The Gravitational Field of a Spatially Non-Local Quantum Superposition

by

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- We are interested in determining what is the gravitational field of a massive particle in a quantum mechanical wave function that is in a spatially non-local superposition.
- Normally we can compute the classical potential of a given particle with an interaction, however, this is not an unambiguous procedure.
- Such a calculation is obtained from the scattering amplitude in the non-relativistic limit.
- Finding the gravitational behaviour of a spatially non-local superposition is interesting for questions of quantum gravitational behaviour, collapse of the quantum mechanical wave function and quantum decoherence.
- The Schrödinger-Newton theory posits that the potential corresponds to a classical superposition of the gravitational potential of half the mass at each position of the non-local superposition.

Schrödinger-Newton Theory

- The S-N theory posits that the quantum mechanical wave function of a particle in the presence of gravitational interactions feels a gravitational self potential that is proportional to the gravitational potential created by a mass density corresponding to the probability density of the particle's wave function itself. This gives rise to a non-linear, non-local Schrödinger equation for the wave function of the particle.

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + m\Phi\Psi$$

$$\nabla^2\Phi = 4\pi G m|\Psi|^2$$

- ie. $$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V - Gm^2\int\frac{|\Psi(t,\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}^3\mathbf{y}\right]\Psi$$

Problems with the S-N formalism

- Diosi has pointed out 4 paradoxes that arise if one embraces the S-N formalism:
- 1.) Action at a distance
- 2.) Superluminal telegraphy
- 3.) Unsuitable superposition
- 4.) Breakdown of the statistical interpretation
- I cannot go into the details of these problems, please look at the paper by Diosi:

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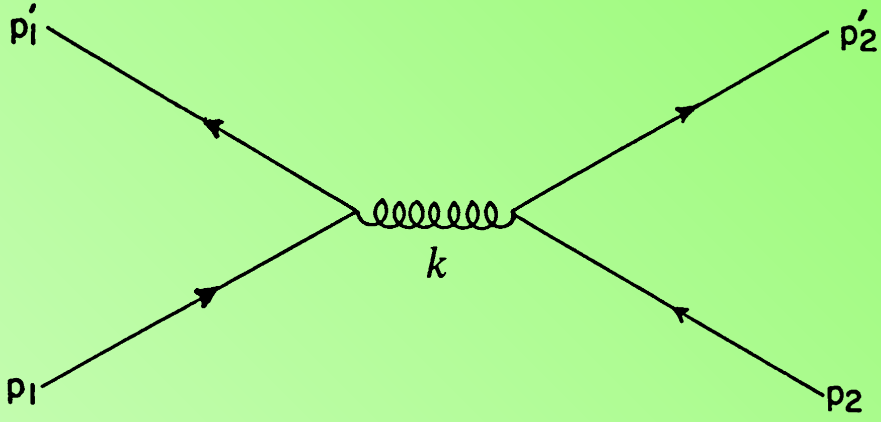
Probing the gravitational field

- The simplest thing to try to compute would be the gravitational potential of a non-local superposition.
- However this is an ambiguous concept.
- Normally, one can extract the potential from the non-relativistic limit of the scattering amplitude.
- One computes the amplitude say:



Gravitons

- Gravitons correspond to quantization of the small fluctuations about a Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \partial^\sigma \partial_\sigma h_{\mu\nu} = 0$$

$$h_{\mu\nu} = \int d^3k \left(\frac{e^{-ik_\mu x^\mu}}{\sqrt{V}} A(\vec{k})\epsilon_{\mu\nu} + \frac{e^{ik_\mu x^\mu}}{\sqrt{V}} \epsilon^*_{\mu\nu} A^\dagger(\vec{k}) \right)$$

$$\left[A(\vec{k}), A^\dagger(\vec{k}') \right] = \delta^3(\vec{k} - \vec{k}')$$

- interaction: $\sim h_{\mu\nu} T^{\mu\nu}$
- amplitude

$$\mathcal{M} = \langle \vec{k}_2, \vec{p}_2 | h_{\mu\nu} T^{\mu\nu} h_{\sigma\tau} T^{\sigma\tau} | \vec{k}_1, \vec{p}_1 \rangle$$

Calculation of the potential from the non-relativistic limit

- the amplitude for non-relativistic potential scattering in the first Born approximation is given by

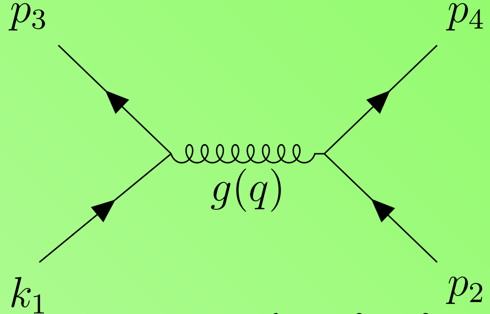
$$f^{(1)}(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi}(2\pi)^3 \frac{2m}{\hbar^2} \langle \mathbf{k}' | V | \mathbf{k} \rangle$$

- therefore the potential can be found from the inverse Fourier transform of the scattering amplitude
- while the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2$$

Potential from amplitudes

- for example from the gravitational scattering of two massive particles



- we have the non-relativistic amplitude

$$\mathcal{M}_{NR}(\boldsymbol{q}) = -\frac{\kappa^2}{2|\boldsymbol{q}|^2}\left(2m_1^2 m_2^2\right)$$

- which gives the Newton potential

$$V(r) = \frac{1}{4m_1 m_2}\int \mathcal{M}_{NR}(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3 r = -\frac{\kappa^2 m_1 m_2}{8\pi|\boldsymbol{r}|}$$

Potential from amplitudes

- We can try the same for a scattering from a non-local wave function. We still use:

$$\langle\phi_f|iT|\phi_i\rangle = -i\tilde{V}(q)(2\pi)\delta(E_i - E_f)$$

- With the initial and final momentum space wave functions:

$$\phi_1(\boldsymbol{k}_1) = \frac{(4\pi\sigma^2)^{3/4}}{\sqrt{2}} \frac{e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}_0} + e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_0}}{\sqrt{1+e^{-|\boldsymbol{r}_0|^2/\sigma^2}}} e^{-\frac{\sigma^2}{2}|\boldsymbol{k}_1|^2}$$

$$\phi_3(\boldsymbol{k}_3) = \frac{(4\pi\sigma^2)^{3/4}}{\sqrt{2}} \frac{e^{i\boldsymbol{k}_3\cdot\boldsymbol{r}_0} + e^{-i\boldsymbol{k}_3\cdot\boldsymbol{r}_0}}{\sqrt{1+e^{-|\boldsymbol{r}_0|^2/\sigma^2}}} e^{-\frac{\sigma^2}{2}|\boldsymbol{k}_3|^2}$$

- which corresponds to the scattering that does not affect the non-local superposition, one finds

$$\tilde{V}(q) = \int \frac{d^3k_1}{(2\pi)^3} \phi_1(\boldsymbol{k}_1)\phi_3(\boldsymbol{k}_1 - \boldsymbol{q}) \frac{\langle k_1 - q, p_4|k_1, p_2\rangle}{4\sqrt{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}}\bigg|_{NR}$$

$$= -\kappa^2 \frac{M\omega}{|\boldsymbol{q}|^2} \frac{\cos(\boldsymbol{q}\cdot\boldsymbol{r}_0) + \cos(2\boldsymbol{q}\cdot\boldsymbol{r}_0)e^{-|\boldsymbol{r}_0|^2/\sigma^2}}{1+e^{-|\boldsymbol{r}_0|^2/\sigma^2}} e^{-\frac{\sigma^2}{4}|\boldsymbol{q}|^2}$$

- which in position space is:

$$V(r) \simeq -\frac{GM\omega}{1+e^{-|r_0|^2/\sigma^2}}\frac{1}{2}\left(\frac{1}{|\boldsymbol{r}+\boldsymbol{r}_0|}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}+\frac{e^{-|\boldsymbol{r}_0|^2/\sigma^2}}{|\boldsymbol{r}+2\boldsymbol{r}_0|}+\frac{e^{-|\boldsymbol{r}_0|^2/\sigma^2}}{|\boldsymbol{r}-2\boldsymbol{r}_0|}\right)$$

- However this result is not meaningful. The choice of final state is arbitrary. The non-relativistic limit for the scattering does not make sense. The momenta implied in the wave function of the superposition are widely spread, and the non-relativistic limit for one pair of scattering momenta is not so for another pair.
- The final state could be the rigid scattering of the superposition, or it could have various excitations, rotations or just a momentum eigenstate etc. All different choices give different forms for the potential.
- To escape this quandary we will compute the inclusive differential cross-section, which is unambiguous.

Amplitude for scattering from a spatially non-local wave function

- We need to use a more sophisticated formalism to describe the scattering.
- This has been described by the work of Kotkin et al and Korvalets which describe the scattering cross section in terms of wave packets.
- They compute scattering cross sections for particles in non-trivial wave functions.

Scattering of wave packets

- Kotkin et al and Korvalets give a proper formalism for scattering of wave packets. They show the cross section is described through the probability of scattering, given by

$$dW = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3b}{(2\pi)^3} d\sigma(\boldsymbol{b},\boldsymbol{k}_{1,2})\mathcal{L}(\boldsymbol{b},\boldsymbol{k}_{1,2})$$

- where the generalized cross section is given by

$$d\sigma(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{b}) = (2\pi)^4\delta\left(\epsilon_1(\boldsymbol{k}_1+\boldsymbol{b}/2)+\epsilon_2(\boldsymbol{k}_2-\boldsymbol{b}/2)-\epsilon_f\right)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p}_f)$$

$$\times T_{PW}(\boldsymbol{k}_1+\boldsymbol{b}/2,\boldsymbol{k}_2-\boldsymbol{b}/2)T^*_{PW}(id.)\frac{1}{v(\boldsymbol{k}_1,\boldsymbol{k}_2)}\Pi_f\frac{d^3p_f}{(2\pi)^3}$$

- where the particle correlation is given by

$$\mathcal{L}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{b}) = v(\boldsymbol{k}_1,\boldsymbol{k}_2)\int d^3r\ d^3R\ dte^{i\boldsymbol{b}\cdot\boldsymbol{R}}n_1(\boldsymbol{r},\boldsymbol{k}_1,t)n_2(\boldsymbol{r}+\boldsymbol{R},\boldsymbol{k}_2,t)$$

- Then the cross section is:

$$d\sigma_{\text{eff.}} = \frac{dW}{L}$$

- Where the luminosity is:

$$L = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}dtd^3rv(\boldsymbol{k}_1,\boldsymbol{k}_2)n_1(\boldsymbol{r},\boldsymbol{k}_1,t)n_2(\boldsymbol{r},\boldsymbol{k}_2,t)$$

Scattering of wave packets

- Where various elements in the expressions before are:

$$v(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}{\varepsilon_1(\boldsymbol{p}_1)\varepsilon_2(\boldsymbol{p}_2)} = \sqrt{(\boldsymbol{u}_1 - \boldsymbol{u}_2)^2 - [\boldsymbol{u}_1 \times \boldsymbol{u}_2]^2},$$

$$\varepsilon_f = \sum_{i=3}^{N_f+2} \varepsilon_i(\boldsymbol{p}_i),\ \varepsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2},\ \boldsymbol{p}_f = \sum_{i=3}^{N_f+2} \boldsymbol{p}_i,\ \boldsymbol{u}_{1,2} = \boldsymbol{p}_{1,2}/\varepsilon_{1,2}(\boldsymbol{p}_{1,2})$$

$$T_{fi}^{(pw)} = \frac{M_{fi}^{(pw)}}{\sqrt{2\varepsilon_1 2\varepsilon_2 \prod_{f=3}^{N_f+2} 2\varepsilon_f}}$$

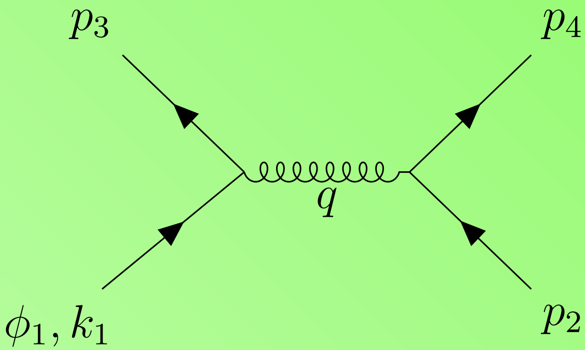
$$\int d^3r\, n(\boldsymbol{r}, \boldsymbol{p}, t) = |\psi(\boldsymbol{p}, t)|^2,\ \int \frac{d^3p}{(2\pi)^3}\, n(\boldsymbol{r}, \boldsymbol{p}, t) = |\psi(\boldsymbol{r}, t)|^2,\ \int \frac{d^3p}{(2\pi)^3}\, d^3r\, n(\boldsymbol{r}, \boldsymbol{p}, t) = 1$$

Scattering from a non-local wave function continued

- where:

$$|\phi_1(\boldsymbol{k}_1)|^2 = 4\left(\pi\sigma^2\right)^{3/2} e^{-\sigma^2|\boldsymbol{k}_1|^2} \frac{2 + e^{2i\boldsymbol{r}\cdot\boldsymbol{k}_1} + e^{-2i\boldsymbol{r}\cdot\boldsymbol{k}_1}}{1 + e^{-|\boldsymbol{r}|^2/(\sigma^2)}}$$

- and the amplitude corresponds to the Feynman diagram



$$\mathcal{M} = \frac{(-i)^2\kappa^2(k_1^\mu p_3^\nu + k_1^\nu p_3^\mu - (p_2\cdot p_4)\eta^{\mu\nu})(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})(p_2^\alpha p_4^\beta + p_2^\beta p_4^\alpha - (p_2\cdot p_4)\eta^{\alpha\beta})}{2q^2}$$

Scattering cross section

- The scattering cross section is unambiguous and given by:

$$\frac{d\sigma}{d\Omega_4} = \int \frac{d^3k_1}{2^9\pi^5} |\phi_1(\boldsymbol{k}_1)|^2 \frac{|\mathcal{M}|^2}{\epsilon_1\omega_2\epsilon_3\omega_4} \frac{\omega_4^2}{|f'_\delta(\omega_4)|}$$

- We essentially have to smear the momentum state cross section with the wave function of the non-local superposition (in momentum space).
- We compute the inclusive cross section for the gravitational scattering of a massless particle on a spatially non-local wave function of a massive particle, scattering to any final state for the massive particle and a specific momentum (scattering angle) for the massless particle.

Multi-pole expansion of the scattering cross section

- The cross section is a complicated mess which we can write as:

$$\frac{d\sigma}{d\Omega_4} = \int \frac{d^3k_1}{2^9\pi^5} |\phi_1(\boldsymbol{k}_1)|^2 g(\boldsymbol{k}_1)\, |\mathcal{M}|^2$$

- Then we can expand the integrand:

$$g(\boldsymbol{k}_1)\, |\mathcal{M}|^2 = g(\boldsymbol{k}_1)\, |\mathcal{M}|^2\Big|_{\boldsymbol{k}_1=0} + \tfrac{1}{2}\, \partial_{\boldsymbol{k}_1^i}\partial_{\boldsymbol{k}_1^j} \left(g(\boldsymbol{k}_1)\, |\mathcal{M}|^2 \right)\Big|_{\boldsymbol{k}_1=0} \boldsymbol{k}_1^i \boldsymbol{k}_1^j + \cdots$$

- Which yields for the cross section:

$$\frac{d\sigma}{d\Omega_4} = \alpha + \frac{\beta^{ij}\delta_{ij}}{3} \left(\frac{3}{2\sigma^2} - \frac{|\boldsymbol{r}|^2}{\sigma^4} \frac{1}{1+e^{|\boldsymbol{r}|^2/\sigma^2}} \right) + \beta^{ij} \left(\frac{|\boldsymbol{r}|^2}{3}\delta_{ij} - r_i r_j \right) \frac{1}{\sigma^4 \left(1+e^{|\boldsymbol{r}|^2/\sigma^2}\right)}$$

- where

$$\alpha = \tfrac{1}{2^6\pi^2} g(\boldsymbol{k}_1)\, |\mathcal{M}|^2\Big|_{\boldsymbol{k}_1=0} \qquad \beta^{ij} = \tfrac{1}{2^7\pi^2}\, \partial_{\boldsymbol{k}_1^i}\partial_{\boldsymbol{k}_1^j} \left(g(\boldsymbol{k}_1)\, |\mathcal{M}|^2 \right)\Big|_{\boldsymbol{k}_1=0}$$

$$\alpha = \frac{1}{\left(1+\frac{3}{4M^2\sigma^2}\right)} \frac{\kappa^4 M^2}{16\pi^2(1-\boldsymbol{\hat{p}_2}\cdot\boldsymbol{\hat{p}_4})^2}$$

$$\beta^{ij} = \frac{1}{\left(1+\frac{3}{4M^2\sigma^2}\right)} \left(\frac{\kappa^4}{16\pi^2(1-\boldsymbol{\hat{p}_2}\cdot\boldsymbol{\hat{p}_4})^2}\right)(\delta_{ij}+3\hat{p}_{2i}\hat{p}_{2j})$$

Multi-pole expansion of the scattering cross section continued

- The α term is simply the monopole contribution for the scattering of a massless particle from a single massive particle at one position. The $\beta^{ij}\delta_{ij}$ term adds to the monopole contribution.
- The quadrupole contribution is given by:

$$\beta^{ij}\left(\frac{|\boldsymbol{r}|^2}{3}\delta_{ij} - r_i r_j\right)\frac{1}{\sigma^4\left(1+e^{|\boldsymbol{r}|^2/\sigma^2}\right)}$$

- We notice that the quadrupole is exponentially small if the spatial separation $|\boldsymbol{r}|$ is large in comparison to the width of the spatial peak σ .

Comparison with the Schrödinger-Newton formalism

- In the S-N formalism, we expect that the gravitational scattering will be proportional to the quadrupole moment of the gravitational potential, which will not be exponentially suppressed:

$$M\left(\frac{|\boldsymbol{r}|^2}{3}\delta_{ij}-r_i r_j\right)$$

- The result we find, putting in all the factors, is:

$$\frac{\kappa^4}{16\pi^2\sigma^4\left(1+e^{|\boldsymbol{r}|^2/\sigma^2}\right)}\frac{3\hat{p}_{2i}\hat{p}_{2j}}{(1-\hat{\boldsymbol{p}}_{\boldsymbol{2}}\cdot\hat{\boldsymbol{p}}_{\boldsymbol{4}})^2}\left(\frac{|\boldsymbol{r}|^2}{3}\delta_{ij}-r_i r_j\right)$$

- which is exponentially suppressed.

Conclusions

- The S-N formalism simply does not give the correct expression for the potential.
- The calculation of the simple scattering cross section from a non-local quantum superposition of a massive particle is a straightforward and unambiguous calculation.
- It shows that the non-local superposition does not gravitate as two point masses that are spatially separated.
- It would be interesting to calculate the gravitational contribution to the self energy of such a superposition. It would be interesting to see if the self energy behaves like 1/(separation) of the two peaks. Such a calculation might shed light on how the non-local peaks gravitate.
- It would be interesting to understand why the two peaks do not gravitate as they might classically.