $t_2 \to 1/t_2$ example!

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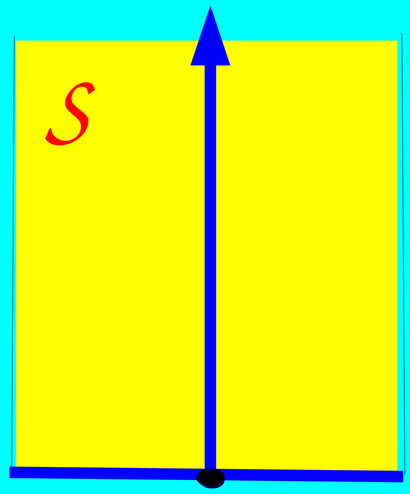
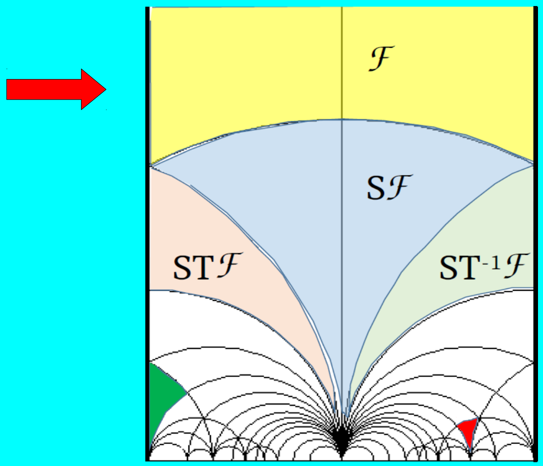




Additional feature!

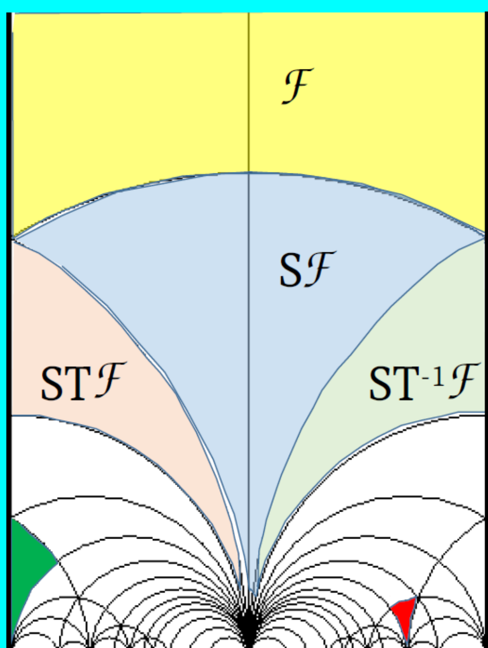
- ***Infinitely many foldings are required!*** We are thus essentially dividing by an infinite gauge "volume"!
- Equivalently, an *infinite number* of field-theory divergences are eliminated, leaving only a single string divergence!
- Thus modular invariance not only relates UV and IR divergences to each other, but also *softens* them since we are dividing out by the infinite number of copies!
- ***Essentially some of the divergences of field theory are reinterpreted as a spurious "gauge" volume and thereby eliminated!***

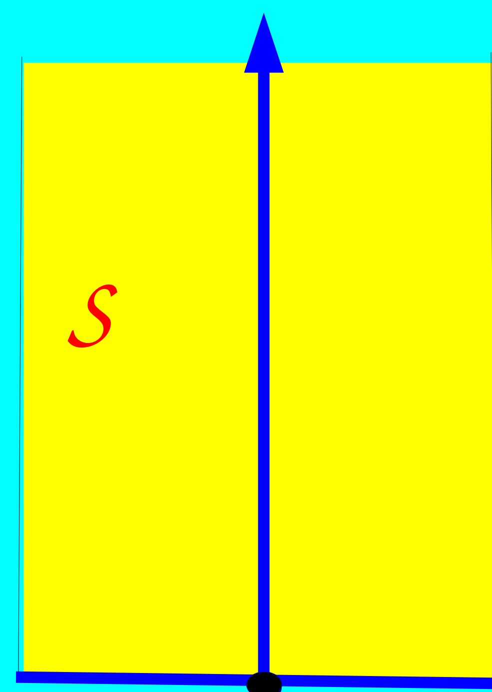
Inverting this procedure, each string divergence "unfolds" into what would appear to be a *combination* of IR and UV divergences in field theory.

Schematically, we therefore have

$$\underbrace{\mathrm{IR}_{\mathcal{F}} = \mathrm{UV}_{S\mathcal{F}}}_{\text{string theory}} \iff \underbrace{(\mathrm{IR}_{\mathcal{S}} \oplus \mathrm{UV}_{\mathcal{S}})}_{\text{field theory}} \,/\, \underbrace{[\Gamma : \Gamma']}_{\infty}$$

\# redundant copies





For example, consider the cosmological constant in string theory.

- Tree-level contribution vanishes by conformal invariance
- Leading contribution is thus actually Λ.

$$\Lambda = -\frac{1}{2}\left(\frac{M_s}{2\pi}\right)^D \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(\tau)$$

$$Z(\tau) = \frac{1}{\tau_2^{D/2-1}} \sum_{\text{states}} (-1)^F \, e^{-\pi\tau_2 M^2/M_s^2} \, e^{-\pi i \tau_1 \Delta M^2/2M_s^2}$$

$$\Delta M^2 \equiv M_L^2 - M_R^2$$

In $\mathcal{F}$-representation, only possible divergence is "IR" from $\tau_2 \to$ infinity region. Thus divergences are governed by lightest states. All consistent string models contain tachyonic "proto-graviton" states with $M_L^2 < 0$ and $M_R^2 = 0$, but these are not level-matched and make no contributions in this region of $\mathcal{F}$. Massless states also do not lead to divergences.

"Proto-graviton theorem": KRD, 1990 (PRL)

$\Rightarrow$ **Λ is actually finite in string theory!**
(not quartically divergent, as would arise in field theory for analogous one-loop diagram)

even without SUSY!

Of course, modular invariance severely constrains the spectrum of states at all mass levels in any consistent closed string theory!

Must demand $$Z(\tau) = Z(\tau+1) = Z(-1/\tau)$$

where
$$Z(\tau) = \frac{1}{\tau_2^{D/2-1}} \sum_{\text{states}} (-1)^F \, e^{-\pi\tau_2 M^2/M_s^2} \, e^{-\pi i \tau_1 \Delta M^2/2M_s^2}$$

What does the resulting spectrum look like?

- In general, very complicated mathematical problem
- Need to exploit properties of sums of products of Poisson resummations
- Insight can also be also gained from mathematical literature on modular-function theory, number theory
- Of course, if SUSY, then Z=0 → trivial solution. What happens otherwise?

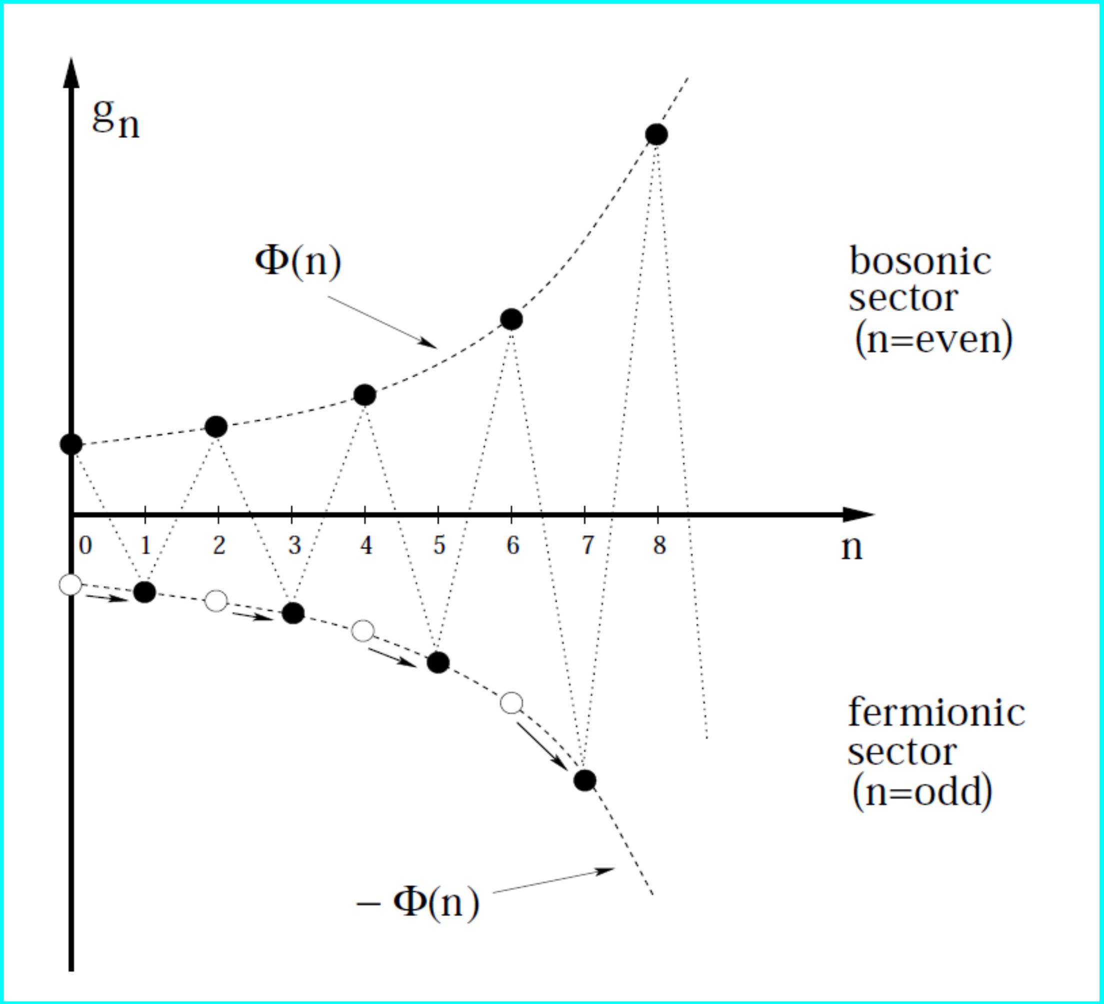
especially “circle method” of Hardy & Ramanujan, 1917

Misaligned SUSY

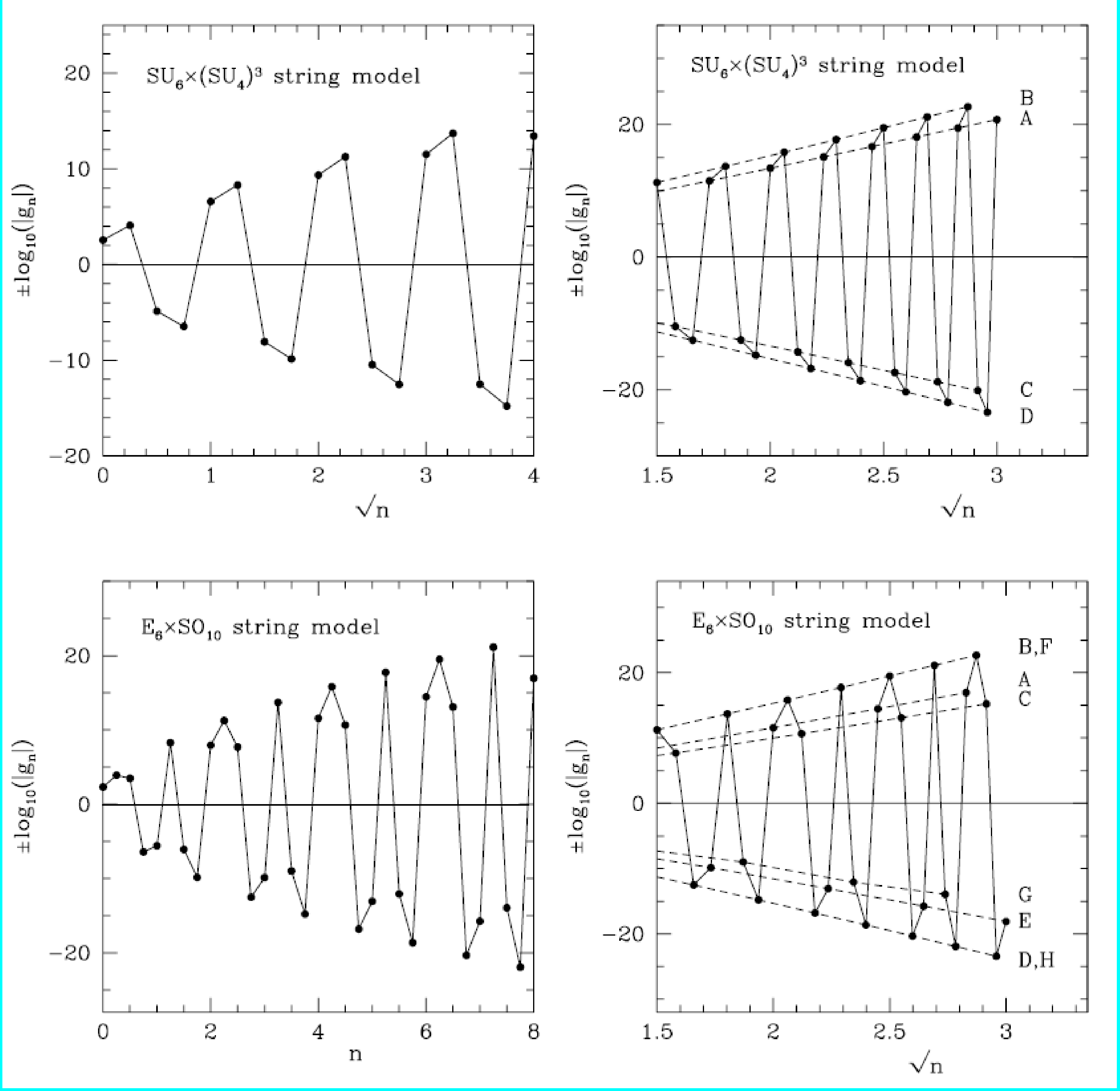
- KRD, 1994 (hep-th/9402006)

In any tachyon-free closed string theory, spacetime SUSY may be broken but a residual “**misaligned SUSY**” must always remain in the string spectrum!

- Functional forms $\Phi(n)$ cancel even if numbers of states do not.
- SUSY is special case where sectors are “aligned”.
- As sectors become misaligned, new states must populate each level to preserve $\Phi(n)$.
- In all cases, all masses (UV/IR) conspire together! --- describes maximum degree to which SUSY may be broken in string theory.
- General feature of all closed string models, and serves as the way in which the spectrum of a given string theory manages to configure itself at all mass levels so as to maintain finiteness --- even without SUSY.



Actual string models...



Thus far we have seen that the strip region $\mathcal{S}$ is "infinitely" larger than the $\mathcal{F}$ region, and we know precisely how to relate them.

- Is there a corresponding way to relate the ***integrals*** over these regions?
- Is there a way to "extract" the infinity and thereby evaluate the $\mathcal{F}$ integral by integrating over the (simpler) strip $\mathcal{S}$?
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Let us first concentrate on situations in which the $\mathcal{F}$ integral is finite.
The corresponding $\mathcal{S}$ integral still diverges, but it turns out that

- R. Rankin, 1939;
- A. Selberg, 1940

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau,\overline{\tau}) = \frac{\pi}{3} \operatorname*{Res}_{s=1} \int_0^\infty d\tau_2 \, \tau_2^{s-2} \left[\int_{-1/2}^{1/2} d\tau_1 \, F(\tau,\overline{\tau}) \right]$$

finite *string* amplitude	=	*residue* of divergent (deformed) *field*-theory amplitude at pole in complex *s*-plane!

only level-matched (physical states) survive τ_1-integral and contribute!

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inverse Mellin transform

$$= \frac{\pi}{3} \lim_{\tau_2\to 0} \int_{-1/2}^{1/2} d\tau_1\, F(\tau,\overline{\tau})$$

- D. Zagier, 1981;
- D. Kutasov & N. Seiberg, 1991

This is an extremely powerful result!

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, F(\tau,\overline{\tau}) = \frac{\pi}{3}\, \mathop{\mathrm{Res}}_{s=1} \int_0^\infty d\tau_2\, \tau_2^{s-2} \left[\int_{-1/2}^{1/2} d\tau_1\, F(\tau,\overline{\tau}) \right]$$

Rankin-Selberg (1939,1940)

Full string amplitude! Depends on all string states in the spectrum, both physical (level-matched) and unphysical (non-level-matched)!

Integral over the strip! Only physical states contribute --- the same states that have field-theory analogues!

This relation allows us express string-theory amplitudes in terms of **supertraces over only physical states!** This will also eventually allow us to rigorously extract EFTs from string theory and analyze their properties.

True for any integrand $F(\tau)$!

Thus, we see that modular invariance is so powerful that the contributions from the off-shell (unphysical) states are already determined once the contributions from the physical states are known! Indeed, both spectra are "locked" together.

For example, applying this to the cosmological constant Λ (*i.e.*, taking $F \to Z$), we find two results:

- KRD, M. Moshe & R.C. Myers, 1995 (PRL)

- $\boxed{\mathrm{Str}\,\mathbf{1} = 0}$ even without SUSY!!

- $\boxed{\Lambda = \frac{1}{24}\mathcal{M}^2\,\mathrm{Str}\,M^2}$

$\boxed{\mathcal{M} \equiv \frac{M_s}{2\pi}}$ reduced string scale

where $\boxed{\mathrm{Str}\,A \equiv \lim_{y\to 0} \sum_{\substack{\text{physical}\\ \text{states}}} (-1)^F A\, e^{-yM^2/M_s^2}}$

supertrace definition appropriate for theories with infinite towers of states, regulated in a modular-invariant manner. ONLY PHYSICAL STATES CONTRIBUTE!

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Very different from field theory, where

- Str **1** governs quartic *<u>divergence</u>* of Λ
- Str M^2 governs quadratic *<u>divergence</u>* of Λ

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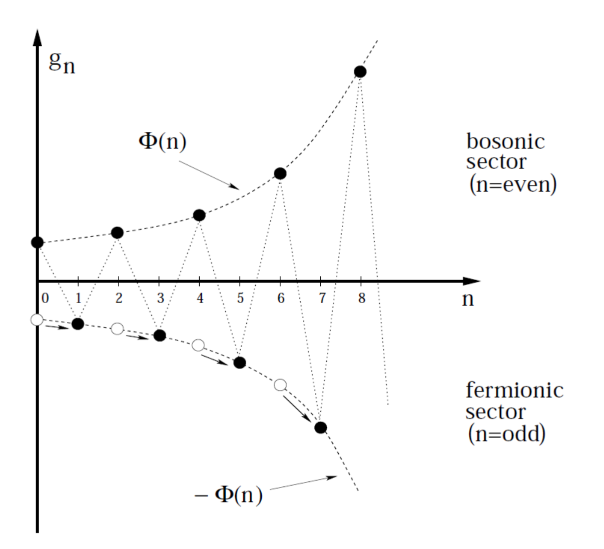


These supertrace relations are realized through misaligned SUSY and thus hold for all closed string theories. Indeed, misaligned SUSY explains how the Hagedorn phenomenon is reconciled with such finite supertraces!

These supertrace relations are thus examples of the ***hidden cancellations*** that run across the entire string spectrum, suppressing divergences and/or ensuring the finiteness of string amplitudes relative to QFT expectations.

It turns out that such relations are only the tip of the iceberg!
Indeed, for *any* modular-invariant operator insertion *X*, we find

- $$\mathrm{Str}\, X \;=\; 0$$

- These *X*-insertions can be combinations of charges, helicities, *etc.*
- These cancellations provide deep constraints on the charges of states at all mass levels, IR to UV!
- Collectively kills the leading operators that would have produced divergent <*X*> amplitudes
- Can even be taken as the *definition* of a modular-invariant operator insertion!

Resulting amplitude is then finite and takes the simple form

- $$\langle X\rangle \;=\; -\frac{1}{12\mathcal{M}^2}\,\mathrm{Str}\big(X M^2\big)$$

amplitude with *X* inserted into sum over states

- S. Abel, KRD, L. Nutricati, to appear

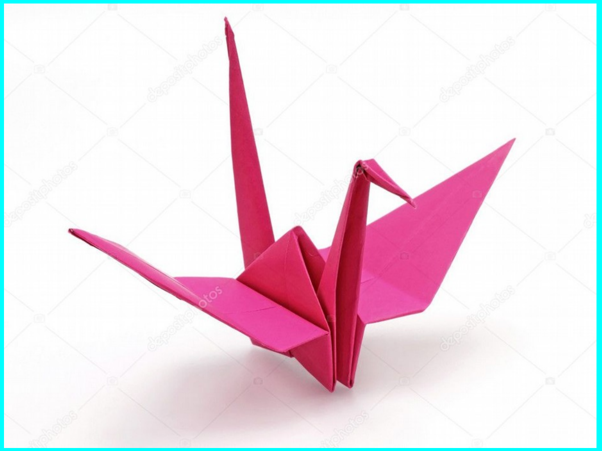
Let's now turn to the Higgs mass!

How can we use this technology to calculate
the Higgs mass in string theory?

- Focus on perturbative closed strings.
- Restrict to *D*=4, with or without SUSY. No restrictions on particle content, gauge symmetries, *etc*. --- we will be completely general.
- Our goal here is *not* to establish a numerical value for the Higgs mass within a particular string model --- rather, our goal is to establish a modular-invariant *framework* for such calculations. This will also allow us to extract general phenomena which will hold across all such string models.

"Higgs" --- any scalar field sitting at the minimum of a potential whose VEV affects the masses of particles in the string spectrum, regardless of the gauge symmetries this VEV may break. Thus we include the SM Higgs, but also include other kinds of Higgses and moduli.

Note: Performing such a calculation now becomes an issue in *string phenomenology* (*i.e.*, extracting "low-energy" phenomenological predictions from string theory)...



However, to study the full UV/IR implications of string theory, we cannot practice string phenomenology in the usual way.

Traditional approach ---

- Start with a suitable vacuum ("string model")
- Enumerate the massless states that arise in such models
- Construct a field-theoretic Lagrangian that describes the dynamics of these states
- Analyze this Lagrangian using all of the regular tools of QFT without further regard for the origins of these states within string theory.

Indeed, this treatment may well be sufficient for certain purposes.

Unfortunately, calculations performed in this manner have a serious shortcoming:

By disregarding the infinite towers of string states that necessarily accompany these low-lying modes within the full string theory, such calculations implicitly disregard many of the underlying string symmetries that ultimately endow string theory with the remarkable UV/IR properties that transcend our field-theoretic expectations.

?

- These states are usually at the Planck scale, or at the scales associated with the compactification geometry! ***How can they ever play an important role for low-energy phenomenology?***
- Can't they just be integrated out, leaving behind higher-dimensional operators suppressed by powers of these heavy scales?
- Wouldn't this justify the usual treatment?

However, there are reasons to take pause...

- We would not be integrating out one or two or three heavy states. We would be integrating out *infinite towers* of states!
- Even more severely, these towers of states have *degeneracies that grow exponentially* with their masses!
- *Can this still leave behind a power-law suppression of higher-dimensional operators? Usual EFT expectations about disregarding these infinite towers of states may not apply.*

Hagedorn

Natural to expect that these infinite towers of states would particularly affect quantities (such as the Higgs mass and cosmological constant) which have positive mass dimension and are therefore sensitive to all mass scales in the theory.

Thus, we should really be doing string calculations by retaining the full towers of string states at all times and manifestly preserving the string symmetries (such as modular invariance) which are critical for the UV/IR mixing properties inherent in string theory and responsible for its finiteness.

Indeed, as we have seen, modular invariance is the ultimate “UV/IR mixer” --- enforces a delicate balance between low-scale and high-scale physics.

In this sense, it would be wrong to take an “EFT approach” in which we integrate out heavy states while treating light states as dynamical.

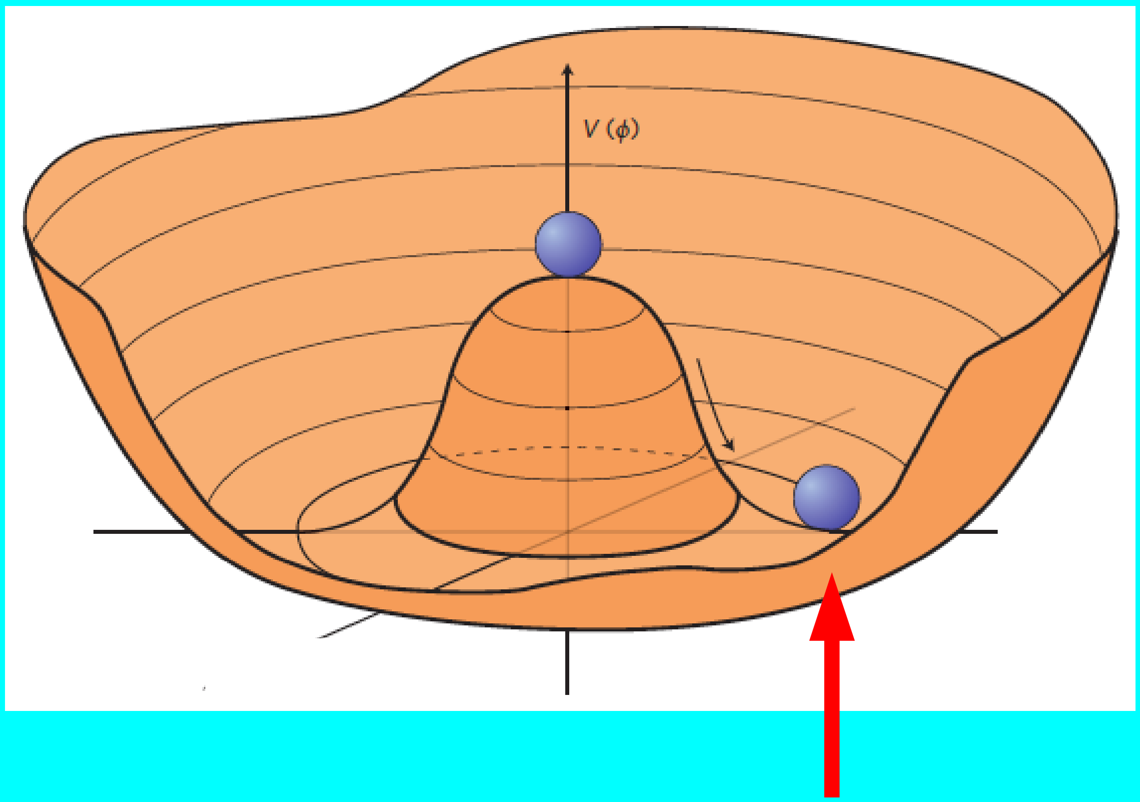
Therefore, if we are to take string theory literally as a theory of physics, we should perform our calculations within the full framework of string theory, incorporating all of the relevant symmetries and infinite towers of states that string theory provides.

We shall now calculate the Higgs mass in string theory.

- Surprisingly, no such calculation ever previously performed.
- We will not choose a particular string model. Instead, our goal will be to establish a framework for doing this calculation rigorously, taking into account all string states and using only techniques that preserve modular invariance. One could then apply our framework within one's favorite string model.
- One goal will be to see how our results agree with EFT-based expectations --- and also where they differ.
- We are not attempting to solve any hierarchy problems. However, we will see some intriguing ideas emerging from our results.

Note that our interest is in calculating the *Higgs mass*, not in studying the *Higgsing phenomenon*.

Accordingly, we shall work exclusively within the Higgsed phase in which the scalar field is already at the new minimum of the potential, and study small radial fluctuations (perturbations) of the scalar field around this minimum.



We work here!

Most importantly, throughout our calculations, we shall be careful to preserve modular invariance.
As we shall see, this will imply a number of unique phenomena.

In general, fluctuations of the Higgs around its VEV will affect the masses of the states across the entire string spectrum.

Let ϕ = fluctuation of Higgs field around its VEV in the Higgsed phase.

For any state s, we can then view $M_{L,R}(\phi)$ as functions of ϕ:

$$\begin{aligned} M_L^{(s)} &\to M_L^{(s)} + \delta M_L^{(s)}(\phi) \equiv M_L^{(s)}(\phi) \\ M_R^{(s)} &\to M_R^{(s)} + \delta M_R^{(s)}(\phi) \equiv M_R^{(s)}(\phi) \end{aligned}$$

These mass shifts describe "response" of the system to Higgs fluctuations around minimum **across the entire infinite towers of states!**

- Not all masses are affected --- only the masses of those states which couple to the Higgs field!
- This becomes a model-dependent question.
- In general, we can treat $M_{L,R}(\phi)$ as functions of ϕ, knowing that in many cases these functions will have no ϕ-dependence if the corresponding state doesn't couple to the Higgs.

Modular invariance:

$$\delta M_L(\phi) = \delta M_R(\phi)$$

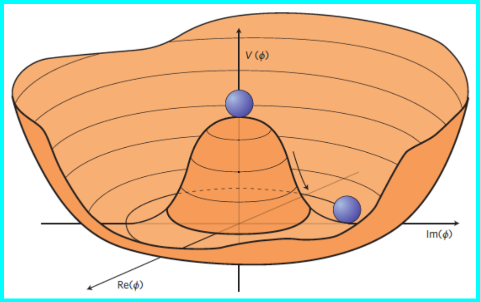
Very important!

Otherwise, fluctuation would break symmetry under T: $\tau \to \tau+1$.

One-loop effective potential also becomes ϕ-dependent:

$$\Lambda(\phi) \equiv -\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\mathcal{Z}(\tau,\overline{\tau},\phi)$$

Z = same as before, only now with masses $M_{L,R}(\phi)$ = functions of ϕ.



$$\Rightarrow \quad m_\phi^2 \equiv \left.\frac{d^2\Lambda(\phi)}{d\phi^2}\right|_{\phi=0}$$

Calculating the second ϕ-derivative, we have

$$\frac{\partial^2 Z(\tau)}{\partial\phi^2} = \tau_2^{-1}\sum_{\text{states}}(-1)^F\, X\, e^{-\pi\tau_2 M^2/M_s^2}\, e^{-\pi i\tau_1 \Delta M^2/2M_s^2}$$

ϕ-independent due to modular invariance!

Insertion into summand!

$$X = -\frac{\pi\tau_2}{M_s^2}\,\partial_\phi^2 M^2 + \left(\frac{\pi\tau_2}{M_s^2}\right)^2 (\partial_\phi M^2)^2$$

<u>*Note*</u>: Insertion X only depends on total M and on τ_2 !
(Higgs-mass contribution for each state)

Truncating to ϕ=0, we therefore generally have

$$X\Big|_{\phi=0} \;=\; \frac{1}{\mathcal{M}^2}\left[\,\tau_2\mathbb{X}_1 + \tau_2^2\,\mathbb{X}_2\,\right]$$

where

$$\boxed{\begin{aligned} \mathbb{X}_1 &= -\frac{1}{4\pi}\partial_\phi^2 M^2\Big|_{\phi=0} \\ \mathbb{X}_2 &= \frac{1}{16\pi^2\mathcal{M}^2}(\partial_\phi M^2)^2\Big|_{\phi=0} \end{aligned}}$$

These two insertions will soon play very different roles!

Thus, defining

$$\langle A\rangle \;\equiv\; \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,\tau_2^{-1}\sum_{\text{states}}(-1)^F\;A\;e^{-\pi\tau_2 M^2/M_s^2}\;e^{-\pi i\tau_1\Delta M^2/2M_s^2}$$

full string amplitude with *A* insertion

we then have

$$m_\phi^2 \;=\; -\,\frac{\mathcal{M}^2}{2}\,\big\langle\tau_2\mathbb{X}_1 + \tau_2^2\,\mathbb{X}_2\big\rangle + \ldots$$

Are we done?

No --- it turns out that these X-insertions into the partition-function sums break modular invariance!   This occurs as the result of a subtle modular "anomaly". Proper modular invariance can only be restored through the addition of an extra term:

$$m_\phi^2 \;=\; -\frac{\mathcal{M}^2}{2}\left\langle \frac{\xi}{4\pi^2} + \tau_2 \mathbb{X}_1 + \tau_2^2\, \mathbb{X}_2 \right\rangle$$

Modular completion: Extra term needed to "complete the square" so that each factor is separately modular-invariant and can be written in terms of modular-covariant derivatives, as required. Needed even though modular-invariant by itself. ξ is a model-dependent *O*(1) coefficient describing how the specific Higgs field is realized in the model.

For the experts, a very rough one-page schematic explanation …

Full details in 2106.04622.

For simplicity and concreteness, let us consider a charge-lattice formulation. Then $M^2 \sim \mathbf{Q}^2$ where $\mathbf{Q}$ = charge lattice vector. Consider holomorphic/anti-holomorphic components separately. Each such component becomes a product of lattice sums, with each sum along a different lattice direction. In general such lattice sums take the forms

$$f(\tau) \;\equiv\; \sum_Q q^{Q^2/2}\;, \quad q \equiv e^{2\pi i\tau}\;.$$

Within such lattice formulations, insertions depend on masses/charges: $X \sim Q^2$. However, with these insertions included these sums now take the form

$$\sum_Q Q^2 q^{Q^2/2} \;\sim\; \frac{d}{d\tau}\left(\sum_Q q^{Q^2/2}\right) \;\sim\; \frac{d}{d\tau} f(\tau)\;.$$

Thus X insertions correspond to τ-derivatives: $X \Longleftrightarrow d/d\tau$.

In general, $Z(\tau)$ is modular-invariant because these sums transform as weight-k reps of the modular group:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \;\sim\; (c\tau+d)^k\, f(\tau)\;.$$

But then
$$\left[\frac{d}{d\tau} f(\tau)\right]_{\tau\to\frac{a\tau+b}{c\tau+d}} \;=\; (c\tau+d)^{k+2}\,\frac{df(\tau)}{d\tau} \,+\, ck(c\tau+d)^{k+1}\, f(\tau)\;.$$

Thus $df/d\tau$ *almost* transforms as a modular function of weight $k+2$ except for an "anomaly" term which appears for $k\neq 0$. *This failure to maintain modular-covariance arises because $d/d\tau$ is not a modular-covariant derivative.* Instead, demand a modular-covariant derivative D_τ for which

$$[D_\tau f(\tau)]_{\tau\to\frac{a\tau+b}{c\tau+d}} \;=\; (c\tau+d)^{k+2}\,[D_\tau f(\tau)] \quad\Longrightarrow\quad D_\tau \;\equiv\; \frac{d}{d\tau} - \frac{ik}{2\tau_2}\;.$$

The extra term in D_τ adjusts for the modular anomalies and restores full modular invariance.

Thus, recombining the holomorphic and anti-holomorphic contributions, our modular-covariant insertions must have the schematic form

$$\begin{aligned}\tau_2^2\,\overline{D}_{\overline{\tau}} D_\tau \;&\sim\; \tau_2^2\,\overline{\mathbf{Q}}^2\mathbf{Q}^2 + \tau_2\left(\mathbf{Q}^2 + \overline{\mathbf{Q}}^2\right) \,+\, \text{constant}\\ &\sim\; \tau_2^2\,\mathbb{X}_2 \,+\, \tau_2\,\mathbb{X}_1 \,+\, \text{constant}\;.\end{aligned}$$

Note that the constant term, although modular-invariant itself, is required in order to build the factorized product of modular-covariant τ-derivatives $\overline{D}_{\overline{\tau}} D_\tau$. The value of the constant ultimately depends on how the holomorphic/anti-holomorphic components are joined together (GSO projections) and other details about how the Higgs scalar is realized across the different lattice directions. Full details in 2106.04622.

So we have $$m_\phi^2 = -\frac{\mathcal{M}^2}{2}\left\langle \frac{\xi}{4\pi^2} + \tau_2 \mathbb{X}_1 + \tau_2^2\,\mathbb{X}_2 \right\rangle$$

How to interpret this new term?

The *X*-terms capture the effects of the mass shifts induced by the fluctuations of the Higgs fields. **However, in string theory, there will also be gravitational back-reactions (specifically deformations of the moduli fields) arising from these fluctuations.** These effects can also make contributions to the Higgs mass. These contributions should be universal, independent of specific *X*-insertions. **Modular invariance has thus automatically given us this extra term!**

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But we recognize this extra term!

$$\Lambda \;=\; -\frac{\mathcal{M}^4}{2}\,\langle \mathbf{1} \rangle$$

Vacuum energy
(cosmological constant)!

We thus have found

$$\boxed{m_\phi^2 \;=\; \frac{\xi}{4\pi^2}\,\frac{\Lambda}{\mathcal{M}^2} - \frac{\mathcal{M}^2}{2}\left\langle \tau_2\,\mathbb{X}_1 + \tau_2^2\,\mathbb{X}_2 \right\rangle}$$

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Editorial comment ---

In ordinary QFT, we would not expect to find such a relation between a Higgs mass and a cosmological constant. Indeed, QFTs do not involve gravity and are thus insensitive to the absolute zero of energy. Even worse, in quantum field theory, the one-loop zero-point function is badly divergent.

String theory, by contrast, not only unifies gauge theories with gravity but also yields a finite Λ (the latter occurring as yet another by-product of modular invariance). Thus, it is only within a string context that such a relation could ever arise.

It is intriguing that such relations join together precisely the two quantities (m_ϕ and Λ) whose values lie at the heart of the two most pressing hierarchy problems in modern physics.

Summary thus far:

To calculate m_ϕ^2 for any given string vacuum and any Higgs scalar therein:

- Calculate one-loop Λ and also calculate the X_i-insertions for each state

$$\mathbb{X}_1 = -\frac{1}{4\pi}\partial_\phi^2 M^2\Big|_{\phi=0}$$
$$\mathbb{X}_2 = \frac{1}{16\pi^2\mathcal{M}^2}(\partial_\phi M^2)^2\Big|_{\phi=0}$$

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where

$$\langle A\rangle \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\tau_2^{-1} \sum_{\text{states}} (-1)^F\, A\; e^{-\pi\tau_2 M^2/M_s^2}\, e^{-\pi i\tau_1 \Delta M^2/2M_s^2}$$

full modular integral

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__Question__: **Is this divergent?**

Λ is presumed finite (no tachyons). Moreover, in "IR" limit, only surviving contributions within integrand $F(\tau)$ are from level-matched ("physical") massless string states!

$$F(\tau) \sim c_0 + c_1 \tau_2 \quad \text{as} \quad \tau_2 \to \infty$$

where

$$c_0 = -\tfrac{1}{2}\mathcal{M}^2 \underset{M=0}{\mathrm{Str}} \; \mathbb{X}_1$$
$$c_1 = -\tfrac{1}{2}\mathcal{M}^2 \underset{M=0}{\mathrm{Str}} \; \mathbb{X}_2$$

Higgs mass is at most *logarithmically* divergent (from X_2-charged massless string states), finite otherwise!

This is already remarkable ---

- Just as the one-loop vacuum energy in any tachyon-free closed string theory is *finite* as a result of modular invariance, the corresponding Higgs mass is at most *logarithmically divergent*.

- Modular invariance has thus induced a significant softening of the Higgs divergence, reducing what would have been a *quadratic* UV Higgs divergence in field theory into a *logarithmic* Higgs divergence in string theory.

But it's still divergent! Need to *regulate* in such a way that...

- Will allow us to isolate divergences and extract finite results
- Will allow us to employ Rankin-Selberg technique to express the full string amplitudes (and not just their divergences) in terms of only physical states at all mass levels, just as with Λ
- Will ultimately allow us to extract an "EFT" description of the Higgs mass and study how it "runs" as a function of scale.

Many different regulators are possible!

- Remove contributions from massless states that cause divergences, handle separately
    - Essentially approach taken by Kaplunovsky in his classic 1987 paper establishing the formalism for calculating string threshold correction for gauge couplings in which there are similar logarithmic divergences.

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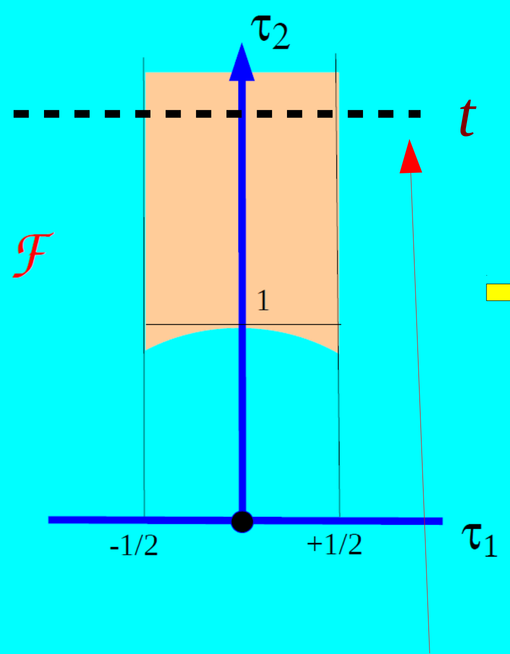
  - Essentially approach taken by Kaplunovsky in his classic 1987 paper establishing the formalism for calculating string threshold correction for gauge couplings in which there are similar logarithmic divergences.

- Remove divergences from dangerous regions of $\mathcal{F}$. For example, for integrands which diverge as $F(\tau) \sim c_0+c_1\tau_2$ as $\tau_2 \to$ infinity, define



$$\widehat{I}(t) \equiv \int_{\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} F + \int_{\mathcal{F}-\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} (F - c_1\tau_2)$$

- D. Zagier (1981)

$$\widehat{I}(t) = \frac{\pi}{3} \operatorname*{Res}_{s=1} \int_0^\infty d\tau_2\, \tau_2^{s-2} \Big[g(\tau_2) - c_0 - c_1\tau_2 \Big] + \frac{\pi}{3} c_0 + \log\left(4\pi\, t\, e^{-\gamma}\right) c_1$$

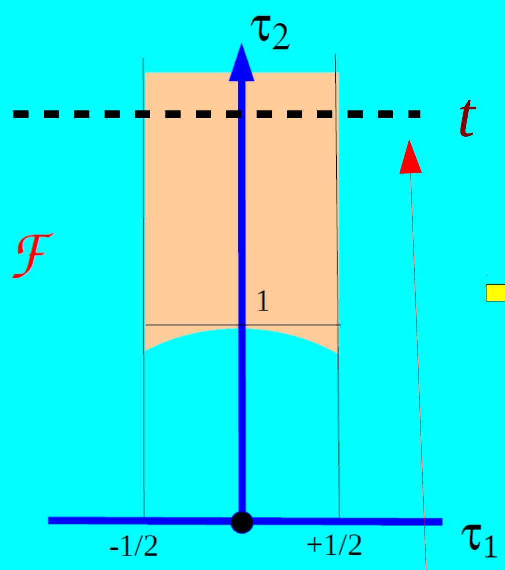
Essentially regulated version of Rankin-Selberg!

t = regularization parameter,
$t \to$ infinity removes regulator

Many different regulators are possible!

- Remove contributions from massless states that cause divergences, handle separately
  - Essentially approach taken by Kaplunovsky in his classic 1987 paper establishing the formalism for calculating string threshold correction for gauge couplings in which there are similar logarithmic divergences.

- Remove divergences from dangerous regions of $\mathcal{F}$. For example, for integrands which diverge as $F(\tau) \sim c_0 + c_1\tau_2$ as $\tau_2 \to$ infinity, define



$$\widehat{I}(t) \equiv \int_{\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} F + \int_{\mathcal{F}-\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} (F - c_1\tau_2)$$

- D. Zagier (1981)

$$\widehat{I}(t) = \frac{\pi}{3} \operatorname*{Res}_{s=1} \int_0^\infty d\tau_2\, \tau_2^{s-2} \left[g(\tau_2) - c_0 - c_1\tau_2 \right] + \frac{\pi}{3} c_0 + \log\left(4\pi\, t\, e^{-\gamma}\right) c_1$$

Essentially regulated version of Rankin-Selberg!

t = regularization parameter,
$t \to$ infinity removes regulator

Logarithmic dependence on t, as expected. Looks like an RGE for an EFT "running" for $I(t)$ with t interpreted as a RG scale μ!

$$\frac{\mu^2}{M_s^2} = 1/t$$

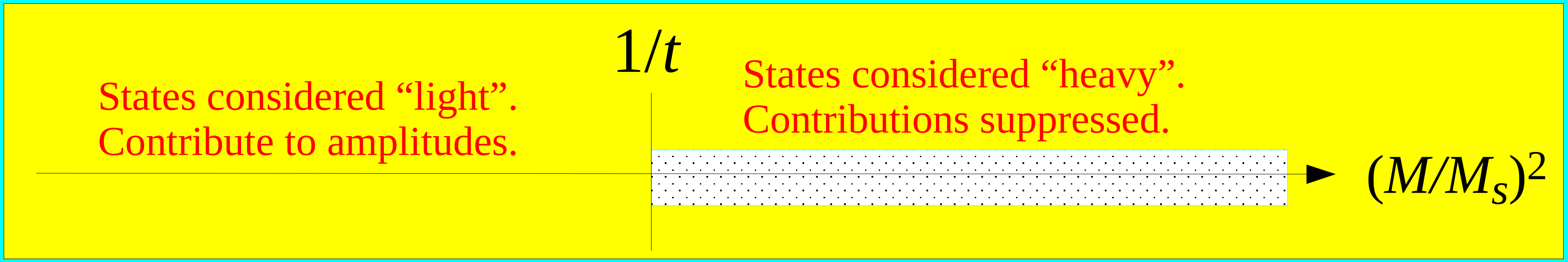
Comment: Such an identification of t with an EFT "floating scale" μ is not an accident.

Recall:

$$Z(\tau) = \frac{1}{\tau_2^{D/2-1}} \sum_{\text{states}} (-1)^F \, e^{-\pi\tau_2 M^2/M_s^2} \, e^{-\pi i\tau_1 \Delta M^2/2M_s^2}$$

Looks like a Boltzmann suppression: contributions suppressed as functions of τ_2

For any demarcation line $\tau_2 = t$, can thus define which states still effectively contribute, which ones do not:



As t increases, increasingly many states no longer contribute!
This is analogous to "integrating out" states with floating scale μ.

$$\frac{\mu^2}{M_s^2} = 1/t$$

General identification for extracting an EFT…

But neither of these regulator approaches is modular-invariant!

- Cannot just remove contributions of massless states while leaving all other states intact --- modular invariance mixes the states at all mass levels!

- Cannot deform integrand/integration region by sharply imposing *t*-cutoffs at certain values of τ_2 --- this is very “field-theoretic” but also breaks modular invariance!

But neither of these regulator approaches is modular-invariant!

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- Cannot deform integrand/integration region by sharply imposing *t*-cutoffs at certain values of τ_2 --- this is very "field-theoretic" but also breaks modular invariance!

Instead, let us deform our entire theory in a modular-invariant way:

$$I \equiv \int_{\mathcal{F}} \frac{d^2\tau_2}{\tau_2^2}\, F(\tau) \quad \longrightarrow \quad \hat{I}_{\mathcal{G}} \equiv \int_{\mathcal{F}} \frac{d^2\tau_2}{\tau_2^2}\, F(\tau)\mathcal{G}(\tau)$$

regulator

where $\mathcal{G}(\tau)$ has certain properties:

- Must be a modular-invariant function (smooth, *etc.*)
- Should have straightforward physical interpretation
- Must suppress power-law divergences at IR cusp (as $\tau_2 \to$ infinity)
- Should have internal parameters (analogues of *t*) with a limit corresponding to removing the regulator (*i.e.*, $\mathcal{G} \to 1$)
- Aside from suppressing IR divergences near IR cusp, should leave the rest of the theory intact as much as possible
- Should be amenable to extraction of physical-state supertraces after regularization
- Must have additional symmetry properties under transformations of the internal parameters (to be discussed later)

Motivated by prior results in the literature, we have developed such a regulator function which satisfies all of these criteria!

$\mathcal{G}_\rho(a,\tau)$: two internal parameters (ρ, a)

For the experts...

$$Z_{\rm circ}(a,\tau) \equiv \sqrt{\tau_2} \sum_{m,n\in\mathbb{Z}} e^{-\pi\tau_2(m^2a^2+n^2/a^2)}\, e^{2\pi i mn\tau_1}$$

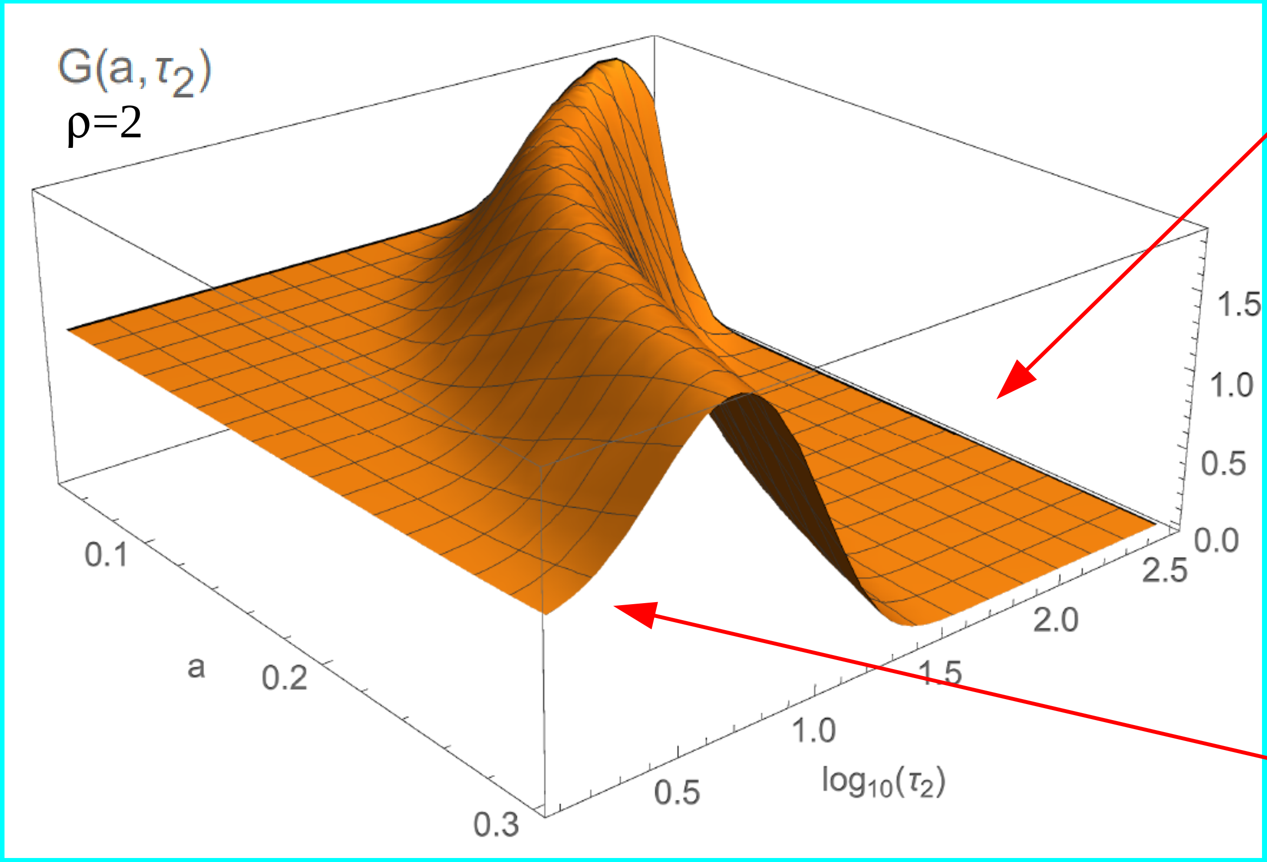
Partition function of WS scalar field on circle $a = 1/(M_s R)$

$$\mathcal{G}_\rho(a,\tau) \equiv \frac{1}{1+\rho a^2}\,\frac{\rho}{\rho-1}\, a^2 \frac{\partial}{\partial a}\Big[Z_{\rm circ}(\rho a,\tau) - Z_{\rm circ}(a,\tau)\Big]$$

Important points...

- $\mathcal{G}_\rho(a,\tau) \to 1$ for all τ as $a \to 0$. Thus $a \to 0$ restores original unregulated theory.
- $\mathcal{G}_\rho(a,\tau) \to 0$ *exponentially rapidly* as $\tau_2 \to \infty$ for any $a > 0$.
- For $\rho = 2$ and $a = 1/\sqrt{k+2}$, has direct physical interpretation.
- Invariant under $\rho a^2 \to 1/(\rho a^2)$. Will be critical.

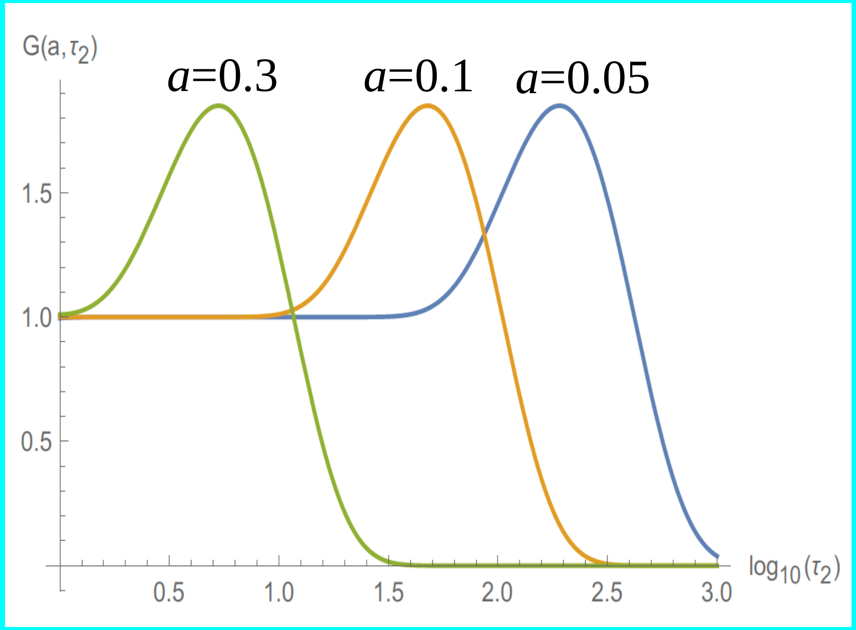
Corresponds to turning on field in background string geometry such that CFT associated with flat 4D spacetime is replaced by that associated with $SU(2)_k$ WZW model. (Kiritsis-Kounnas,'95)



$\mathcal{G} \to 0$ exponentially rapidly: kills "IR" divergences here.

Good regulator for 0< a <<1.

$\mathcal{G}$ =1: preserves theory here.



Locations of transition regions scale as $1/(\rho a^2)$.

$$\boxed{\frac{\mu^2}{M_s^2} = \rho a^2}$$

So we use our modular-invariant regulator to regulate the Higgs mass!

$$m_\phi^2 \;=\; -\,\frac{\mathcal{M}^2}{2}\left\langle \frac{\xi}{4\pi^2} + \tau_2 \mathbb{X}_1 + \tau_2^2\, \mathbb{X}_2 \right\rangle_{\mathcal{G}}$$

String amplitude deformed by $\mathcal{G}$.

Contributions from $X_{\rm i}$-charged states

Contributions from universal gravitational backreactions (Λ)

So we use our modular-invariant regulator to regulate the Higgs mass!

$$m_\phi^2 \;=\; -\,\frac{\mathcal{M}^2}{2}\left\langle \frac{\xi}{4\pi^2} + \tau_2\mathbb{X}_1 + \tau_2^2\,\mathbb{X}_2 \right\rangle_{\mathcal{G}}$$

String amplitude deformed by $\mathcal{G}$.

Now finite! We can therefore use the Rankin-Selberg relation, taking residues, *etc.*, in order to express this amplitude in terms of supertraces over only physical states! Calculations are a mess...

Contributions from X_i-charged states

Contributions from universal gravitational backreactions (Λ)

Results are...

$$
\begin{aligned}
\widehat{m}_\phi^2(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{
&\underset{M=0}{\mathrm{Str}}\ \mathbb{X}_1 \left[-\frac{\pi}{6}\left(1+\mu^2/M_s^2\right)\right] \\
&+ \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right] \\
&+ \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_1 \left\{-\frac{\pi}{6} - \frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times \right. \\
&\qquad \left. \times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\right\} \\
&+ \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_2 \left[2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) - \mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}
\end{aligned}
$$

+

$$
\begin{aligned}
\widehat{\Lambda}(\mu) = \frac{1}{1+\mu^2/M_s^2}\Bigg\{ &\frac{\mathcal{M}^2}{24}\,\mathrm{Str}\, M^2 \\
&- \frac{7}{960\pi^2}\left(n_B - n_F\right)\mu^4 \\
&- \frac{1}{2\pi^2}\underset{M>0}{\mathrm{Str}}\ M^4 \left[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right. \\
&\qquad + 4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) \\
&\qquad \left. + \mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}
\end{aligned}
$$

… multiplied by $\xi/(4\pi^2\mathcal{M}^2)$

where

$$
\mathcal{K}_\nu^{(n,p)}(z) \equiv \sum_{r=1}^{\infty} (rz)^n \left[K_\nu(rz/\rho) - \rho^p K_\nu(rz)\right]
$$

combinations of infinite sums of modified Bessel functions of the second kind...

Bessel functions!

$$
\begin{aligned}
\widehat{m}_\phi^2(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{ & \\
& \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_1 \left[-\frac{\pi}{6}\left(1+\mu^2/M_s^2\right)\right] \\
& + \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right] \\
& + \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_1 \Bigg\{-\frac{\pi}{6} - \frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times \\
& \qquad \times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\} \\
& + \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_2 \left[2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) - \mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}
\end{aligned}
$$

$$+$$

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& - \frac{7}{960\pi^2}\boxed{(n_B - n_F)}\,\mu^4 \\
& - \frac{1}{2\pi^2}\,\boxed{\underset{M>0}{\mathrm{Str}}\ M^4}\,\Bigg[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) \\
& \qquad + 4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) \\
& \qquad + \mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\Bigg]\Bigg\}
\end{aligned}
$$

… multiplied by $\xi/(4\pi^2\mathcal{M}^2)$

Supertraces over spectrum weighted by combinations of Bessel functions.

Supertraces of *X-charges*

Supertraces of *masses*

$$\widehat{m}_\phi^2(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{$$

$$\underset{M=0}{\mathrm{Str}}\ \mathbb{X}_1 \left[-\frac{\pi}{6}\boxed{(1}+\mu^2/M_s^2)\right]$$

$$+\ \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right]$$

$$+\ \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_1 \left\{\boxed{-\frac{\pi}{6}} - \frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times\right.$$

$$\left.\times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\right\}$$

$$+\ \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_2 \left[2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) - \mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}$$

$+$

$$\widehat{\Lambda}(\mu) = \frac{1}{1+\mu^2/M_s^2}\Bigg\{\boxed{\frac{\mathcal{M}^2}{24}\,\mathrm{Str}\,M^2}$$

$$-\frac{7}{960\pi^2}(n_B-n_F)\,\mu^4$$

$$-\frac{1}{2\pi^2}\underset{M>0}{\mathrm{Str}}\ M^4\left[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right.$$

$$+4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)$$

$$\left.+\mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}$$

… multiplied by $\xi/(4\pi^2\mathcal{M}^2)$

"IR" limit:

$$\boxed{\lim_{\mu\to 0}\widehat{m}_\phi^2(\mu) = \frac{\xi}{4\pi^2}\frac{\Lambda}{\mathcal{M}^2} - \frac{\pi}{6}\mathcal{M}^2\,\mathrm{Str}\,\mathbb{X}_1}$$

assuming $\underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 = 0$

<u>*All*</u> states contribute, even in deep IR !

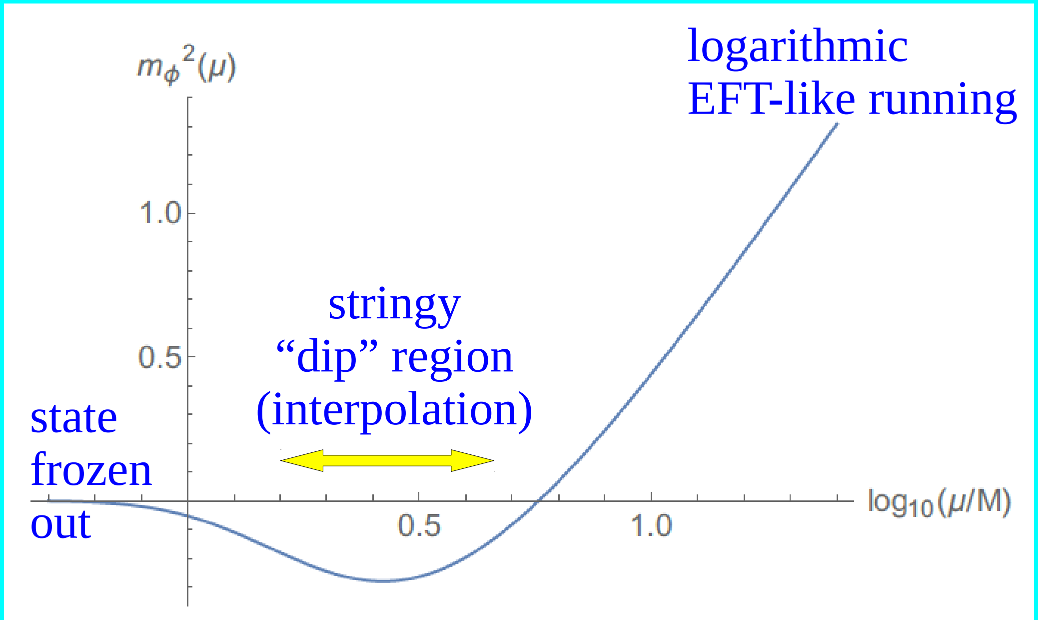
$$\widehat{m}_\phi^2(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{ \underset{M=0}{\mathrm{Str}}\, \mathbb{X}_1 \left[-\frac{\pi}{6}\left(1+\mu^2/M_s^2\right)\right] + \underset{M=0}{\mathrm{Str}}\,\mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right] + \underset{M>0}{\mathrm{Str}}\,\mathbb{X}_1\left\{-\frac{\pi}{6} - \frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\right\} + \underset{M>0}{\mathrm{Str}}\,\mathbb{X}_2 \left[\boxed{2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) - \mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)}\right]\Bigg\}$$

$$+$$

$$\widehat{\Lambda}(\mu) = \frac{1}{1+\mu^2/M_s^2}\Bigg\{\frac{\mathcal{M}^2}{24}\,\mathrm{Str}\,M^2 - \frac{7}{960\pi^2}\left(n_B - n_F\right)\mu^4 - \frac{1}{2\pi^2}\underset{M>0}{\mathrm{Str}}\,M^4\left[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + 4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}$$

… multiplied by $\xi/(4\pi^2\mathcal{M}^2)$

Contribution from each bosonic state of mass *M* per unit X_2 charge:



(Must then evaluate supertrace of this curve over the full string spectrum.)

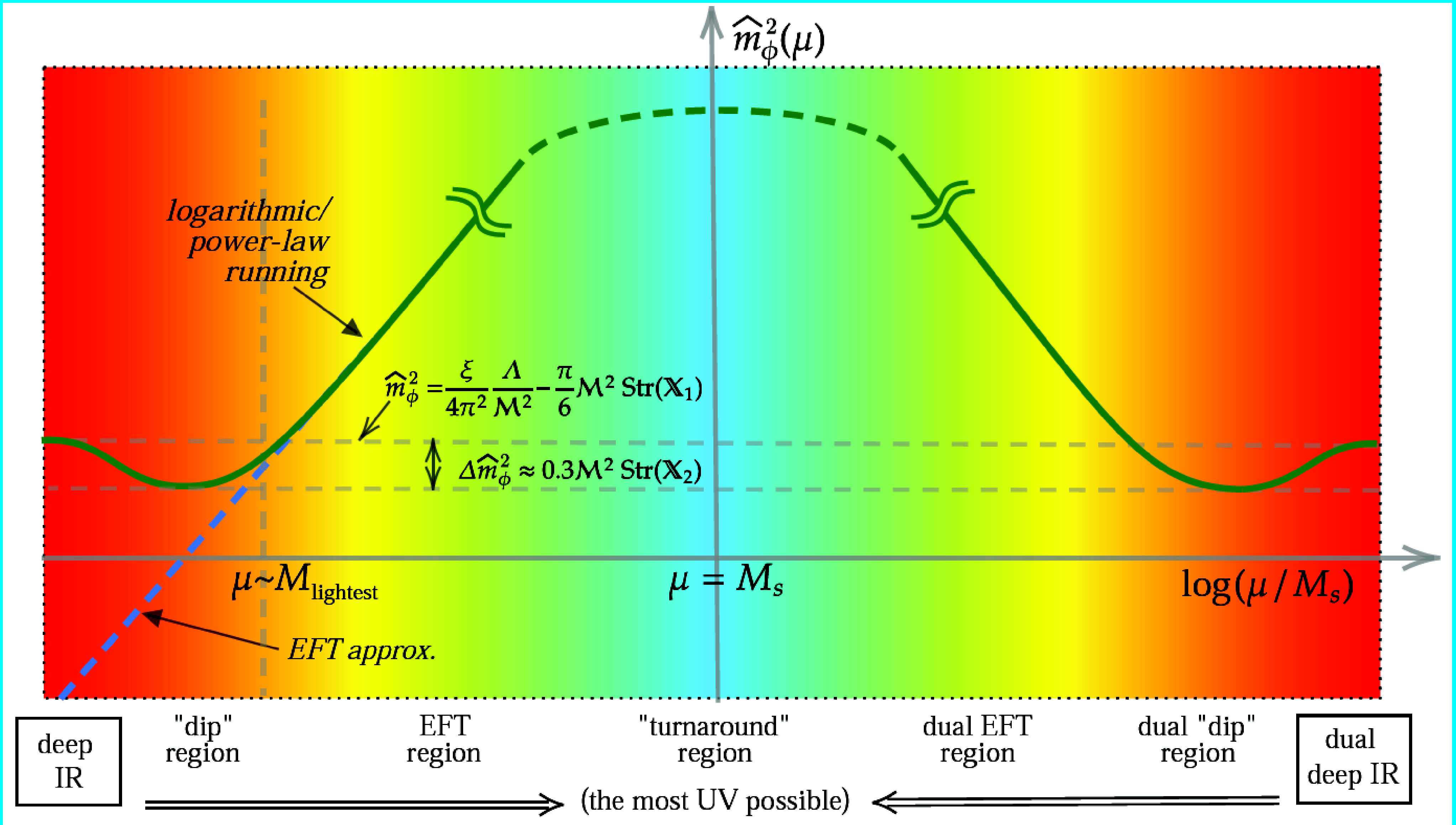
$$\widehat{m}_\phi^2(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{ \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_1 \left[-\frac{\pi}{6}\left(1+\mu^2/M_s^2\right)\right] + \underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right] + \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_1 \left\{-\frac{\pi}{6} - \frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times \times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\right\} + \underset{M>0}{\mathrm{Str}}\ \mathbb{X}_2 \left[2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) - \mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}$$

$$+$$

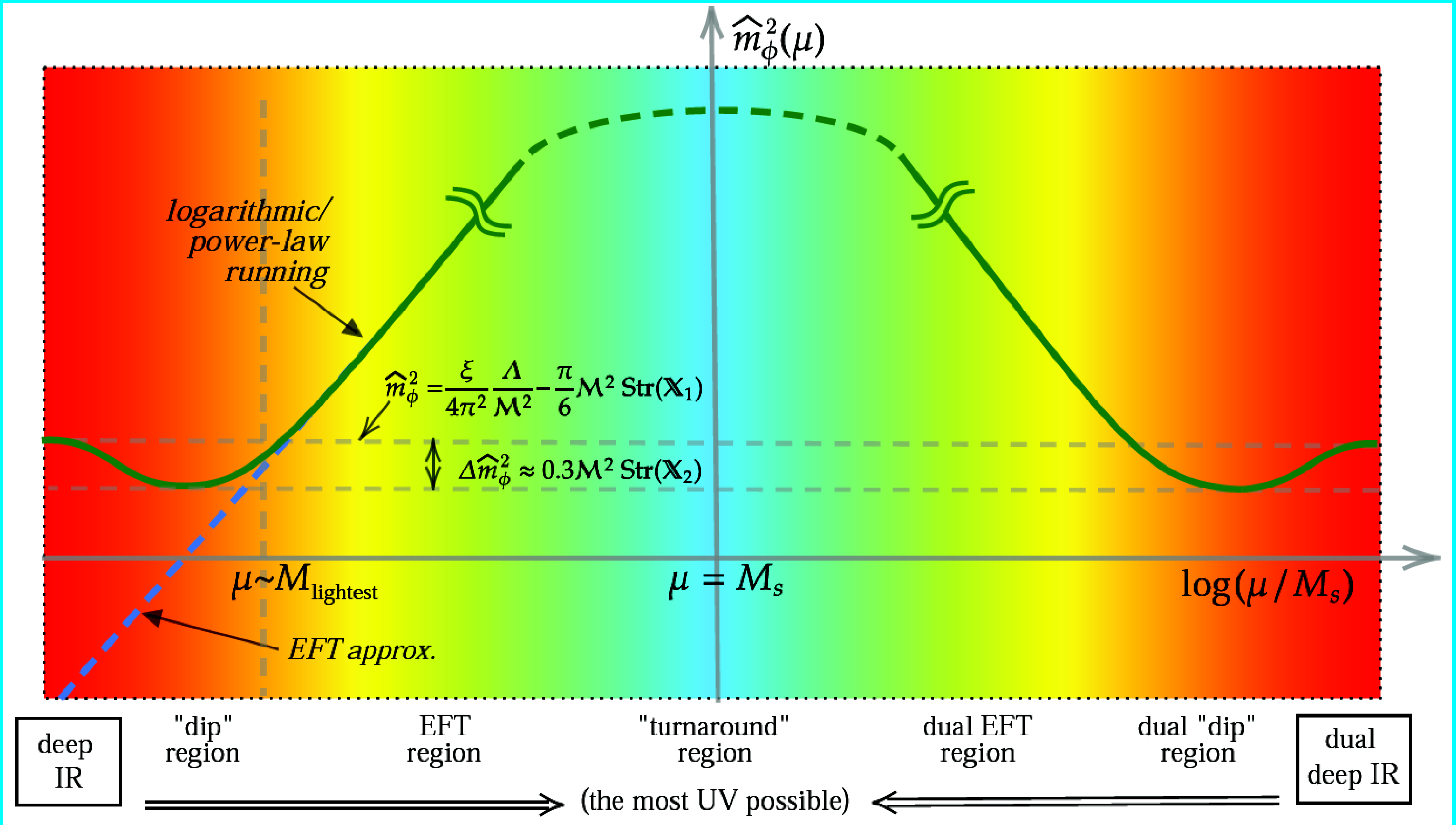
$$\widehat{\Lambda}(\mu) = \frac{1}{1+\mu^2/M_s^2}\Bigg\{\frac{\mathcal{M}^2}{24}\,\mathrm{Str}\,M^2 - \frac{7}{960\pi^2}(n_B-n_F)\,\mu^4 - \frac{1}{2\pi^2}\underset{M>0}{\mathrm{Str}}\ M^4\left[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + 4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) + \mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}$$

All other contributions are significantly subleading.

Thus, putting the pieces together, we obtain...



“Anatomy” of the running of the Higgs mass!

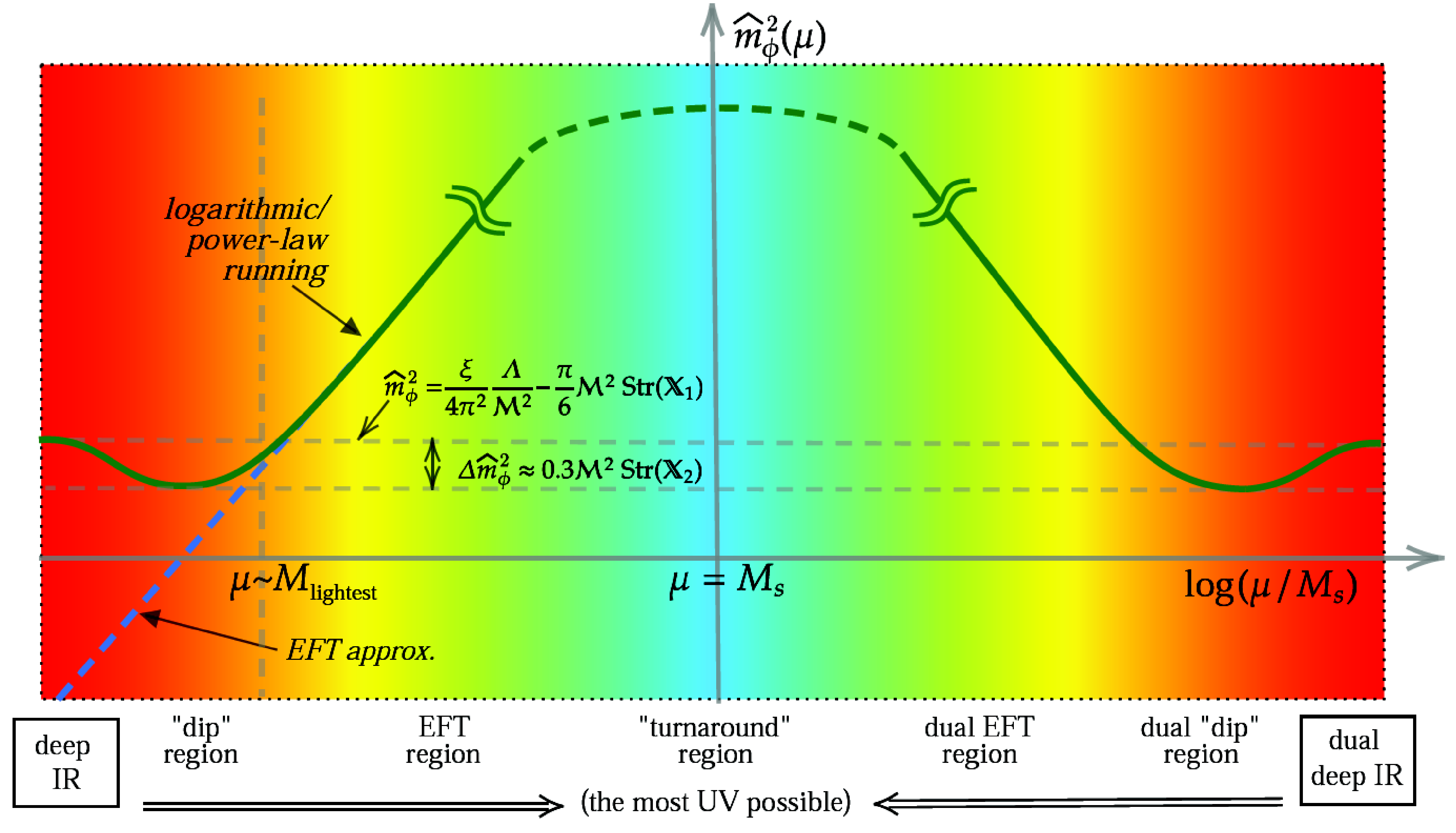


Blue curve only if non-zero net # of massless X_2-charged states

M_{lightest} ~ mass of lightest massive X_2-charged states

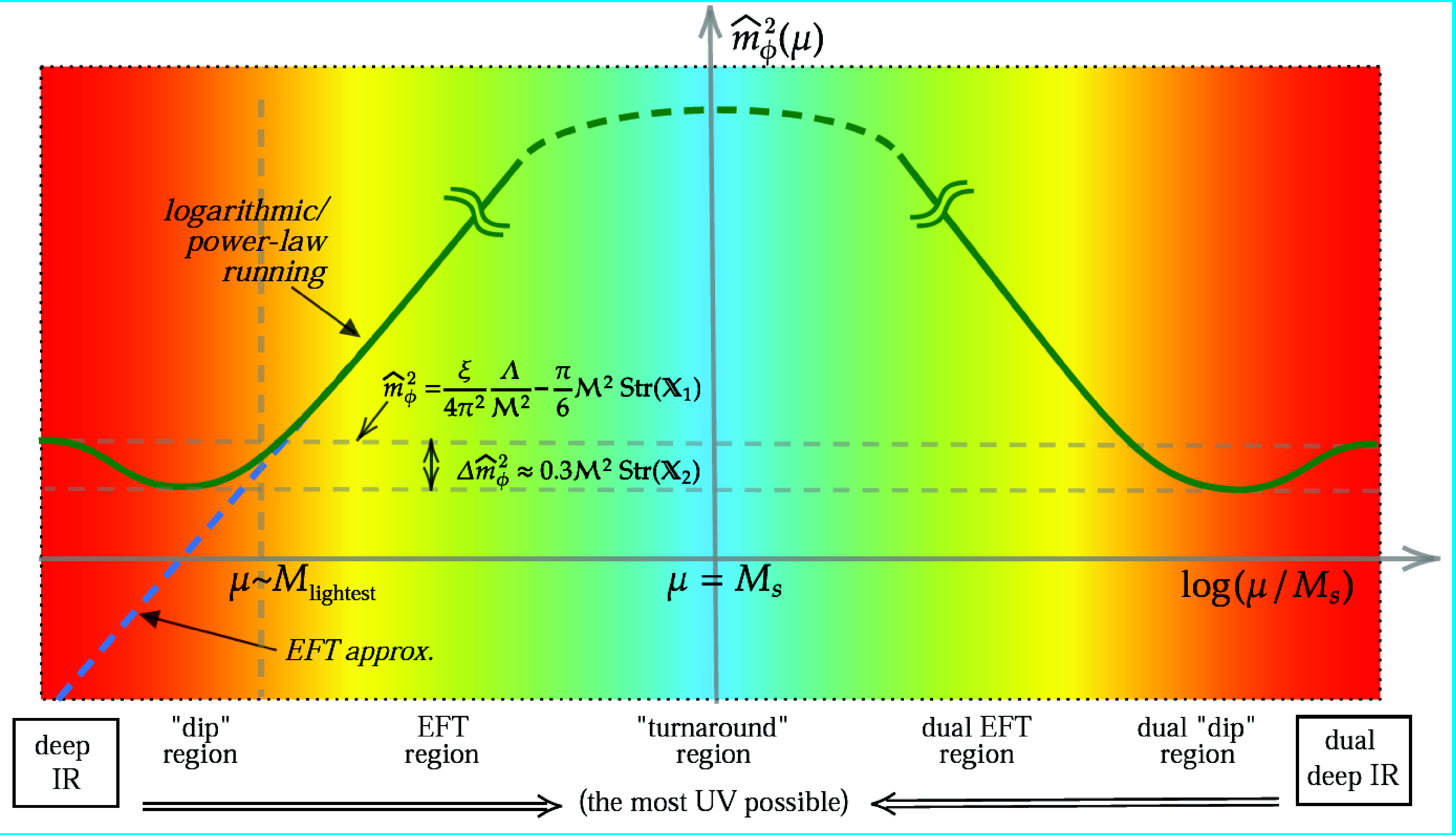
Generally logarithmic, can become effectively power-law if density of X_2-charged states is high

“Dual” region!



$$\mu \;\rightarrow\; M_s{}^2 / \mu$$

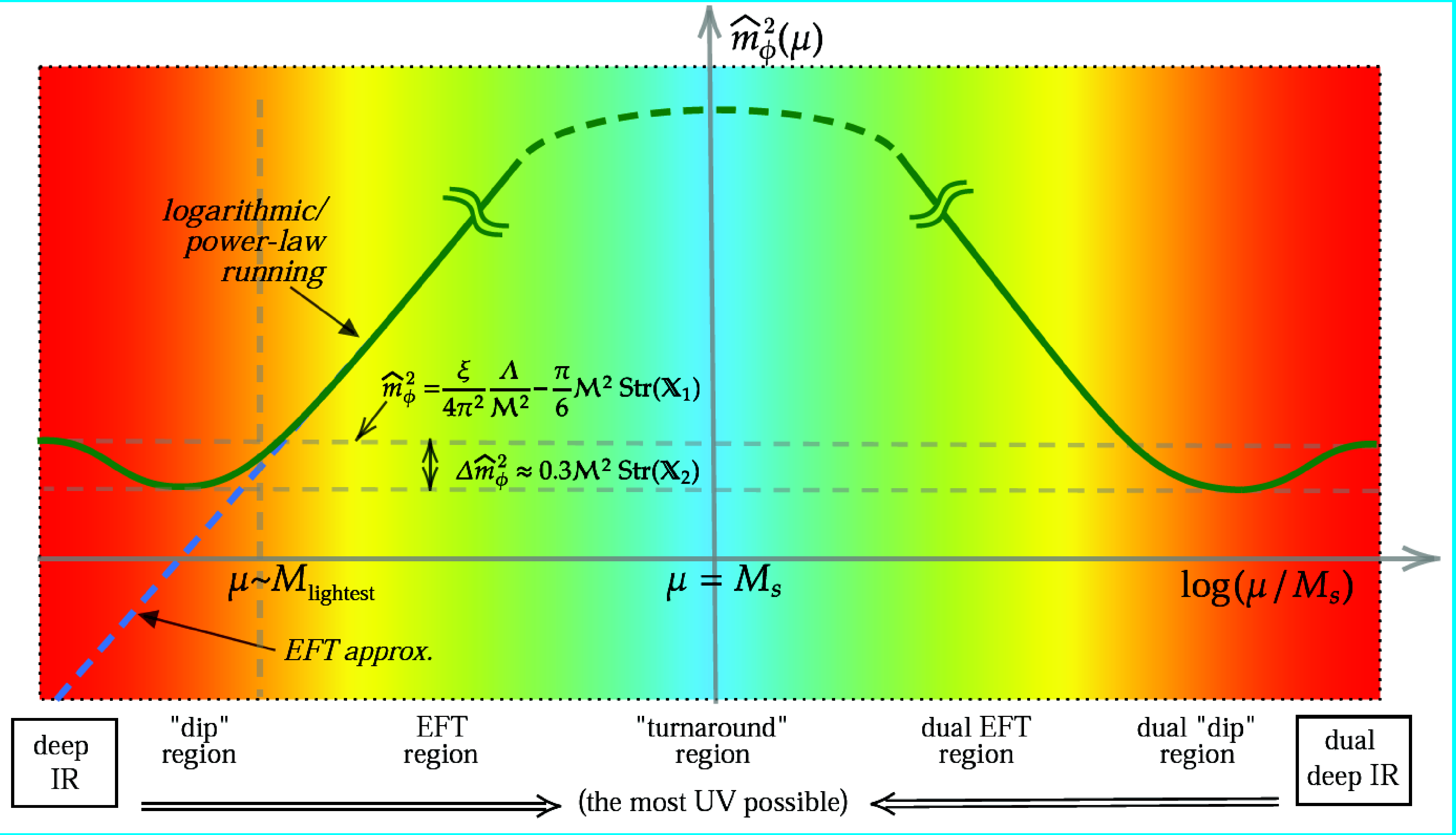
Scale duality!



Background colors indicate this duality

There is a maximum degree to which we can probe "UV" behavior --- increasing μ further only re-introduces IR-like behavior!

Theory must have vanishing β-function at self-dual scale!



Scale duality requires that even the "deep IR" (which would ordinarily only care about light states) must know about the "dual deep IR" (in which all states contribute)! Both must be determined/regulated together!

(*e.g.*, "depth" of tachyons determines Hagedorn temp!)

$\mu \rightarrow M_s^2 / \mu$

Whence scale duality?

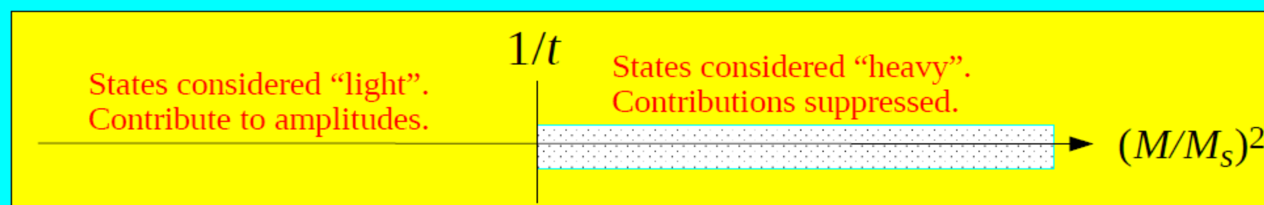
- Inevitable consequence of using a modular-invariant regulator through which to extract EFT-like behavior

Recall:

$$Z(\tau) = \frac{1}{\tau_2^{D/2-1}} \sum_{\text{states}} (-1)^F \, e^{-\pi\tau_2 M^2/M_s^2} \, e^{-\pi i \tau_1 \Delta M^2 / 2M_s^2}$$

Looks like a Boltzmann suppression: contributions suppressed as functions of τ_2

For any demarcation line $\tau_2 = t$, can define which states still effectively contribute, which ones do not according to this factor:



As t increases, increasingly many states no longer contribute!
This is analogous to "integrating out" states with floating scale μ.

$$\frac{\mu^2}{M_s^2} = 1/t$$

General identification for extracting an EFT…

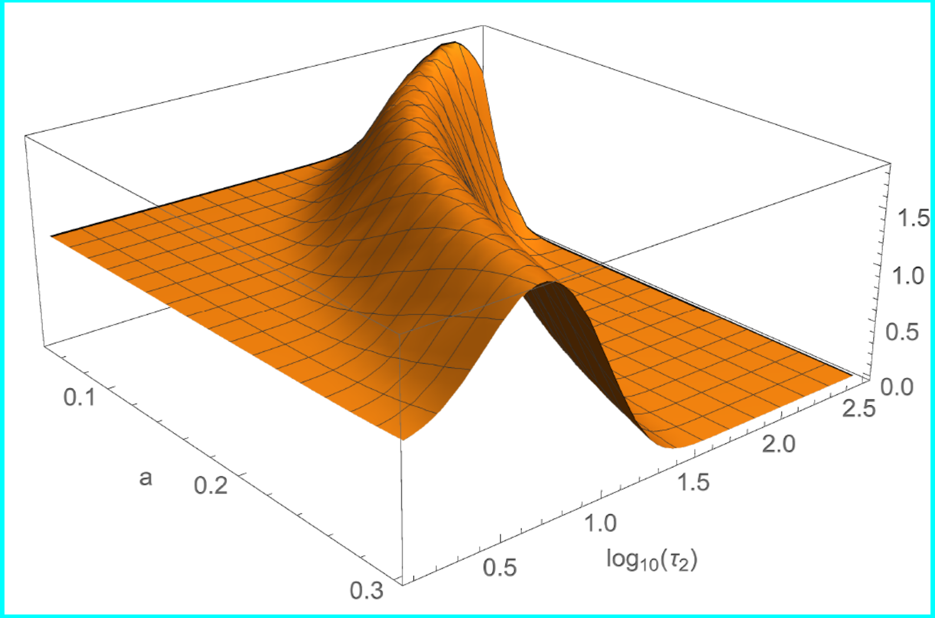
- But modular invariance along the $\tau_1 = 0$ axis implies symmetry under $\tau_2 \rightarrow 1/\tau_2$, or equivalently $t \rightarrow 1/t$.

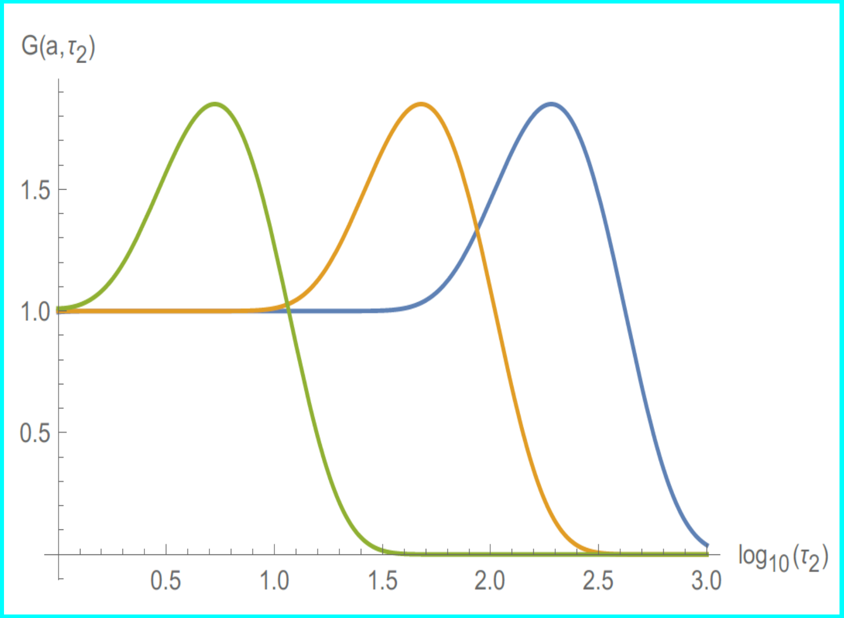
Scale duality!

$\mu \to M_s^2 / \mu$

Whence scale duality?

- Equivalently, because our $\mathcal{G}$-function is modular-invariant, it also suppresses contributions in the opposite direction!





Thus we could have chosen to identify our "*t*" cutoff with *1/t* instead! Once again implies scale duality!

Our modular-invariant regulator simultaneously regulates *both* the IR and the UV because they are the same!

Does scale duality also emerge directly from the Bessel-function expressions?

- Our calculations leading to these expressions were careful to respect modular invariance as long as all leading and subleading terms are retained.
- Thus, these expressions *do* reflect scale duality!

However, this duality is not *manifest* because it relies on misaligned SUSY and non-trivial identities between the different supertraces. (Indeed, this becomes a way of *deriving* such identities.)

e.g., trivial example:

$$\mathrm{Str}\,\mathbf{1} ~=~ 0$$

Holds for all tachyon-free string models, even without SUSY! Constraint applies across *entire* string spectrum.

Thus supertraces over light states (EFT) must be balanced against supertraces over heavy states (dual EFT), *etc*...

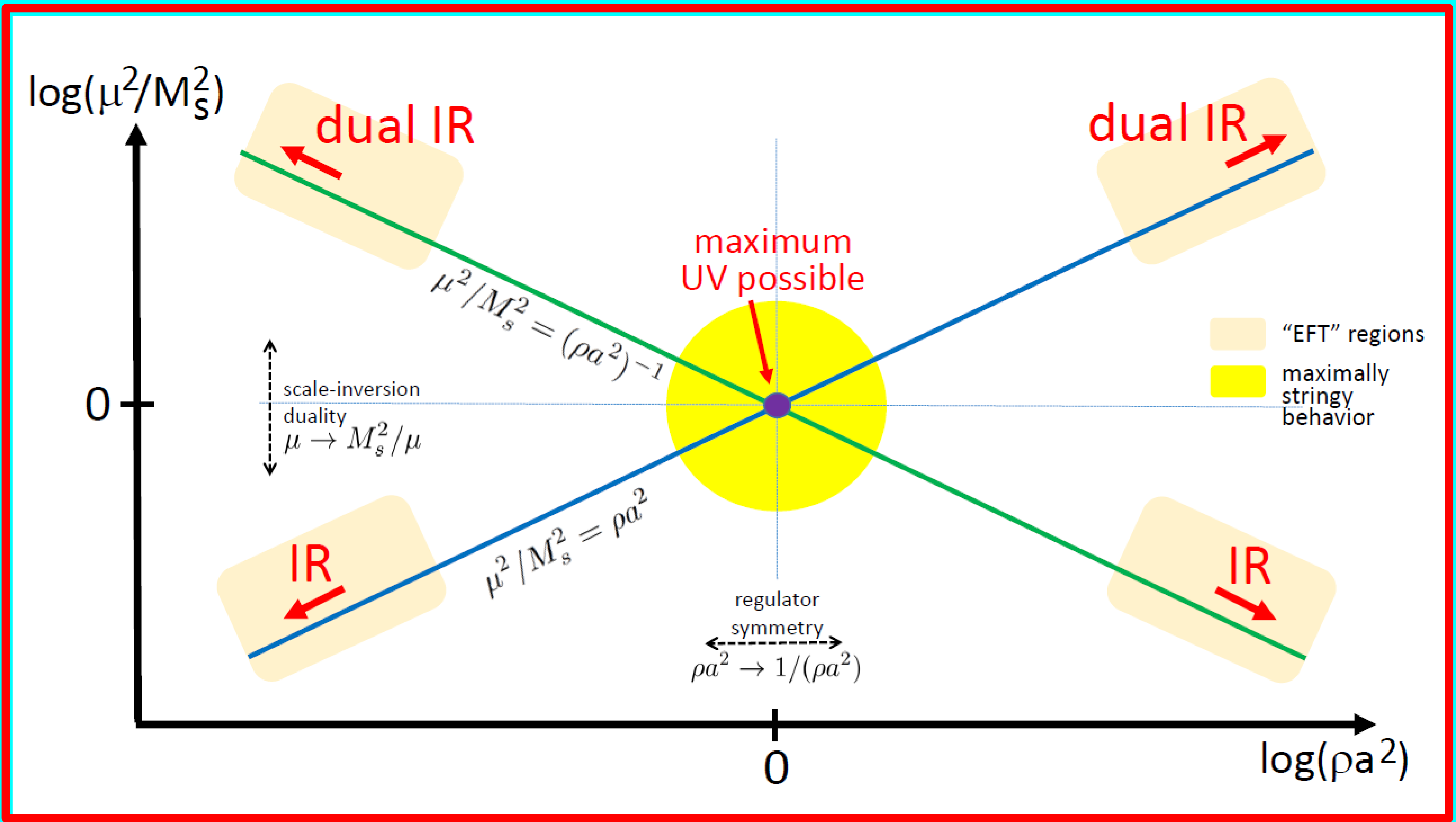
Similar identities relate different X_i-weighted supertraces as well!

Indeed, inherent in our attempt to extract an EFT description from a modular-invariant UV/IR mixed theory is a **choice of direction** as to

- what constitutes “UV” (integrate out);
- what constitutes “IR” (retain).

Making such a choice (in order to establish an EFT) therefore inherently breaks modular invariance!

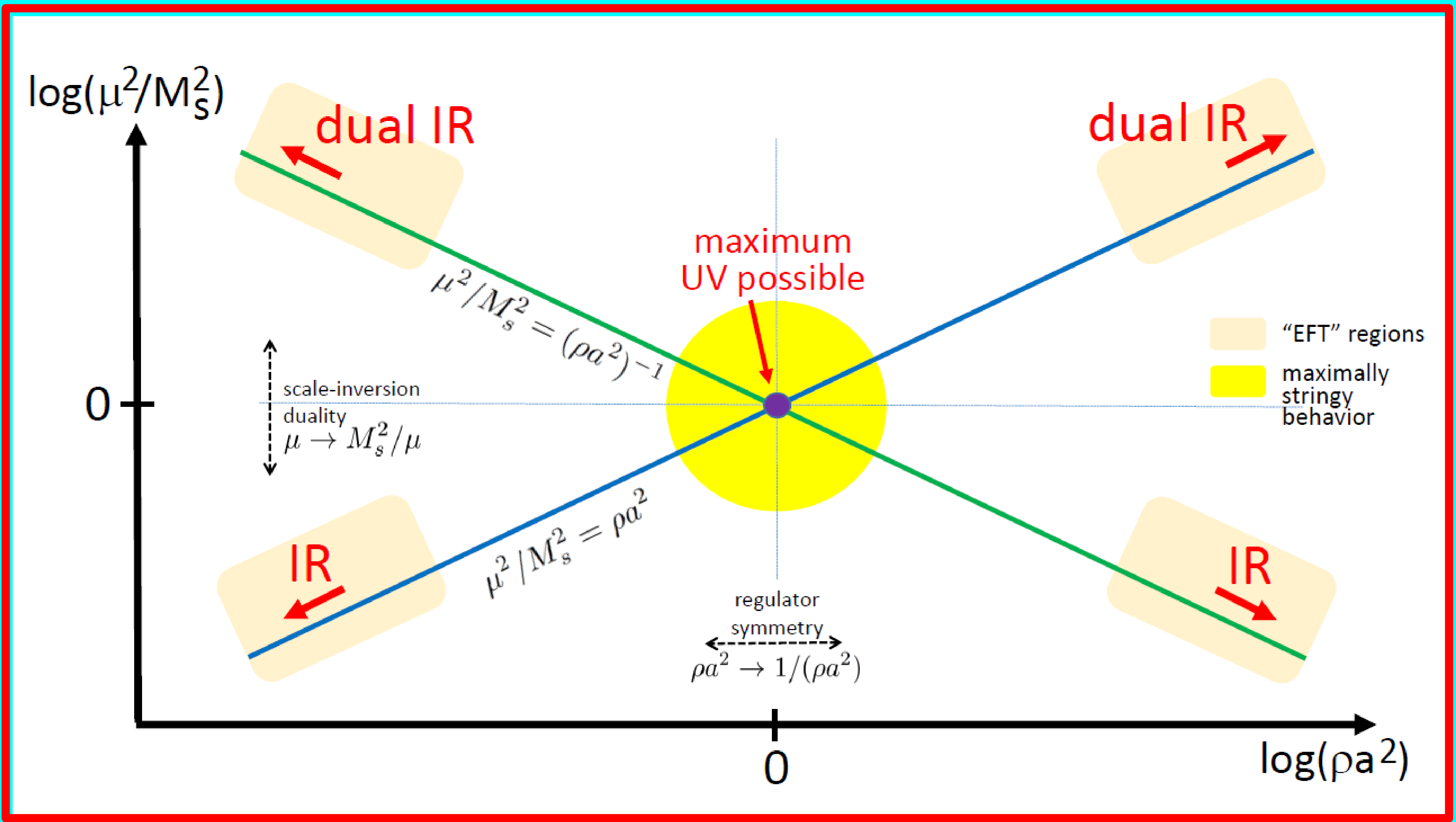
Scale duality is thus part of a deeper structure which exposes the role of EFTs in modular-invariant UV/IR mixed theories…..



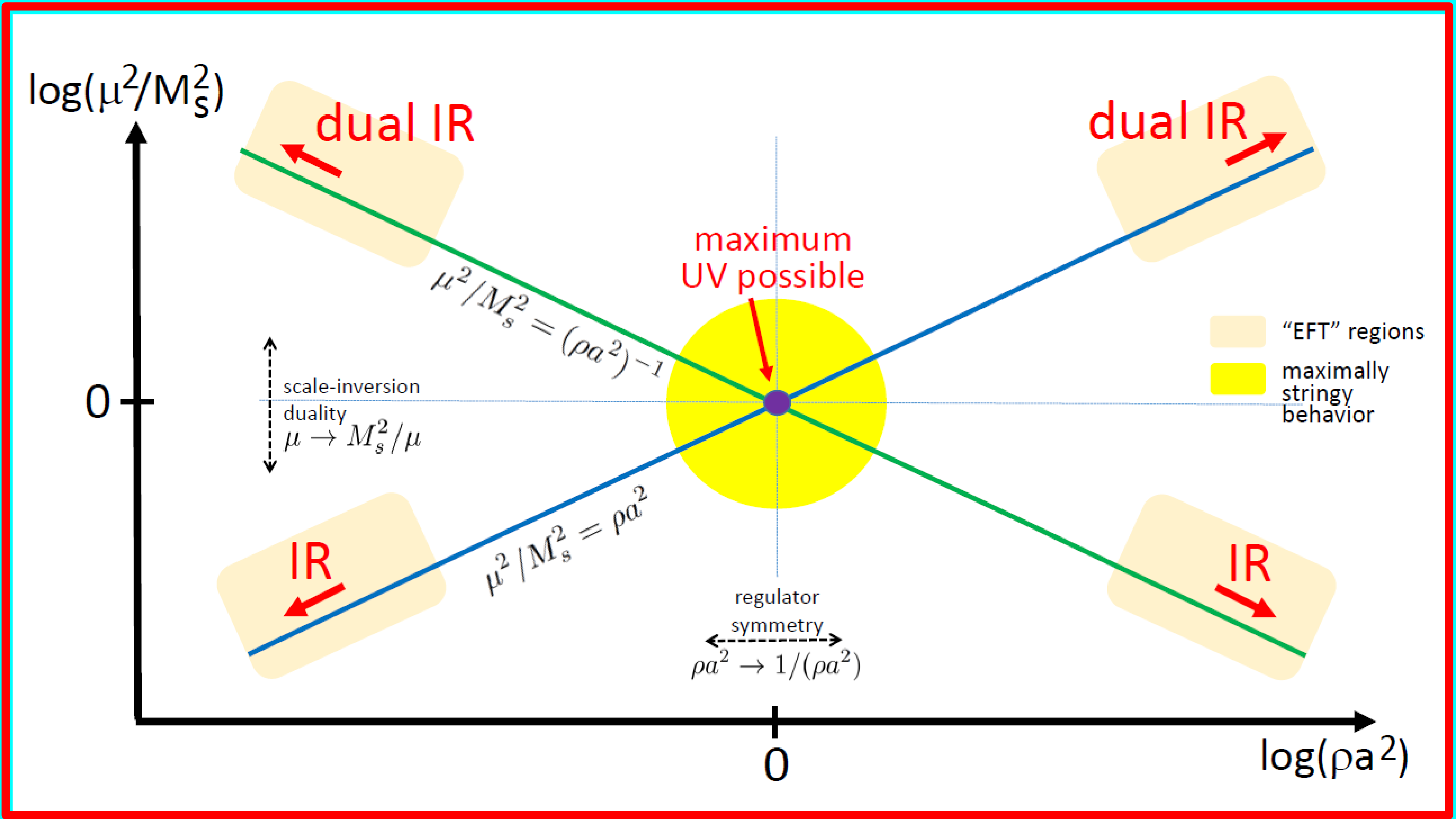
Identify $\mu^2/M_s^2 = \rho a^2$

Scale duality then requires $(\mu^2/M_s^2)^{-1} = \rho a^2$

Mapping between WS and ST physics has *two* branches!
→ *Four-fold* symmetry!

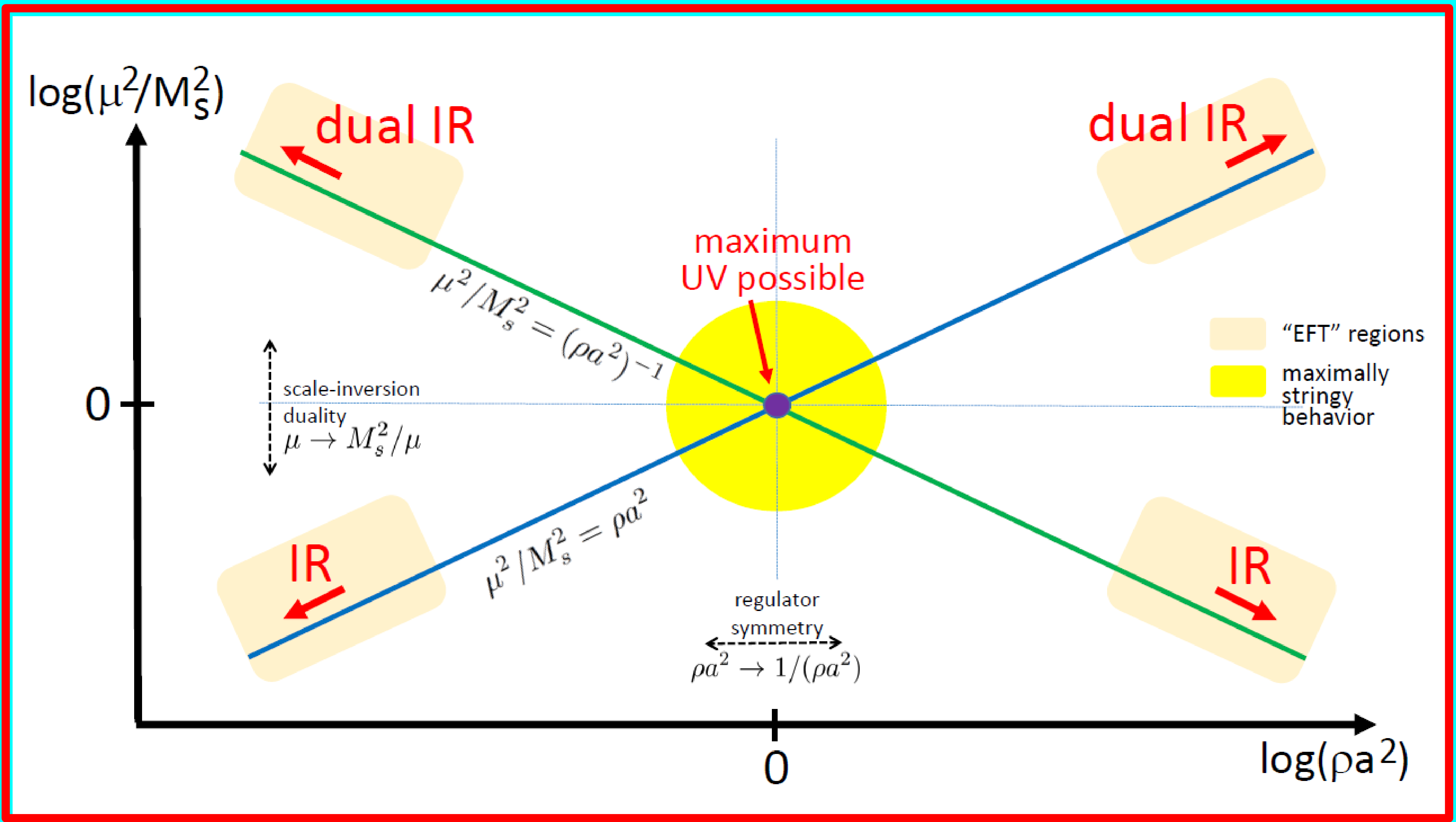


Actually a *four*-fold symmetry structure relating
(WS) internal regulator parameter $\boldsymbol{\rho a^2}$ and (ST) RG scale $\boldsymbol{\mu}$!

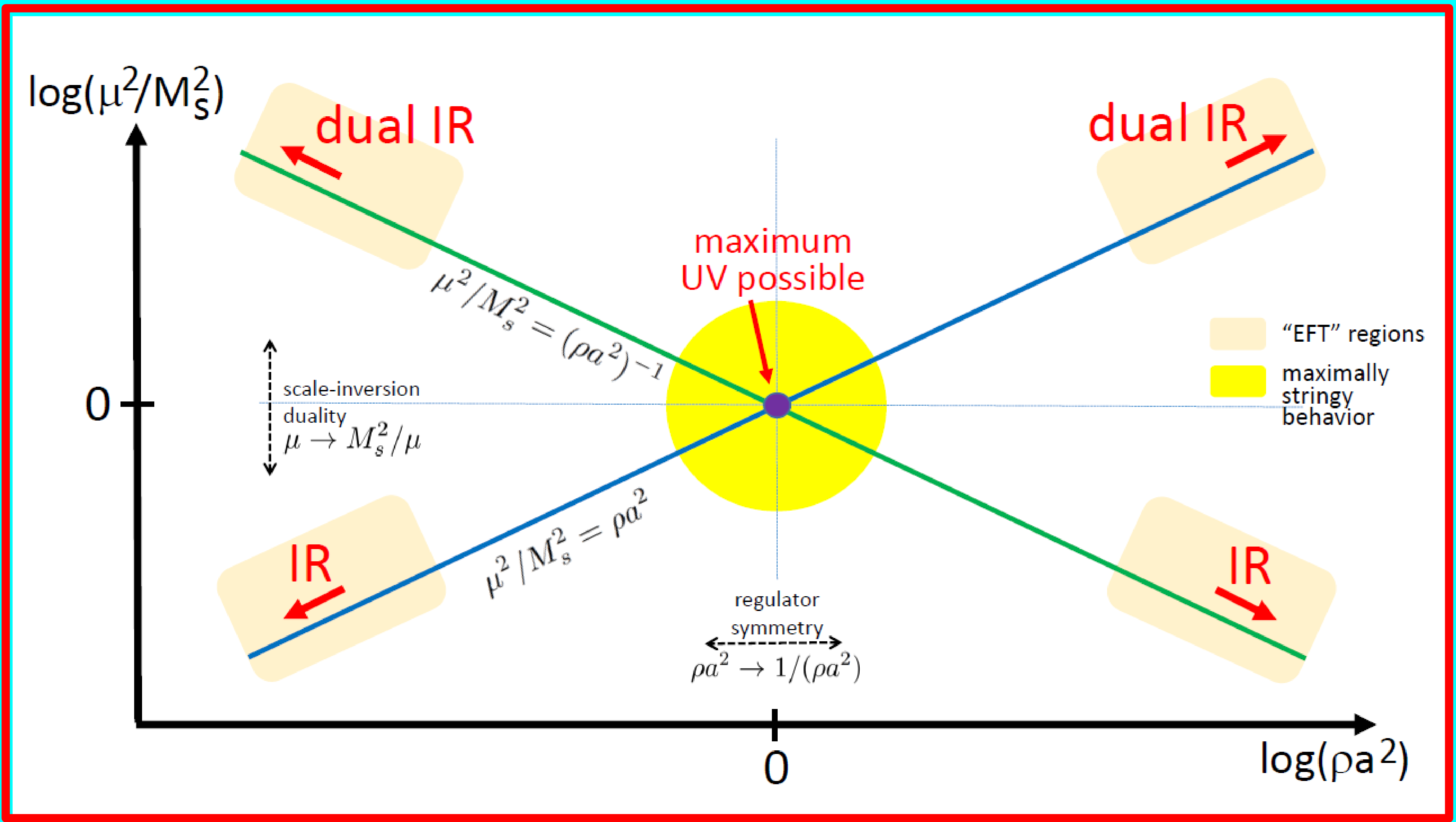


This, then, becomes an additional self-consistency requirement on acceptable modular-invariant regulators $\mathcal{G}(\tau)$:

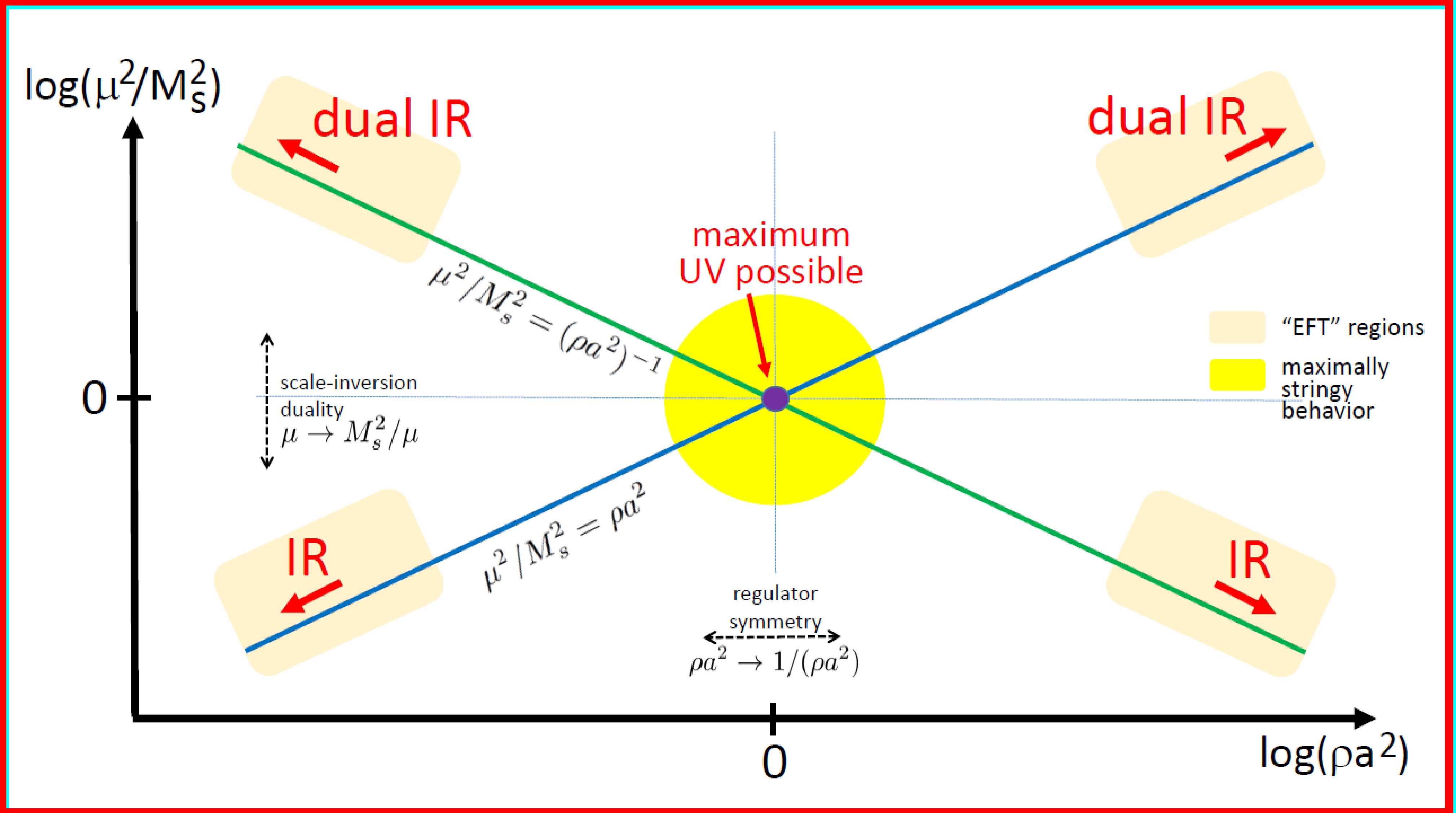
If we wish to use such a regulator to identify a spacetime scale μ,
it must also have an internal duality symmetry in the appropriate WS variable.

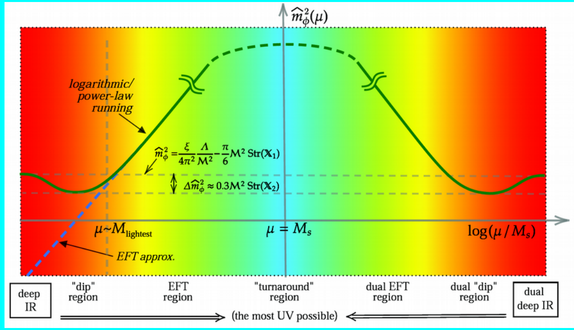


For any internal regulator parameter ρa^2, identifying a spacetime scale μ corresponds to choosing an identification branch. These branches have opposite spacetime UV/IR directions!

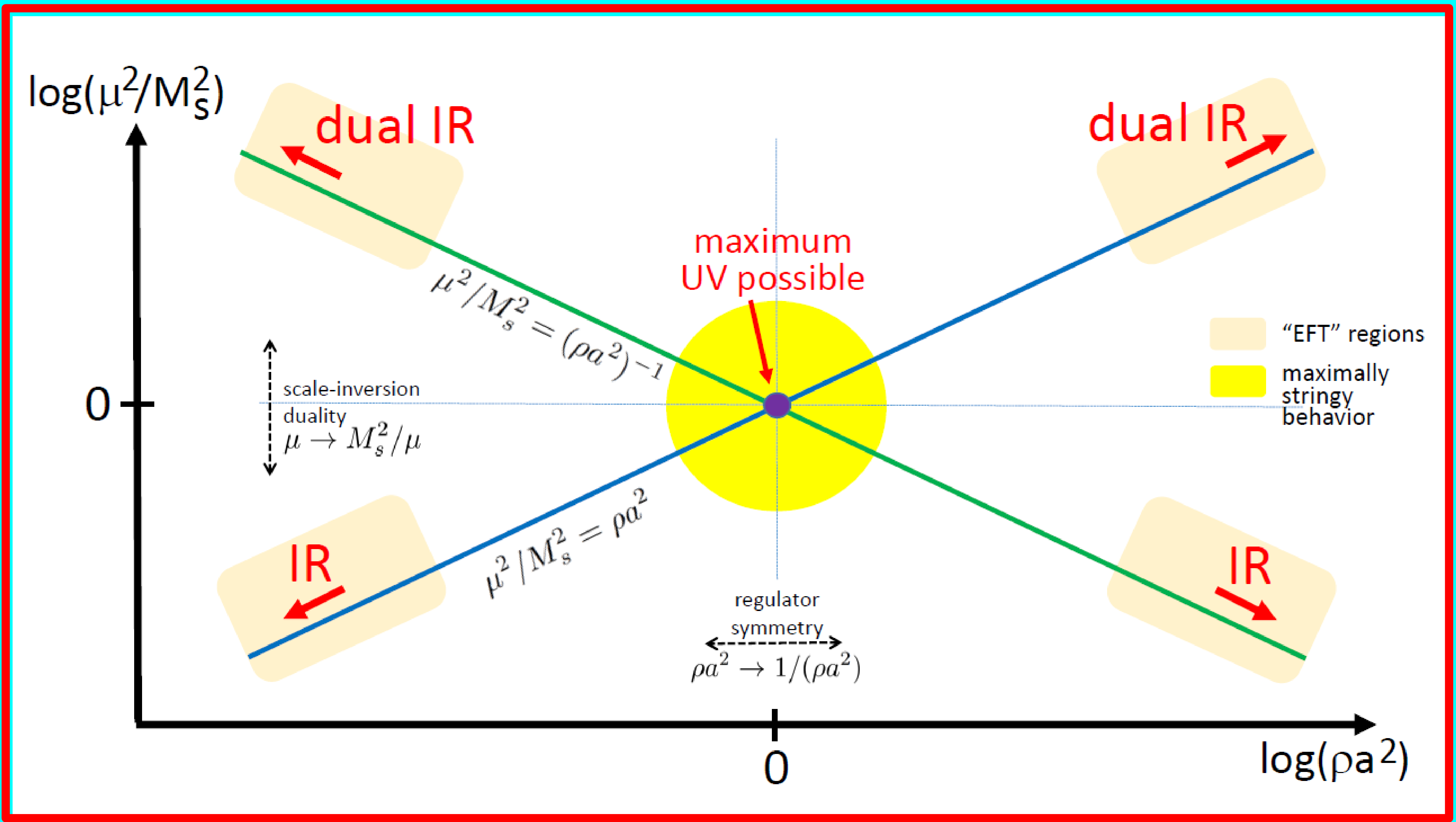


EFT regions (beige) lie out along the spokes. Thus, within such strings, the passage to an EFT requires choosing a spoke and thereby breaking the four-fold symmetry associated with modular invariance.



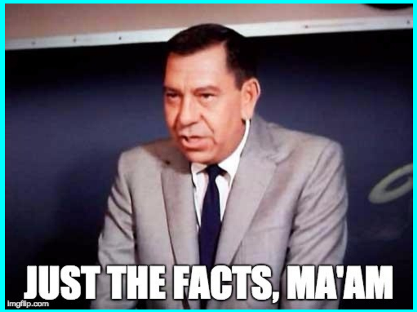


Viewed this way, rainbow plot starts in lower-left corner, reaches the center, and then flows out again along either upward spoke.



In this sketch, IR/UV labels follow usual conventions of lower-left spoke.
Could have equivalently chosen any of the other three spokes.
Thus, can freely relabel IR ⇔ UV everywhere in this sketch.

Whither EFTs?



To what extent do EFTs provide relevant low-energy descriptions of string theory?

- As discussed, one must break modular invariance (choose a branch) in order to build an appropriate EFT.
- Certainly for $\mu \ll M_s$, the features associated with scale duality are "far away", not directly relevant.
- Thus, within certain range of scales, the theory then behaves as one would expect for an EFT ***<u>except</u>***
    - Divergences are softened, running is different (*e.g.*, log running for Higgs)
    - Even in this region the theory is still sensitive to the *infinite* towers of states. Running governed by supertraces over *all* states.
    - EFT-like behavior also cuts off as one approaches the deep IR --- required since theory must remain sensitive to infinite towers and match the "dual" deep IR in which all states contribute. For example, "IR" limit of Higgs mass becomes finite! Thus new IR behavior induces new features (such as the "dip" region) which are entirely stringy.

Caution advised, must understand the context and purpose.

(Consult a professional near you.)

Finally, an extra bonus!

Let us consider cases with $\underset{M=0}{\mathrm{Str}}\ \mathbb{X}_2 = 0$

Within such cases, the Higgs mass is actually *finite*, just like Λ.

It then turns out...

$$
\begin{aligned}
\widehat{m}^2_\phi(\mu)\Big|_{\mathcal{X}} = \frac{\mathcal{M}^2}{1+\mu^2/M_s^2}\Bigg\{ & \\
& \underset{M=0}{\mathrm{Str}}\, \mathbb{X}_1 \left[-\frac{\pi}{6}\left(1+\mu^2/M_s^2\right)\right] \\
+ & \underset{M=0}{\mathrm{Str}}\, \mathbb{X}_2 \left[\log\left(\frac{\mu}{2\sqrt{2}eM_s}\right)\right] \\
+ & \underset{M>0}{\mathrm{Str}}\, \mathbb{X}_1 \left\{-\frac{\pi}{6}-\frac{1}{2\pi}\left(\frac{M}{\mathcal{M}}\right)^2 \times \right. \\
& \qquad \left. \times \left[\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)+\mathcal{K}_2^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\right\} \\
+ & \underset{M>0}{\mathrm{Str}}\, \mathbb{X}_2 \left[2\mathcal{K}_0^{(0,1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)-\mathcal{K}_1^{(1,2)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Lambda}(\mu) = \frac{1}{1+\mu^2/M_s^2}\Bigg\{ & \frac{\mathcal{M}^2}{24}\,\mathrm{Str}\, M^2 \\
& - \frac{7}{960\pi^2}\left(n_B-n_F\right)\mu^4 \\
& - \frac{1}{2\pi^2}\,\underset{M>0}{\mathrm{Str}}\, M^4 \left[\mathcal{K}_1^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right. \\
& \qquad + 4\,\mathcal{K}_2^{(-2,-1)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right) \\
& \qquad \left. + \mathcal{K}_3^{(-1,0)}\left(\frac{2\sqrt{2}\pi M}{\mu}\right)\right]\Bigg\}
\end{aligned}
$$

… multiplied by $\xi/(4\pi^2\mathcal{M}^2)$

Undo truncation to ϕ=0:
Put back $M \to M(\phi)$

… then …

This is the double ϕ -derivative of **that!**

This implies

$$\widehat{m}_\phi^2(\mu) = \left(\partial_\phi^2 + \frac{\xi}{4\pi^2\mathcal{M}^2}\right)\widehat{\Lambda}(\mu,\phi)\Big|_{\phi=0}$$
$$= D_\phi^2\,\widehat{\Lambda}(\mu,\phi)\Big|_{\phi=0}$$

$\Lambda(\mu,\phi)$ is a stringy effective potential for the Higgs mass!

(at least locally)

where $D_\phi^{\,2}$ is the modular-covariant double ϕ–derivative

$$D_\phi^2 \equiv \partial_\phi^2 + \frac{\xi}{4\pi^2\mathcal{M}^2}$$

This implies

$$\boxed{\begin{aligned} \widehat{m}_\phi^2(\mu) &= \left(\partial_\phi^2 + \frac{\xi}{4\pi^2\mathcal{M}^2}\right) \widehat{\Lambda}(\mu,\phi)\Big|_{\phi=0} \\ &= D_\phi^2\, \widehat{\Lambda}(\mu,\phi)\Big|_{\phi=0} \end{aligned}}$$

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$\Lambda(\mu,\phi)$ is a stringy effective potential for the Higgs mass!
(at least locally)

We thus have

$$\begin{cases} \widehat{\Lambda}(\mu) &= \widehat{\Lambda}(\mu,\phi)\big|_{\phi=0} \\ \widehat{m}_\phi^2(\mu) &= D_\phi^2\, \widehat{\Lambda}(\mu,\phi)\big|_{\phi=0} \end{cases} \xrightarrow{\mu \to 0} \boxed{\begin{cases} \Lambda &= \Lambda(\phi)\big|_{\phi=0} \\ m_\phi^2 &= D_\phi^2\, \Lambda(\phi)\big|_{\phi=0} \end{cases}}$$

Both Λ and Higgs mass from same potential!

Exact relations! All quantities finite, no divergences!

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$$\widehat{m}_\phi^2(\mu) = \left(\partial_\phi^2 + \frac{\xi}{4\pi^2\mathcal{M}^2}\right)\widehat{\Lambda}(\mu,\phi)\Big|_{\phi=0}$$
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Both Λ and Higgs mass from same potential!

$\mu \to 0$

$$\boxed{\begin{cases} \Lambda = \Lambda(\phi)\big|_{\phi=0} \\ m_\phi^2 = D_\phi^2\,\Lambda(\phi)\big|_{\phi=0} \end{cases}}$$

Exact relations! All quantities finite, no divergences!

Moreover, had previously shown

$$\Lambda = \frac{1}{24}\mathcal{M}^2\,\mathrm{Str}\,M^2$$

(KRD, 1995)

We now have corresponding result for Higgs mass!

$$\boxed{m_\phi^2 = \frac{1}{24}\mathcal{M}^2\,\mathrm{Str}\left[D_\phi^2 M^2(\phi)\right]\Big|_{\phi=0}}$$

At first glance, these relations might seem somewhat trivial / expected. After all, we started by defining

$$\Lambda(\phi) \equiv -\frac{\mathcal{M}^4}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \mathcal{Z}(\tau,\overline{\tau},\phi)$$

$$\Rightarrow \quad m_\phi^2 \equiv \left.\frac{d^2\Lambda(\phi)}{d\phi^2}\right|_{\phi=0}$$

However,

- No guarantee that this structure would hold across the Rankin-Selberg transition to purely physical string states!
- In field theory, similar structure is anticipated (Coleman-Weinberg) but divergences require cutoffs which can change ϕ behavior and disturb such relations. CW also needed to eliminate certain quadratic divergences "by fiat" (subtracted from origin of field space).
- In string theory, all quantities are finite, divergences are killed through modular invariance! Yet this structure is preserved. These are now thus precise mathematical relations between finite quantities, with each taking the form of a supertrace over infinite towers of *physical* string states!
- This structure holds as a function of scale μ --- even in our UV/IR mixed theory, and even in the presence of scale duality!
- Appearance of modular-covariant derivative $D_\phi{}^2$ is also unexpected.

- S. Abel, KRD, L. Nutricati, to appear

Similar phenomena arise for one-loop contributions to gauge couplings!
For any group G, we have

$$\mathcal{L} \;=\; -\frac{1}{4g_G^2} F^{(G)}_{\mu\nu} F^{(G)\mu\nu}$$

$$\left.\frac{16\pi^2}{g_G^2}\right|_{\substack{\text{one-loop}\\ \text{order}}} \;=\; \left.\frac{16\pi^2}{g_G^2}\right|_{\text{tree}} + \Delta_G$$

one-loop contributions

$$\Delta_G \;=\; -2\left\langle \tau_2^2 \left(\overline{Q}_H^2 - \frac{1}{12}\overline{E}_2\right)\left(Q_G^2 - \frac{\xi}{4\pi\tau_2}\right)\right\rangle$$

Spacetime helicity charge

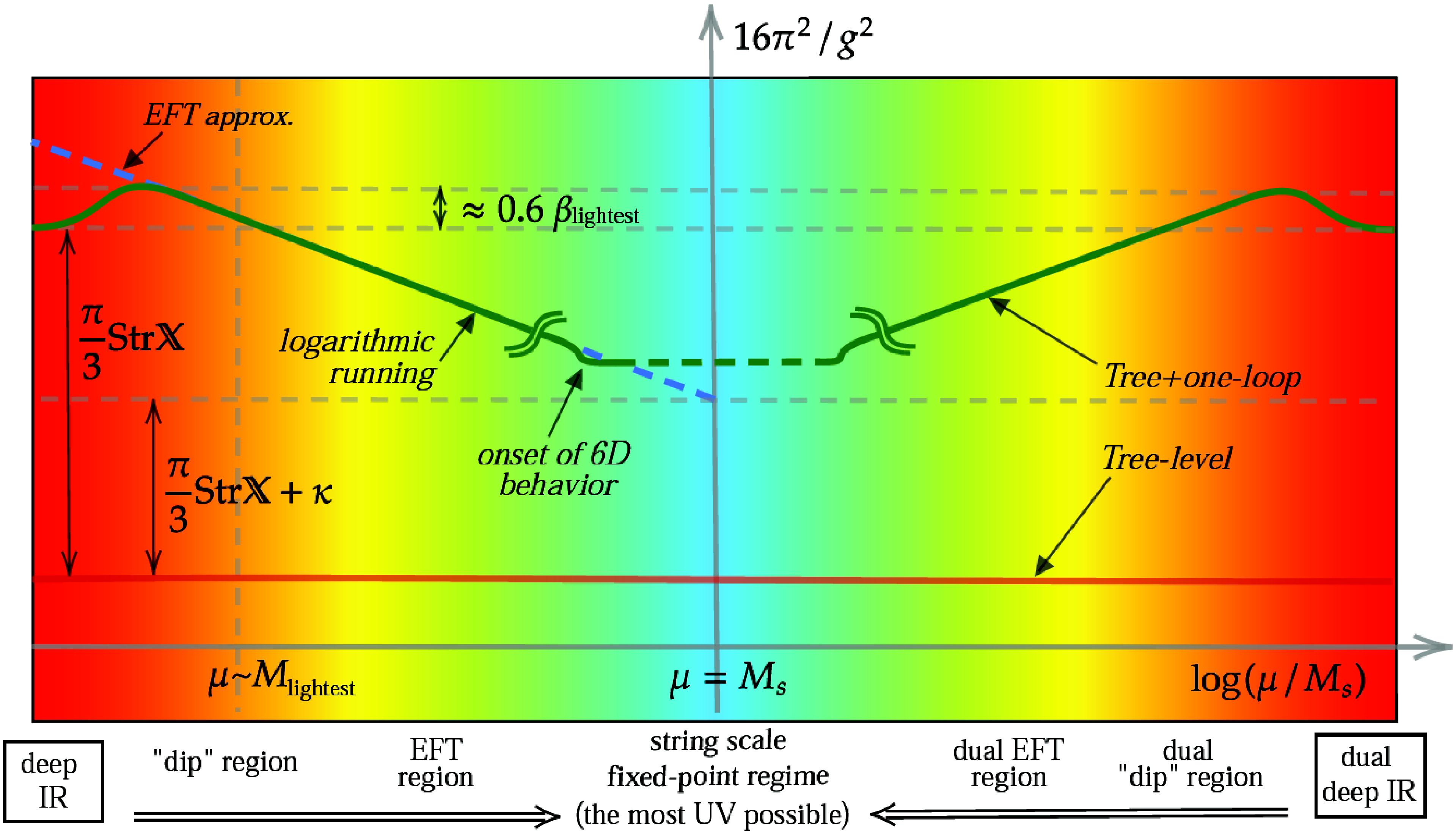
Eisenstein function = modular completion of 1 !!!!

quadratic Casimir

modular completion of Casimir

These are the *X*-insertions for the gauge-coupling calculation!

The rest of the calculation then proceeds as before...



- S. Abel, KRD, L. Nutricati, to appear

S. Abel, KRD, L. Nutricati, in progress.

Future work

- Non-renormalization theorems via misaligned SUSY and UV/IR mixing! The protections of SUSY without SUSY!
- Misaligned SUSY for off-shell string states, not just on-shell!
- Interactions, vertices? How do these behave as functions of scale?
- Implications of scale duality? Coefficients of all operators should run to zero slope at the self-dual point ($\mu=M_s$). Is this a fixed point in some theories?
- Extension to open strings? Remnants of modular invariance and misaligned SUSY survive orientifolding procedure.
- Finally, implications/applications to **gauge hierarchy problem**, or hierarchy problems in general? *Three comments*:

1

Our results suggest that Λ and m_ϕ problems can be reformulated to the form:

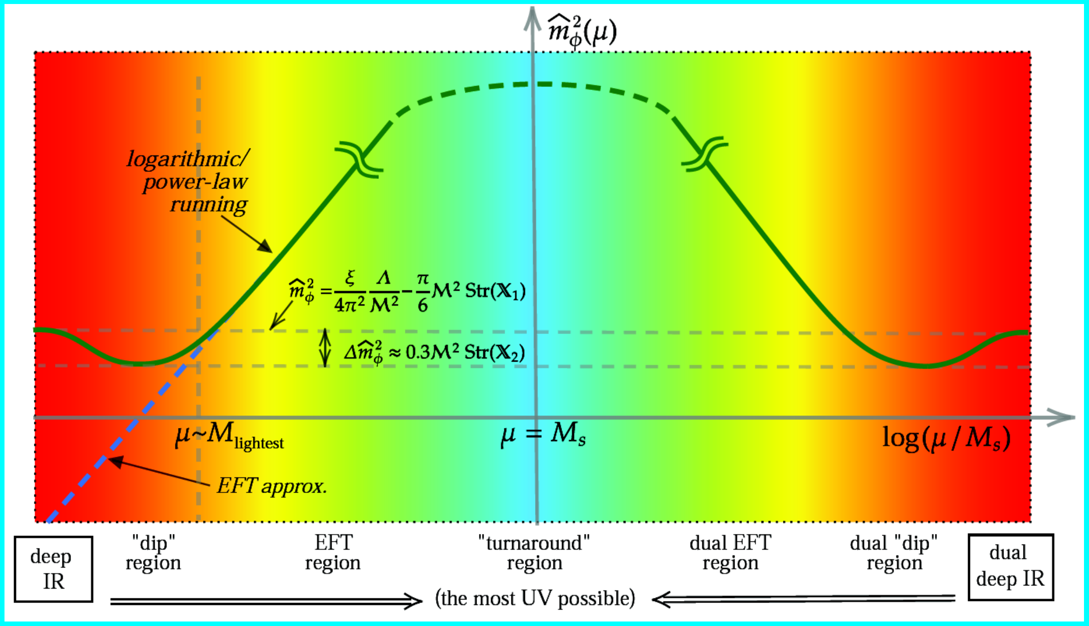
$$\begin{cases} \left. \mathrm{Str}\, M^2 \right|_{\phi=0} \sim 24\, M_\Lambda^4 / \mathcal{M}^2 \\ \left. \partial_\phi^2\, \mathrm{Str}\, M^2 \right|_{\phi=0} \sim 24\, M_{\mathrm{EW}}^2 / \mathcal{M}^2 \end{cases}$$

Reminiscent of Veltman conditions, but with supertraces over *all* states!

2

Can we exploit the “dip” region to make $m_\phi^2 < 0$ to trigger EWSB?? Would require:

$$\frac{\pi}{6}\,\mathrm{Str}\,\mathbb{X}_1 + \frac{3}{10}\,\mathrm{Str}\,\mathbb{X}_2 \;\gtrsim\; \frac{\xi}{4\pi^2}\,\frac{\Lambda}{\mathcal{M}^4}$$



3

But overall: Hierarchy problems assume traditional field-theory relationships between UV and IR. By contrast, string theory tells us that we have UV/IR mixing, softened divergences (even finiteness), scale duality, *etc*. Thus hierarchy problems may not be fundamental or survive in the manner we normally assume.