PhD School on QCD in Extreme Conditions

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Lattice Field Theory for Extreme QCD

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lattice virtual academy

Lattice Field Theory for Extreme QCD - 1

Basics

- From real to imaginary time - field theory thermodynamics
- From continuum to the lattice and back
- Importance sampling and basic observables

Symmetry and pattern of breaking at high T

- Yang Mills
- Massless QCD - light quarks
- Scaling window
- Interplay of chiral symmetry and confinement ?

QCD - why Lattice Field Theory

QED vs QCD

- Photons do not carry charge
- Free electrons and free photons exist
- Interactions are strong at short distance - Coulomb force

- Gluons are charged
- Free quarks and gluons do not exist: confinement?
- Interactions are faible at short distance: asymptotic freedom

A theory with only photons
Is free

A theory with gluons only
is interacting and Interesting

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g_s f_{abc} A^b_\mu A^c_\nu}$$

QED vs QCD vs Yang-Mills

- Photons do not carry charge
- Free electrons and free photons exist
- Interactions are strong at short distance - Coulomb force

A theory with only photons
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A theory with gluons only (Yang-Mills)
is interacting and Interesting

$$F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g_s f_{abc} A^b_\mu A^c_\nu}$$

From QED to Yang-Mills theories

Electrodynamics:

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi$$

Infinite mass

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cancel{\bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Free photons

Yang-Mills

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\mu\nu}$$

Gluons

$$F^{(a)}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g_s f_{abc} A^b_\mu A^c_\nu}$$

Self-interacting gluons

QCD : why Lattice Field Theory

Confinement: quarks and gluons are not observed as asymptotic states

Breaking of chiral symmetry: due to the coupling becoming large at large distance

Topological properties: non-existent at any order in perturbation theory

..why and where

Extreme QCD

Confinement: quarks and gluons are not observed as asymptotic states

Deconfinement : quark and gluon dynamics

Breaking of chiral symmetry: due to the coupling becoming large at large distance

Coupling does not grow enough to break symmetry

Topological properties: non-existent at any order in perturbation theory

Topology becomes ‘simpler’

Calculational schemes from real to imaginary time

Correlators, Correlation lengths, Masses

General calculation scheme: Gran Canonical formalism

Rotate to imaginary time $x_0 \equiv t \rightarrow -ix_4 \equiv -i\tau$

GCPF —-> $$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-S}$$ <— note: Euclidean space time

$$\mathcal{S} = \int d^4x \, (\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}M\psi) \, .$$

A RFT in d space dimensions becomes a statistical field theory in d+1 dimensions

Integrate out fermions

$$Z = \int \mathcal{D}A_\mu \, \det M \, e^{\int d^4x \, (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})} .$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S} \, .$$

$$S = S_{gauge} + S_{quarks} = \int d^4x \, \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \sum_i \log(\mathrm{Det}M_i)$$

Minkowski —> Euclidean

Green functions —> Correlation functions

In many cases correlation functions decay exponentially at large distance:

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t/t_0}$$

t_0 correlation length

Back to Minkwoski

$$\int dt e^{ip_0 t} \frac{e^{-t/t_0}}{2t_0} = \frac{1}{p_0^2 + \frac{1}{t_0^2}}$$

$$\to_{p0 \to iE} = \frac{1}{1/t_0^2 - E^2}$$

Mass = inverse correlation length

Minkowski —> Euclidean

Green functions —> Correlation functions

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t\,M}$$

$\boxed{\text{M = lowest excitation in the channel which couples to } O}$

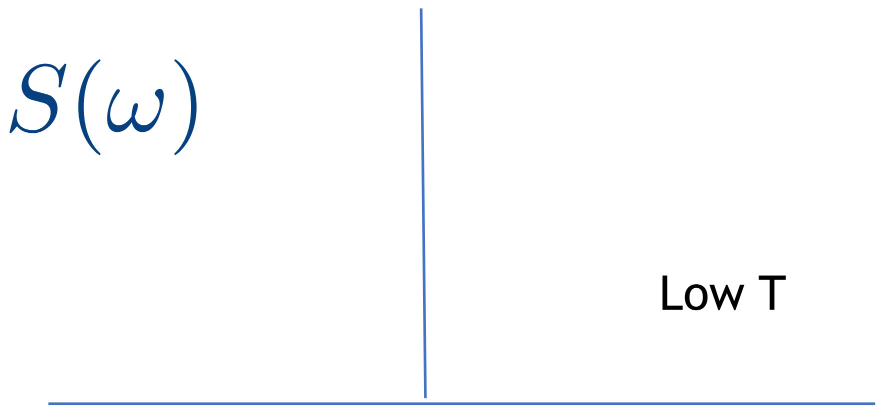
From real time to real frequency space:

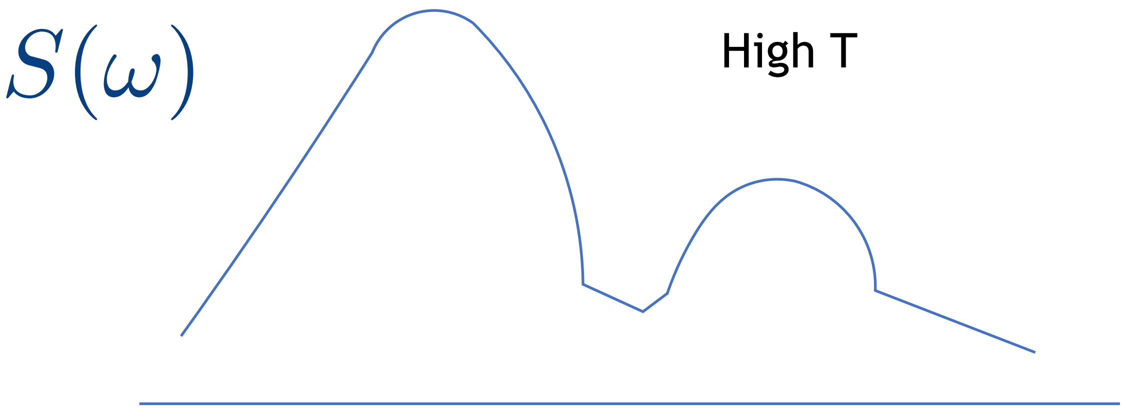
In imaginary time G(t) $$G(t) = \int \delta(M-\omega)e^{-\omega t} \propto e^{-Mt}$$

In real frequency space: $$\delta(M-\omega)$$

Spectral functions and two point functions : a challenge for LFT

(more later)

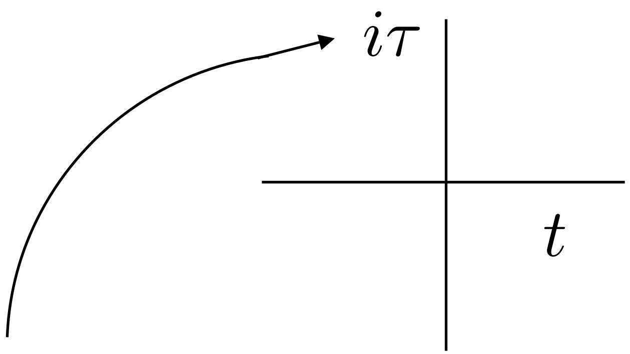




$$G(t) = \int \delta(M - \omega) e^{-\omega t} \propto e^{-Mt}$$

$$G(t) = \int S(\omega) e^{-\omega t}$$

Objects of interest: Spectral functions



Computed on the lattice: Euclidean (imaginary) Time Correlators

Functions of real, continuous frequency Spectral functions

Integral inverse transform

Euclidean Time Correlators

Analytic continuation

Fourier transform

Euclidean correlator in imaginary (Matsubara) frequency space

Field Theory in Euclidean space — summing up so far

Complete equivalence between Minkowski FT in d space dimension with statistical field theory in d+1 dimension

The Grand Canonical Partition Function defines all the observables of the theory

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \; \mathcal{O} \; e^{-S}$$

Exponential decays of Euclidean two point functions —> mass of the lowest excitation in that channel

More general functional forms may appear, which require a dedicated analysis

Extreme conditions:

Temperature, density, termodynamics, dimensional reduction

Euclidean Field Theory —> Classical Statistical System

$$< \phi_a | e^{-iHt} | \phi_a > = \int d\pi \int_{\phi(x,0)=\phi_a(x)}^{\phi(x,t)=\phi_a(x)} d\phi e^{i \int_0^t dt \int d^3x (\pi(\vec{x},t) \frac{\partial \phi(\vec{x},t)}{\partial t} - H(\pi,\phi))}$$

$$\mathcal{Z} = Tr e^{-\beta(H-\mu\hat{N})} = \int d\phi_a < \phi_a | e^{-\beta(H-\mu N)} | \phi_a >$$

$$\boxed{\beta \equiv \frac{1}{T} \rightarrow it}$$

Temperature

Temperature and Density

Gran Canonical Formalism: introduce chemical potential for conserved charge

$$\mu J_0, \qquad J_0 = \bar{\psi}\gamma_0\psi$$

$$S_{QCD} = F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(\not{\partial} + m + \mu\gamma_0)\psi = S_G + \bar{\psi}M\psi$$

$$\mathcal{Z}(\mu, T) = \int_0^{1/T} dt \int e^{S(\bar{\psi},\psi,U)} d\bar{\psi} d\psi dU$$

$$\mathcal{Z}(T,\mu) = \int dU det M e^{-S_G} = \int dU e^{-\boxed{\underbrace{(S_g - log(det M))}_{=\; S_{eff}}}}$$

Boundary conditions for the fields

$$\mathcal{Z} = \int d\phi d\psi e^{-S(\phi,\psi)}$$

$$S(\phi,\psi) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\psi)$$

Bosons : periodic

$$G_B(\vec{x},\vec{y};\tau,0) = Tr\{\hat{\rho} T_\tau[\hat{\phi}(\vec{x},\tau)\hat{\phi}(\vec{y},0)]\}/\mathcal{Z}$$

where T_τ is the imaginary time ordering operator:

$$T_\tau[\hat{\phi}(\tau_1)\hat{\phi}(\tau_2)] = \hat{\phi}(\tau_1)\hat{\phi}(\tau_2)\theta(\tau_1-\tau_2)+\hat{\phi}(\tau_2)\hat{\phi}(\tau_1)\theta(\tau_2-\tau_1)$$

Use now the commuting properties of the imaginary time ordering evolution and H:

$$[T_\tau, e^{-\beta H}] = 0$$

togheter with the Heisenberg time evolution

$$e^{\beta H}\phi(\vec{y},0)e^{-\beta H} = \phi(\vec{y},\beta)$$

to get:

$$G_B(\vec{x},\vec{y};\tau,0) = G_B(\vec{x},\vec{y};\tau,\beta)$$

Fermions

$$T_\tau[\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)] = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1 - \tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$\hat{\psi}(\vec{x}, 0) = -\hat{\psi}(\vec{x}, \beta)$$

Antiperiodic

Fermions

$$T_\tau[\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)] = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1 - \tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$\hat{\psi}(\vec{x}, 0) = -\hat{\psi}(\vec{x}, \beta)$$ Antiperiodic

Bosons

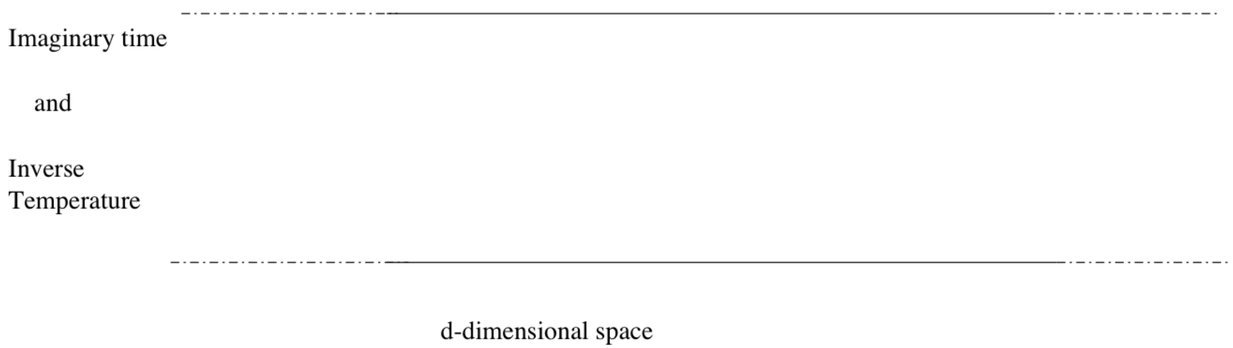
$$T_\tau[\hat{\phi}(\tau_1)\hat{\phi}(\tau_2)] = \hat{\phi}(\tau_1)\hat{\phi}(\tau_2)\theta(\tau_1 - \tau_2) + \hat{\phi}(\tau_2)\hat{\phi}(\tau_1)\theta(\tau_2 - \tau_1)$$

$$G_B(\vec{x}, \vec{y}; \tau, 0) = G_B(\vec{x}, \vec{y}; \tau, \beta)$$ Periodic

Thermodynamics

$$\begin{aligned}
P &= T\frac{\partial ln\mathcal{Z}}{\partial V} \\
N &= T\frac{\partial ln\mathcal{Z}}{\partial \mu} \\
S &= \frac{\partial T ln\mathcal{Z}}{\partial T} \\
E &= -PV + TS + \mu N
\end{aligned}$$

Dimensional Reduction



+ coarse graining

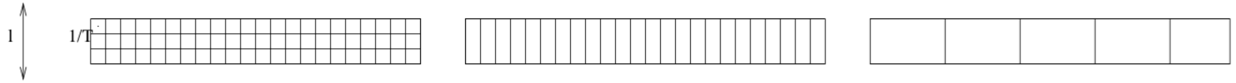


Figure 2: Sketchy view of the dimensional reduction from d+1 to d dimensions(from the leftmost to the middle picture) and subsequent coarse graining (from middle to rightmost)

Cases for Dimensional Reduction

1) T >>> any mass —> High T Electroweak transition

2) Diverging correlation length —> second order transition, basis for universality
High T QCD

Mode expansion and Decoupling

$$\phi(x,t) = \sum_{\omega_n = 2n\pi T} e^{i\omega_n t}\phi_n(x)$$

Bosons

$$\psi(x,t) = \sum_{\omega_n = (2n+1)\pi T} e^{i\omega_n t}\psi_n(x)$$

Fermions

In the expression for the Action

$$S(\phi,\psi) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\psi)$$

the integral over time can then be traded with a sum over modes, and we reach the conclusion that a d+1 statistical field theory at $T > 0$ is equivalent to a d-dimensional theory with an <u>infinite</u> number of fields.

When dimensional reduction is possible, only one boson field survives

Finite temperature at a glance

- The partition function $\mathcal{Z}$ has the intepretation of the partition function of a statistical field theory in d+1 dimension, where the temperature has to be identified with the reciprocal of the (imaginary) time.

- The fields' boundary conditions follows from the Bose and Fermi s

$$
\begin{aligned}
\phi(t=0,\vec{x}) &= \phi(t=1/T,\vec{x}) \\
\psi(t=0,\vec{x}) &= -\psi(t=1/T,\vec{x})
\end{aligned}
$$

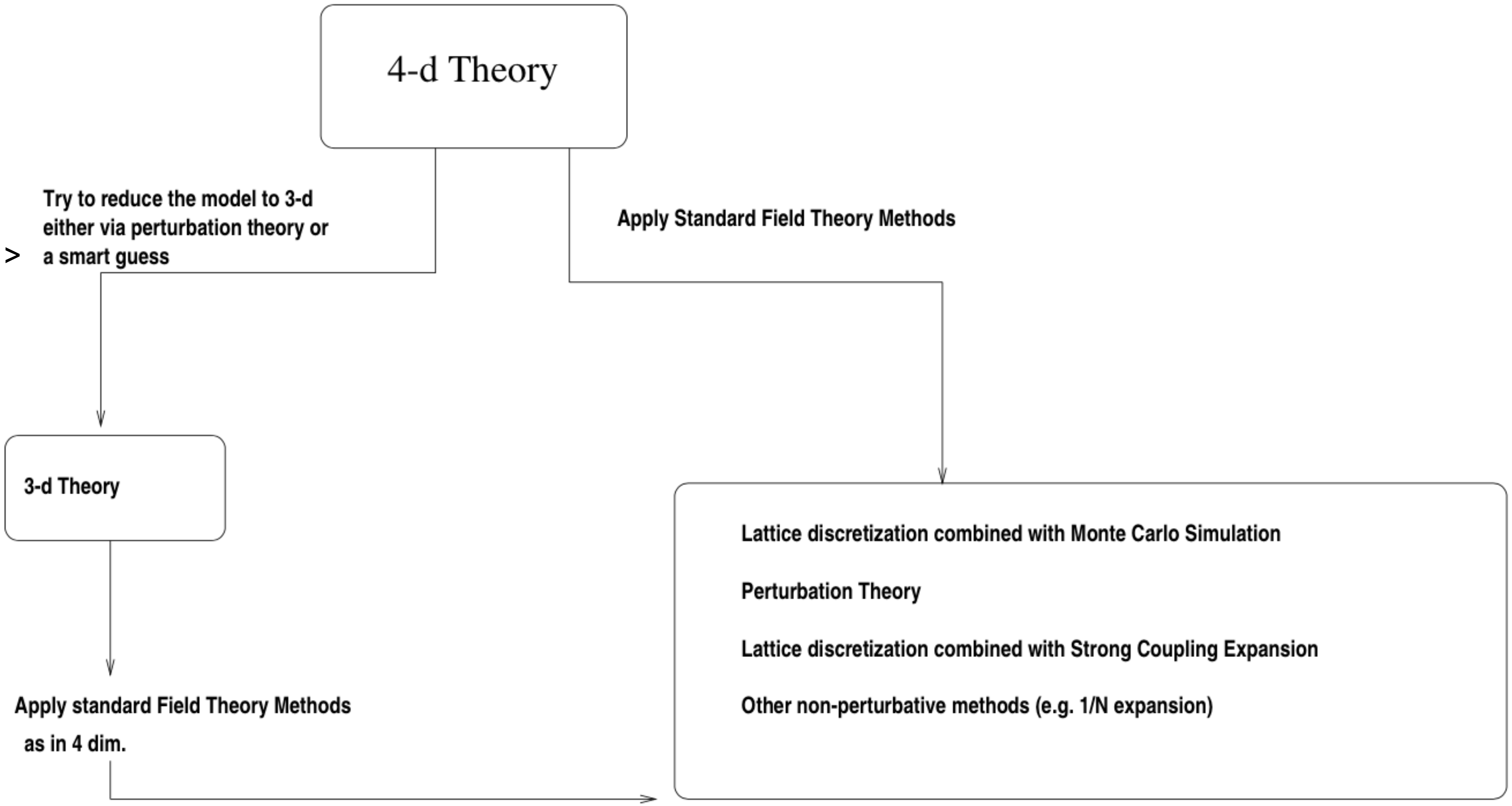
  i.e. fermionic and bosonic fields obey antiperiodic and periodic boundary conditions in time.

- "Dimensional reduction", when 'true' means that the system become effectvely 3-dimensional. In this case only the Fourier component of each Bose field with vanishing Matsubara frequency will contribute to the dynamics, while Fermions would decouple.

- The scenario above is very plausible and physically well founded, but it is by no means a theorem. Ab initio calculations can confirm or disprove it.

Computational schemes at a glance

Universality ->

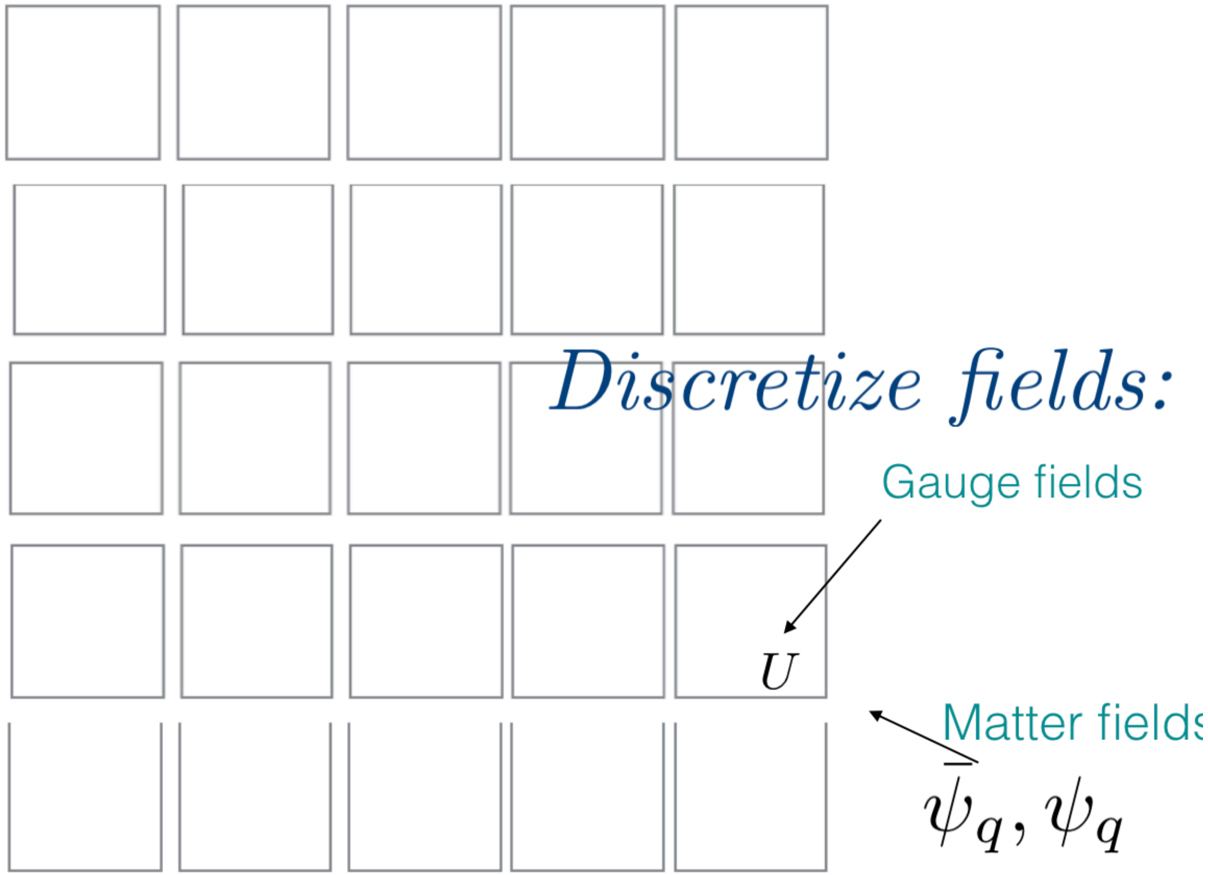


On the lattice - general issues

Computational Strategy:

1) Rotation to imaginary time + discretisation

Lattice Gauge Theory



Computational Strategy:

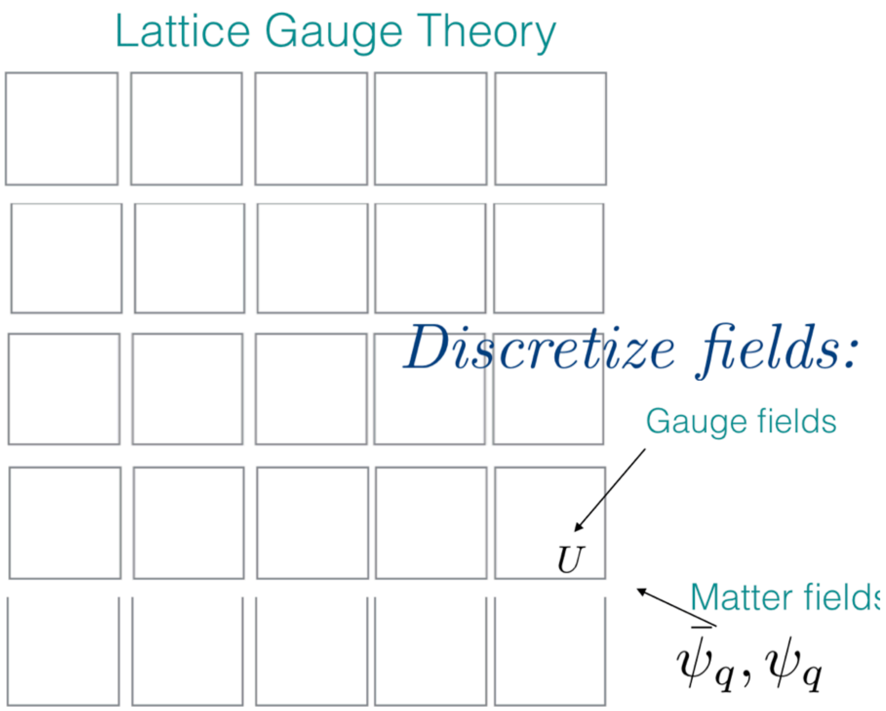
1) Rotation to imaginary time + discretisation

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S}$$

2) Monte Carlo Simulation

Performing the integration

$$\boxed{\langle 0|\mathcal{O}|0\rangle = \frac{\sum_\alpha \mathcal{O}_\alpha e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}.} \rightarrow \boxed{\frac{\sum_{i=1}^{N} \mathcal{O}_i}{N}}$$



—Discretization: from continuum space to a grid

—Why? Two standard motivations:

1. Physical system intrinsically discrete (i.e. spin models)
2. **Make it amenable to a numerical study —> QCD**

— Discretization is in principle trivial:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2} + \sum_{k=1}^{n-1}\left(f\left(a + k\frac{b-a}{n}\right)\right) + \frac{f(b)}{2}\right)$$

— Already in this simple example:

.Strategies for improvement?

.How to check the ‘**continuum limit?’**

.Suppose a, b $\rightarrow \infty$

How to check **convergence to infinite volume?**

— Slightly more complicated: increase the dimensionality, make the function less smooth..

Computational costs??

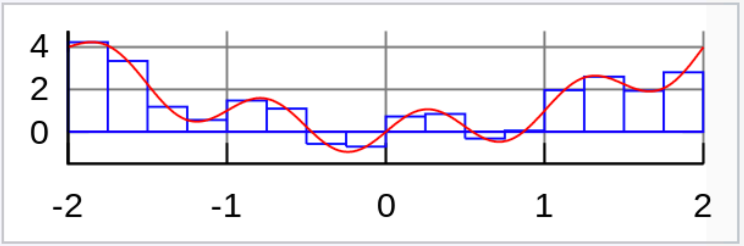


Illustration of the rectangle rule.

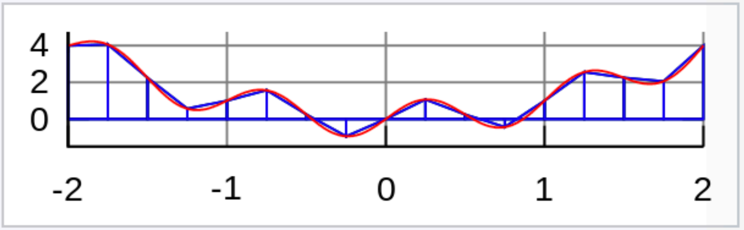


Illustration of the trapezoidal rule.

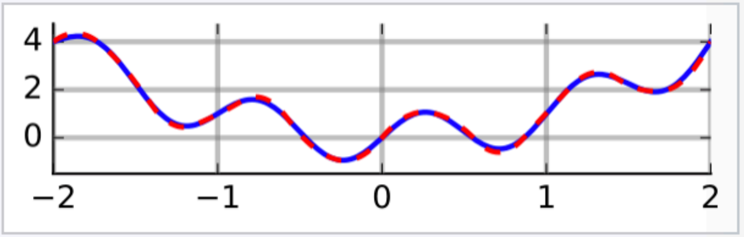


Illustration of Simpson's rule.

Matter (scalar) fields: on sites

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4.$$

$$S = \sum_n a^4 \left(\frac{1}{2} \sum_{\mu=1}^{4} \left[\frac{\phi(n+1_\mu) - \phi(n-1_\mu)}{2a} \right]^2 + \frac{1}{2}m^2\phi^2(n) + \lambda\phi^4(n) \right)$$

a

n, n+1

Gauge fields

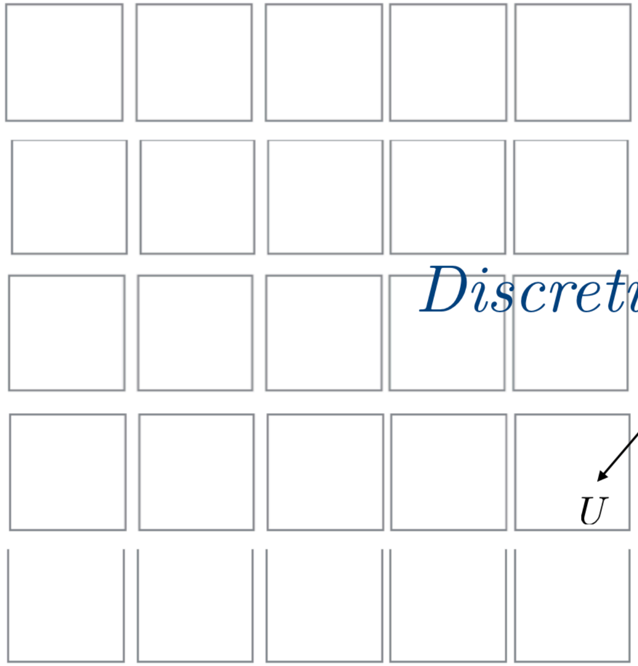
$$U_\mu(n) = \exp(igaT^a A^a_\mu(n))$$

SU(3) Matrix

$$\phi_s(y) \to P(\exp\ ig \int_s dx_\mu A_\mu)\phi(x) \equiv U(y,x)\phi(x)$$

Parallel transport:

Lattice Gauge Theory : gauge invariance 'by fiat'



Discretize fields:

Gauge fields

Matter fields

$$\left(\bar\psi_q, \psi_q\right)$$

$$\mathrm{Tr}\ldots U_\mu(x)U_\mu(x+\hat\mu)\ldots \to \mathrm{Tr}\ldots U_\mu(x)V^\dagger(x+\hat\mu)V(x+\hat\mu)U_\mu(x+\hat\mu)\ldots .$$

Gauge invariant

Build the Action 'by guessing'..

$$S = \frac{2}{g^2} \sum_x \sum_{\mu > \nu} \mathrm{Re}\ \mathrm{Tr}\ (1 - U_{P_{\mu\nu}}(x)).$$

? Does it work?

$$U_{P_{\mu\nu}}(x) = U_\mu(x) U_\nu(x + \hat{\mu} a) U_\mu^\dagger(x + \hat{\nu} a) U_\nu^\dagger(x)$$

Check continuum limit a-> 0

$$A_\mu(x + \hat{\nu} a) = A_\mu(x) + a \partial_\nu A_\mu(x) + \ldots$$

$$U_P(x) = 1 + i a^2 F_{\mu\nu} + \ldots$$

$$\beta = 2N/g^2$$

Continuum limit OK

Continuum limit

Coming back to correlation functions:

$$\lim_{t\to\infty} < O(t)O(0) > \propto e^{-t\,M}$$

M = dimensionless quantity, expressed in lattice units = $1/\xi$

M = M_{phys} * a = $1/\xi$

$$\boxed{a \to 0, \xi \to \infty}$$

Continuum limit: singularity !

‘g’ in the Lattice Lagrangian is the coupling at the scale ‘a’

$$a\Lambda_L = \left(\frac{1}{b_0 g^2}\right)^{b_1/2b_0^2} e^{-1/2b_0 g^2}$$

Physical scale — dimensional transmutation

$$b_0 = \left(\tfrac{11}{3}N_c - \tfrac{2}{3}N_f\right)/16\pi^2 \text{ and } b_1 = \left(\tfrac{34}{3}N_c^2 - \left(\tfrac{10}{3}N_c + \tfrac{N_c^2-1}{N_c}\right)N_f\right)/(16\pi^2)^2.$$

QCD Asymptotic freedom allows a rigorous continuum limit

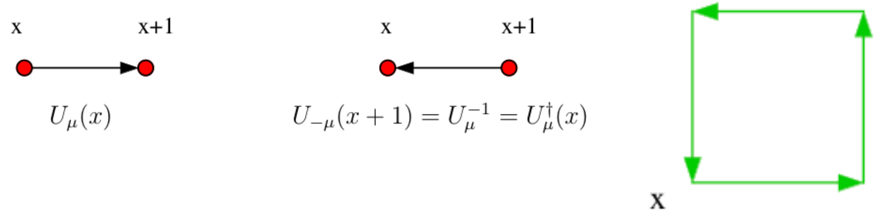
In perturbation theory, a(g) is known.

Yang-Mills, continuum and lattice

$$S_{\rm cont} = \int d^4x \frac{1}{4g^2}\,{\rm Tr}\,F_{\mu\nu}F^{\mu\nu}$$

$$S_{\rm latt} = \beta \sum_p \left(1 - \frac{1}{3}{\rm Re}\,\{{\rm Tr}\,U_p\}\right)\,; \quad \beta = \frac{6}{g^2}.$$

Not unique — improvement



U : SU(3) matrix

Det = 1

U^{-1} = U*

a: 0.1 fm? 0.003fm? 1cm???

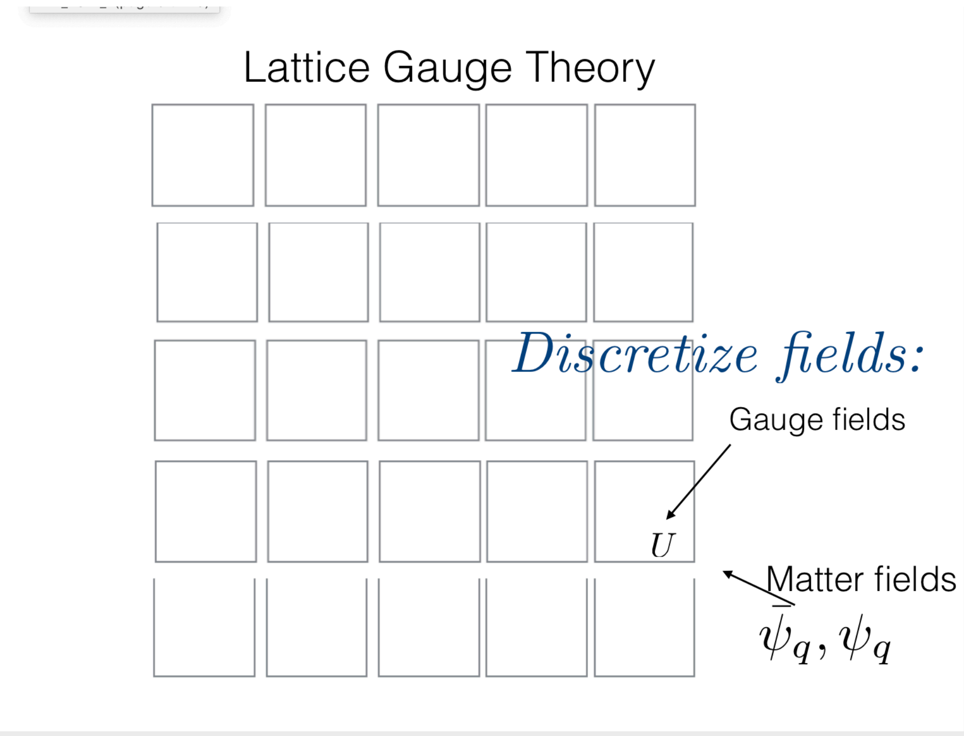
g only parameter. —> where is the spacing?

After discretising -

Back to the continuum

Planning a simulation

Parameters: $N_\sigma\,, N_\tau\,, g$



Assume we have the lattice results for some masses

M1Lat
M2Lat
M3Lat

How do we get results in physical units?

Issues

— Scale setting : one physical value needed as input!

—Scaling : how strong are the discretisation effects?

—Asymptotic scaling : are we sensitive to the g=0 singularity?

Scaling : repeat for different couplings and check consistencies of results -
or, which is the same, check that dimensionless ratios do not depend on g

Asymptotic scaling : repeat for different couplings, and check consistency with the two-loop universal scaling — implies scaling, but much harder to get

$$a\Lambda_L = \left(\frac{1}{b_0 g^2}\right)^{b_1/2b_0^2} \mathrm{e}^{-1/2b_0 g^2}$$

Improvement: in general, a program aimed at controlling lattice artifacts, so to reach faster and with more confidence scaling (hence continuum limit)

The functional integral

Sampling the phase space

Task:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S}$$

from Mike Creutz:

A direct evaluation of such an integral has pitfalls. At first sight, the basic size of the calculation is overwhelming. Considering a 10^4 lattice, small by today standards, there are 40,000 links. For each is an $SU(3)$ matrix, parametrized by 8 numbers. Thus we have a $10^4 \times 4 \times 8 = 320,000$ dimensional integral. One might try to replace this with a discrete sum over values of the integrand. If we make the extreme approximation of using only two points per dimension, this gives a sum with

$$2^{320,000} = 3.8 \times 10^{96,329} \tag{6}$$

terms! Of course, computers are getting pretty fast, but one should remember that the age of universe is only $\sim 10^{27}$ nanoseconds.

!!!

Monte Carlo methods:

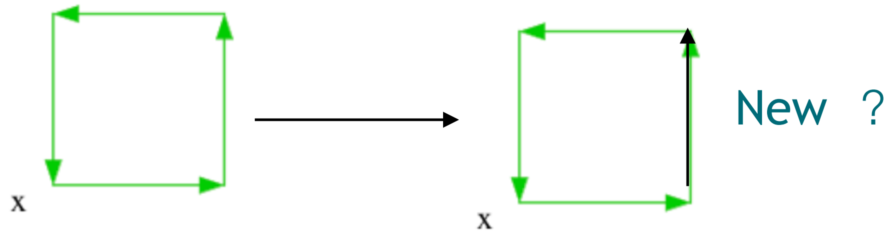
Create a sample of configurations distributed according to

$$e^{-S}$$

The functional integral may then be traded with an average over configurations

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \, \mathcal{O} \, e^{-S} \quad \boxed{\rightarrow \frac{\sum_{i=1}^{N} \mathcal{O}_i}{N}}$$

Monte Carlo time ‘evolution’



Different methods use different strategies for choosing the new link

Crucial point: positive Action!

1. Metropolis

$$\Delta S = S(\hat{U}_{ji}) - S(U_{ji}) \text{ (all other variables kept fixed)}.$$

If $\Delta S \leq 0$ the move is accepted

Otherwise: pick random number r 0 < r < 1 and accept if

$$r \leq e^{-\Delta S}.$$

Maximise distance between configurations, at the price of high rejection rate

2. Heath Bath

Choose new U with the appropriate weight:

$$\exp\{-\bar{S}(U_{ji}')\}$$

Accept step always satisfied by construction

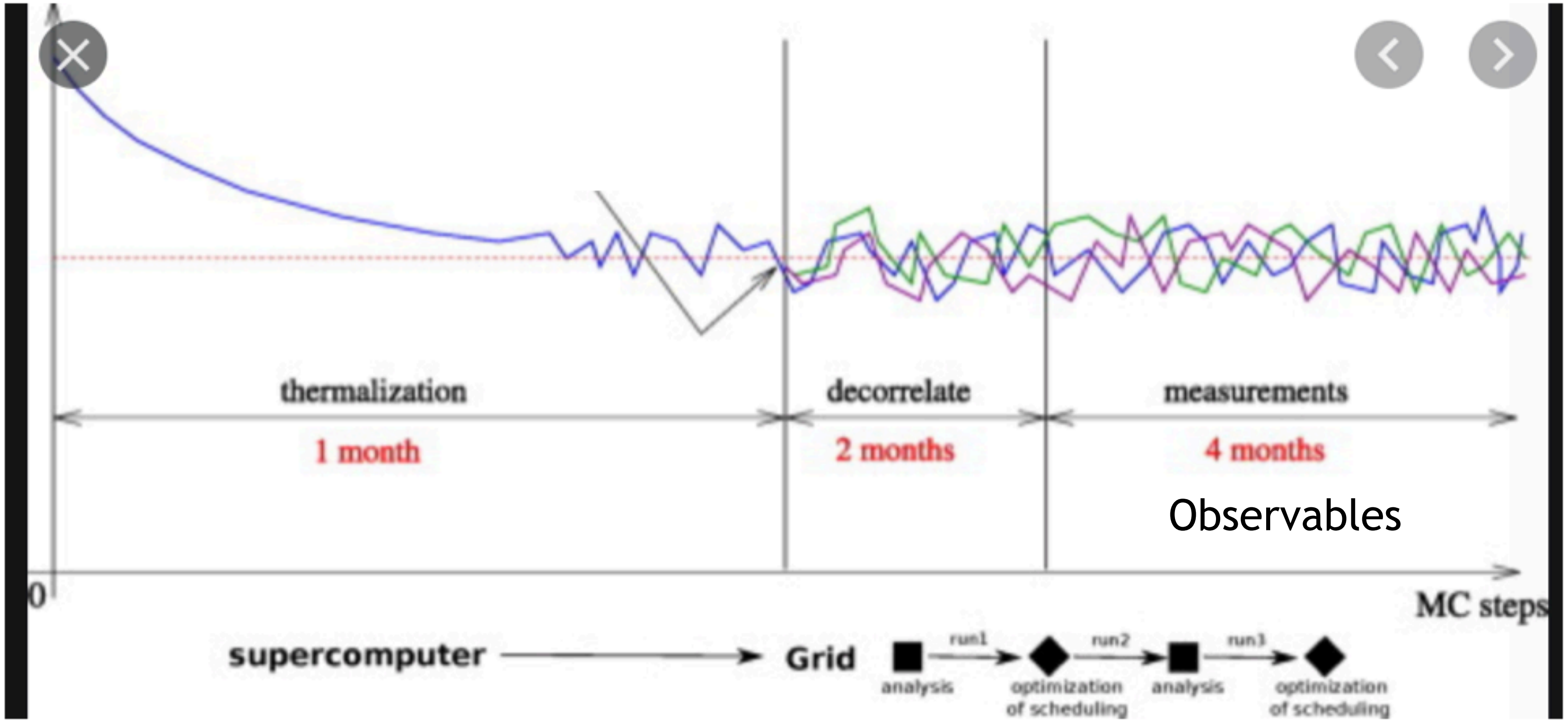
It may be expensive

3. Modified Metropolis

Same as Metropolis, but make several hits
with the same link :
“n upgrading per step”

It ‘interpolates’ between the two previous cases - optimal n to be determined

A typical Monte Carlo simulation:



Basic observables

Thinking in abstract terms - i.e. let us consider the discretised theory as a statistical system in d+1 dimension – these are basic measurable quantities:

1) Wilson loops

2) Polyakov loop

3) Topological charge

Difficult to draw ‘butterly operator’ .

$F\tilde{F}$ In the continuum

(more later)

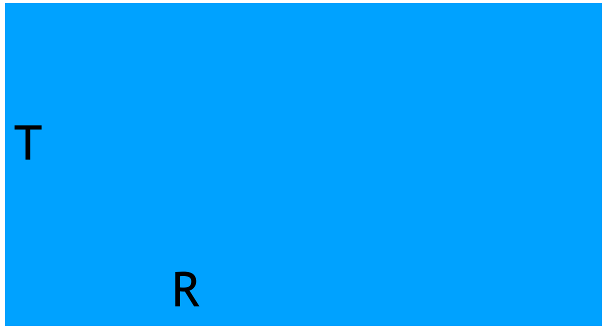
4) Two point functions of any of the above

5) Two point functions of composite fermion operators

Wilson loop

String tension - Interquark potential

$$V(\boldsymbol{R}) = -\lim_{T\to\infty} \frac{1}{T} \ln W(\boldsymbol{R}, T)$$



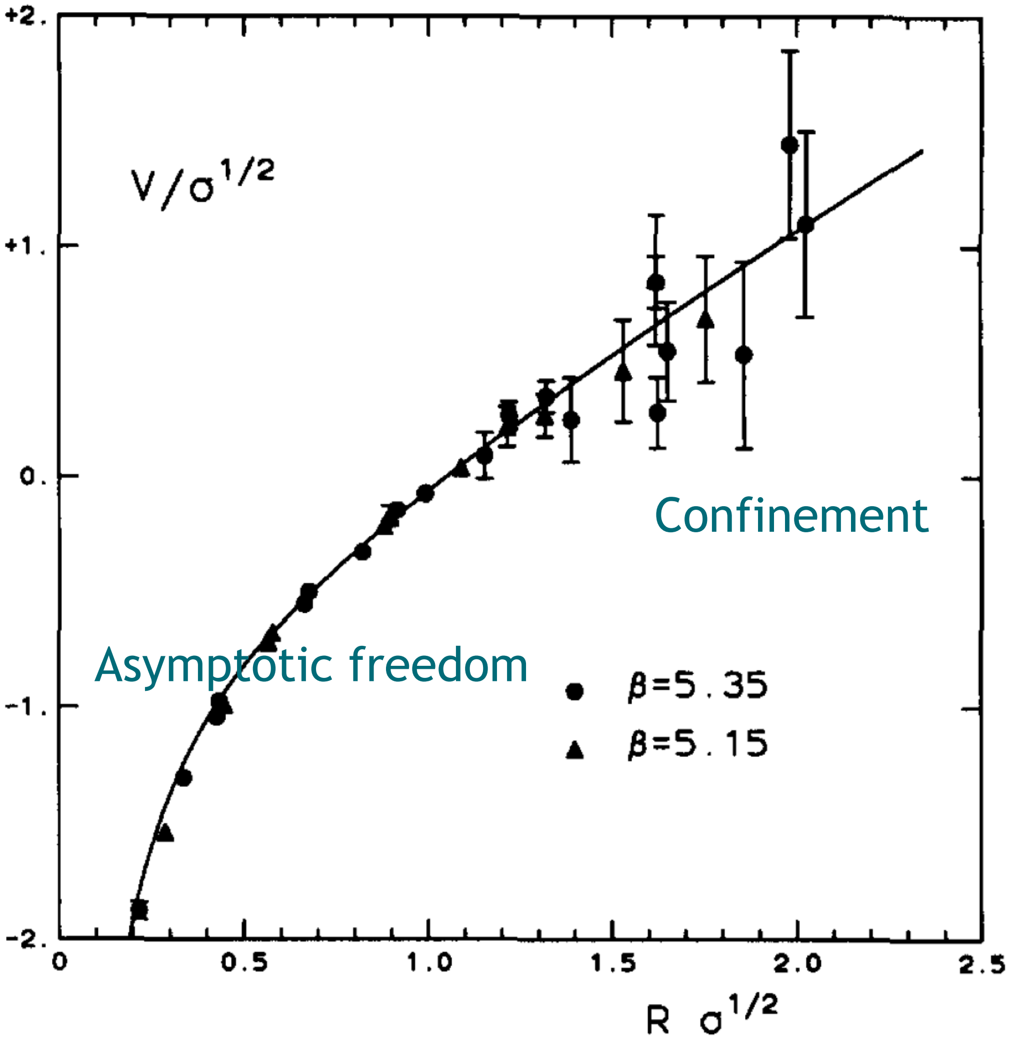


Fig. 1. The interquark potential measured for two values $\beta = 5.15$ and $\beta = 5.35$ and mapped onto each other by setting the scale from the fitted string tension.

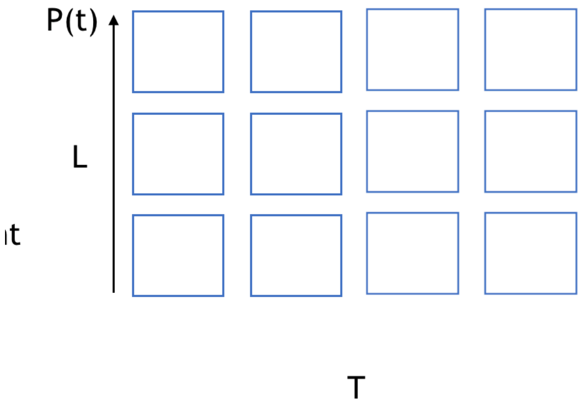
$$V_I(R) = c_0 - \frac{e}{R} + \sigma R$$

Cornell form

String tension $\sqrt{\sigma} = 420\,\mathrm{MeV}$, Physical value

Two point functions of Polyakov loop: alternative extraction of the potential

$$e^{-V(R,T)/T} \propto < P(\vec{0}) P^{\dagger}(\vec{R}) > \xrightarrow[R \to \infty]{} \propto e^{-\sigma R}$$



Parametro d'ordine per $Z(N_c)$. —> Confinement as a symmetry

For two colours: Z(2): Ising model!! —> Universality

'Build' Polyakov loop system: it is a cube of spins!! —> dimensional reduction at Tc

Checking the universality class of Yang Mills

Order parameter : L Analogous to magnetisation In 3D

$$M \qquad \begin{cases} A\left(\frac{T_c^0 - T}{T_c^0}\right)^\beta & , \quad T < T_c^0 \\ 0 & , \quad T \geq T_c^0 \end{cases}$$

Note: high and low temperatures interchanged

Some references:

Basics, from QCD to lattice:

Thomas Schaefer https://arxiv.org/pdf/1608.05459.pdf

Lattice pedagogical reviews:

Christine Davies, https://arxiv.org/pdf/hep-ph/0205181.pdf
Tom de Grand https://arxiv.org/pdf/1907.02988.pdf
Martin Luescher https://arxiv.org/pdf/hep-lat/9802029.pdf
Michael Creutz https://arxiv.org/pdf/hep-lat/0406007.pdf