



Hyperon physics at BESIII

Lake Louise Winter Institute 2024

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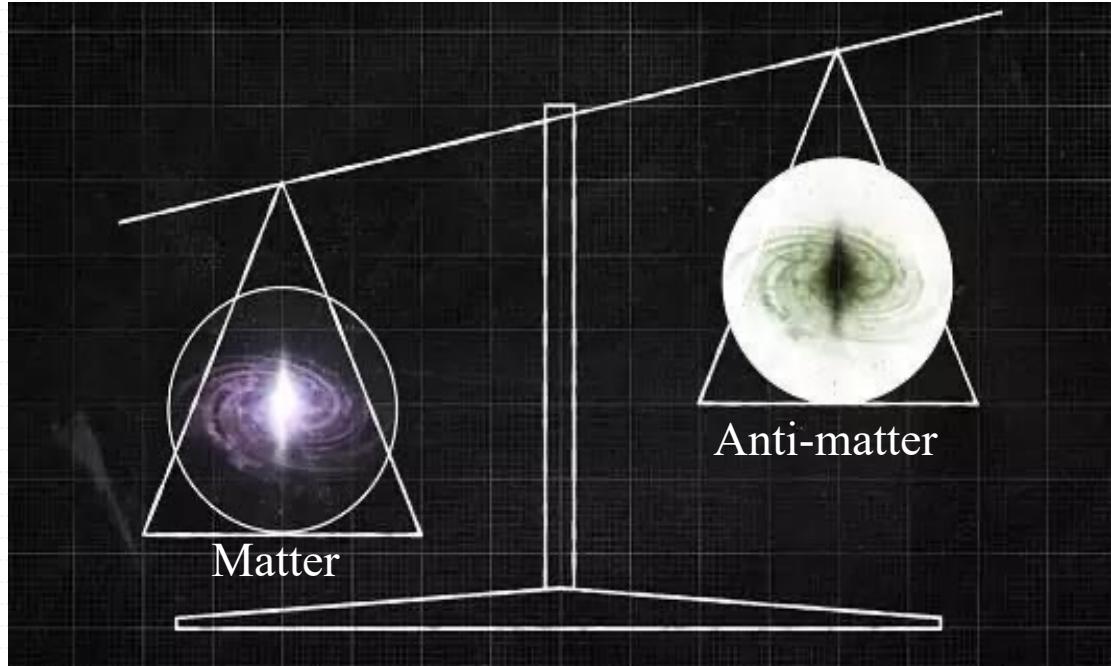
Summary

CPV in hyperon decays

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/// Mystery of matter-antimatter asymmetry



- According to the Big Bang theory:
 - Matter and anti-matter have the same amount
- The observed universe is matter dominant:
$$(n_B - n_{\bar{B}})/n_\gamma \sim 10^{-10}$$

New Journal of Physics 14 (2012) 095012

- The standard model predicted value:
$$(n_B - n_{\bar{B}})/n_\gamma \sim 10^{-20}$$
- Why has the anti-matter disappeared?
- Sakharov's three conditions :
 1. Baryon number violation
 2. C and CP violation
 3. Thermal non-equilibrium

Gavela, Hernandez, Orloff, Pene, CERN 93/7081



Pisma Zh. Eksp. Teor. Fiz., 1967,
5: 32-35

/// CP violation (CPV)

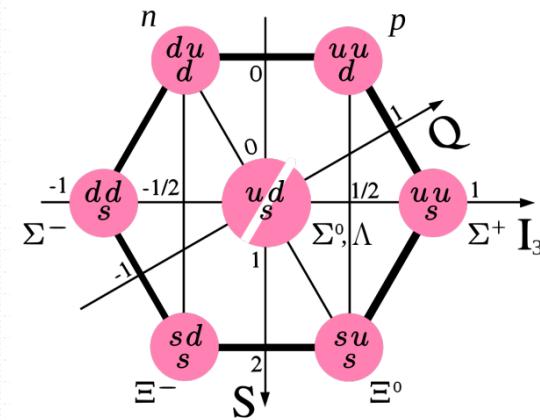
$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c \end{pmatrix}$$

$\delta_{13} \rightarrow V_{CKM}^* \neq V_{CKM} \rightarrow CPV!$

CPV has been observed at K , B , D meson systems, but not enough to explain the matter-dominant Universe.

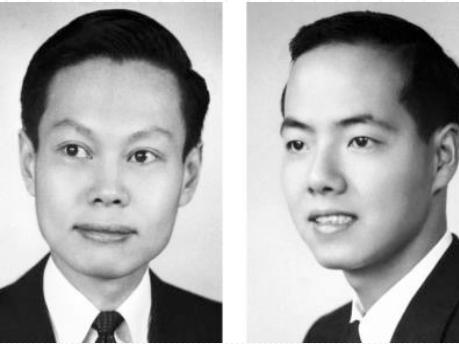
- [1] Phys. Rev. 104 (1956) 254-258
- [2] Phys. Rev. 105 (1957) 1413-1414
- [3] Phys. Rev. Lett., 1964, 13: 138-140
- [4] Phys. Rev. Lett., 2001, 87: 091801
- [5] Phys. Rev. Lett., 2001, 87: 091802
- [6] Phys. Rev. Lett., 122, 211803 (2019)

A new possibility: search for CPV in entangled hyperon-antihyperon pairs.



+ $\Omega^- (sss)$ spin- $\frac{3}{2}$

/// CPV in hyperon non-leptonic weak decays



General Partial Wave Analysis of the Decay of a Hyperon of Spin $\frac{1}{2}$

T. D. LEE* AND C. N. YANG

Institute for Advanced Study, Princeton, New Jersey

(Received October 22, 1957)

Phys. Rev. 108, 1645 (1957)

The amplitude of spin-1/2 hyperon B_i decays to a spin-1/2 baryon B_f and a pion can be described by three decay asymmetry parameters:

$$\alpha_Y = \frac{2 \operatorname{Re}(S^* P)}{|S|^2 + |P|^2}, \quad \beta_Y = \frac{2 \operatorname{Im}(S^* P)}{|S|^2 + |P|^2}, \quad \gamma_Y = \frac{|S|^2 - |P|^2}{|S|^2 + |P|^2}$$

$$\alpha_Y^2 + \beta_Y^2 + \gamma_Y^2 = 1$$
$$\beta_Y = (1 - \alpha_Y^2)^{\frac{1}{2}} \sin \phi_Y, \quad \gamma_Y = (1 - \alpha_Y^2)^{\frac{1}{2}} \cos \phi_Y$$

CP conservation: $\alpha_Y = -\bar{\alpha}_Y, \quad \phi_Y = -\bar{\phi}_Y$

CPV can be searched by comparing the decay parameters between the hyperon and anti-hyperon

Recent results from BESIII

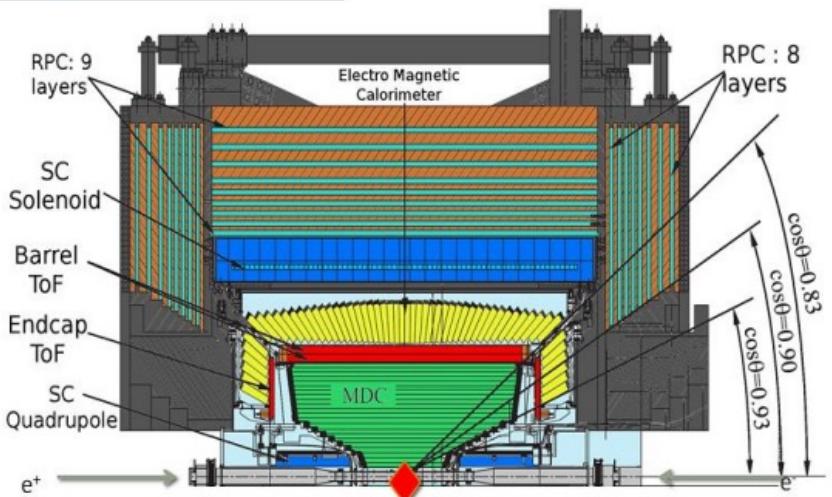
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/// Study hyperons at BESIII

Electromagnetic Calorimeter
CsI(Tl): L=28 cm
Barrel $\sigma_E = 2.5\%$
Endcap $\sigma_E = 5.0\%$

Muon Counter
RPC
Barrel: 9 layers
Endcap: 8 layers
 $\sigma_{\text{spatial}} = 1.48 \text{ cm}$



Main Drift Chamber
Small cell, 43 layer
 $\sigma_{xy} = 130 \mu\text{m}$
 $dE/dx \sim 6\%$
 $\sigma_p/p = 0.5\% \text{ at } 1 \text{ GeV}$

Time Of Flight
Plastic scintillator
 $\sigma_T(\text{barrel}) = 80 \text{ ps}$
 $\sigma_T(\text{endcap}) = 110 \text{ ps}$
(update to 65 ps with MRPC)

With 10 billion J/ψ and 2.7 billion $\psi(3686)$ collected at BESIII. About 10^7 entangled hyperon pairs can be produced, which enables precise studies of the hyperon physics.

Front. Phys. 12(5), 121301 (2017)

Decay mode	$B (\times 10^{-3})$	$N_B (\times 10^6)$
$J/\psi \rightarrow \Lambda \bar{\Lambda}$	1.89 ± 0.09	~ 18.9
$J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$	1.172 ± 0.032	~ 11.7
$J/\psi \rightarrow \Sigma^+ \bar{\Sigma}^-$	1.07 ± 0.04	~ 10.7
$J/\psi \rightarrow \Xi^0 \bar{\Xi}^0$	1.17 ± 0.04	~ 11.7
$J/\psi \rightarrow \Xi^- \bar{\Xi}^+$	0.97 ± 0.08	~ 9.7
$\psi(2S) \rightarrow \Omega^- \bar{\Omega}^+$	0.057 ± 0.003	~ 0.17

$$e^+e^- \rightarrow J/\psi \rightarrow \Lambda\bar{\Lambda}, \Lambda(\bar{\Lambda}) \rightarrow p\pi$$

Differential cross-section of $e^+e^- \rightarrow J/\psi \rightarrow \Lambda\bar{\Lambda}$:

Nuovo Cim. A 109, 241 (1996)
 Phys. Rev. 185 D 75, 074026 (2007)
 Nucl. Phys. A190 771, 169 (2006)
 Phys. Lett. B 772, 16(2017)

$$d\sigma \sim 1 + \alpha_\psi \cos^2 \theta_\Lambda + (\alpha_\psi + \cos^2 \theta_\Lambda) s_\Lambda^z s_{\bar{\Lambda}}^z +$$

$$\sin^2 \theta_\Lambda s_\Lambda^x s_{\bar{\Lambda}}^x - \alpha_\psi \sin^2 \theta_\Lambda s_\Lambda^y s_{\bar{\Lambda}}^y + \sqrt{1 - \alpha_\psi^2} \cos \Delta\Phi \sin \theta_\Lambda \cos \theta_\Lambda (s_\Lambda^x s_{\bar{\Lambda}}^z +$$

$$s_\Lambda^z s_{\bar{\Lambda}}^x) + [1 - \alpha_\psi^2] \sin \Delta\Phi \sin \theta_\Lambda \cos \theta_\Lambda (s_\Lambda^y + s_{\bar{\Lambda}}^y)$$

**SPIN
CORRELATIONS**

POLARIZATIONS

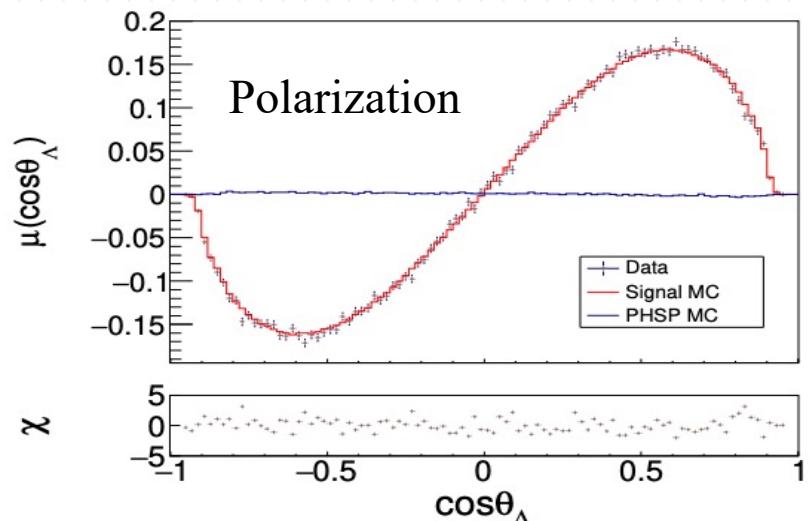
10B J/ψ

1.3B J/ψ

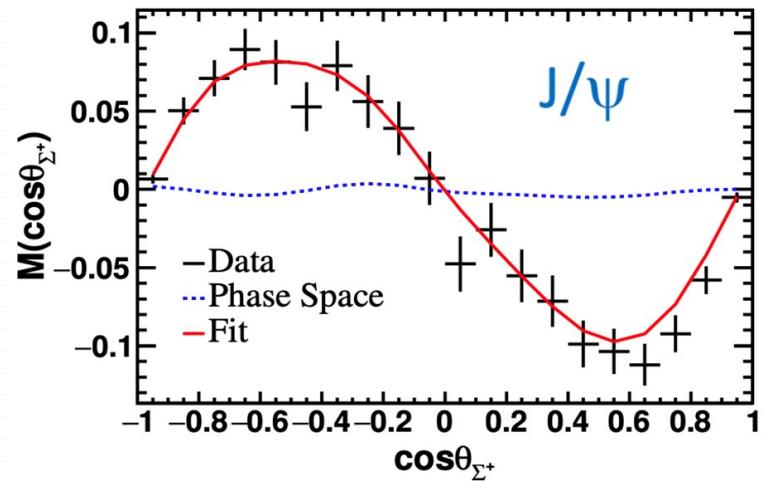
Par.	Phys. Rev. Lett. 129, 131801 (2022)	Nature Phys. 15(2019)631
$\alpha_{J/\psi}$	$0.4748 \pm 0.0022 \pm 0.0031$	$0.461 \pm 0.006 \pm 0.007$
$\Delta\Phi$	$0.7521 \pm 0.0042 \pm 0.0066$	$0.740 \pm 0.010 \pm 0.009$
α_-	$0.7519 \pm 0.0036 \pm 0.0024$	$0.750 \pm 0.009 \pm 0.004$
α_+	$-0.7559 \pm 0.0036 \pm 0.0030$	$-0.758 \pm 0.010 \pm 0.007$
A_{CP}	$-0.0025 \pm 0.0046 \pm 0.0012$	$0.006 \pm 0.012 \pm 0.007$
α_{avg}	$0.7542 \pm 0.0010 \pm 0.0024$	-

Most precise CP asymmetry, $A_{CP} = \frac{\alpha_- + \bar{\alpha}_+}{\alpha_- - \bar{\alpha}_+}$, in hyperon sector.

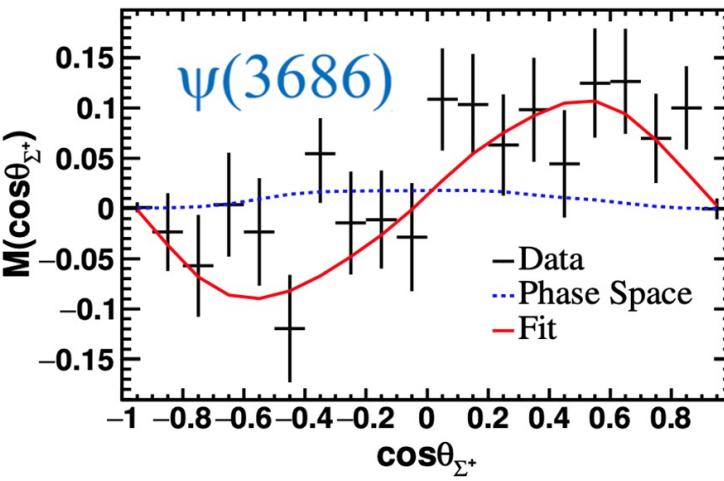
$\alpha_- : \Lambda \rightarrow p\pi^-$
 $\alpha_+ : \bar{\Lambda} \rightarrow \bar{p}\pi^+$



Σ^+ Polarization and CP tests in Σ^+ decays



Phys. Rev. Lett. 125, 052004 (2020)
1.3B J/ψ and 0.45B $\psi(3686)$



Phys. Rev. Lett. 131, 191802 (2023)
10B J/ψ

Polarizations of Σ^+

Opposite directions of the Σ^+ polarization in J/ψ and $\psi(3686)$!

Parameter	$\Sigma^+ \rightarrow p\pi^0, \bar{\Sigma}^- \rightarrow \bar{p}\pi^0$	$\Sigma^+ \rightarrow p\pi^0, \bar{\Sigma}^- \rightarrow \bar{n}\pi^- + c.c.$
$\alpha_{J/\psi}$	$-0.508 \pm 0.006 \pm 0.004$	$-0.5156 \pm 0.0030 \pm 0.0061$
$\Delta\Phi_{J/\psi}$	$-0.270 \pm 0.012 \pm 0.009$	$-0.2772 \pm 0.0044 \pm 0.0041$
$\alpha_{\psi(3686)}$	$0.682 \pm 0.03 \pm 0.011$	-
$\Delta\Phi_{\psi(3686)}$	$0.379 \pm 0.07 \pm 0.014$	-
$\alpha_0(\Sigma^+ \rightarrow p\pi^0)$	$-0.998 \pm 0.037 \pm 0.009$	-
$\bar{\alpha}_0(\bar{\Sigma}^- \rightarrow \bar{p}\pi^0)$	$0.990 \pm 0.037 \pm 0.011$	-
$\alpha_+(\Sigma^+ \rightarrow n\pi^+)$	-	$0.0481 \pm 0.0031 \pm 0.0019$
$\bar{\alpha}_-(\bar{\Sigma}^- \rightarrow \bar{n}\pi^-)$	-	$-0.0565 \pm 0.0047 \pm 0.0022$

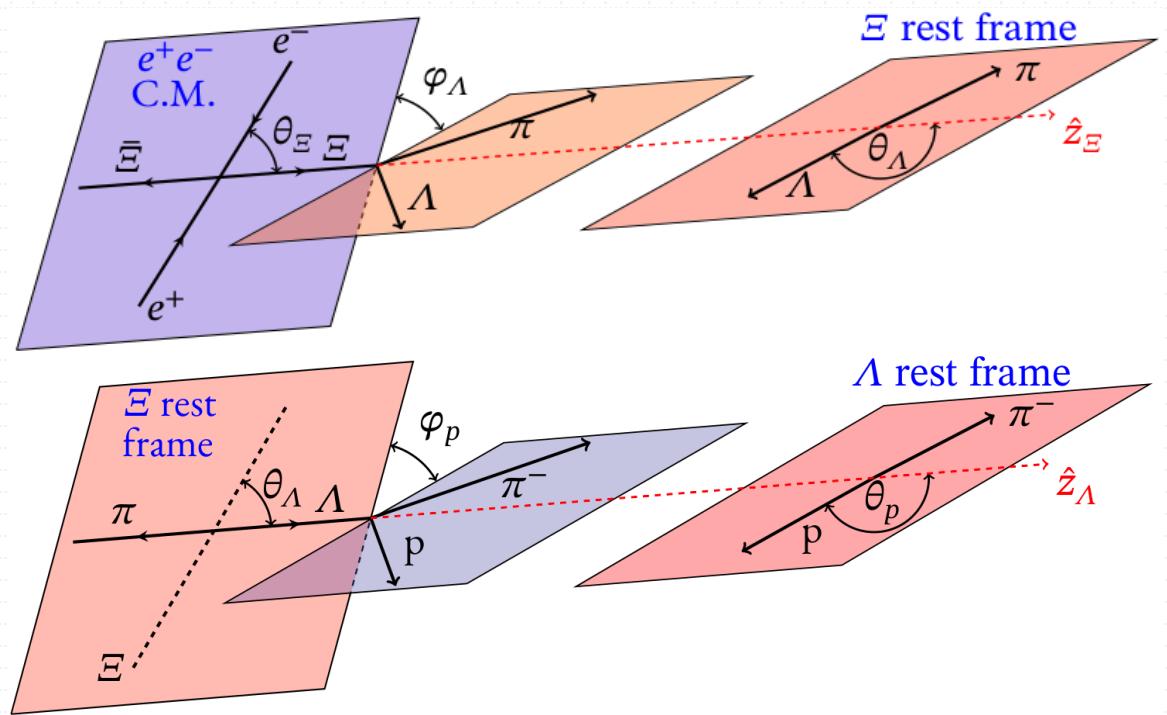
Two CP tests are performed in the decays of Σ^+ :

$$A_{CP}(\Sigma^+ \rightarrow p\pi^0) = \frac{\alpha_0 + \bar{\alpha}_0}{\alpha_0 - \bar{\alpha}_0} = -0.004 \pm 0.037 \pm 0.010$$

$$A_{CP}(\Sigma^+ \rightarrow n\pi^+) = \frac{\alpha_+ + \bar{\alpha}_-}{\alpha_+ - \bar{\alpha}_-} = -0.080 \pm 0.052 \pm 0.028$$

CP tests in Ξ decays

Joint angular distribution of $e^+e^- \rightarrow J/\psi \rightarrow \Xi\bar{\Xi}, \Xi \rightarrow \Lambda(\rightarrow p\pi^-)\pi, \bar{\Xi} \rightarrow \bar{\Lambda}(\rightarrow \bar{p}\pi^+)\pi$:



$$W = \sum_{\mu,\bar{\nu}=0}^3 C_{\mu\bar{\nu}} \sum_{\mu',\bar{\nu}'=0}^3 a_{\mu,\mu'}^{\Xi} a_{\bar{\nu},\bar{\nu}'}^{\bar{\Xi}}, a_{\mu',0}^{\Lambda} a_{\bar{\nu}',0}^{\bar{\Lambda}}$$

Phys. Rev. D 99, 056008 (2019)

$d\Gamma \propto W(\xi, \omega)$, ξ : 9 kin. variables

8 parameters:

$$\omega = \begin{matrix} \text{Production} \\ (\alpha_\Psi, \Delta\Phi, \alpha_\Xi, \phi_\Xi, \alpha_\Lambda, \bar{\alpha}_\Xi, \bar{\phi}_\Xi, \bar{\alpha}_\Lambda) \end{matrix} \begin{matrix} \text{Decay} \\ \end{matrix}$$

$$A_{CP} = \frac{\alpha + \bar{\alpha}}{\alpha - \bar{\alpha}} \approx -\tan(\delta_P - \delta_S) \tan(\xi_P - \xi_S)$$

$$\Delta\phi_{CP} = \frac{\phi + \bar{\phi}}{2} \approx \frac{\alpha}{\sqrt{1 - \alpha^2}} \cos\phi \tan(\xi_P - \xi_S)$$

If the strong phase difference $(\delta_P - \delta_S)$ is close to 0, A_{CP} will be suppressed, however, $\Delta\phi_{CP}$ won't suffer from this problem.

$\Delta\phi_{CP}$ can be directly determined through the sequential decays of the hyperon. This is a big advantage of $\Xi^{0(-)}$ hyperon compared to Λ or Σ^\pm hyperon

CP tests in Ξ decays

$\Xi^- (1.3B J/\psi)$

Parameter	Nature 606 (2022) 64-69	Previous result
a_ψ	$0.586 \pm 0.012 \pm 0.010$	$0.58 \pm 0.04 \pm 0.08$
$\Delta\Phi$	$1.213 \pm 0.046 \pm 0.016$ rad	-
a_Ξ	$-0.376 \pm 0.007 \pm 0.003$	-0.401 ± 0.010
ϕ_Ξ	$0.011 \pm 0.019 \pm 0.009$ rad	$-0.042 \pm 0.011 \pm 0.011$
\bar{a}_Ξ	$0.371 \pm 0.007 \pm 0.002$	-
$\bar{\phi}_\Xi$	$-0.021 \pm 0.019 \pm 0.007$ rad	-
a_Λ	$0.757 \pm 0.011 \pm 0.008$	$0.750 \pm 0.009 \pm 0.004$
\bar{a}_Λ	$-0.763 \pm 0.011 \pm 0.007$	$-0.758 \pm 0.010 \pm 0.007$
$\xi_P - \xi_S$	$(1.2 \pm 3.4 \pm 0.8) \times 10^{-2}$ rad	-
$\delta_P - \delta_S$	$(-4.0 \pm 3.3 \pm 1.7) \times 10^{-2}$ rad	$(10.2 \pm 3.9) \times 10^{-2}$ rad
A_{CP}^Ξ	$(6 \pm 13 \pm 6) \times 10^{-3}$	-
$\Delta\phi_{CP}^\Xi$	$(-5 \pm 14 \pm 3) \times 10^{-3}$ rad	-
A_{CP}^Λ	$(-4 \pm 12 \pm 9) \times 10^{-3}$	$(-6 \pm 12 \pm 7) \times 10^{-3}$
$\langle\phi_\Xi\rangle$	$0.016 \pm 0.014 \pm 0.007$ rad	

Phys. Rev. Lett. 93 (2004) 011802

The precision of our analysis (73K events) is comparable with the measurement from HyperCP (144M events), which means that the accuracy of a single event is more than 1000 times higher than HyperCP!

$\Xi^0 (10B J/\psi)$

Parameter	Phys. Rev. D 108 (2023) 3, L031106	Previous result
$\alpha_{J/\psi}$	$0.514 \pm 0.006 \pm 0.015$	0.66 ± 0.06 [34]
$\Delta\Phi$ (rad)	$1.168 \pm 0.019 \pm 0.018$	-
α_Ξ	$-0.3750 \pm 0.0034 \pm 0.0016$	-0.358 ± 0.044 [18]
$\bar{\alpha}_\Xi$	$0.3790 \pm 0.0034 \pm 0.0021$	0.363 ± 0.043 [18]
ϕ_Ξ (rad)	$0.0051 \pm 0.0096 \pm 0.0018$	0.03 ± 0.12 [18]
$\bar{\phi}_\Xi$ (rad)	$-0.0053 \pm 0.0097 \pm 0.0019$	-0.19 ± 0.13 [18]
α_Λ	$0.7551 \pm 0.0052 \pm 0.0023$	0.7519 ± 0.0043 [13]
$\bar{\alpha}_\Lambda$	$-0.7448 \pm 0.0052 \pm 0.0017$	-0.7559 ± 0.0047 [13]
$\xi_P - \xi_S$ (rad)	$(0.0 \pm 1.7 \pm 0.2) \times 10^{-2}$	-
$\delta_P - \delta_S$ (rad)	$(-1.3 \pm 1.7 \pm 0.4) \times 10^{-2}$	-
A_{CP}^Ξ	$(-5.4 \pm 6.5 \pm 3.1) \times 10^{-3}$	$(-0.7 \pm 8.5) \times 10^{-2}$ [18]
$\Delta\phi_{CP}^\Xi$ (rad)	$(-0.1 \pm 6.9 \pm 0.9) \times 10^{-3}$	$(-7.9 \pm 8.3) \times 10^{-2}$ [18]
A_{CP}^Λ	$(6.9 \pm 5.8 \pm 1.8) \times 10^{-3}$	$(-2.5 \pm 4.8) \times 10^{-3}$ [13]
$\langle\alpha_\Xi\rangle$	$-0.3770 \pm 0.0024 \pm 0.0014$	-
$\langle\phi_\Xi\rangle$ (rad)	$0.0052 \pm 0.0069 \pm 0.0016$	-
$\langle\alpha_\Lambda\rangle$	$0.7499 \pm 0.0029 \pm 0.0013$	0.7542 ± 0.0026 [13]

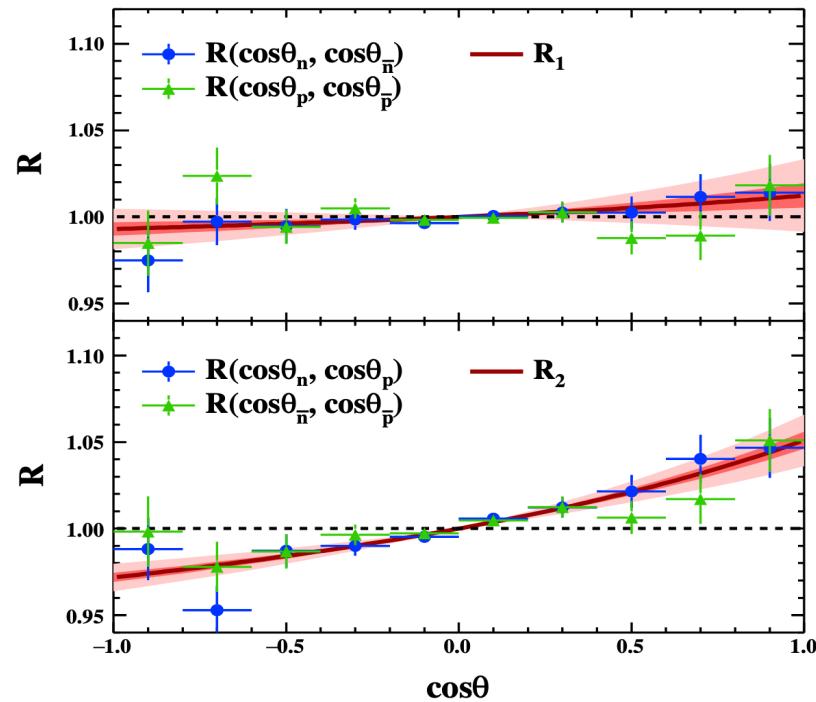
First measurements of the weak phase difference in Ξ^-/Ξ^0 decays

Three CP tests in Ξ^-/Ξ^0 decays

Test the CP symmetry and $\Delta I = 1/2$ rule in Ξ^- decays

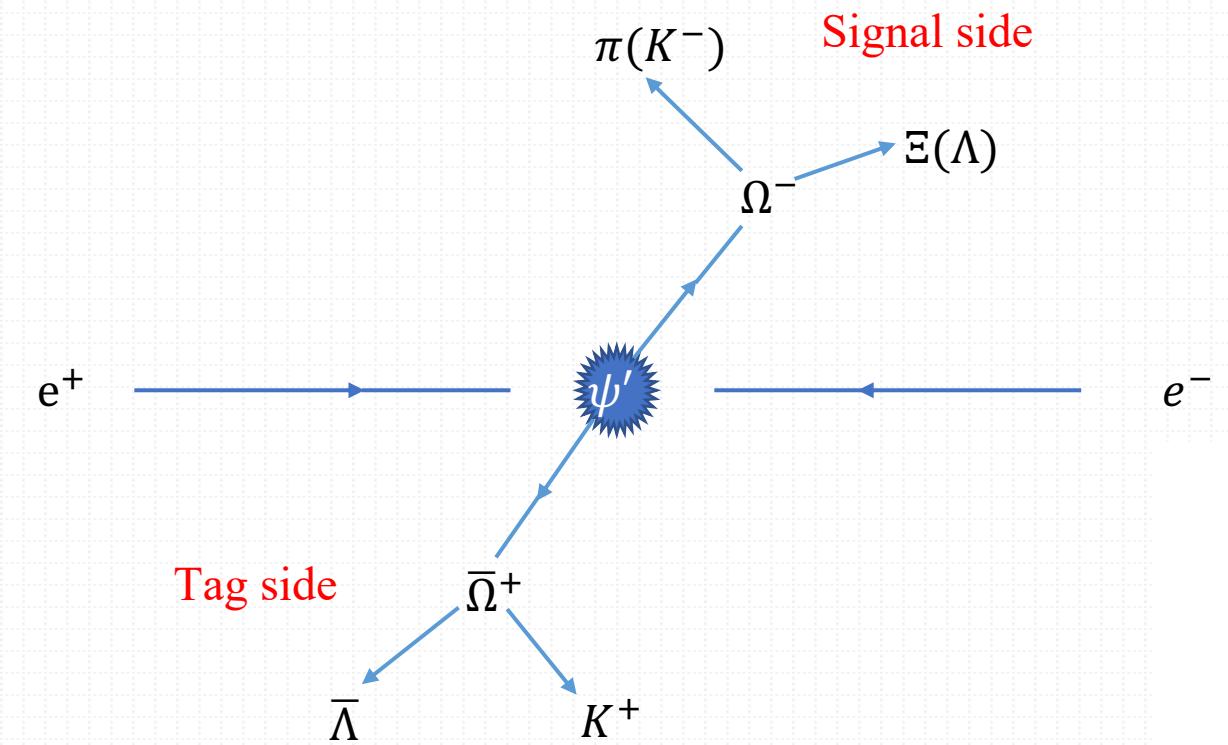
$e^+e^- \rightarrow J/\psi \rightarrow \Xi^-\bar{\Xi}^+, \Xi^- \rightarrow \Lambda(\rightarrow p\pi^-)\pi^-, \bar{\Xi}^+ \rightarrow \bar{\Lambda}(\rightarrow \bar{n}\pi^0)\pi^+$

Parameters	arXiv:2309.14667 (Accepted by PRL)	Previous result
$\alpha_{J/\psi}$	$0.611 \pm 0.007^{+0.013}_{-0.007}$	$0.586 \pm 0.012 \pm 0.010$ [17]
$\Delta\Phi_{J/\psi}$ (rad)	$1.30 \pm 0.03^{+0.02}_{-0.03}$	$1.213 \pm 0.046 \pm 0.016$ [17]
α_Ξ	$-0.367 \pm 0.004^{+0.003}_{-0.004}$	$-0.376 \pm 0.007 \pm 0.003$ [17]
ϕ_Ξ (rad)	$-0.016 \pm 0.012^{+0.004}_{-0.008}$	$0.011 \pm 0.019 \pm 0.009$ [17]
$\bar{\alpha}_\Xi$	$0.374 \pm 0.004^{+0.003}_{-0.004}$	$0.371 \pm 0.007 \pm 0.002$ [17]
$\bar{\phi}_\Xi$ (rad)	$0.010 \pm 0.012^{+0.003}_{-0.013}$	$-0.021 \pm 0.019 \pm 0.007$ [17]
α_{Λ^-}	$0.764 \pm 0.008^{+0.005}_{-0.006}$	$0.7519 \pm 0.0036 \pm 0.0024$ [37]
α_{Λ^+}	$-0.774 \pm 0.009^{+0.005}_{-0.005}$	$-0.7559 \pm 0.0036 \pm 0.0030$ [37]
α_{Λ^0}	$0.670 \pm 0.009^{+0.009}_{-0.008}$	0.75 ± 0.05 [29]
$\bar{\alpha}_{\Lambda^0}$	$-0.668 \pm 0.008^{+0.006}_{-0.008}$	$-0.692 \pm 0.016 \pm 0.006$ [18]
$\delta_P - \delta_S$ (rad)	$0.033 \pm 0.020^{+0.008}_{-0.012}$	$-0.040 \pm 0.033 \pm 0.017$ [17]
$\xi_P - \xi_S$ (rad)	$0.007 \pm 0.020^{+0.018}_{-0.005}$	$0.012 \pm 0.034 \pm 0.008$ [17]
A_{CP}^Ξ	$-0.009 \pm 0.008^{+0.007}_{-0.002}$	$0.006 \pm 0.013 \pm 0.006$ [17]
$\Delta\phi_{CP}^\Xi$ (rad)	$-0.003 \pm 0.008^{+0.003}_{-0.007}$	$-0.005 \pm 0.014 \pm 0.003$ [17]
A_{CP}^-	$-0.007 \pm 0.008^{+0.002}_{-0.003}$	$-0.0025 \pm 0.0046 \pm 0.0012$ [37]
A_{CP}^0	$0.001 \pm 0.009^{+0.005}_{-0.007}$	-
A_{CP}^Λ	$-0.004 \pm 0.007^{+0.003}_{-0.004}$	-
$\alpha_{\Lambda^0}/\alpha_{\Lambda^-}$	$0.877 \pm 0.015^{+0.014}_{-0.010}$	1.01 ± 0.07 [29]
$\bar{\alpha}_{\Lambda^0}/\alpha_{\Lambda^+}$	$0.863 \pm 0.014^{+0.012}_{-0.008}$	$0.913 \pm 0.028 \pm 0.012$ [18]



$R_1 = (1 + \alpha_\Lambda \alpha_\Xi \cos \theta) / (1 + \bar{\alpha}_\Lambda \bar{\alpha}_\Xi \cos \theta)$
 $R_2 = (1 + \langle \alpha_{\Lambda^0} \rangle \langle \alpha_\Xi \rangle \cos \theta) / (1 + \langle \alpha_{\Lambda^-} \rangle \langle \alpha_\Xi \rangle \cos \theta)$
 5 CP tests are performed.
 According to $\Delta I = 1/2$ rule, the ratio between the decay parameter α_{Λ^0} ($\Lambda \rightarrow n\pi^0$) and α_{Λ^-} ($\Lambda \rightarrow p\pi^-$) should be 1.
 Our result: $\langle \alpha_{\Lambda^0} \rangle / \langle \alpha_{\Lambda^-} \rangle = 0.870 \pm 0.012^{+0.011}_{-0.010}$ indicate the $\Delta I = 1/2$ rule is violated in the decays of Λ .

Absolute BF measurement of the Ω^- and test of the $\Delta I = 1/2$ rule



If $\Delta I=1/2$ rule is valid in Ω^- decay, the BF ratio between two decay channels $\Omega^- \rightarrow \Xi^0\pi^-$, $\Omega^- \rightarrow \Xi^-\pi^0$ should be 2. This ratio in PDG is 2.74 ± 0.15 , inconsistent with the prediction of the $\Delta I=1/2$ rule.

The result was only measured by one experiment, and therefore, received some skepticism!

Partial reconstruction to improve the efficiencies, only a π^- , a π^0 and a K^- in the signal side for the decay channels $\Omega^- \rightarrow \Xi^0\pi^-$, $\Omega^- \rightarrow \Xi^-\pi^0$, and $\Omega^- \rightarrow \Lambda K^-$, respectively.

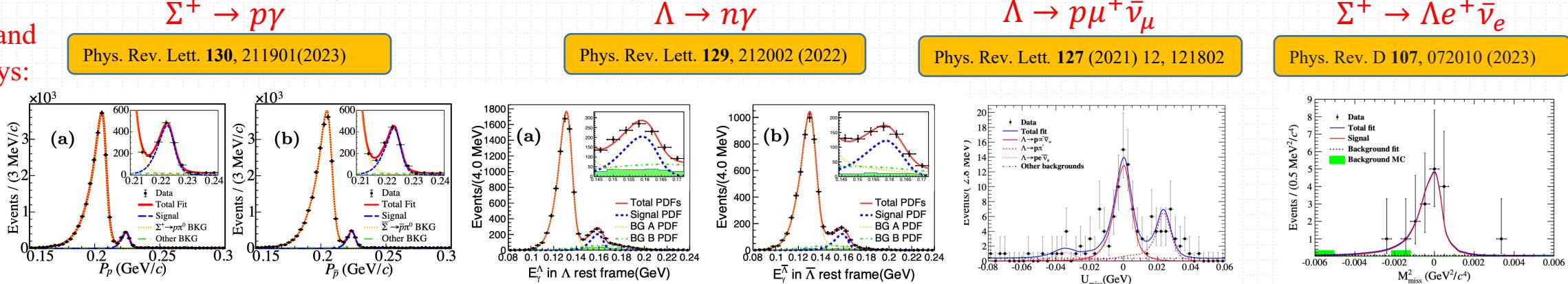
BFs	$\mathcal{B}_{\Omega^- \rightarrow \Xi^0\pi^-}$	$\mathcal{B}_{\Omega^- \rightarrow \Xi^-\pi^0}$	$\mathcal{B}_{\Omega^- \rightarrow \Lambda K^-}$
This work	$25.03 \pm 0.44 \pm 0.53$	$8.43 \pm 0.52 \pm 0.28$	$66.3 \pm 0.8 \pm 2.0$
PDG	23.6 ± 0.7	8.6 ± 0.4	67.8 ± 0.7

Phys. Rev. D 108 (2023) 9, L091101

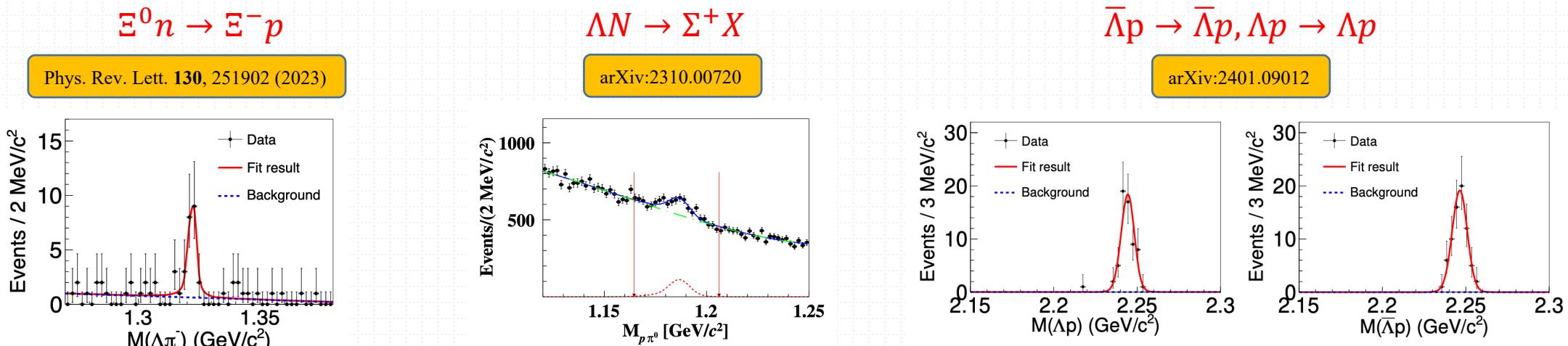
Our result: $R = 2.97 \pm 0.19 \pm 0.11$, consistent with the PDG value, confirms the $\Delta I=1/2$ rule is not suitable for describing the decays of the Ω^- hyperon.

More results of hyperon physics at BESIII...

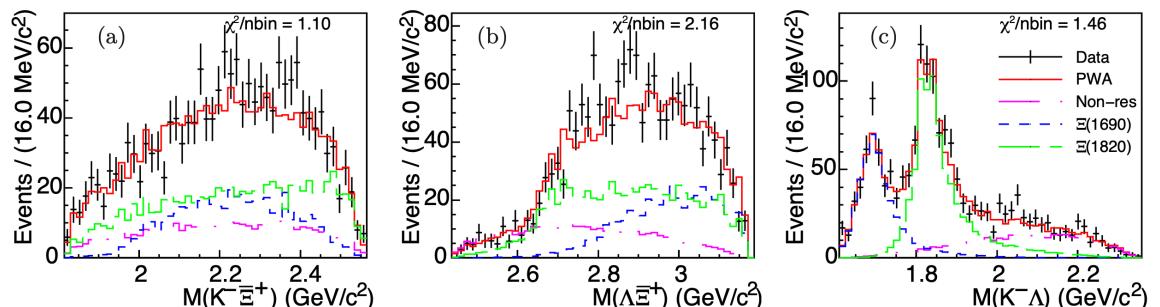
Semileptonic and radiative decays:



Hyperon-nucleus interactions:



Study of excited Ξ :



arXiv:2308.15206

Summary

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Summary

- ✓ BESIII has made fruitful achievements in the studies of hyperon physics:
 - ✓ Precise CP tests are performed in hyperon decays;
 - ✓ The $\Delta I = 1/2$ rule is found to be unsuitable for describing the decays of the Λ and Ω^- hyperons.
- ✓ More exciting hyperon results will come in the near future.

THANKS!

Backup

Hyperon Polarization

$$P_y(\cos \theta_Y) = \frac{\sqrt{1 - \alpha_\psi^2} \sin \Delta\Phi_\psi \cos \theta_Y \sin \theta_Y}{1 + \alpha_\psi \cos^2 \theta_Y}$$

$$\mu(\cos \theta_Y) = \frac{\alpha - \bar{\alpha}}{2} \frac{1 + \alpha_\psi \cos^2 \theta_Y}{3 + \alpha_\psi} P_y(\cos \theta_Y)$$

$$\mu^k(\cos \theta_\Xi) = \frac{1}{N^k} \sum_i^{N^k} (\sin \theta_\Lambda^i \sin \varphi_\Lambda^i + \sin \theta_{\bar{\Lambda}}^i \sin \varphi_{\bar{\Lambda}}^i)$$