Generalizing Binary Black Hole Systems

SUM UP PRESENTATION

Università degli studi di Perugia and Niels Bohr Institute

Daniele Pica

Supervisors:

Troels Harmark
Marta Orselli

Tutor:

Gianluca Grignani

28/10/2022

Tidal deformation of a binary system induced by a Kerr black hole

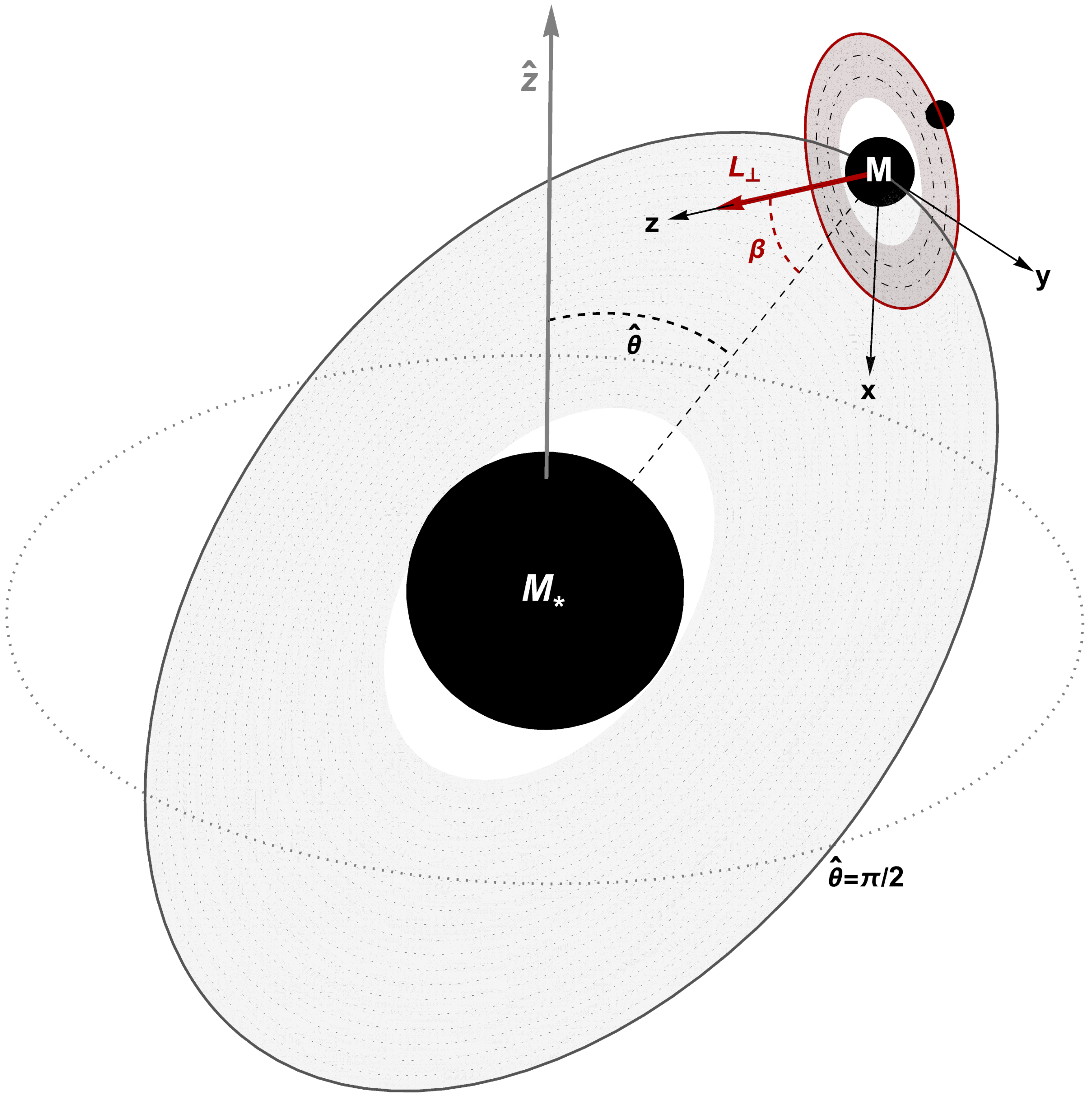
[Camilloni, Grignani, Harmark, Oliveri, Orselli, Pica]

- **Three body system**: a binary system under the gravitational interaction of a supermassive black hole.

- Hierarchical in scale: $m \ll M \ll M_*$

- $m \to$ test particle, $M \to$ Schwarzschild BH, $M_* \to$ Kerr BH

- $m \to 0 \Rightarrow$ Extreme mass ration inspirals **(EMRIs)**



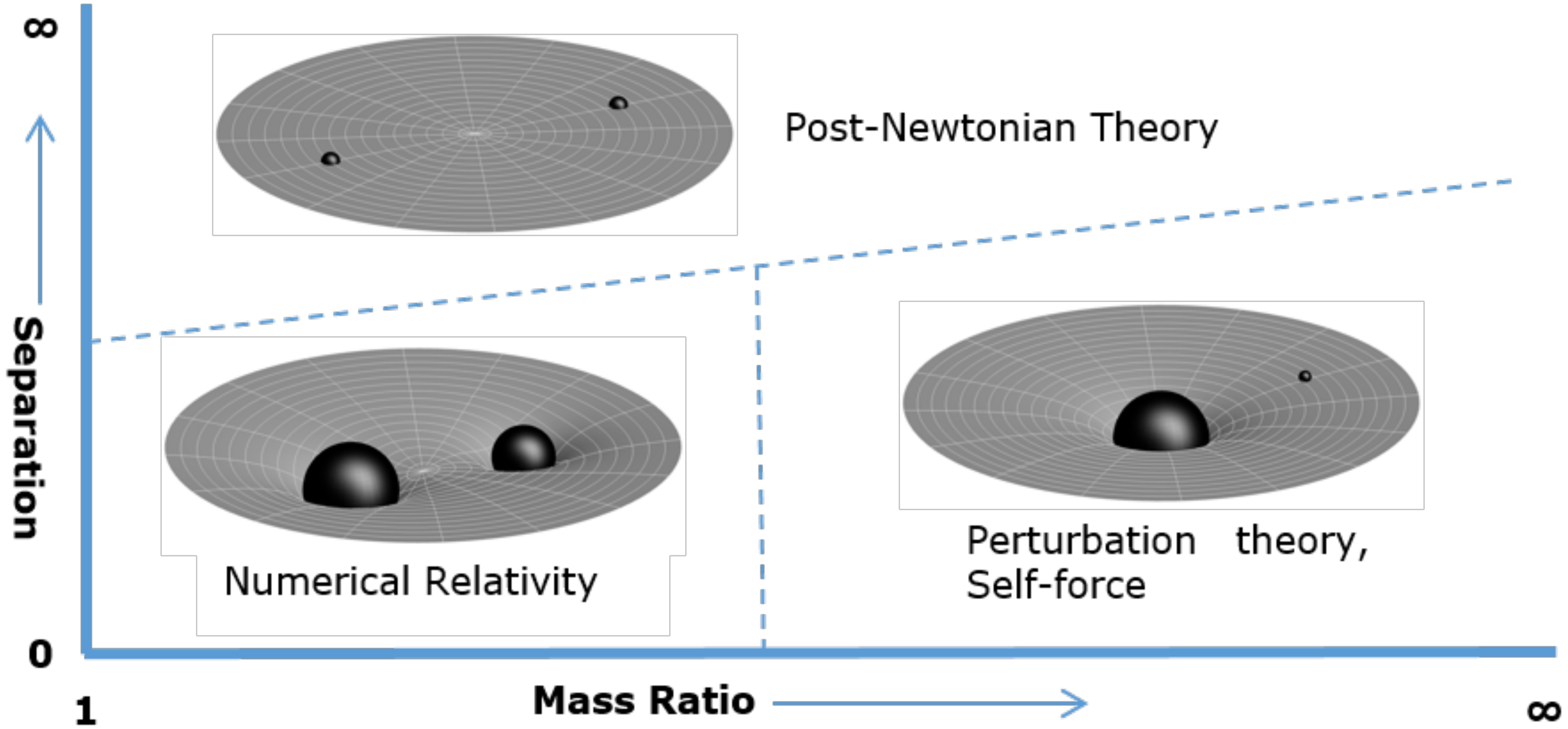
Extreme Mass Ratio Inspirals

- EMRIs are primary target for **future GWs observations**
  - Binary system orbital frequency in the inspiral regime $\omega = \sqrt{G\frac{m_1 + m_2}{d^3}}$
  - $\omega \sim 10^{-2} \div 10$ kHz $\Rightarrow$ **LIGO-VIRGO**
  - $\omega \sim 0.1 \div 100$ mHz $\Rightarrow$ **LISA**
- Abundance of sources
- Interesting dynamics $\rightarrow \mu = \frac{m}{M} \ll 1 \Rightarrow$ natural perturbative approaches!

$$\mu = \frac{m_2}{m_1} \sim 10^{-4} \div 10^{-6}$$

Ex

$$m_1 \sim 10^6\ M_\odot, \quad m_2 \sim 10\ M_\odot \implies \omega \sim 10 \text{ mHZ}$$



Tidal Scales

- **Two different length scales**: the mass of the intermediate black hole M and the radius of the curvature generated by the third body evaluated in the position of the binary system $\mathscr{R}$

- In order to be able to study the physics on the scale of M we need to require that the tidal interaction is weak → **small-tide approximation $M \ll \mathscr{R}$**

- If d is the distance between the binary system and the big black hole, V is the orbital velocity

$V \sim \sqrt{\frac{M+M_*}{d}}$ and $\mathscr{R} \sim \sqrt{\frac{d^3}{M+M_*}}$ then $\frac{M}{\mathscr{R}} \sim \frac{M}{M+M_*}V^3$

- $\frac{M}{\mathscr{R}}$ should be a small quantity in order to have a weak tidal interaction. To satisfy this condition we will use the **small-hole approximation $M \ll M_*$** $\longrightarrow$ $\frac{M}{\mathscr{R}} \ll 1$ independently of V and d !

- This allows us to study how the "external spacetime" affects the binary system, not only when $d \to \infty$

[Yang, Casals (2017)]

Tidally Deformed Black Hole

- In the tidal approximation the **metric** which we use to describe the **tidally deformed black hole** is expressed as a cover expansion of $s \ll \mathscr{R}$, with s being the distance from the black hole M.

- We are only interested in the **quadrupole order** of the perturbed metric, meaning that we will consider only terms up to order $\mathcal{O}\left(\frac{s}{\mathscr{R}}\right)^2$

- The metric of any vacuum spacetime can be constructed in the neighborhood of any geodesic world line and express in term of two sets of **tidal multiple moments** [Poisson, Vlasov (2010)]

$$ds^2 \simeq -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)+h_{\mu\nu}dx^\mu dx^\nu+\mathcal{O}(s/\mathscr{R})^3$$

$$h_{tt}=-r^2\left(1-\frac{2M}{r}\right)^2\mathscr{E}^q \qquad h_{tr}=\left(1-\frac{2M}{r}\right)h_{rr}=-r^2\left(1-\frac{2M}{r}\right)\mathscr{E}^q$$

$$h_{tA}=\left(1-\frac{2M}{r}\right)h_{rA}=-\frac{2}{3}r^3\left(1-\frac{2M}{r}\right)\left(\mathscr{E}^q_A-\mathscr{B}^q_A\right) \qquad h_{AB}=-\frac{1}{3}r^4\left[\left(1-\frac{2M^2}{r^2}\right)\mathscr{E}^q_{AB}-\left(1-3\frac{2M^2}{r^2}\right)\mathscr{B}^q_{AB}\right]$$

Tidal Moments of a Kerr perturber

- **Fermi normal coordinates**
- Tidal moments are defined through the Weyl tensor once it is evaluated on the Kerr geodesic

$$C_{ab} \equiv C_{a0b0} = C_{\rho\sigma\mu\nu}\omega^{\rho}{}_{(a)}\omega^{\sigma}{}_{(b)}\omega^{\mu}{}_{(c)}\omega^{\nu}{}_{(d)}e_a^{(a)}e_0^{(b)}e_b^{(c)}e_0^{(d)}$$
$$C_{abc} \equiv C_{abc0} = \omega^{\rho}{}_{(a)}\omega^{\sigma}{}_{(b)}\omega^{\mu}{}_{(c)}\omega^{\nu}{}_{(d)}e_a^{(a)}e_b^{(b)}e_c^{(c)}e_0^{(d)}$$

- where $\omega_{\mu}^{(a)}$ is the Carter tetrad for Kerr spacetime.
- From the symmetries of Weyl tensor $\Rightarrow$ 10 independent components encoded in two symmetric-trace-free (STF) tensors

$$\mathscr{E}_{ab} := \left(C_{a0b0}\right)^{\mathrm{STF}}$$
$$\mathscr{B}_{ab} := \frac{1}{2}\left(\epsilon_{acd}C^{cd}{}_{b0}\right)^{\mathrm{STF}}$$

- We refer to $\mathscr{E}_{ab}$ and $\mathscr{B}_{ab}$ as the **tidal quadrupole moments**.

Tidal Moments of a Kerr perturber

- The construction of tidal multipole moments stems from the identification of a local orthonormal tetrad which is designed to be an inertial frame parallel transported along the Kerr geodesic motion [Marck (1983)]

- The tetrad considerably simplifies for circular equatorial geodesics in Kerr
$\hat{r} = d, \quad \dot{\hat{r}} = 0, \quad \hat{\theta} = \pi/2$

$$e^{\mu}_{(0)} = \frac{1}{d\sqrt{\Delta}}\left(\hat{E}(d^2+a^2) - a\hat{L}\right)\delta^{\mu}_{0} + \frac{\left(a\hat{E}-\hat{L}\right)}{d}\delta^{\mu}_{3},$$

$$e^{\mu}_{(3)} = \sqrt{\frac{1}{\hat{K}}}\left(\hat{L} - a\hat{E}\right)\delta^{\mu}_{2},$$

$$e^{\mu}_{(1)} = \tilde{e}^{\mu}_{(1)}\cos\Psi + \tilde{e}^{\mu}_{(2)}\sin\Psi,$$

$$e^{\mu}_{(2)} = \tilde{e}^{\mu}_{(1)}\sin\Psi + \tilde{e}^{\mu}_{(2)}\cos\Psi$$

$$\tilde{e}^{\mu}_{(1)} = \frac{\left(\hat{E}(d^2+a^2) - a\hat{L}\right)}{\sqrt{\left(d^2+\hat{K}\right)\Delta}}\delta^{\mu}_{1}, \qquad \hat{K} = \left(a\hat{E}-\hat{L}\right)^2$$

$$\tilde{e}^{\mu}_{(2)} = \sqrt{\frac{\hat{K}}{\left(d^2+\hat{K}\right)\Delta}}\frac{\left(\hat{E}(d^2+a^2) - a\hat{L}\right)}{d}\delta^{\mu}_{0} + \sqrt{\frac{d^2+\hat{K}}{\hat{K}}}\frac{\left(a\hat{E}-\hat{L}\right)}{d}\delta^{\mu}_{3}$$

- Ψ is an angle introduced in order to parallel transport the tetrad $(e^{\mu}_{(0)}, e^{\mu}_{(1)}, e^{\mu}_{(2)}, e^{\mu}_{(3)})$ along the geodesic

Spherical Coordinates

- It is convenient to introduce the **unit radial vector** Ω^a, oriented in the $\hat{r}$-direction and written in Cartesian components

- In order to study **all possible orientations** for the binary system around the Kerr BH, we introduce the **Euler angles**

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad R_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} , \quad R_\chi = \begin{pmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

- We define the unit radial vector Ω^a as

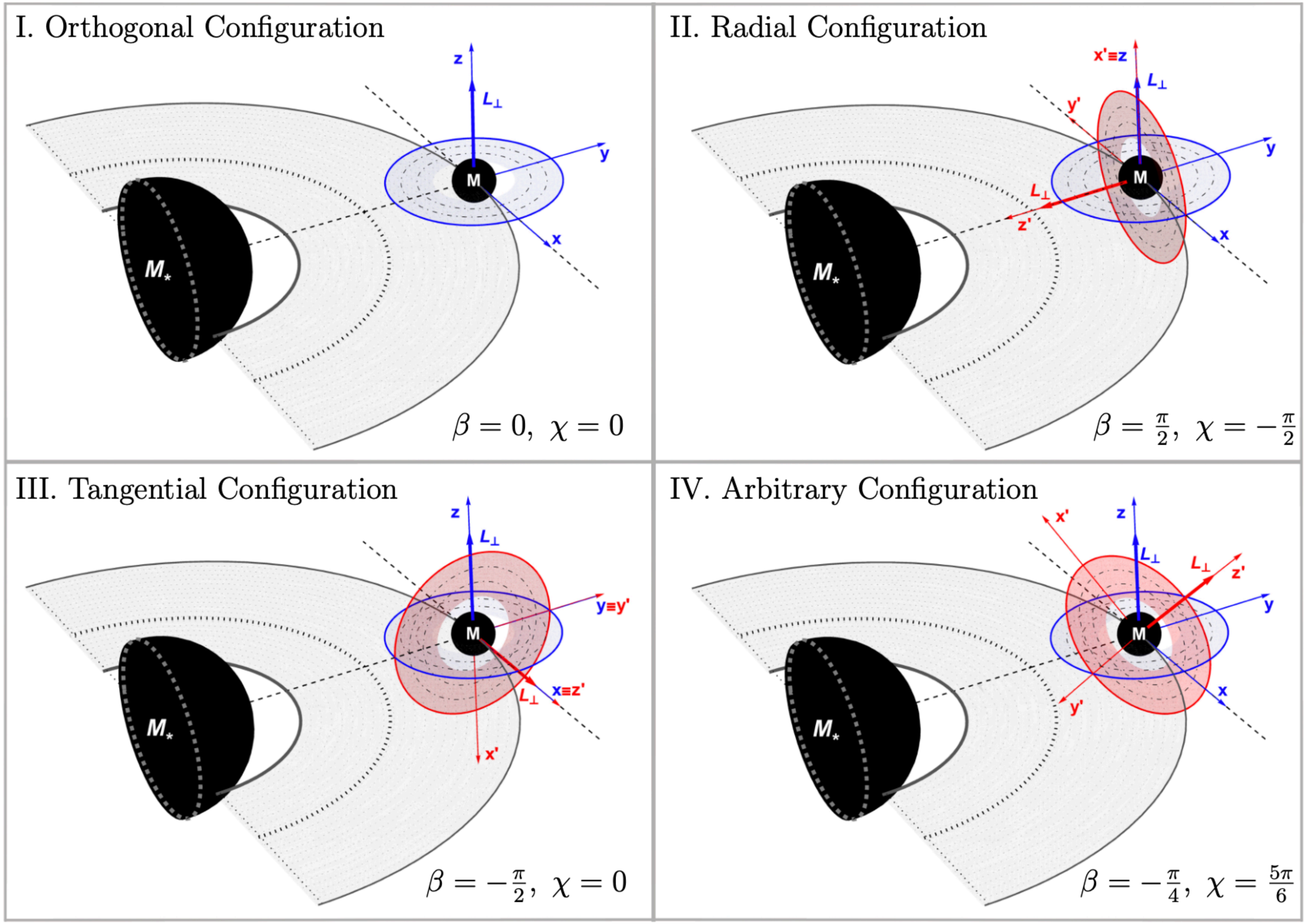
$$\Omega^a = R_\chi R_\beta R_\alpha \Omega^a_*, \quad \Omega^a_* = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

- This is only one among the possible 12 equivalent combinations of Euler matrices

- Equatorial orbit in BBH $\Rightarrow$ α-angle can be eliminated by redefinition of the Schwarzschild azimuthal angle ϕ

Possible Orientations

Any orientation of a Schwarzschild orbit with respect to the Kerr perturber is specified only by the two angles β and χ



Tidal Shift

- It is useful to introduce the **dimensionless perturbative parameter** $\epsilon = \dfrac{M_* M^2}{d^3}$, under the assumption $M \ll M_* \Rightarrow \epsilon \ll 1$

- Explicitly $\mathscr{E} \sim \mathcal{O}\left(\epsilon\right)$ and $\mathscr{B} \sim \mathcal{O}\left(\epsilon\right)$

- The tidal fields generated by the outer body deform the orbits of the unperturbed Schwarzschild metric

- If $\bar{x}^\mu(\tau)$ solves the geodesic equation in the unperturbed Schwarzschild geometry, $\ddot{\bar{x}}^\mu = -\Gamma^\mu_{\nu\rho}|_{\bar{x}}\dot{\bar{x}}^\nu\dot{\bar{x}}^\rho$, the effect of the tidal deformation will reflect in a deviation from the unperturbed curve $\bar{x}^\mu$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}\left(\epsilon^2\right) \qquad x^\mu(\tau) + \epsilon y^\mu(\tau)$$

- In general the tidal moments depend on $\phi \Rightarrow$ **circular orbits are deformed into elliptic orbits**

- We want to construct the **Hamiltonian** per unit mass squared H of the EMRI + tidal interaction system

- The **radial correction** enters the Hamiltonian only with terms of order $\mathcal{O}\left(\epsilon^2\right)$, as a consequence it is possible to replace the true trajectory in the perturbed spacetime with the "**mean**" circular trajectory

Secular Hamiltonian

- Define

$$\langle A \rangle = \frac{1}{2\pi} \int_0^{2\pi} A \,|_\gamma \, d\phi$$

where γ is the mean circular orbit on $G = g + \epsilon \; h$

- This averaging procedure allows one to consider the **secular dynamics** of bound orbits in the tidally deformed spacetime

- The Hamiltonian per unit mass squared H of the EMRI + tidal interaction system is given by

$$H = \frac{1}{2} p^\mu p^\nu \langle G_{\mu\nu} \rangle = \frac{1}{2} \left(g_{\mu\nu} + \epsilon \; \langle h_{\mu\nu} \rangle \right) + \mathcal{O}\left(\epsilon^2\right)$$

- Starting from an **unperturbed circular geodesic**, the total momentum p^μ can be written as

$$p^\mu = \left(\frac{E}{(1 - r_0/r) - \epsilon \langle h_{tt} \rangle}, 0, 0, \frac{L}{r^2 + \epsilon \langle h_{\phi\phi} \rangle} \right)$$

Secular Hamiltonian

- Using the expressions for $\langle h_{tt}\rangle$ and $\langle h_{\phi\phi}\rangle$ the **secular Hamiltonian** becomes

$$H = \frac{1}{2}\left(\frac{L^2}{r^2} - \frac{E^2}{1 - r_0/r}\right) - 4\eta\frac{r^2}{r_0^2}\left[\frac{L^2}{r^2}\left(1 - \frac{r_0^2}{2r^2}\right) + E^2\right]$$

- Where the **effective perturbative parameter** η contains all the information regarding how the binary system is oriented, as well as where it is located with respect to the Kerr perturber

- In the **equatorial plane** of the Kerr black hole ($\hat{\theta} = \pi/2$), η reduces to

$$\eta = \frac{\epsilon\left(6\sin^2\beta\cos 2\chi\left(a^2 - 2M_*\hat{r} + \hat{r}^2\right) + 3\cos 2\beta\left(3a^2 - 4a\sqrt{M_*\hat{r}} + \hat{r}^2\right) + 3a^2 - 4a\sqrt{M_*\hat{r}} + \hat{r}^2\right)}{16\left(2a\sqrt{M_*\hat{r}} - 3M_*\hat{r} + \hat{r}^2\right)}$$

ISCO Shifts

- The **ISCO** (Innermost stable circular orbit) can be obtained upon demanding the Hamiltonian H to be on-shell, the orbit to be circular and that the radial perturbations become stationary

$$H = -\frac{1}{2}, \quad \frac{\partial H}{\partial r} = 0, \quad \frac{\partial^2 H}{\partial r^2} = 0$$

- Using these conditions and expanding at the first order in ϵ, it is possible to determine the **secular shifts** caused by the tidal perturbations to the energy angular momentum and radius of Schwarzschild

$$r = r_{isco} + \eta \; r_1 + \mathcal{O}(\eta^2), \quad E = E_{isco} + \eta \; E_1 + \mathcal{O}(\eta^2), \quad L = L_{isco} + \eta \; L_1 + \mathcal{O}(\eta^2)$$

- At the leading order one finds $r_{isco}, E_{isco}, L_{isco}$, namely the values for the radius, energy and angular momentum of the ISCO for an unperturbed Schwarzschild black hole

$$r_{isco} = 3r_0, \quad E_{isco} = \frac{\sqrt{8}}{3}, \quad L_{isco} = \sqrt{3}r_0$$

- At the **first order in η** one determines the corrections to the three quantities

$$r_1 = 1536 r_0, \quad E_1 = -\frac{152\sqrt{2}}{3}, \quad L_1 = -174\sqrt{3}r_0$$

[Yang, Casals (2017)]

ISCO Shifts

- The effective secular perturbative parameter η is new! It allows to specify not only the orientation of the binary system but also its location with respect to a spinning tidal perturber

- We also computed the shift in the **ISCO orbital frequency**. In general for quasi-circular orbits the orbital frequency can be determined by means of the ratio

$$\Omega = \frac{p^\phi}{p^t} \quad \xrightarrow{\text{Expanding}} \quad \Omega = \Omega_{isco} + \eta\, \Omega_1 + \mathcal{O}\left(\eta^2\right)$$

where

$$r_0\, \Omega_{isco} = \frac{1}{3\sqrt{6}}, \quad r_0\, \Omega_1 = -\sqrt{\frac{2}{3}}\frac{491}{3}$$

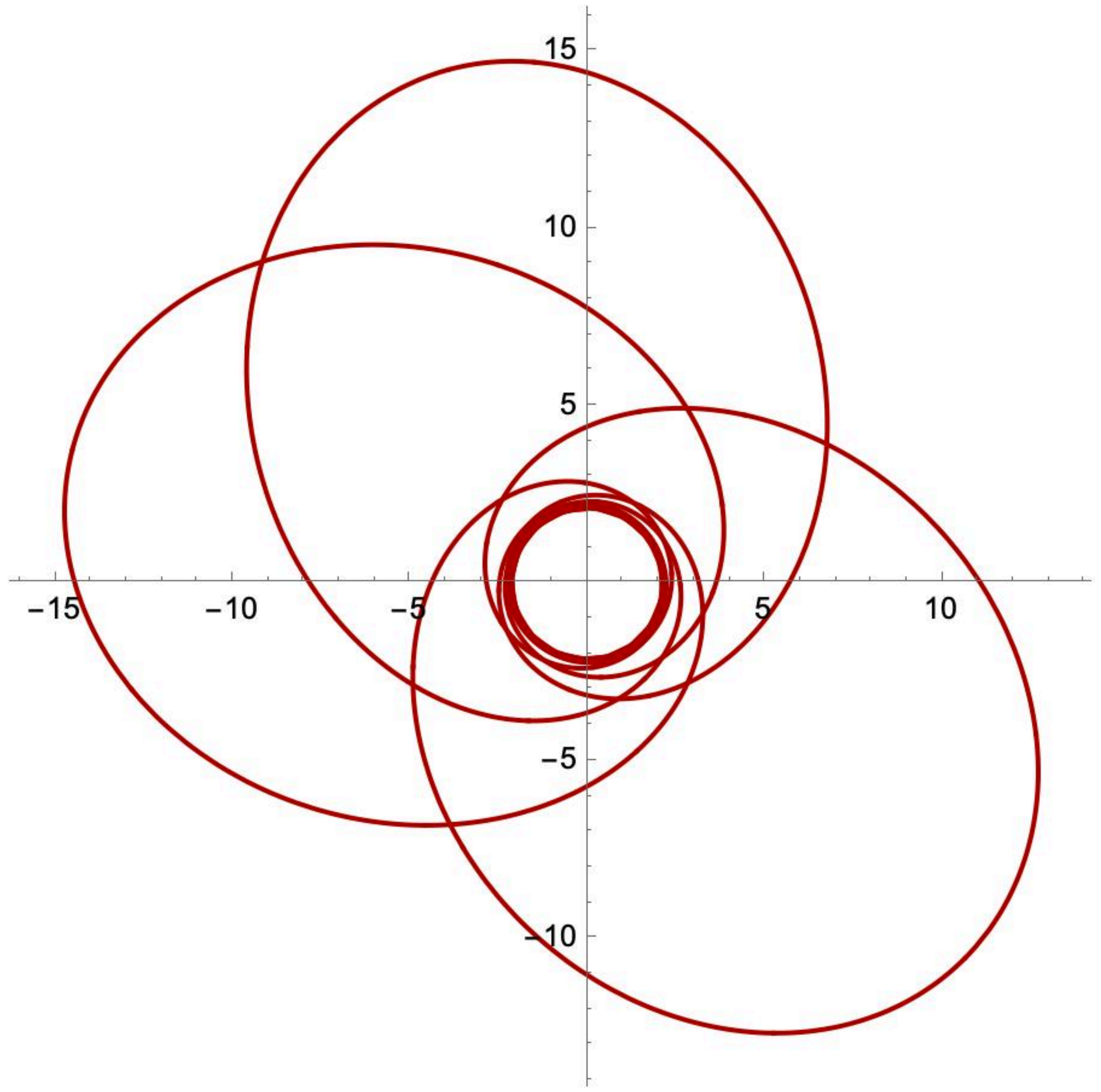
- In the case in which the binary system is located asymptotically away from the Kerr perturber, $d \gg M_*$, every information about the black hole spin parameter a is lost and we recover the results already known in the literature [Yang, Casals (2017) & Cardoso, Foschi (2021)]

Summary

- We computed the **ISCO shifts** induced by the tidal moments of a Kerr black hole on an EMRI
- ISCO shifts of the energy, angular momentum, radius and angular velocity for any value of the distance, the spin of the Kerr black hole and the inclination of the orbit of the test particle around the Schwarzschild black hole

Perspectives

- It is possible to compute the frequencies of the motion through the **action angle variables**
- Non circular orbits, eccentricity, resonances!
- Corrections to the orbital precession
- **ISO** (Innermost Stable Orbit)!



My Activities

- ✓ “Responsible Conduct of Research” course (Passed)
- ✓ “Introduction to String Theory” course (Passed)
- ✓ “Theoretical Astrophysics” course (Currently following)
- ✓ TA for “Introduction to String Theory”
- ✓ Talk for “NBI Section 9 mini-conference” (14/06/2022)
- ✓ Conference “What’s new in Gravity” (9/08/2022 - 12/08/2022)
- ✓ PhD School “Gravity@Prague” (19/09/2022-23/09/2022)
- ✓ Published paper “Event Horizon of a Charged Black Hole binary merger” [Marin Pina, Orselli, Pica] DOI: 10.1103/PhysRevD.106.084012

Thank you for the attention!