

## Heavy Quarks (cont'd from Part 2)

- Quarkonia
- Potential models
- Temperature effects
- NRQCD

# Quarkonia: heavy quarks $\Rightarrow$ non-relativistic potential theory

Jacobs et al. 1986

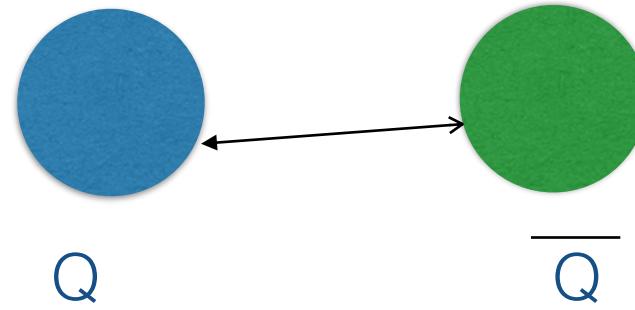
Schrödinger equation  $\left\{ 2m_c - \frac{1}{m_c} \nabla^2 + V(r) \right\} \Phi_i(r) = M_i \Phi_i(r)$

with confining (“Cornell”) potential  $V(r) = \sigma r - \frac{\alpha}{r}$

$$E(r) = 2m + \frac{1}{2mr^2} + V(r),$$

Minimizing the energy:

$$\frac{1}{mr_{J/\psi}^3} - \frac{\alpha_{\text{eff}}}{r_{J/\psi}^2} - \sigma = 0$$



Relation between  
the parameters of  
the potential, the mass  
and the radius

Quarkonia: heavy quarks  $\Rightarrow$  non-relativistic potential theory

Jacobs et al. 1986

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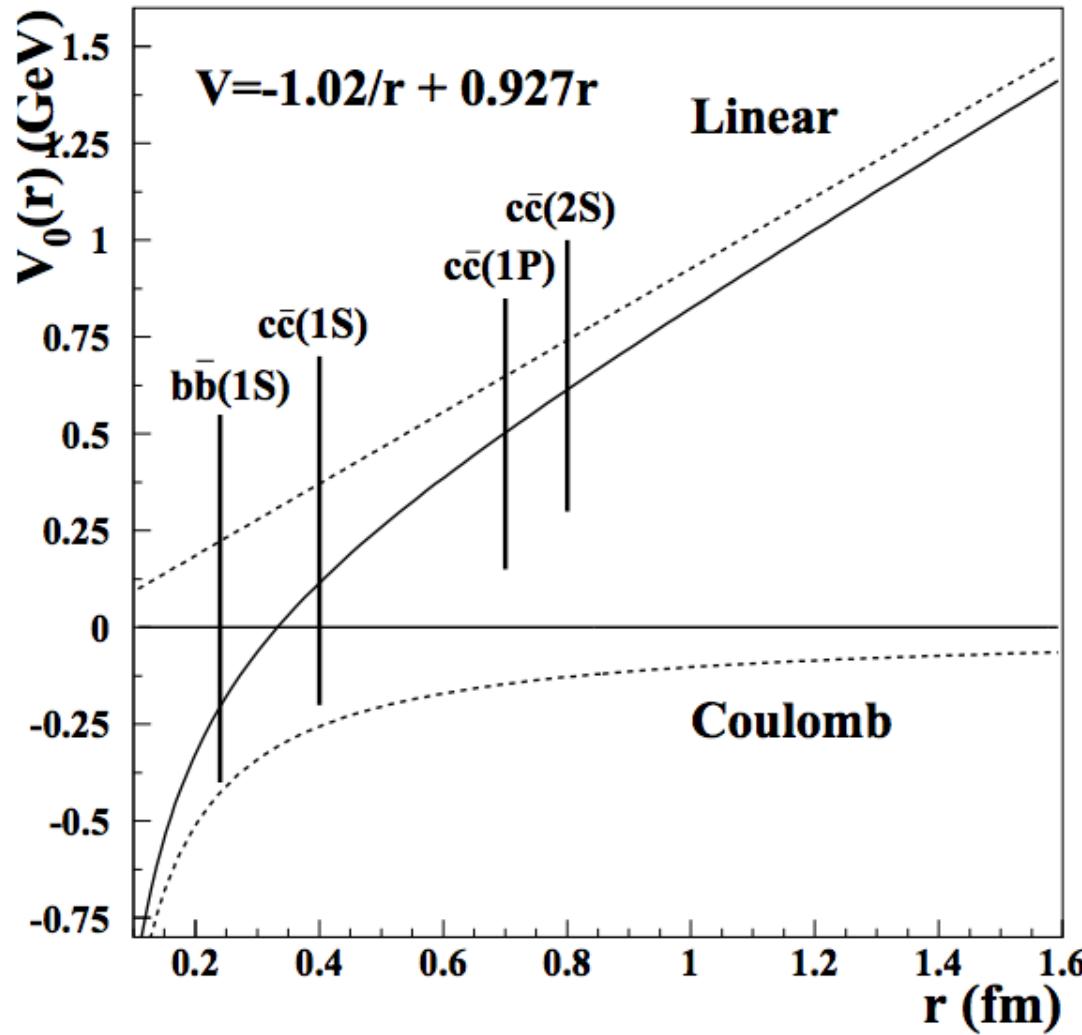
with confining (“Cornell”) potential  $V(r) = \sigma r - \frac{\alpha}{r}$

.

state	$J/\psi$	$\chi_c$	$\psi'$	$\Upsilon$	$\chi_b$	$\Upsilon'$	$\chi'_b$	$\Upsilon''$
mass [GeV]	3.10	3.53	3.68	9.46	9.99	10.02	10.26	10.36
radius [fm]	0.25	0.36	0.45	0.14	0.22	0.28	0.34	0.39

# Bound by Coulombic part vs confining binding

Potential probes gauge dynamics



Seth 2009

Potential model recipe:

- 1) Find Potential as a function of temperature  $\leftrightarrow$  Lattice
- 2) Solve Schrödinger eq. as a function of  $T$
- 3) Check who remains bound (binding energy)

Various uncertainties, including theoretical ones

# Compromise: Bottomonium via NRQCD

Zero temperature NRQCD works beautifully for the spectrum

$n^{S+1}L_J$	State	$a_\tau M$	$E_0 + M$ (MeV)	$M_{\text{expt}}$ (MeV)
$1^1S_0$	$\eta_b$	0.20549(4)	9409(12)	9398.0(3.2)
$2^1S_0$	$\eta'_b$	0.311(3)	10004(21)	9999(4)
$1^3S_1$	$\Upsilon$	0.21460(5)	9460*	9460.30(26)
$2^3S_1$	$\Upsilon'$	0.318(3)	10043(22)	10023.26(31)
$1^1P_1$	$h_b$	0.2963(4)	9920(15)	9899.3(1.0)
$1^3P_0$	$\chi_{b0}$	0.2921(4)	9896(15)	9859.44(52)
$1^3P_1$	$\chi_{b1}$	0.2964(4)	9921(15)	9892.78(40)
$1^3P_2$	$\chi_{b2}$	0.2978(4)	9928(15)	9912.21(40)

Relativistic

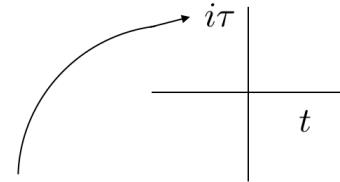
$$D(\tau) = \int_0^\infty \frac{e^{-\tau\omega} + e^{-(\beta-\tau)\omega}}{1 - e^{-\beta\omega}} S(\omega) d\omega$$

Non-relativistic

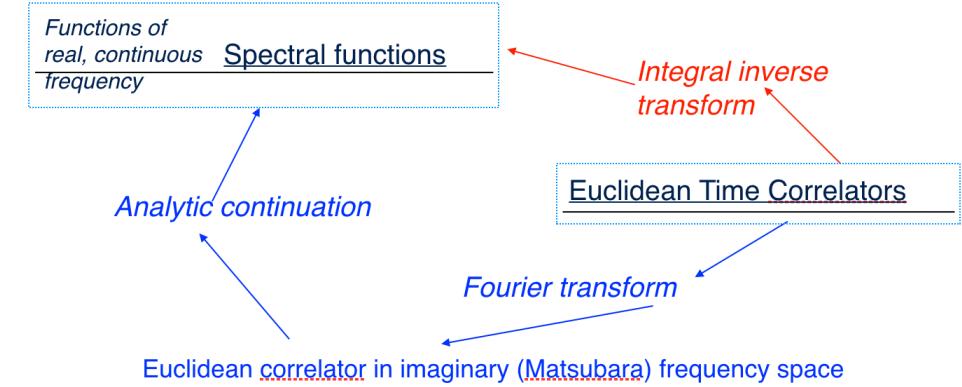
$$D(\tau) = \int_{-M_0}^\infty e^{-\tau\omega} S(\omega) d\omega$$

Inverse Laplace:  
makes life easier..

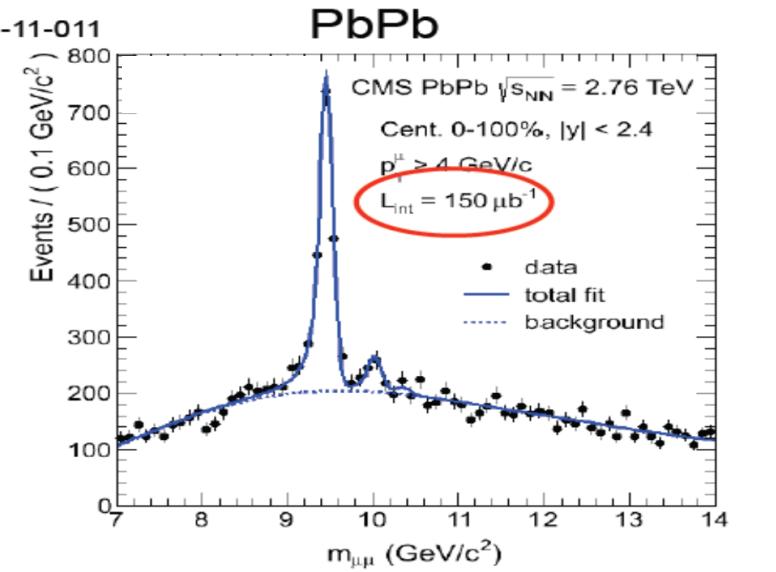
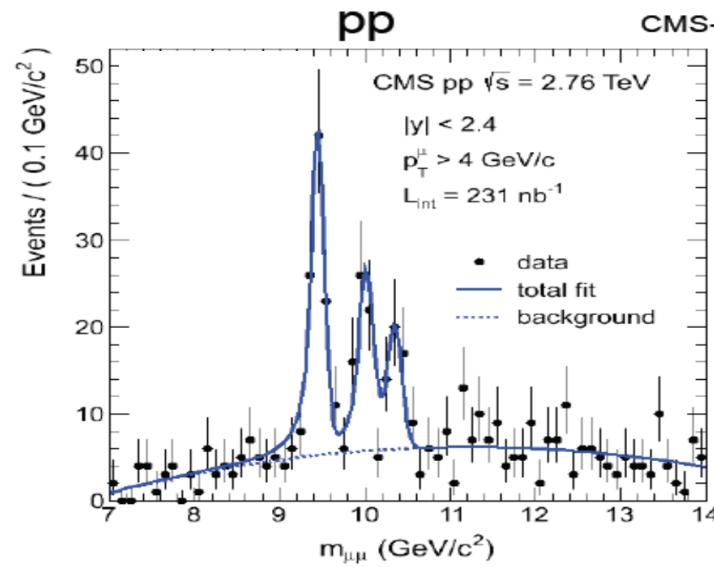
Objects of interest: [Spectral functions](#)



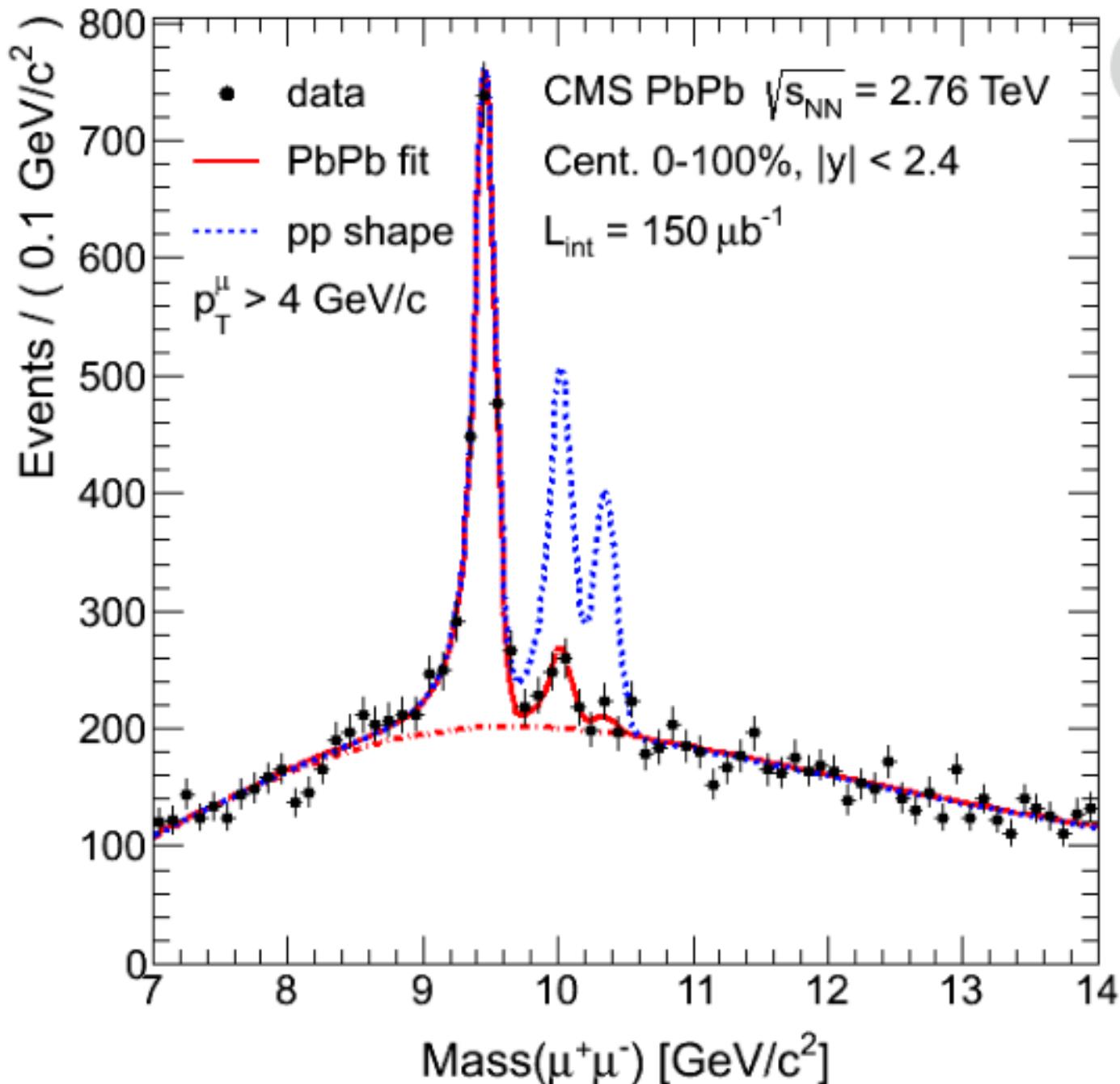
Computed on the lattice: [Euclidean \(imaginary\) Time Correlators](#)



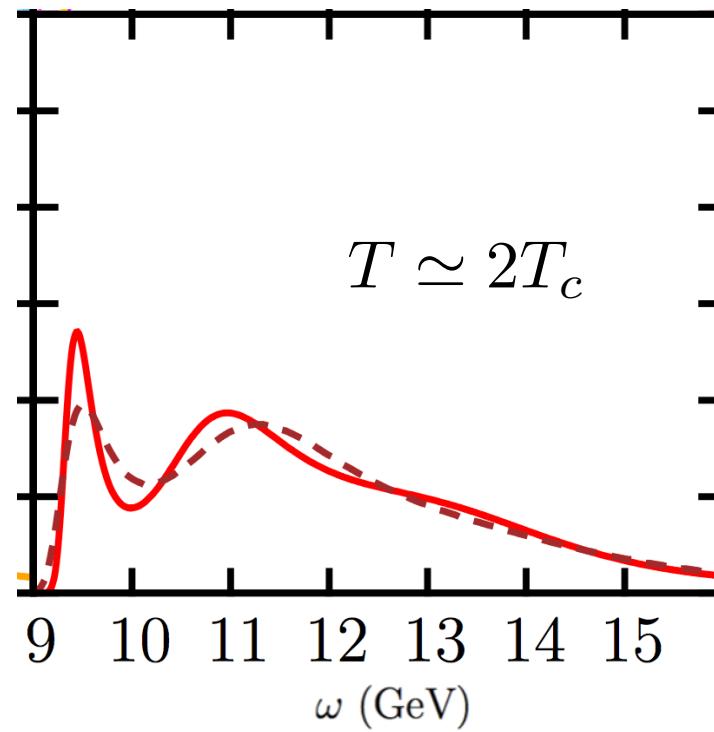
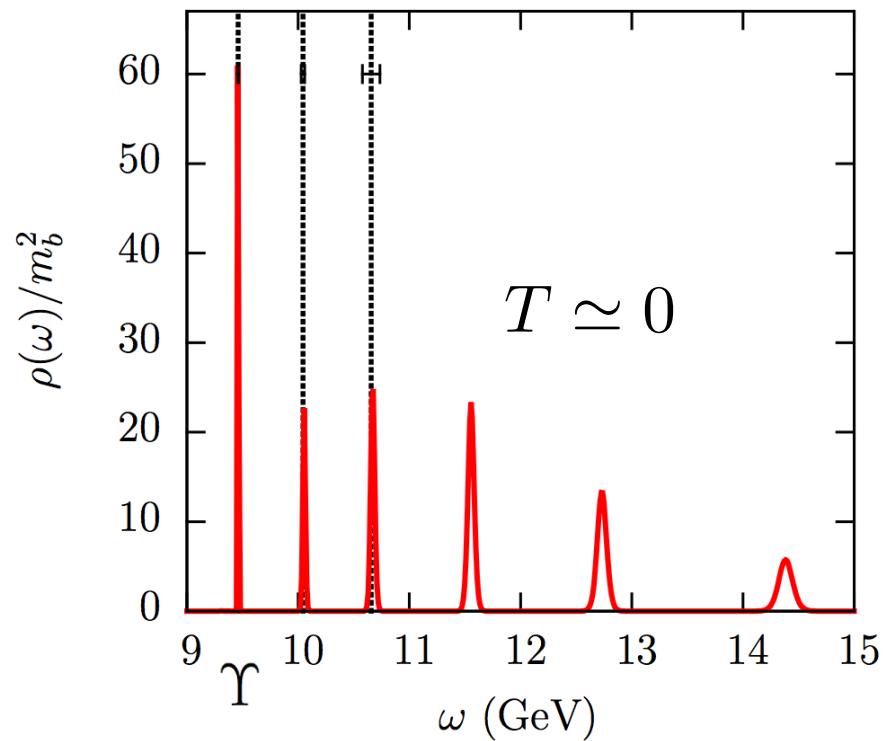
# Bottomonium as a probe of QGP



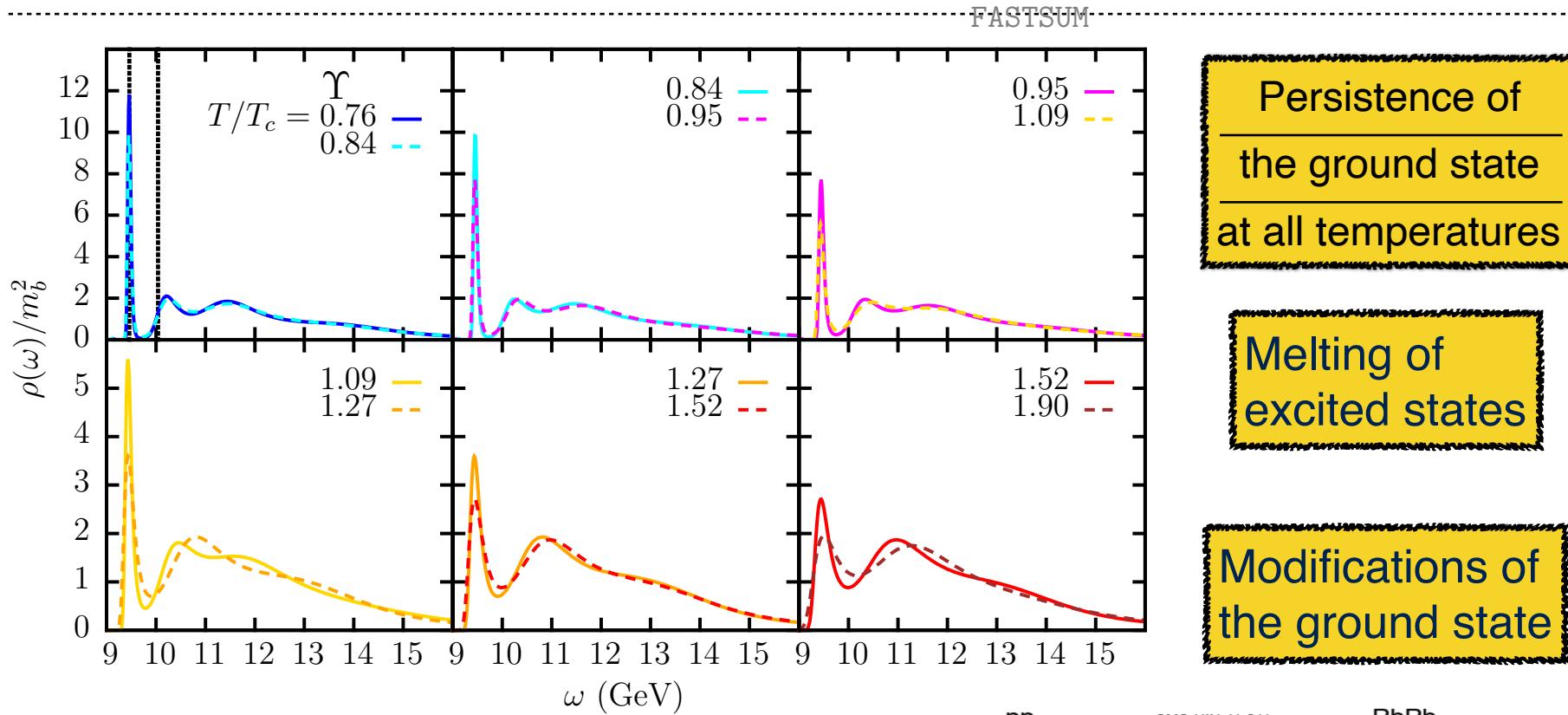
**CMS**  
**Eur.Phys.J. C76 (2016) no.3, 107**



## Bottomonium spectral functions from the lattice



# Upsilon's spectral functions from MEM (NRQCD)

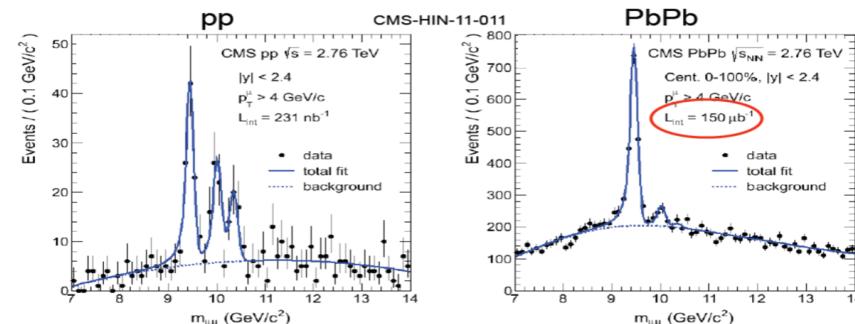


Persistence of  
the ground state  
at all temperatures

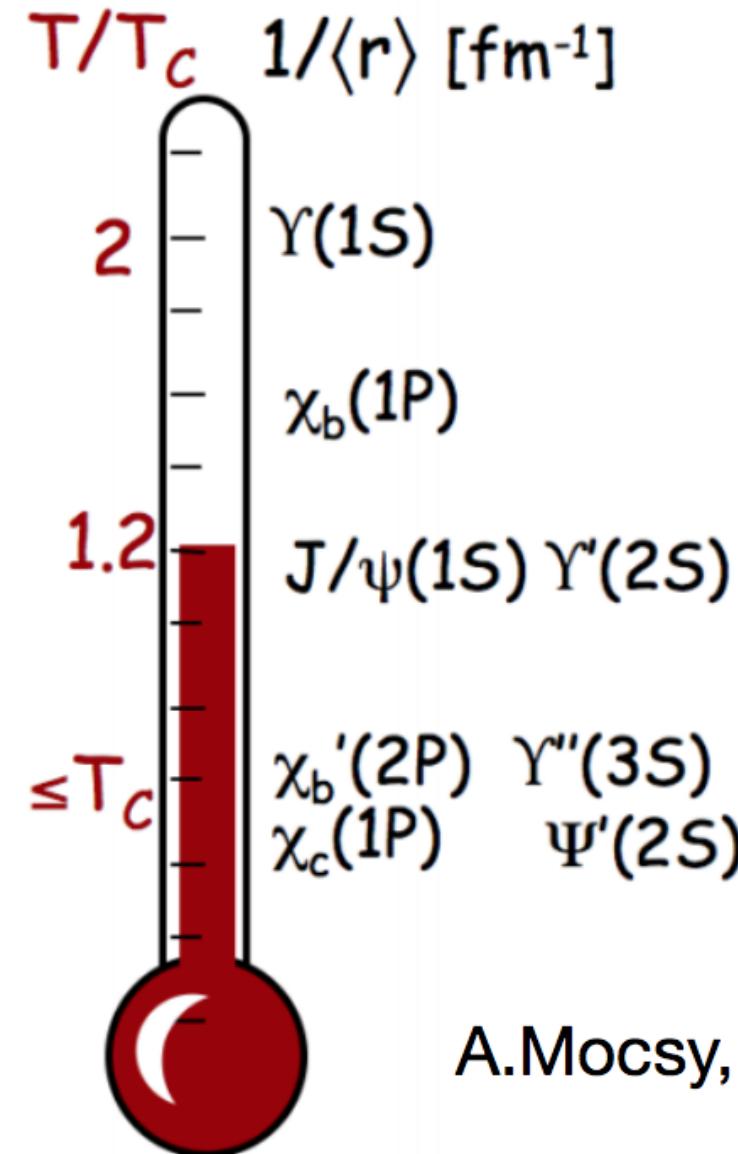
Melting of  
excited states

Modifications of  
the ground state

Pattern reminiscent of experimental observations



The picture of sequential melting is qualitatively correct, but quantitative uncertainties remain – cross checks from different methods important



A.Mocsy, arXiv:0811.0337

## Review by C.Allton:

<https://www.ggi.infn.it/talkfiles/slides/slides5843.pdf>

# Study of Numerical Methods

- |   |   |                          |
|---|---|--------------------------|
| 1. Exponential (Conventional $\delta$ f'ns)       | } | Maximum Likelihood       |
| 2. Gaussian Ground State (+ $\delta$ f'n excited) |   | (Minimise $\chi^2$ )     |
| 3. Moments of Correlation F'ns                    |   | Direct Method - “no” fit |
| 4. BR Method                                      | } | Bayesian Approaches      |
| 5. Maximum Entropy Method                         |   |                          |
| 6. Kernel Ridge Regression                        |   | Machine Learning         |
| 7. Backus Gilbert                                 |   | from Geophysics          |



# PhD School on QCD in Extreme Conditions

## Lattice Field Theory for Extreme QCD - Part 3

Maria Paola Lombardo

INFN Firenze [lombardo@fi.infn.it](mailto:lombardo@fi.infn.it)



# Lattice Field Theory for Extreme QCD - Part 2

## Topology

UA(1) problem and strong CP problem

QCD axion and high temperature topology

Topological susceptibility in the QGP

Limits on axion mass from lattice QCD topology

## Lattice methods for finite density (and high temperature)

Sign problem and complex chemical potential

## Methods

The quest for the critical point

Focus on the  $\theta$  term

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$$

$$\frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} = q(x)$$

$$Q = \sum q(x)$$



Q — topological charge

**CP-violating term**

|

## GCPF and $\theta$

The GCPF of QCD is now a function of  $\theta$ :

$$\mathcal{Z}(\theta, T) = \int \mathcal{D}[\Phi] e^{-T \sum_t \int d^3x \mathcal{L}(\theta)} = e^{-VF(\theta, T)}.$$

The energy density  $F(\theta, T)$  is related with the probability of finding configurations with given topological charge  $Q = \int d^4x q(x)$ :

$$P_Q = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} e^{-VF(\theta)},$$

so the coefficients  $C_n$  of the Taylor expansion

$$F(\theta, T) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\theta^{2n}}{2n!} C_n$$

are given by the cumulants of the topological charge:

$$C_n = (-1)^{n+1} \frac{d^{2n}}{d\theta^{2n}} F(\theta, T) \Big|_{\theta=0} = \langle Q^{2n} \rangle_{conn}.$$

# Topology, $\eta'$ and solution of the $U_A(1)$ problem

It can be proven that

and

$$\frac{1}{32\pi^2} \int d^4x F \tilde{F} = Q \quad \text{Gluonic definition}$$

$$Q = n_+ - n_- \quad \text{Fermionic definition}$$

The  $\eta'$  mass may now be computed from the decay of the correlation

$$\langle \partial_\mu j_5^\mu(x) \partial_\mu j_5^\mu(y) \rangle \propto \frac{1}{N^2} \langle F(x) \tilde{F}(x) F(y) \tilde{F}(y) \rangle$$

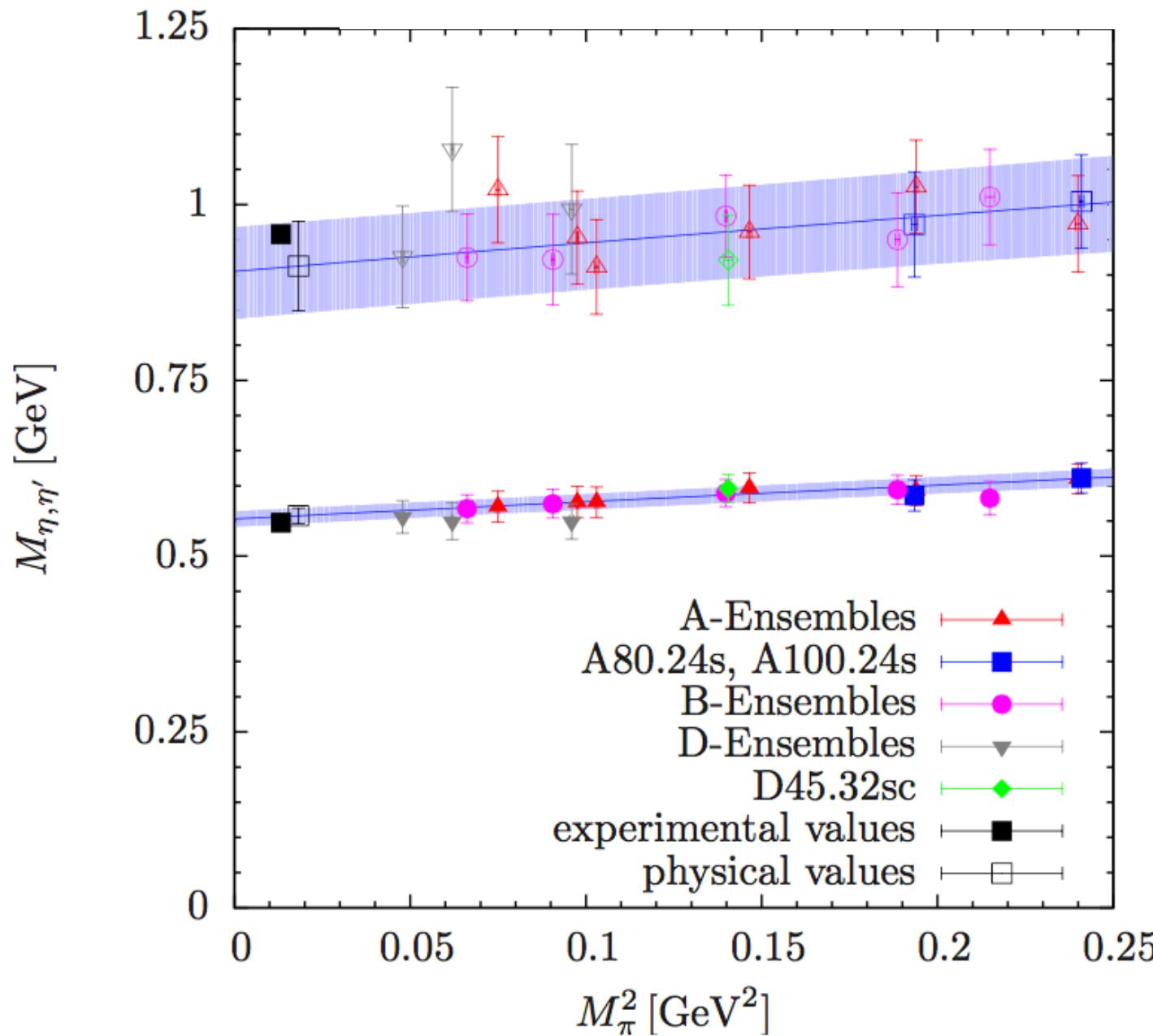
which at leading order gives the Witten-Veneziano formula

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2} \chi_t^{\text{qu}}$$

Successful  
at  $T=0$

# Topology observable effects:

# EVIDENCES OF THE EXPLICIT $U(1)_A$ BREAKING.

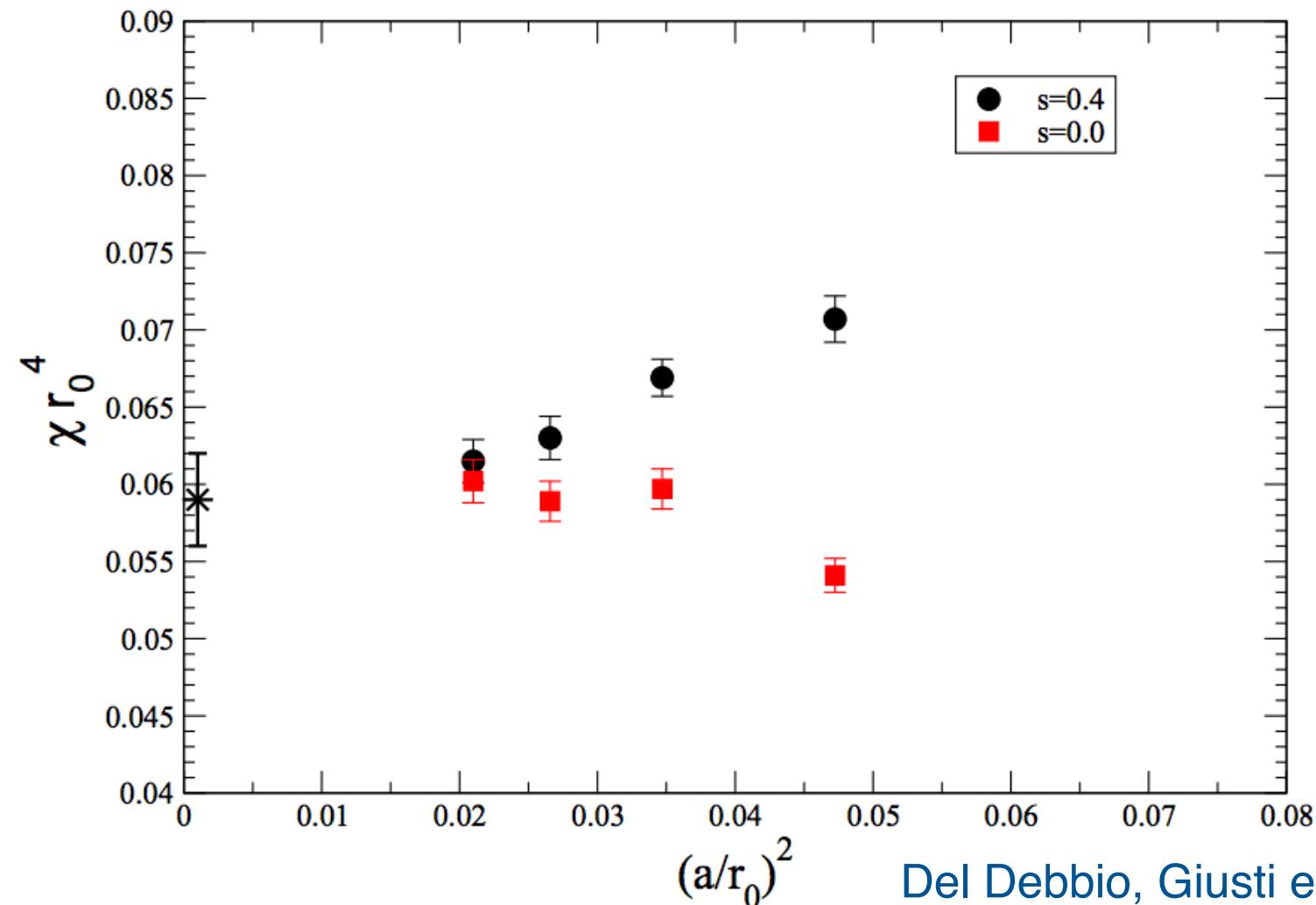


ETMC 2017

# Topology observable effects:

EVIDENCES OF THE EXPLICIT  $U(1)_A$  BREAKING.

$$\chi = (191 \pm 5 \text{ MeV})^4, \quad \text{Yang-Mills Topological Susceptibility}$$



Del Debbio, Giusti et al. (2005)

## Strong CP problem and the QCD axion

# Topology and the Strong CP problem

How ‘large’ is  $\theta$  ?

The QCD Lagrangian admits a CP violating term

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu},$$

$\frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$  is the topological charge density  $q(x)$ ,

$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ , and  $\theta q(x)$  is known as the  $\theta$ -term.

Without the  $\theta$ -term strong interactions conserve CP. With the  $\theta$ -term the neutron acquires an electric dipole moment  $d_n$ :

with QCD sum rules:  $d_n = 2.4 \times 10^{-16} \theta$  e cm

chiral perturbation theory:  $d_n = 3.3 \times 10^{-16} \theta$  e cm

Experiments:  $|d_n| < 1.8 \times 10^{-26}$  e cm at a 90% C.L,  
 $\theta < 0.5 \times 10^{-10}$ .

## Solution of the strong CP problem: the axion

Suppose  $\theta$  were a dynamical parameter: in such a case, dynamics would force its value to zero, thus solving the strong CP problem.

Postulate Axion! a pseudo-Goldstone boson of a spontaneously broken symmetry known as the Peccei-Quinn (PQ) symmetry, which couples to the QCD topological charge, with a coupling suppressed by a scale  $f_A$ .

Axion field  $a(x) = f_A\theta(x)$  is now a space-time dependent  $\theta$  parameter.

The axion–QCD Lagrangian:

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} + \partial_\mu^2 a^2 + \frac{a}{f_A} \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}.$$

Assume a shift symmetry:  $a \rightarrow a + \alpha$ . The  $\theta$  dependence has been traded with a dependence on the axion field, whose minimum is at zero:

This solves the strong CP problem. !

## The axion mass

At leading order in  $1/f_A$  – well justified as  $f_A \gtrsim 4 \times 10^8$  GeV – the axion can be treated as an external source, and its mass is given by

$$m_A^2(T)f_A^2 = \left. \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \right|_{\theta=0} \equiv \chi_{top}(T).$$

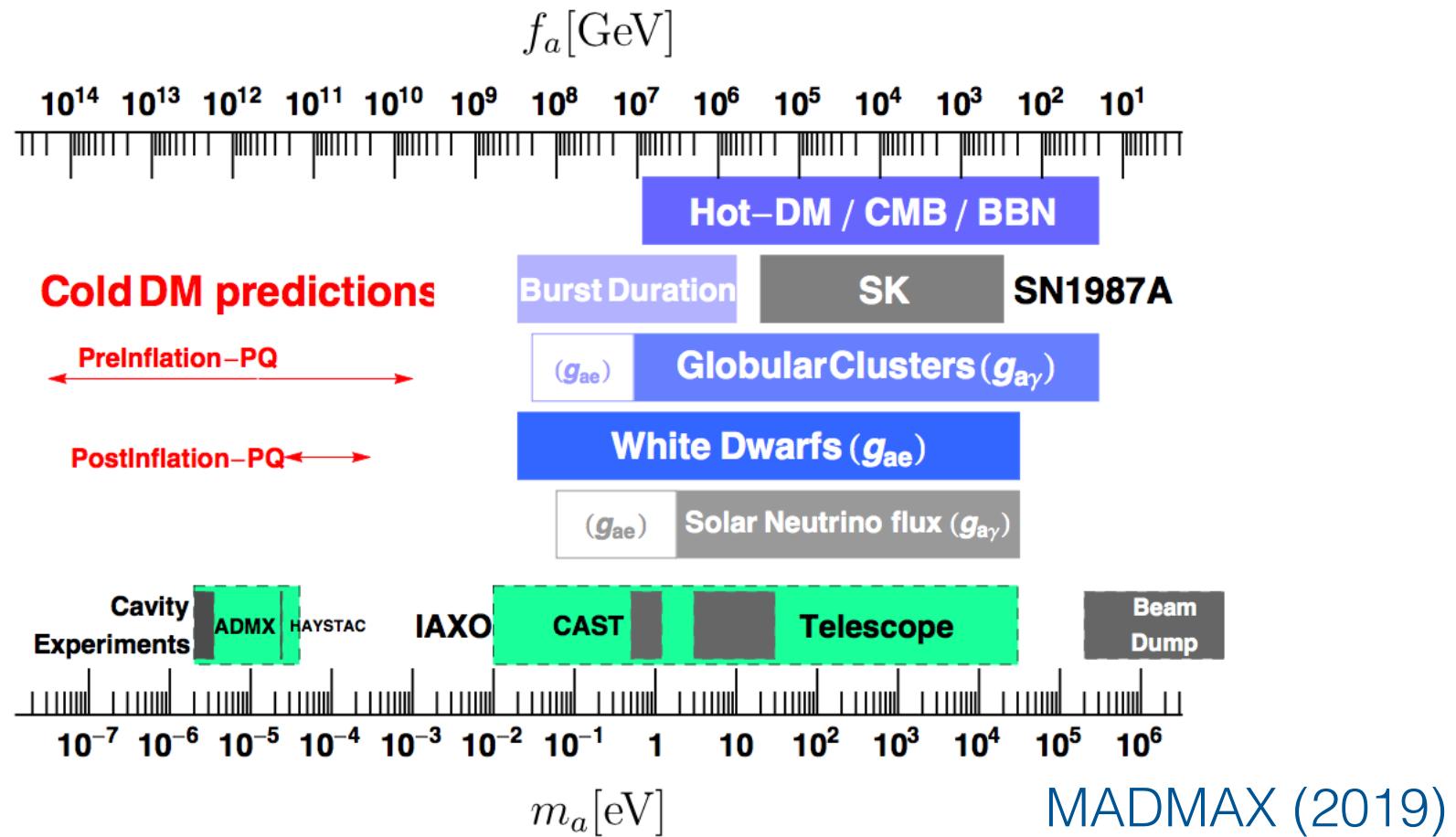
At zero and low temperature, chiral perturbation theory gives:

$$m_A^2 = \frac{m_u m_d}{(m_u + m_d)^2} \frac{m_\pi^2 f_\pi^2}{f_A^2},$$

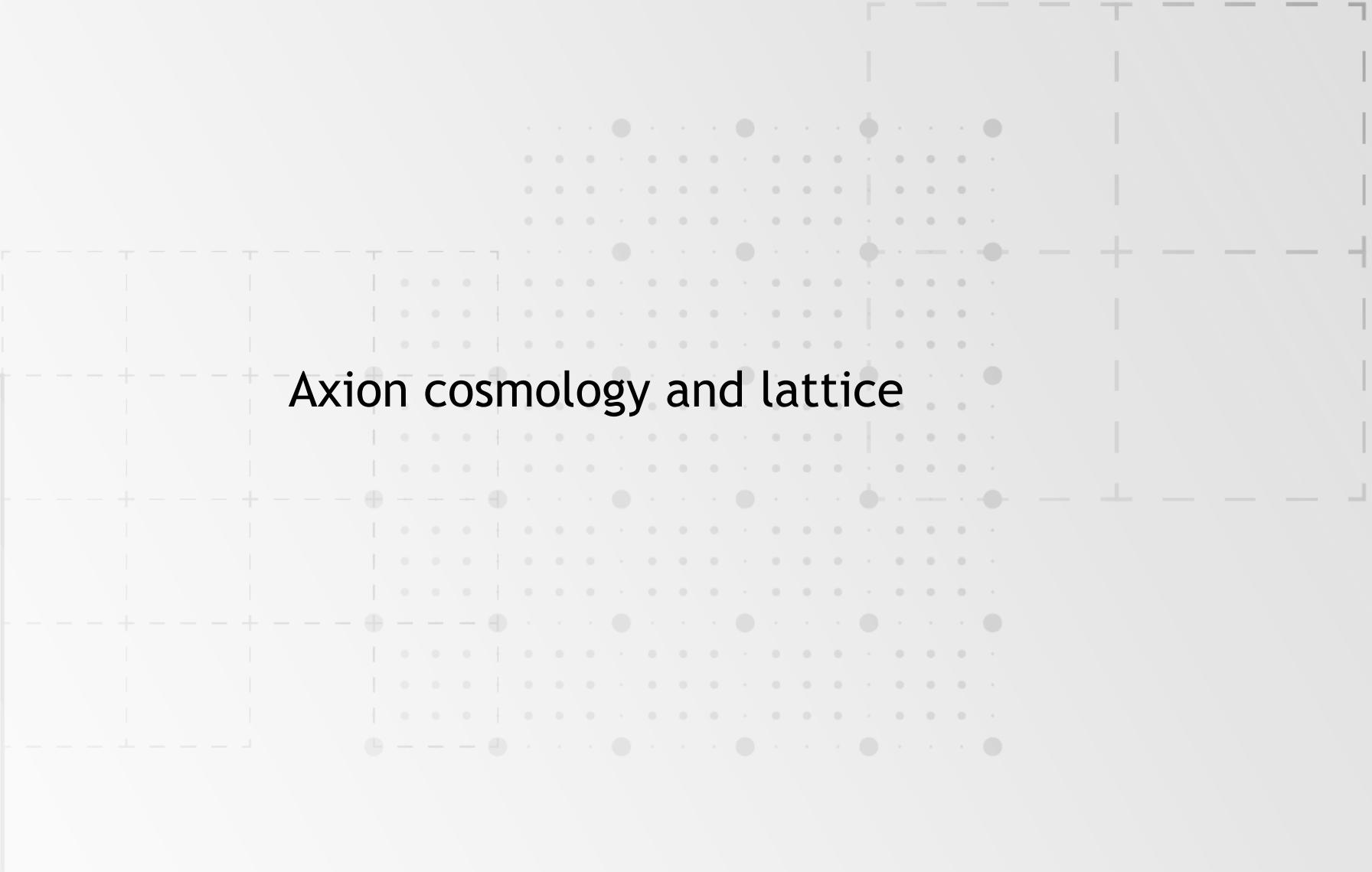
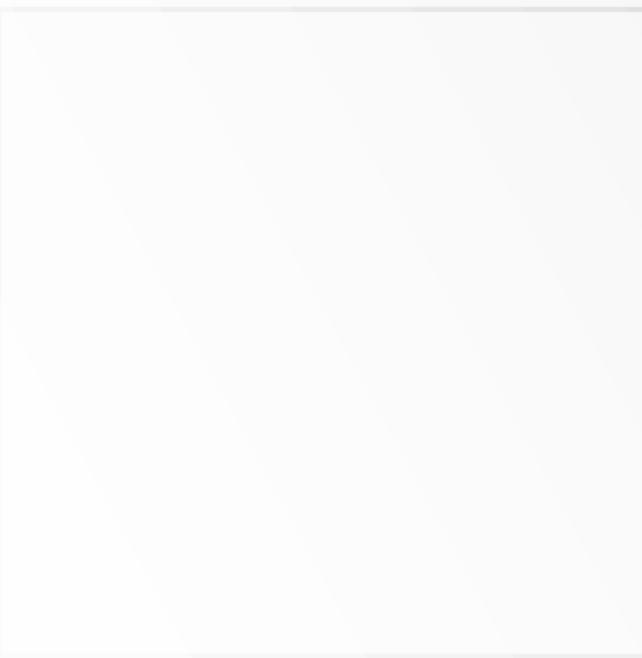
In very brief summary, the essence of this discussion is the close relation between axion mass and topological susceptibility:

$$m_A^2 f_A^2 = \chi_{top},$$

# QCD axion mass landscape

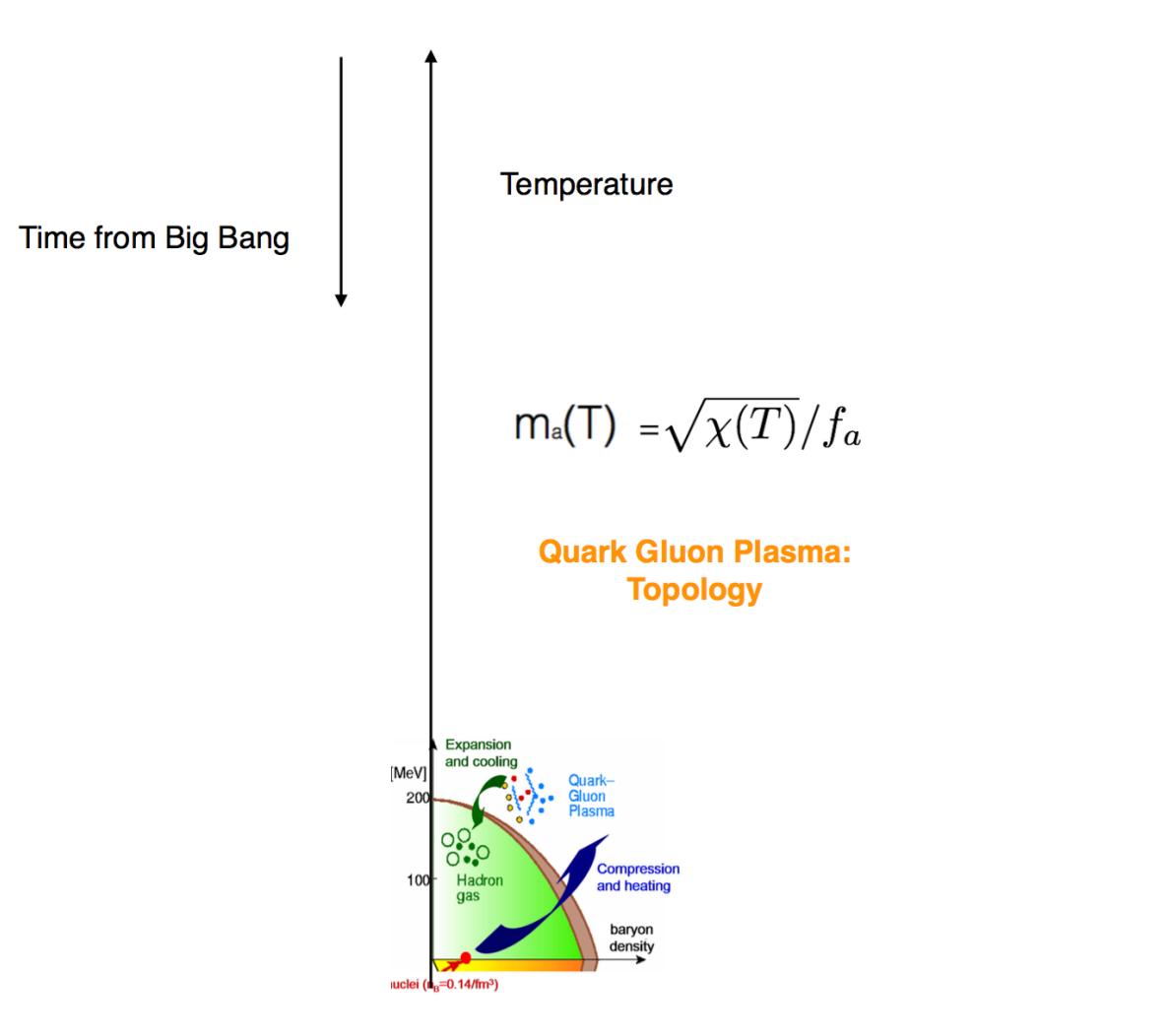


# Axion cosmology and lattice



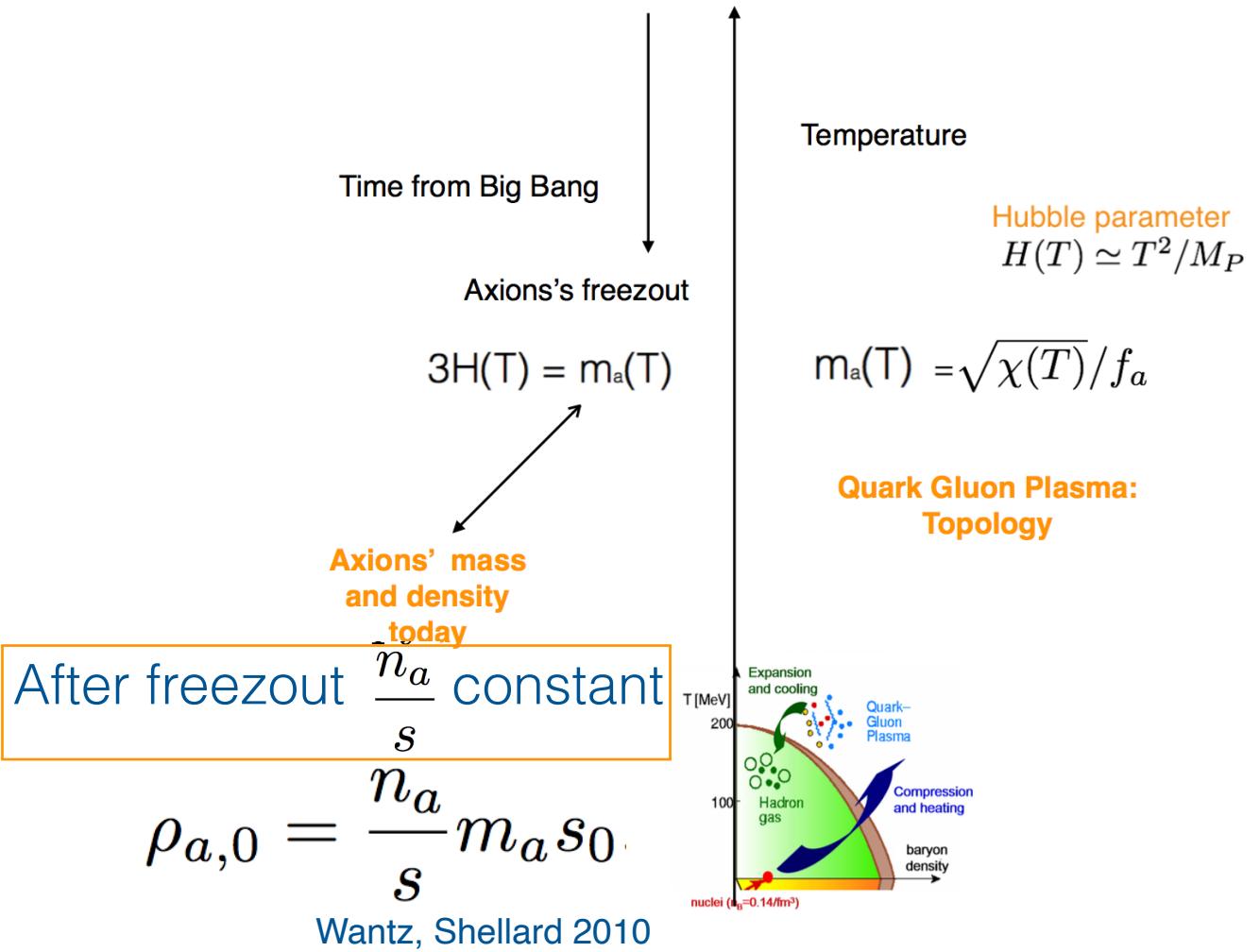
$T_c$  Peccei Quinn  $\gtrsim 10^7$  - $10^8$  GeV

$T_c$  Electroweak  $\simeq 160$  GeV (SM)



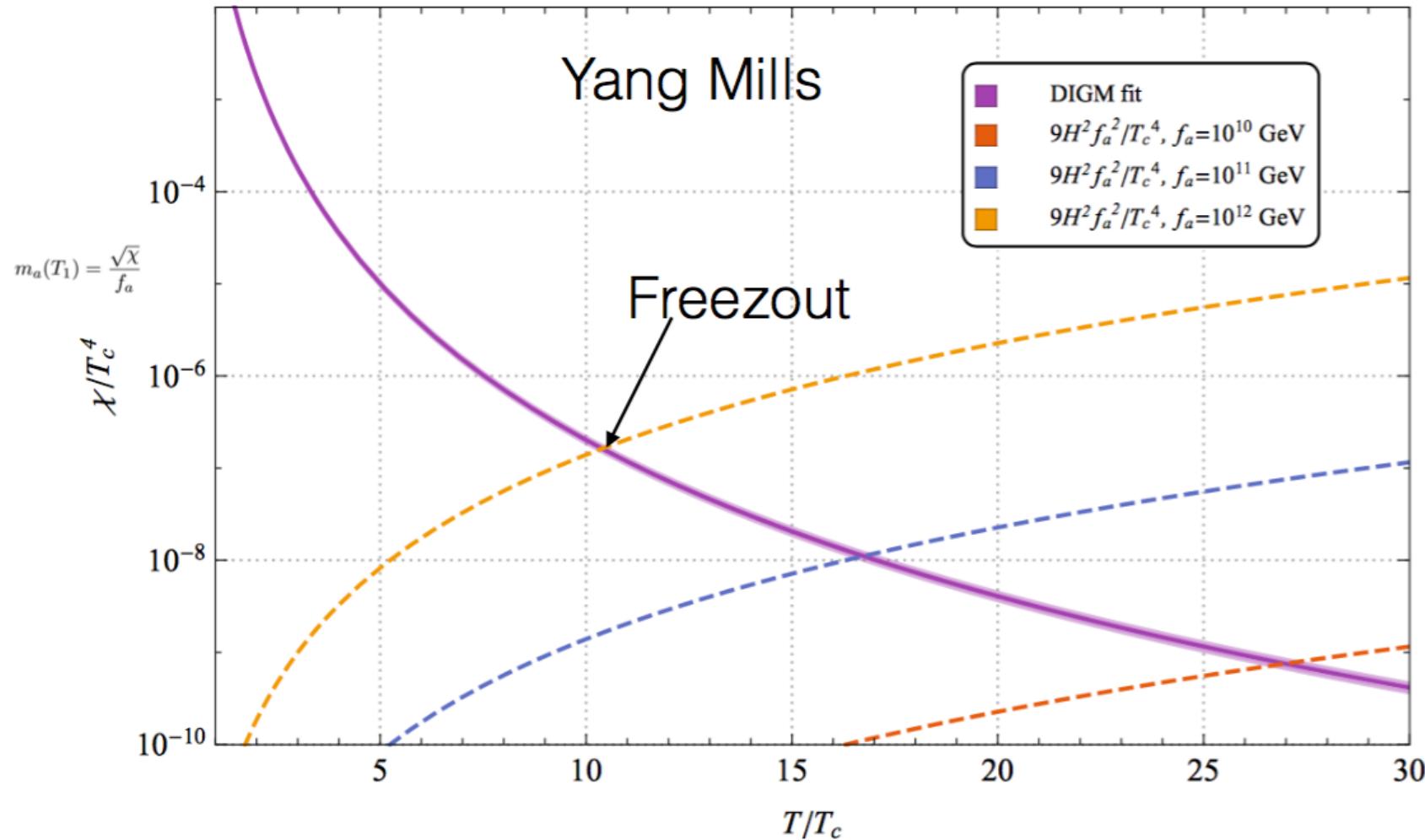
$T_c$  Peccei Quinn  $\gtrsim 10^7$  - $10^8$  GeV

$T_c$  Electroweak  $\simeq 160$  GeV (SM)



Axion freezout :  $3H(T) = m_a(T) = \sqrt{\chi(T)}/f_a$

Berkowitz Buchoff Rinaldi 2015



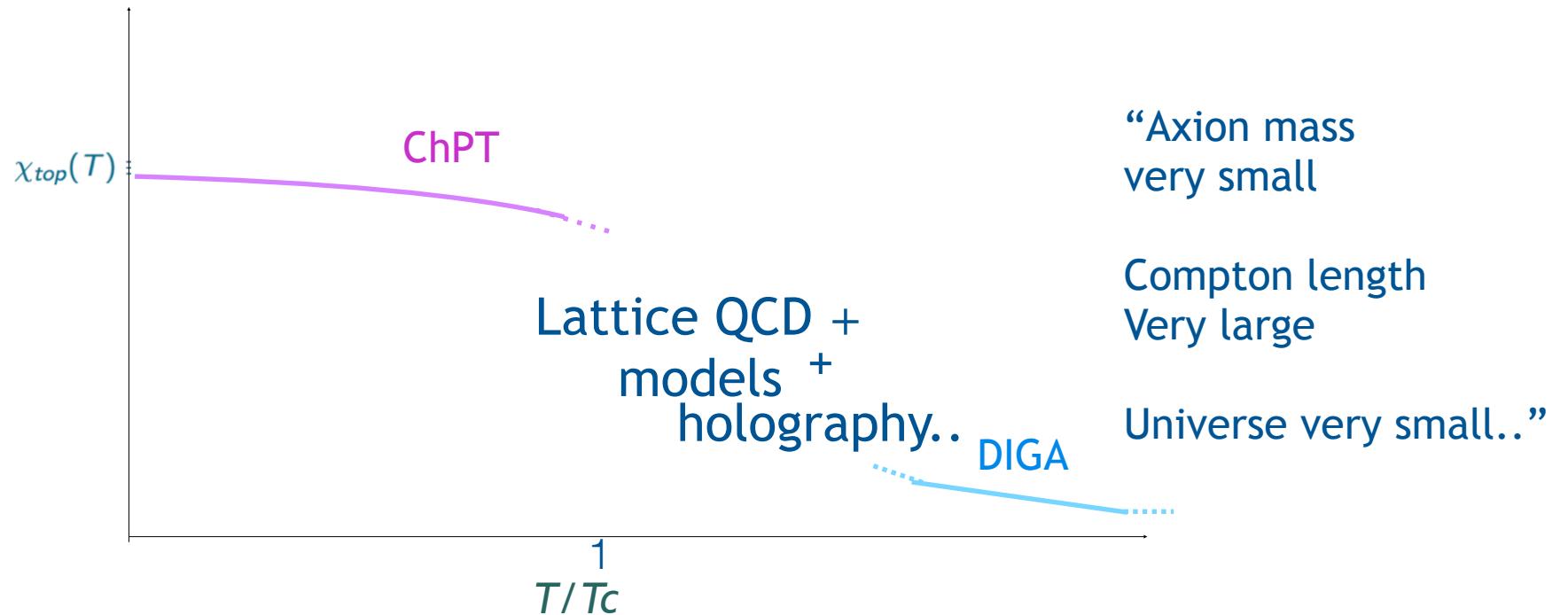
Axion density at freezout controls axion density today

Topological susceptibility around and above the critical temperature



# What do we know about

$$\chi_{top}(T) \equiv \frac{\partial^2 F(\theta, T)}{\partial \theta^2} \Big|_{\theta=0}$$



$$\chi(T) \sim T^4 \left(\frac{m}{T}\right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4 - \frac{11}{3}N_c - \frac{1}{3}N_f}$$

For axion applications we need  $T$  approx. 500-600 MeV

1) Comparison with DIGA:      Only instanton-anti-instanton pair contribute

$$F(\theta, T) - F(0, T) \simeq T^{4-\beta_0} \left( \frac{m_l}{T} \right)^{N_{f,l}} (1 - \cos \theta),$$

where  $\beta_0 = 11N_c/3 - 2N_f/3$  and  $N_{f,l}$  is the number of light flavors.

$$\chi(T) \sim T^4 \left( \frac{m}{T} \right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4 - \frac{11}{3}N_c - \frac{1}{3}N_f}$$

# Lattice topology

Michael Mueller-Preussker(2015)

- ▶ Gluonic:Luscher(2010), Bonati,d'Elia e al (2014),Alexandrou et al . (2015)

$$Q = \frac{a^4}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \sum_n \text{Tr}[F_{lat}^{\mu\nu}(n) F_{lat}^{\rho\sigma}(n)],$$

Need smooth configurations,using smearing,cooling, gradient flow..

$$\dot{V}_\mu(n, \tau) = -g^2 [\partial_{n,\mu} S_G(V(\tau))] V_\mu(n, \tau), \quad V_\mu(n, 0) = U_\mu(n),$$

Pros: Easy

Cons: suffers very much from lattice artifacs

- ▶ Fermionic:Atiyah Singer(1971,1984)

$$Q = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int \text{Tr}[F^{\mu\nu}(x) F^{\rho\sigma}(x)] d^4x = n_+ - n_-$$

Pros: not affected but UV fluctuations

Cons: very high computational cost

- ▶ Fermionic - simple but approximate: Kogut et al.(1996),Petreczky, Sharma(2016)

$$\chi_{top} = \frac{\langle Q^2 \rangle}{V} = m_I^2 \chi_{5,disc}$$

$$\chi_{top}(T \gtrsim T_c) = m_I^2 \chi_{disc} = m_I^2 \frac{V}{T} (\langle (\bar{\psi}\psi)^2 \rangle_I - \langle \bar{\psi}\psi \rangle_I^2).$$

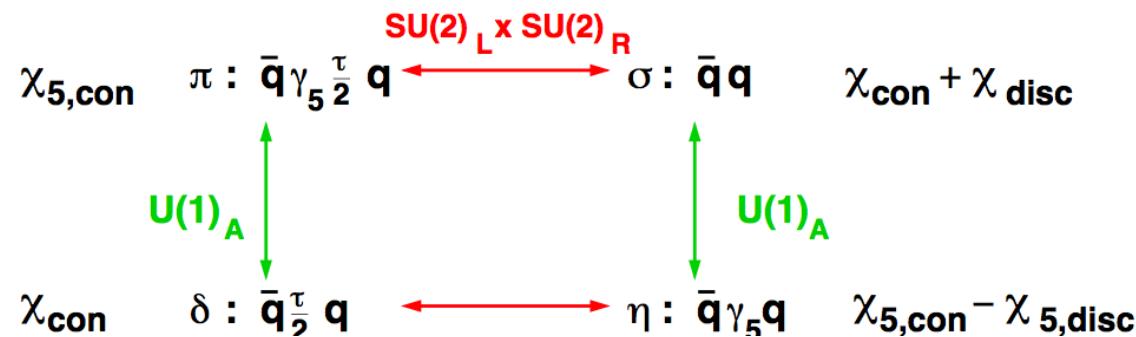
# Topological and chiral susceptibility

Kogut, Lagae, Sinclair 1999

HotQCD, 2012

$$\chi_{top} = \langle Q_{top}^2 \rangle / V = m_l^2 \chi_{5,disc}$$

From:  
 $m \int d^4x \bar{\psi} \gamma_5 \psi = Q_{top}$

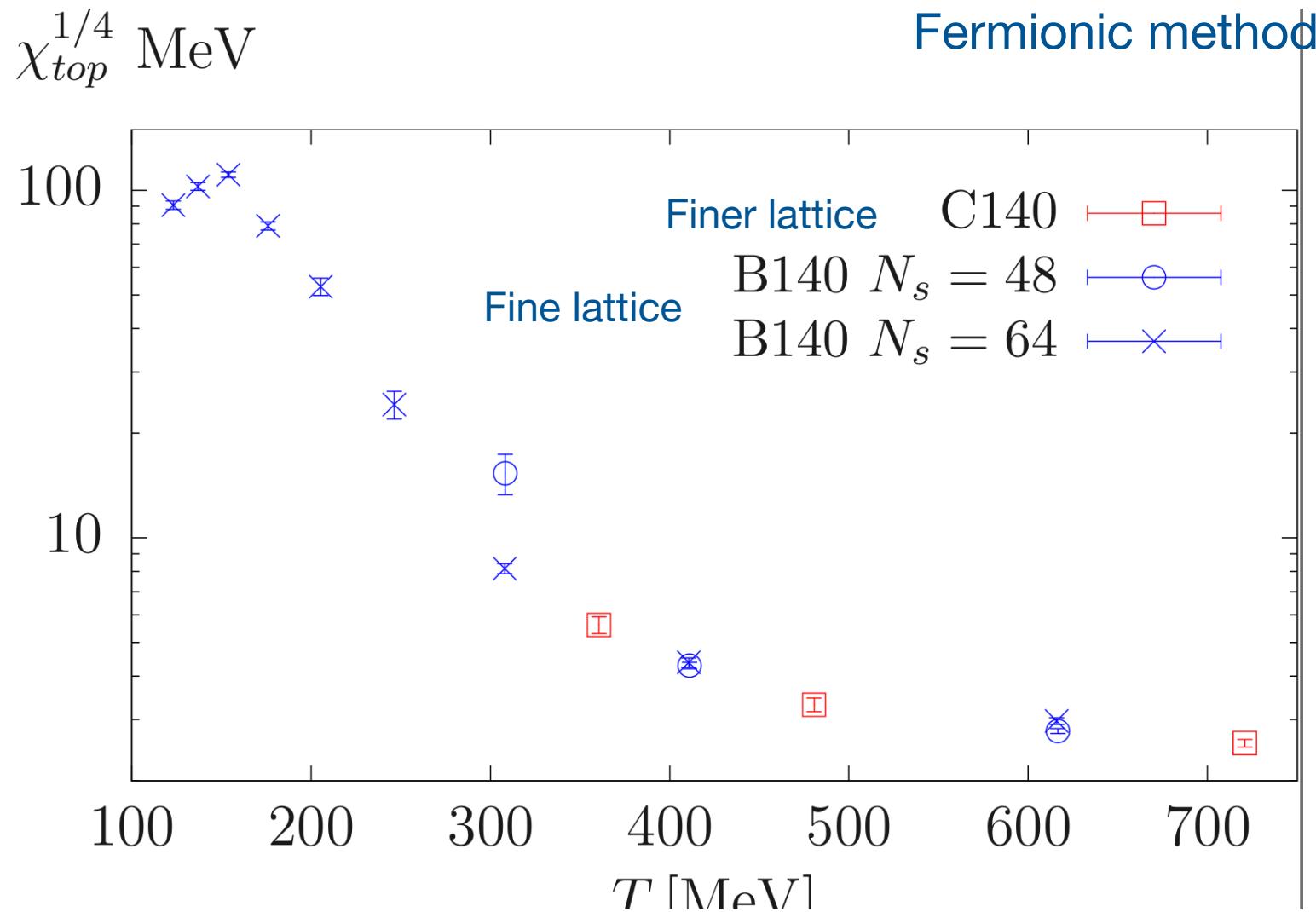


$$\chi_\pi - \chi_\delta = \chi_{disc} = \chi_{5,disc} , \quad \text{for } T \geq T_c , \ m_l \rightarrow 0$$

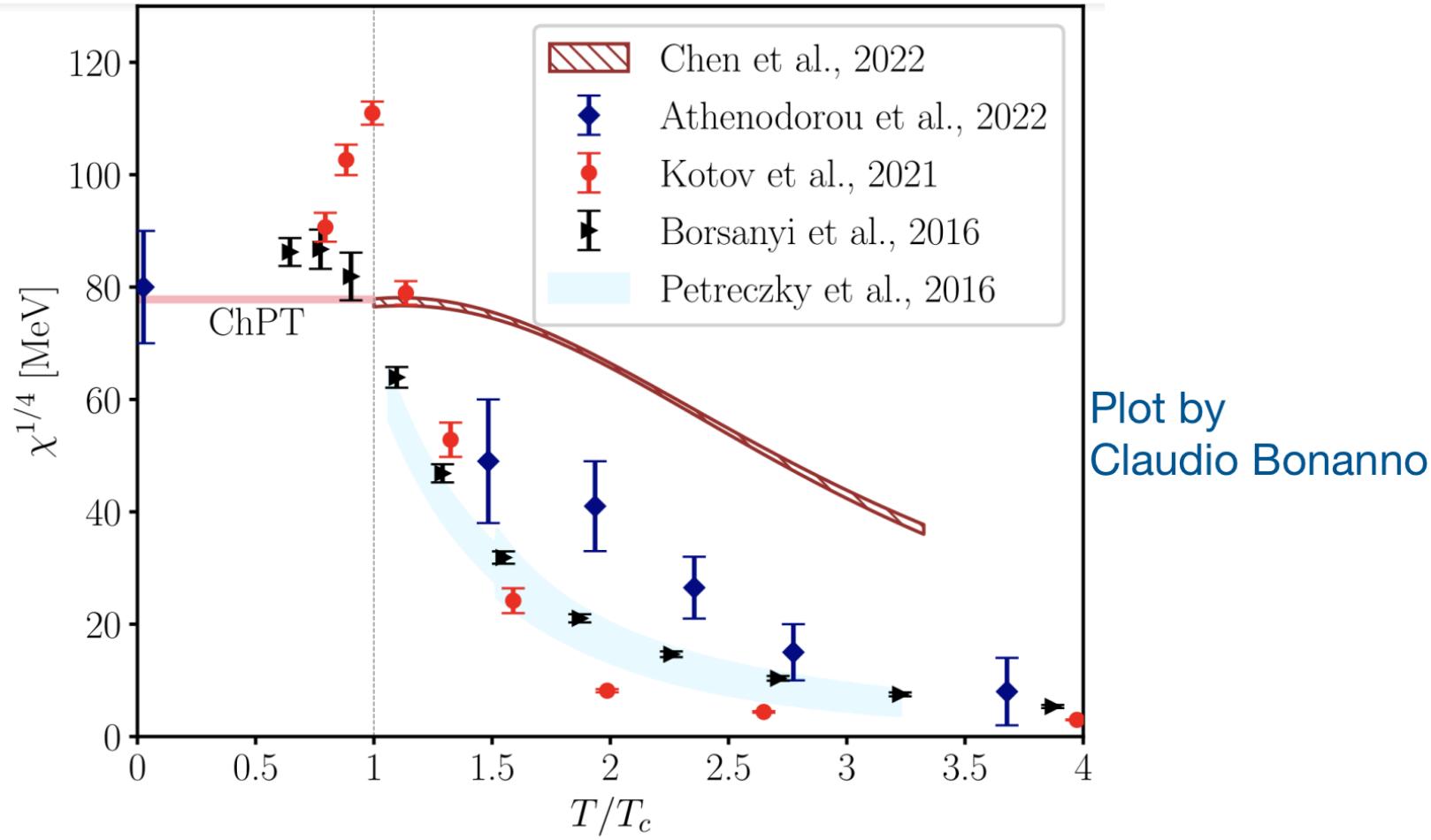
$$\chi_{top} = \langle Q_{top}^2 \rangle / V = m_l^2 \chi_{disc}$$

# Systematics from twisted mass Wilson fermions

2+1+1 flavours

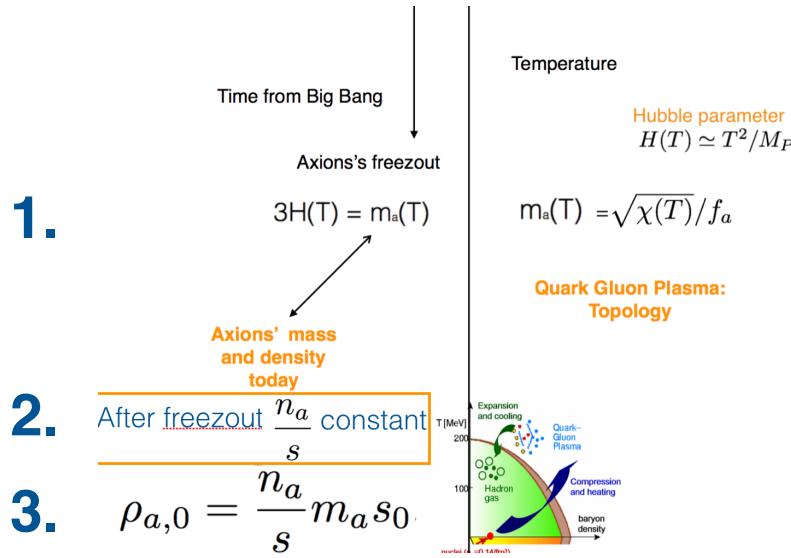


# QCD - summary



Na6-Strong2020, MpL et al, 2023

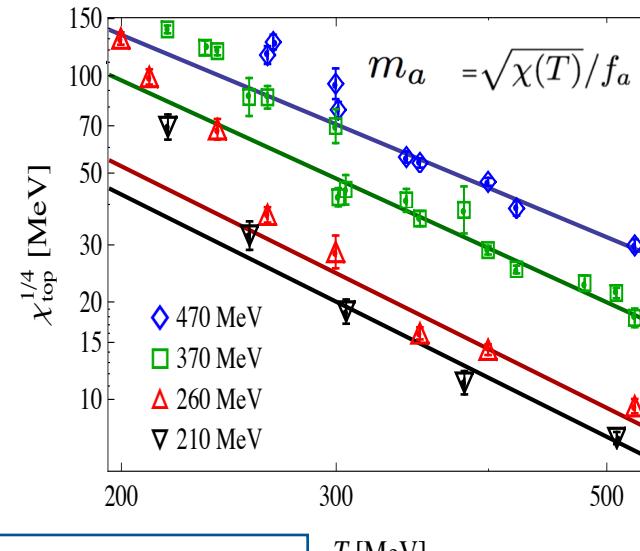
# From exponent $d$ to axion mass in three steps



$$\chi_{\text{top}} \simeq A T^{-d}$$

$$d = (6.26, 6.88, 7.52, 7.48)$$

$$m_\pi = (470, 370, 260, 210) \text{ MeV}$$

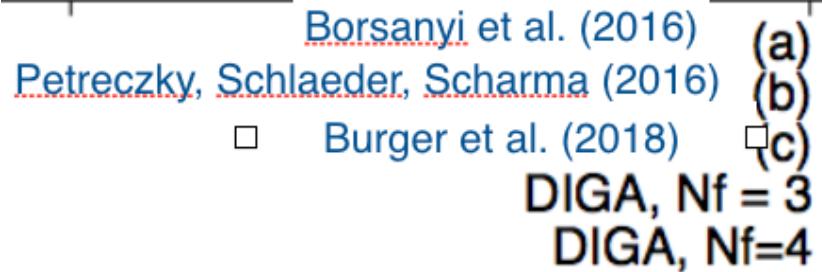


$$\rho_a(m_a) \propto m_a^{-\frac{3.053+d/2}{2.027+d/2}}$$

For axion freeze-out extrapolation needed  $\chi(T) = A T^d$

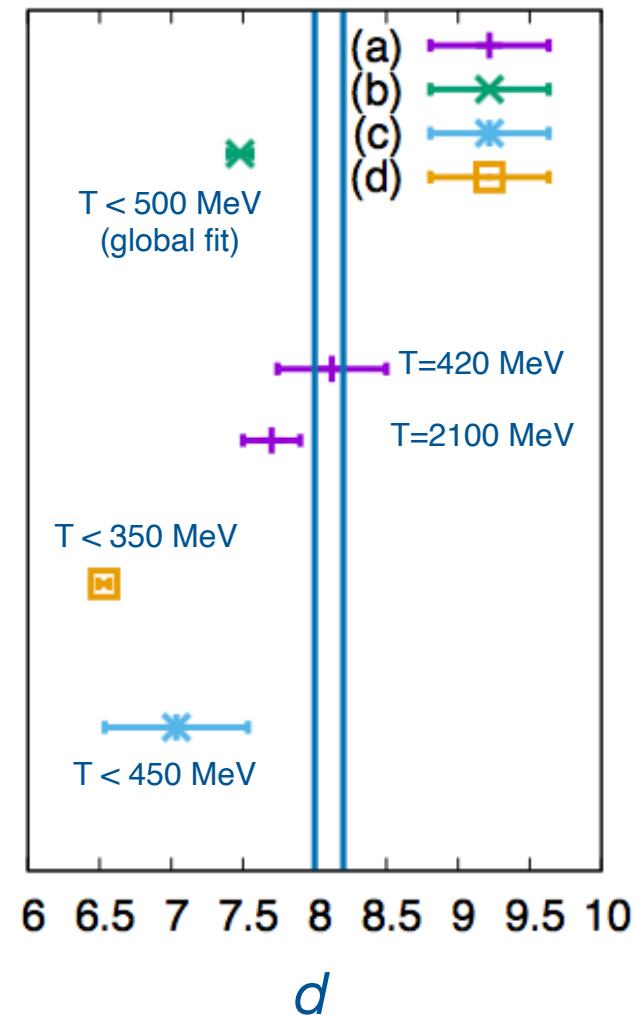
### Summary of the $d$ parameters

Y. Taniguchi, K. Kanaya, H. Suzuki and T. Umeda (2017) (d),



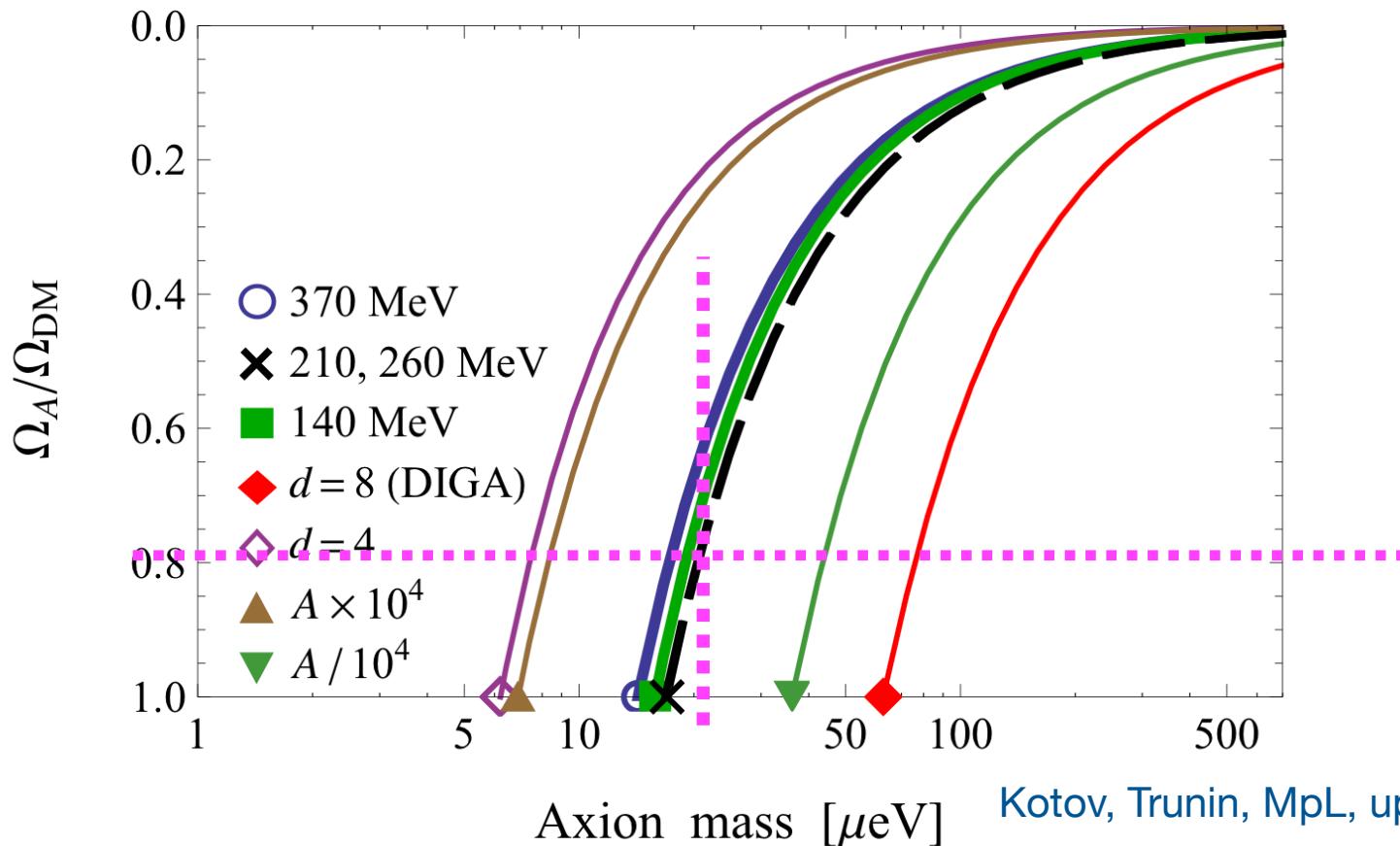
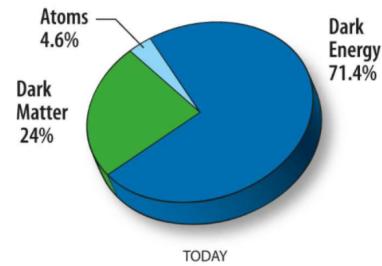
For  $T > 300$  MeV the DIGA exp  
is approached from below

$T_c < T < 250 - 300$  MeV ??



# Limits on the (post-inflationary) axion mass

$$\Omega_A = F(A, d, \dots) m_A^{-\frac{3.053+d/2}{2.027+d/2}}$$



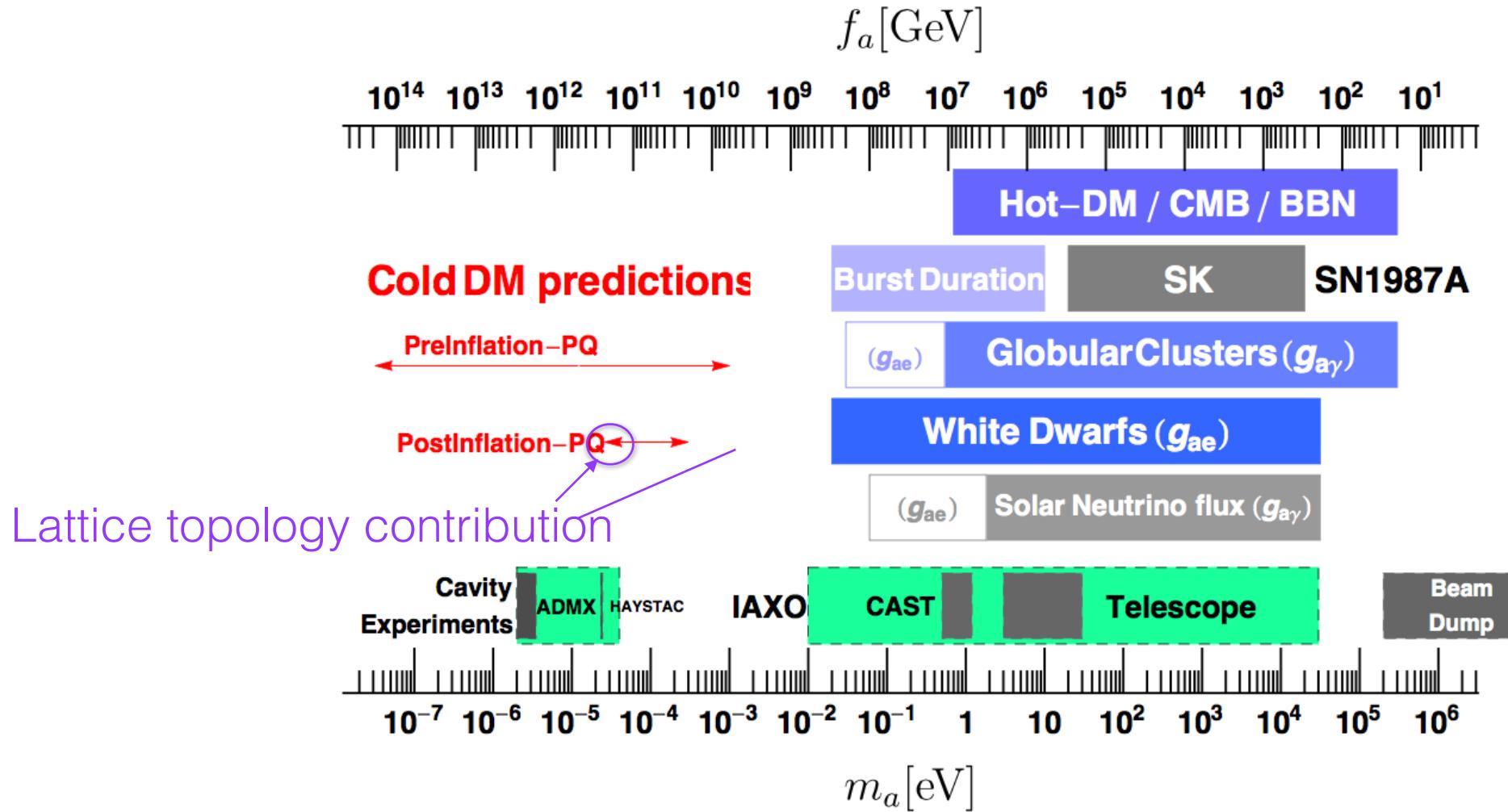
$$\Omega_a = \frac{\rho_{a,0}}{\rho_c};$$

Example: if axions constitute 80% DM,  
our results give a lower bound for the  
axion mass of  $\simeq 30 \mu\text{eV}$

Kotov, Trunin, MpL, updated 2023

Burger,Trunin, Ilgenfritz,Mueller-Preussker,MpL 2019

# Limits on the axion mass

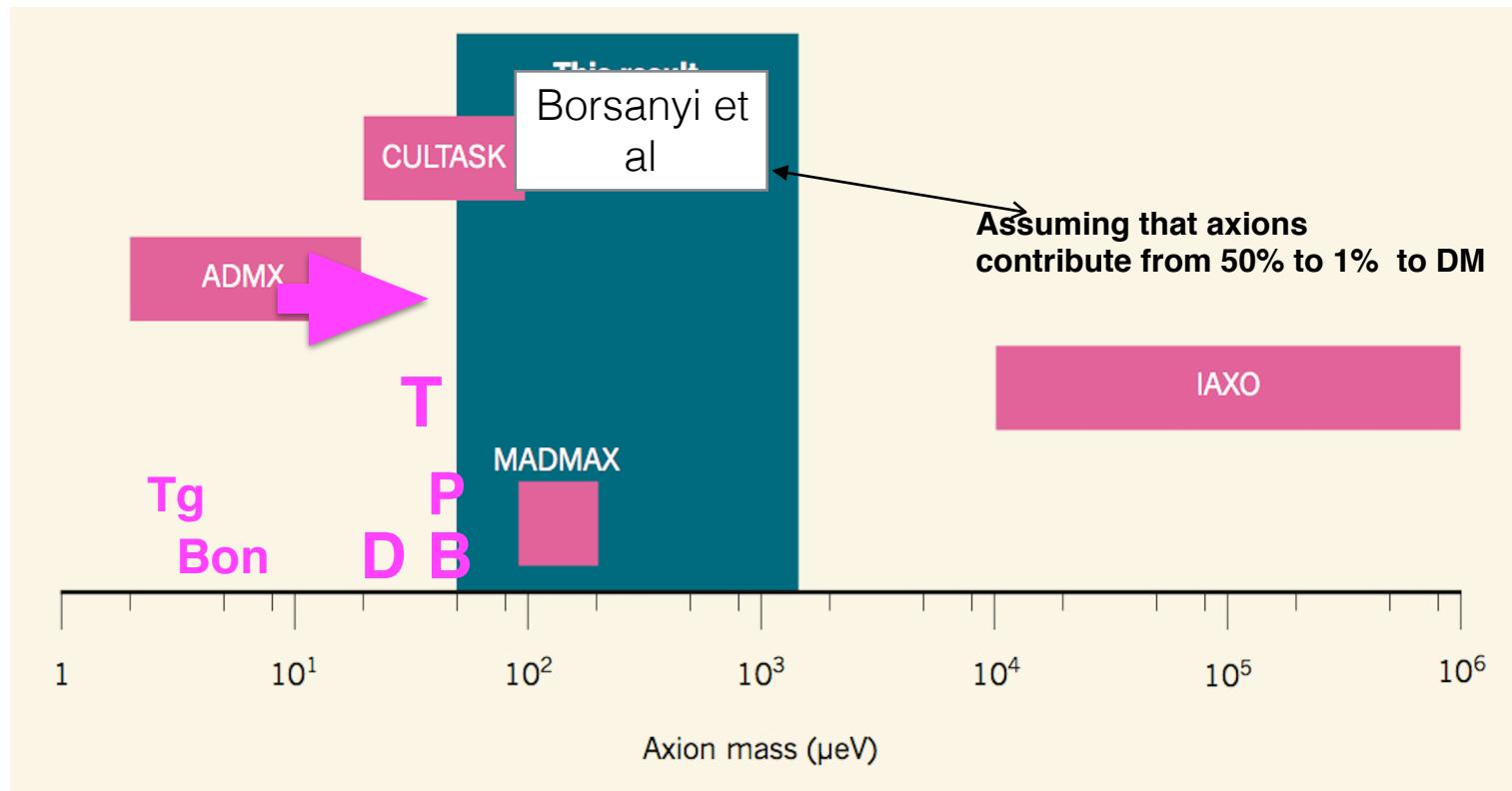


Issue: string contribution ??

## Lower limits on the axion mass assuming that axions make 100% of DM:

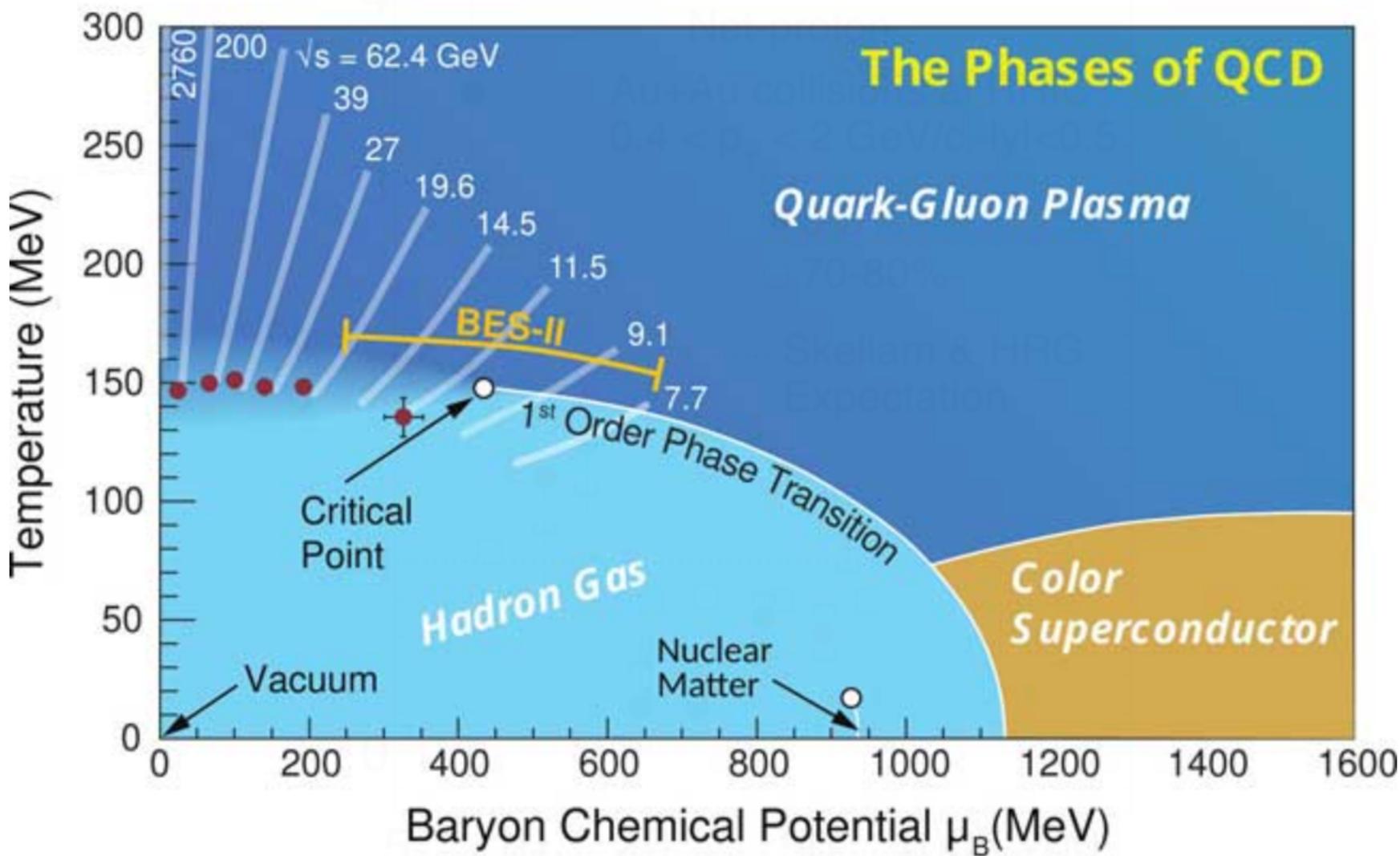
Lower limits

Tg: TWEXT gluonic; Bon: Bonati et al.; D: DIGA, B: Borsanyi et al.,  
P: Petreczky et al., T: TWEXT, fermionic



Updated from Nature N&V

## Lattice methods for finite density (and high temperature)



# Sign problem

$$\int dU d\psi d\bar{\psi} \mathcal{L}(T, \mu, \bar{\psi}, \psi, U) = \int dU e^{-(S_{YM}(U) - \log(\det M))}$$

$$M^\dagger(\mu_B) = -\bar{M}(-\mu_B)$$

Forget the log, keep determinant as an observable

Keep chemical potential zero, works compute derivatives

Purely imaginary chemical potential and analytic continuation

$$M^\dagger(\mu_B) = -\bar{M}(-\mu_B) \quad M^\dagger = -M^*$$

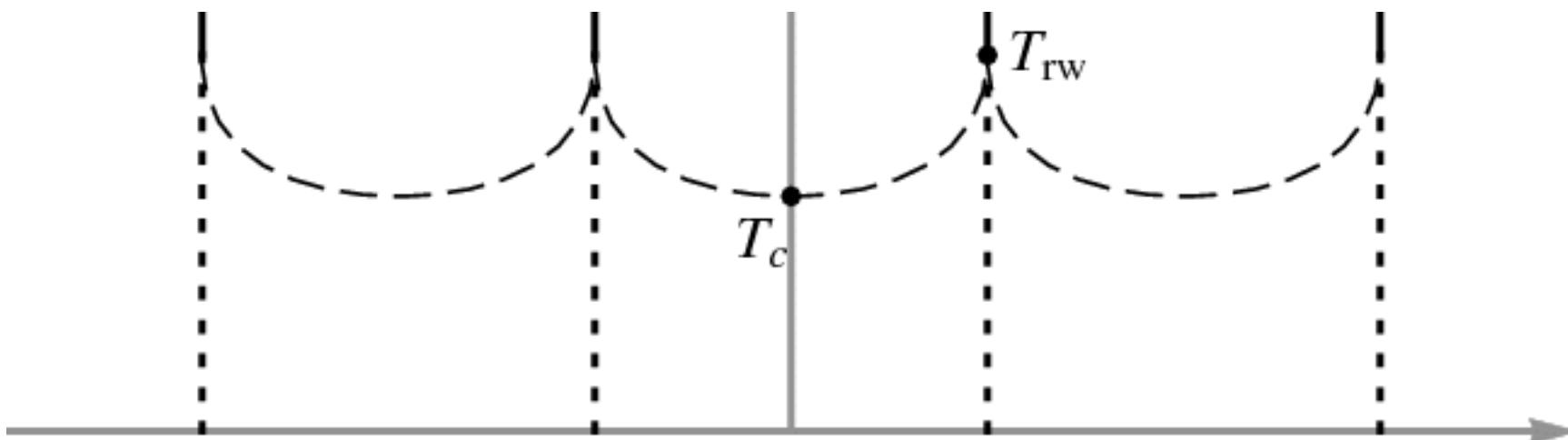
# Phases at imaginary chemical potential

$$\mathcal{Z}(i\mu/T) = \mathcal{Z}(i\mu/T + 2k\pi/3)$$

The periodicity is smooth at low temperature

Phase transitions at high temperature at  $T = 2\pi/3(k+1/2)$

Roberge-Weiss 1981

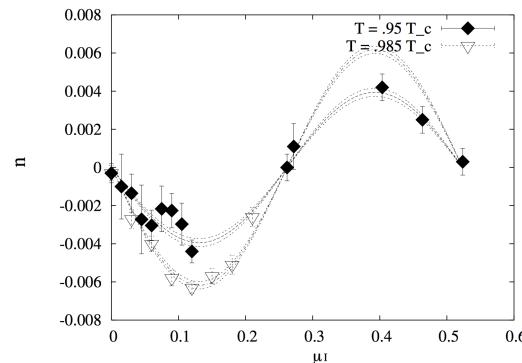


# Pressure and its derivatives from analytic continuation

- Strategy: analytically continue from imaginary  $\mu$

MpL 00 de Forcrand Philipsen 02 d'Elia MpL 02

- Observation: optimal parametrization depends on T



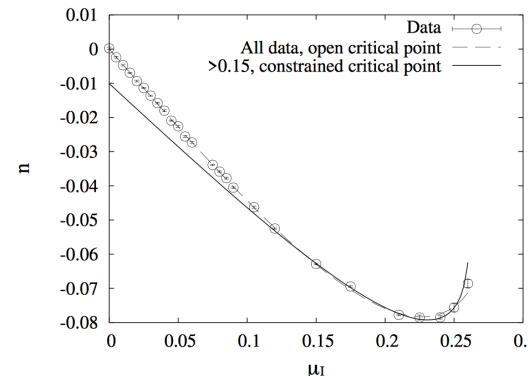
D'Elia, MpL 04

Below  $T_c$ :

**Fourier**



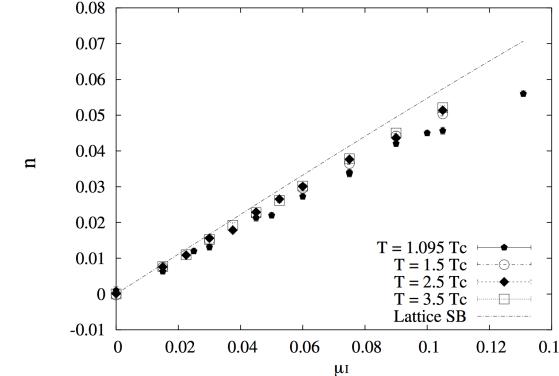
**Virial expansion natural**



D'Elia, di Renzo, MpL 05

$T_c < T < T_{RW}$

Singular

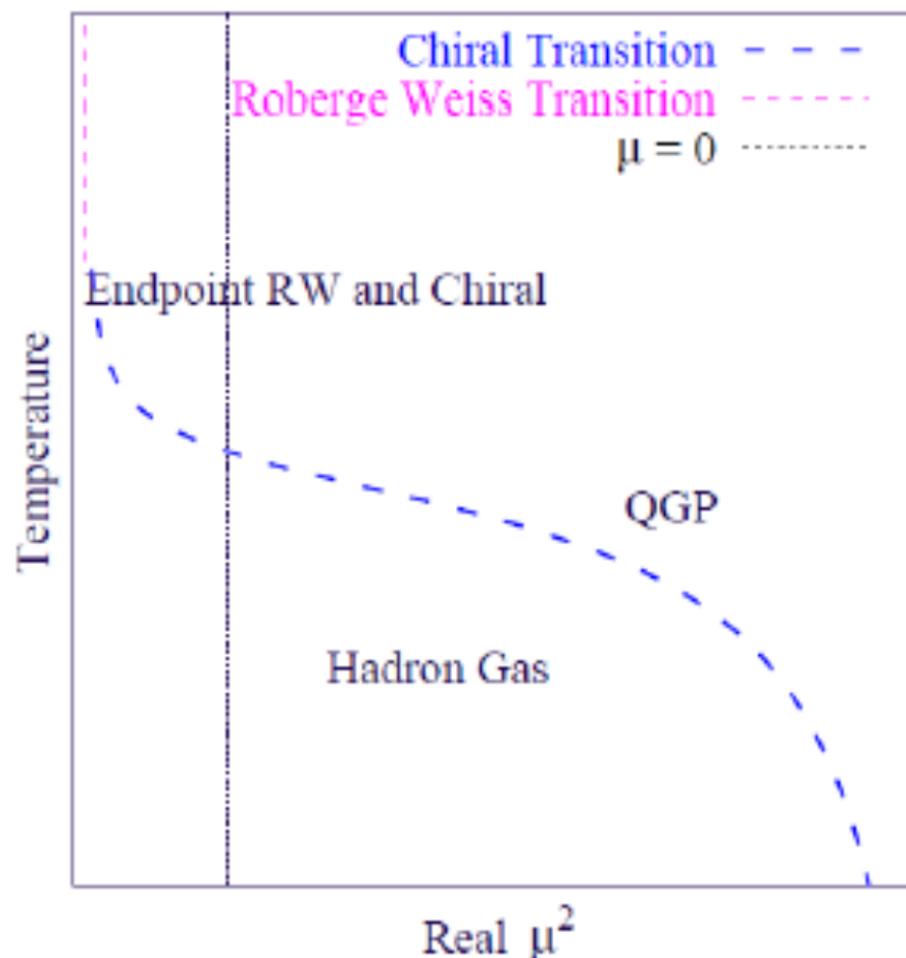


D'Elia, MpL 04

$T > T_{RW}$

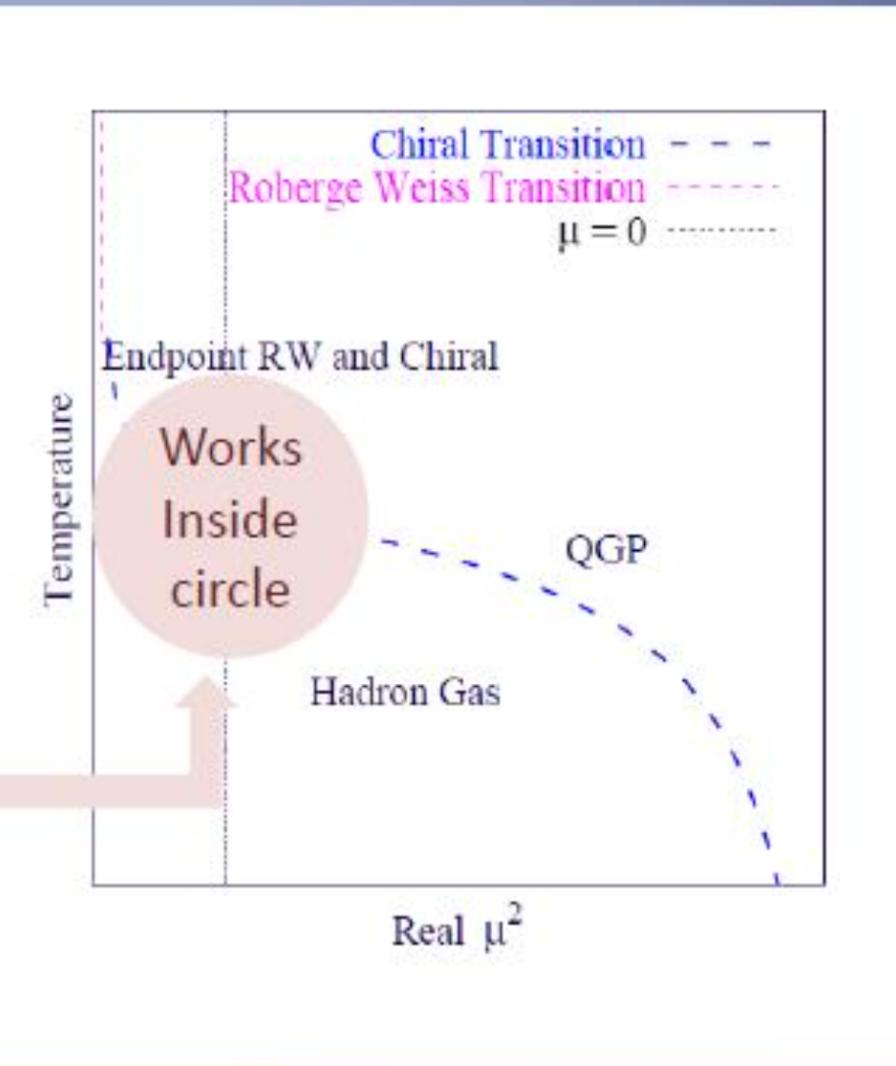
Taylor

Useful to consider  
the QCD phase  
diagram in the  
temperature,  
 $\mu^2$  plane



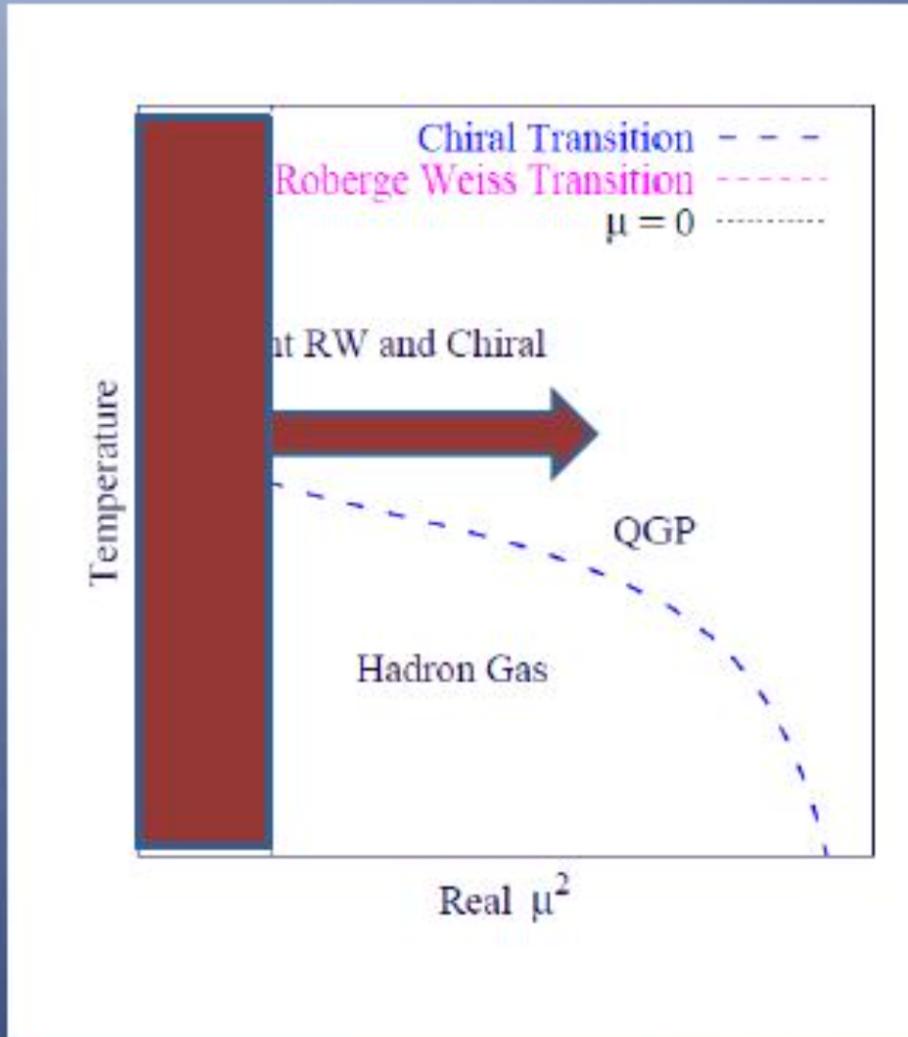
# Taylor expansion

- Taylor expansion:  
main problem,  
control of the  
convergence



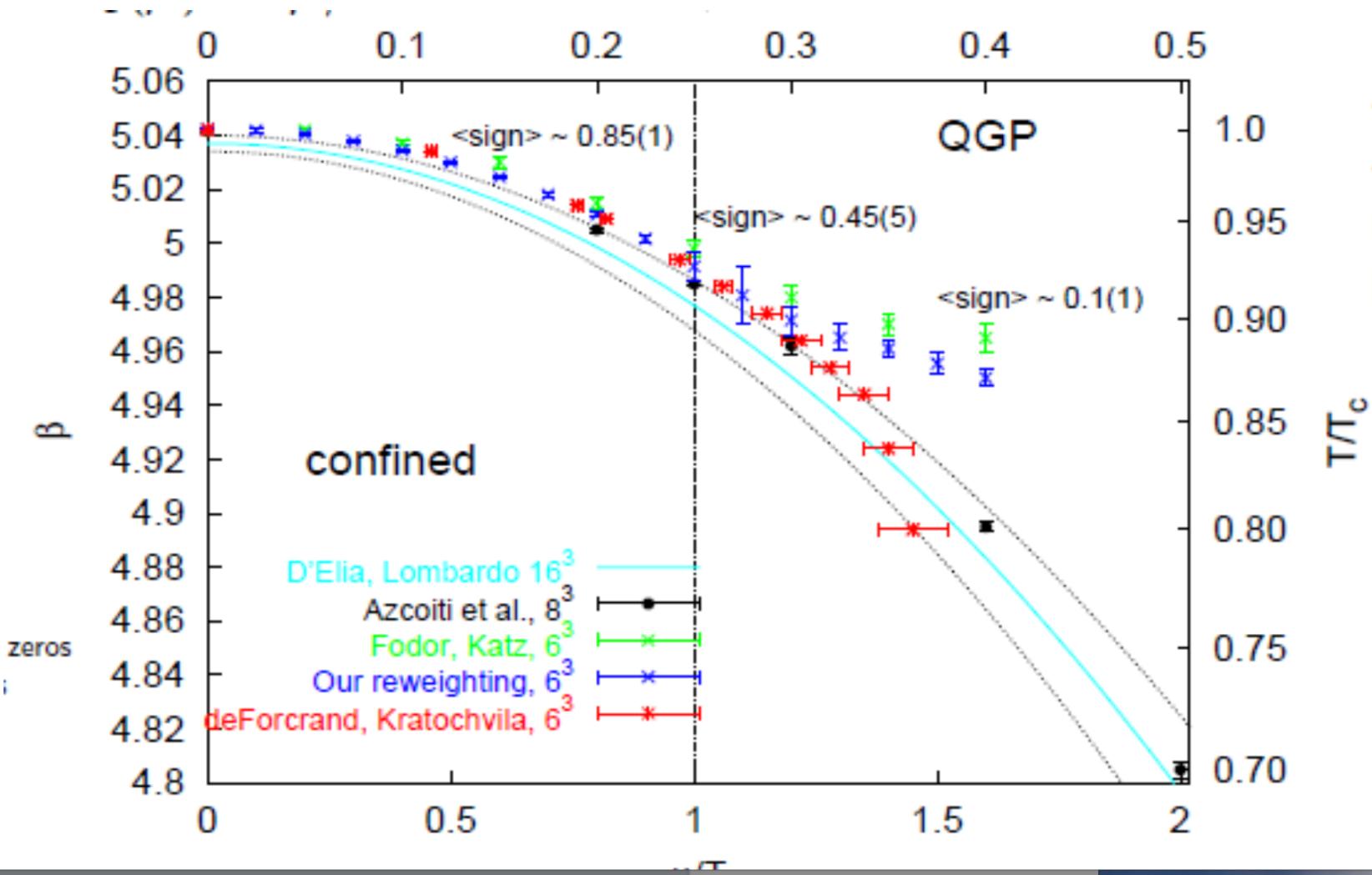
# Imaginary chemical potential

- **Imaginary chemical potential: main problem, control over analytic continuation**



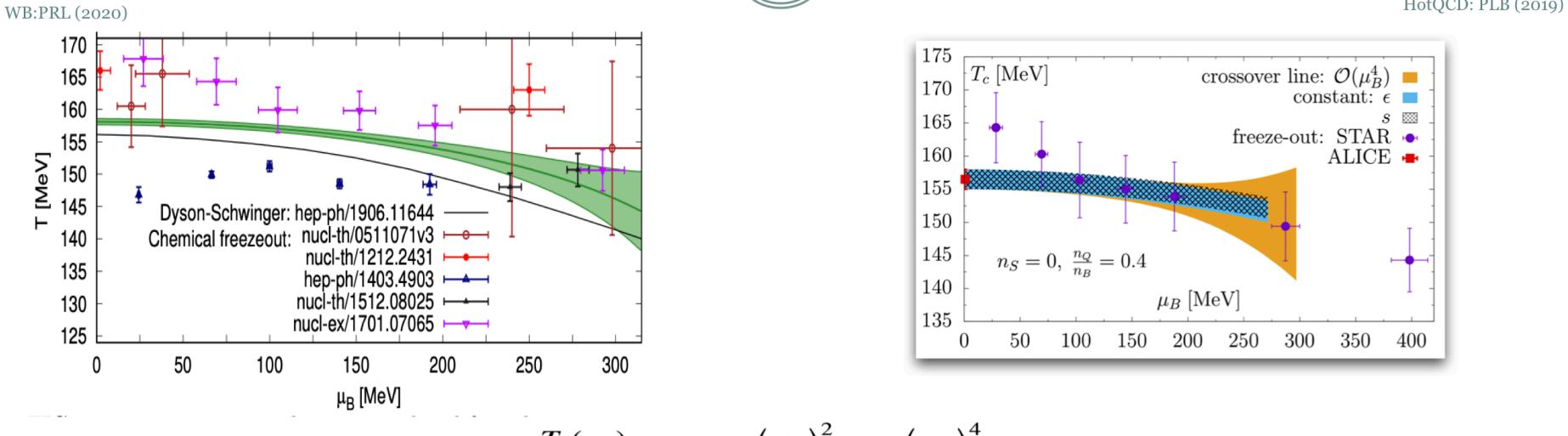
# All methods work at high T

oldish results



# Current status for the crossover line

## QCD crossover temperature



$$\frac{T_c(\mu_B)}{T_0} = 1 - \kappa_2 \left( \frac{\mu_B}{T_0} \right)^2 - \kappa_4 \left( \frac{\mu_B}{T_0} \right)^4 + \mathcal{O}(\mu_B^6)$$

$$T_0 = 158.0 \pm 0.6 \text{ MeV}$$
$$\kappa_2 = 0.0153 \pm 0.0018$$
$$\kappa_4 = 0.00032 \pm 0.00067$$

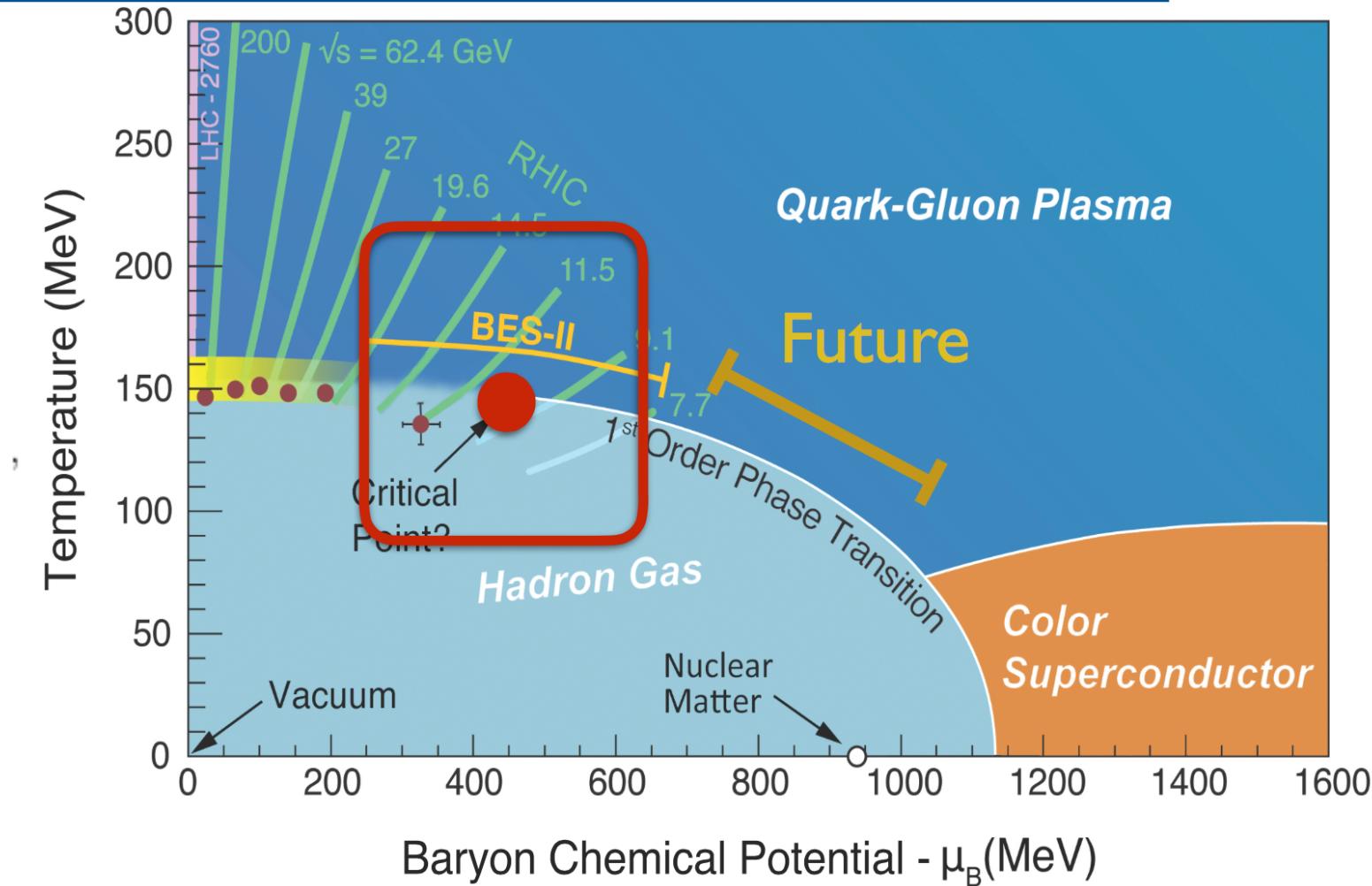
$$T_0 = 156.5 \pm 1.5 \text{ MeV}$$
$$\kappa_2 = 0.012 \pm 0.004$$
$$\kappa_4 = 0.000 \pm 0.004$$

## The QCD critical point



# Search for the critical point of QCD -

(and there are more in the complex  $\mu_B$  plane )



# Searching for the QCD critical point

Series expansions are useful tools to study the phase diagram in the region which is not directly accessible to MonteCarlo simulations due to the sign problem.

Typical parameters for the expansion are the the chemical potential  $\mu$  and the fugacity  $e^{\mu/T}$ , while the observables to be expanded include the Grand Canonical Partition Function or the associated thermodynamic quantities, derivatives of the GCPF

$$Z(\mu) = \sum_{k=-3n_s^3}^{3n_s^3} \langle b_i \rangle e^{k\mu T} = \sum_{k=-3n_s^3}^{3n_s^3} Q_k f^k$$

GCPF  
on the lattice  
Glasgow-style

I.M. Barbour, N. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone, H. Wyld, Nuclear Physics **B275** (FS17)(1986) 296.

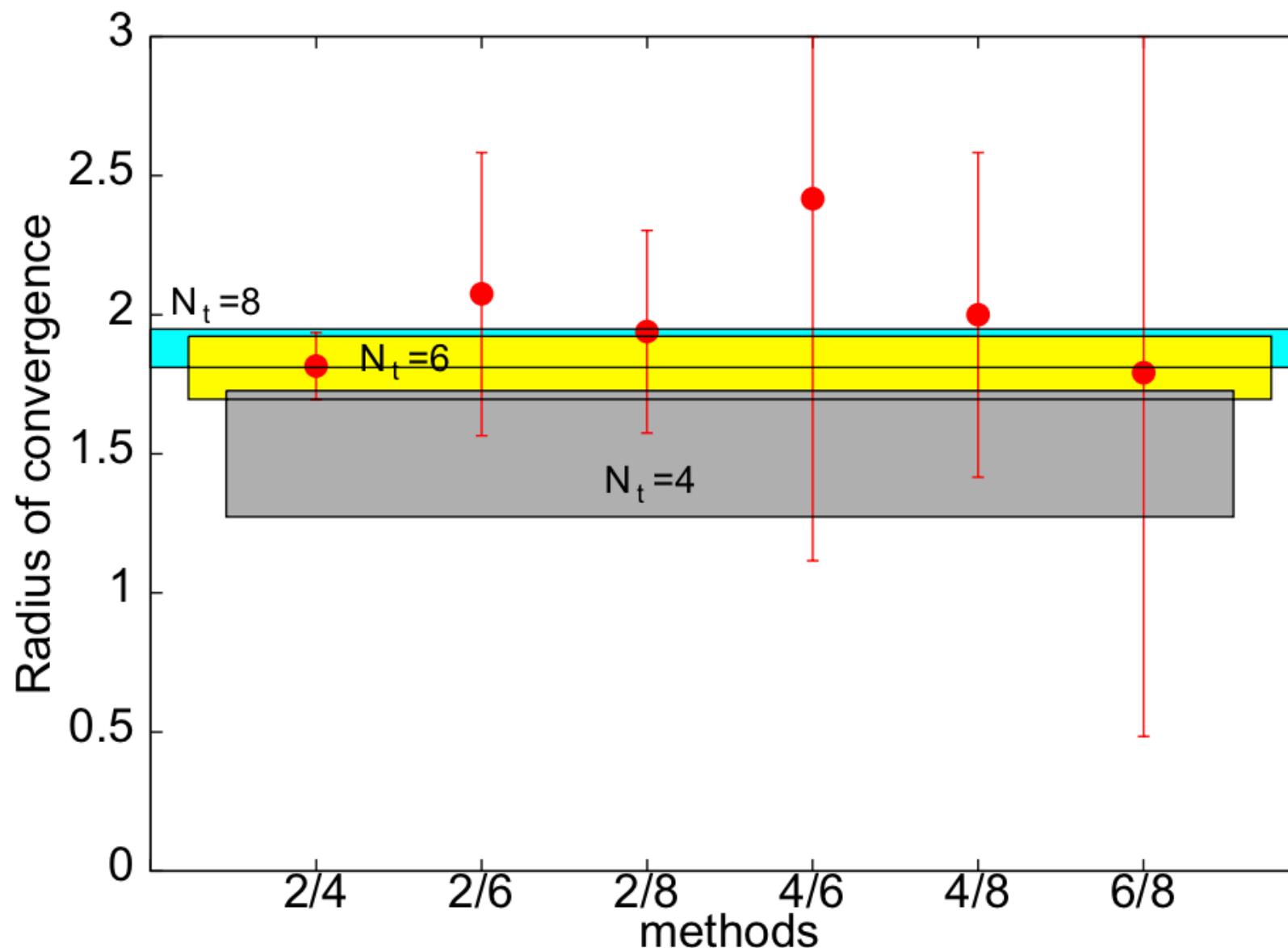
$$\frac{p(\mu/T)}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu}{T}\right)^n$$

$$p = (T/V) \ln \mathcal{Z}$$

Pressure  
on the lattice Bielefeld-  
Swansea -style

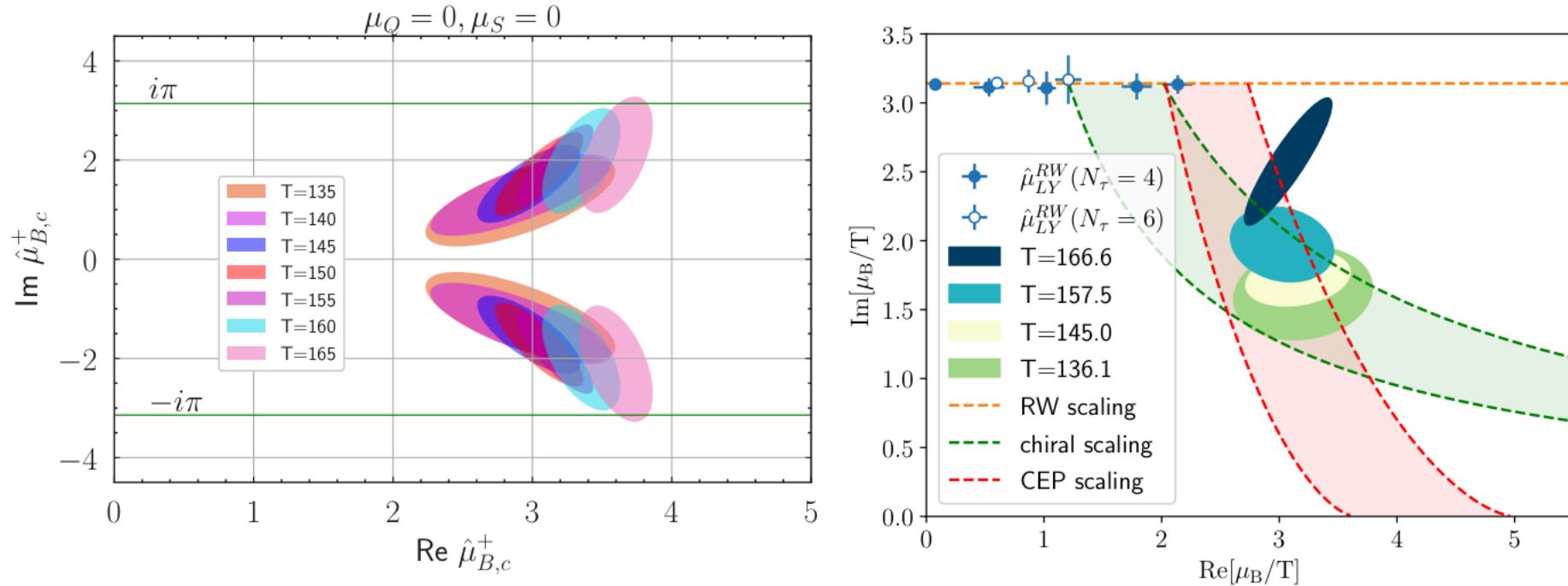
C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and C. Schmidt, Phys. Rev. **D66**, 074507 (2002); **D68**, 014507 (2003); C. R. Allton, M. Doering, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, **D71**, 054508 (2005).

# Estimate of the position of the critical point via radius of convergence



Gavai-Gupta

# Attempts at a direct identification of the poles



**Figure 4.1:** Poles in the complex  $\hat{\mu}_B$  plane from the  $[4,4]$ -Padé re-summation of the Taylor series about  $\hat{\mu}_B = 0$  (left) and from the multi-point Padé approach applied to lattice QCD data at imaginary  $\mu_B$  (right). Also shown in the right panel is the expected scaling behavior of the Lee-Yang edge singularities for different critical points, indicated by dashed lines/bands.

# The Canonical Approach

Beating the Sign problem by Canonical Approach  
(Hasenfraz-Toussant,1992)

Calculation in pure  $\mu_I$  regions,  
where no sign problem.

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(T, \theta \equiv \frac{\text{Im } \mu}{T})$$

Then calculate  $Z$  in real  $\mu$  regions.

$$Z(\xi, T) = \sum_n Z_n(T) \xi^n$$
$$\xi \equiv e^{\mu/T}$$

- A. Nakamura, Mod. Phys. Lett. A **22** (2007) 473; V. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V. I. Zakharov, Nucl. Phys. A **956** (2016) 809.  
V. G. Bornyakov, D. L. Boyda, V. A. Gov, A. V. Molochkov, A. Nakamura, A. A. Nikolaev and V. I. Zakharov, Phys. Rev. D **95** (2017) no.9, 094506. V. G. Bornyakov *et al.*, arXiv:1712.02830 [hep-lat].

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where no sign problem.

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(T, \theta \equiv \frac{\text{Im } \mu}{T}) = e^{-\int_0^{\mu_I/T} n^n(x) dx}$$

Then calculate  $Z$  in real  $\mu$  regions.

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$$\xi \equiv e^{\mu/T}$$

- A. Nakamura, Mod. Phys. Lett. A **22** (2007) 473; V. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V. I. Zakharov, Nucl. Phys. A **956** (2016) 809.  
V. G. Bornyakov, D. L. Boyda, V. A. Goy, A. V. Molochkov, A. Nakamura, A. A. Nikolaev and V. I. Zakharov, Phys. Rev. D **95** (2017) no.9, 094506. V. G. Bornyakov *et al.*, arXiv:1712.02830 [hep-lat].

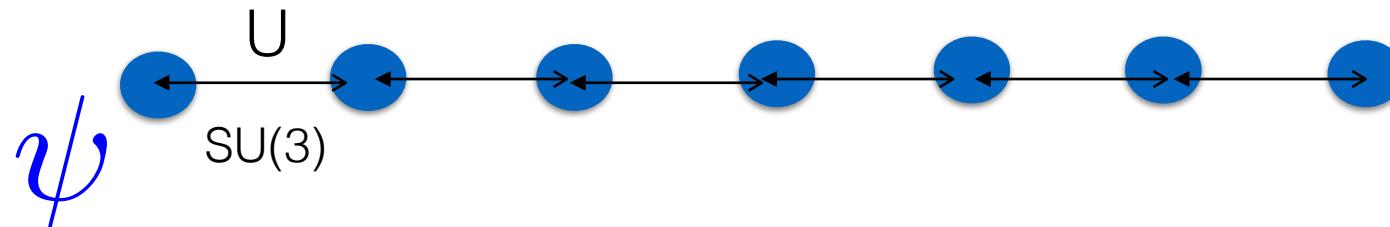
## A simple model : QCD in 1d

One dimensional QCD is an interesting, exactly solvable model. There is no spontaneous symmetry breaking, but there are baryons. Its partition function is formally the same as the one obtained in 4d QCD at strong coupling, once an explicit mass term is identified with the 4d dynamically generated mass. For any temperature  $T$  and varying  $\mu$  there is a crossover to a baryon rich phase, which turns into a first order transition at zero temperature.

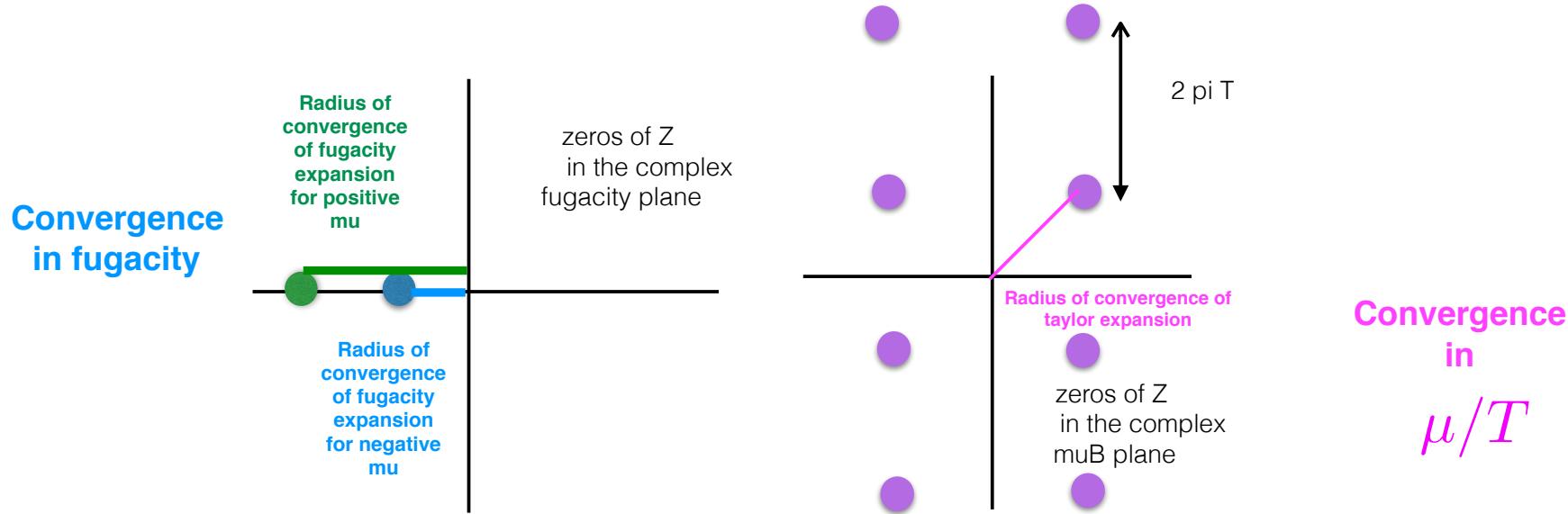
The partition function reads

$$\mathcal{Z}(\mu, T) = 2 \cosh(\mu/T) + \sinh(4m'/T)/\sinh(m'/T) = \sum_{n=-1,0,1} A_n \xi^n$$

$A_0 = \sinh(4m'/T)/\sinh(m'/T) \equiv A$ ,  $A_1 = A_{-1} = 1$ ,  $\mu$  is understood to be  $\mu_B$ ,  $m' = sh^{-1}m$ , with  $m$  the quark mass.



# Zeros of the GCPF and radius of convergence in QCD 1d



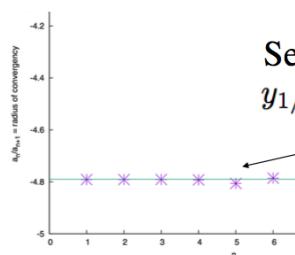
$4 < A < \infty \rightarrow y_1 = 1/y_2$  are two negative real roots.

$$\Re e^{\mu/T} = (-A + / - \sqrt{A^2 - 4})/2 \equiv y_{1/2}$$

$$\mu = \pm T(\log|y_1|) + i(2k + 1)T\pi.$$

The fugacity series converges for:

$$1/y_1 < e^{\mu/T} < y_1,$$



Setting  $A=5$ .  
 $y_{1/2} = -4.79, -0.208$ .

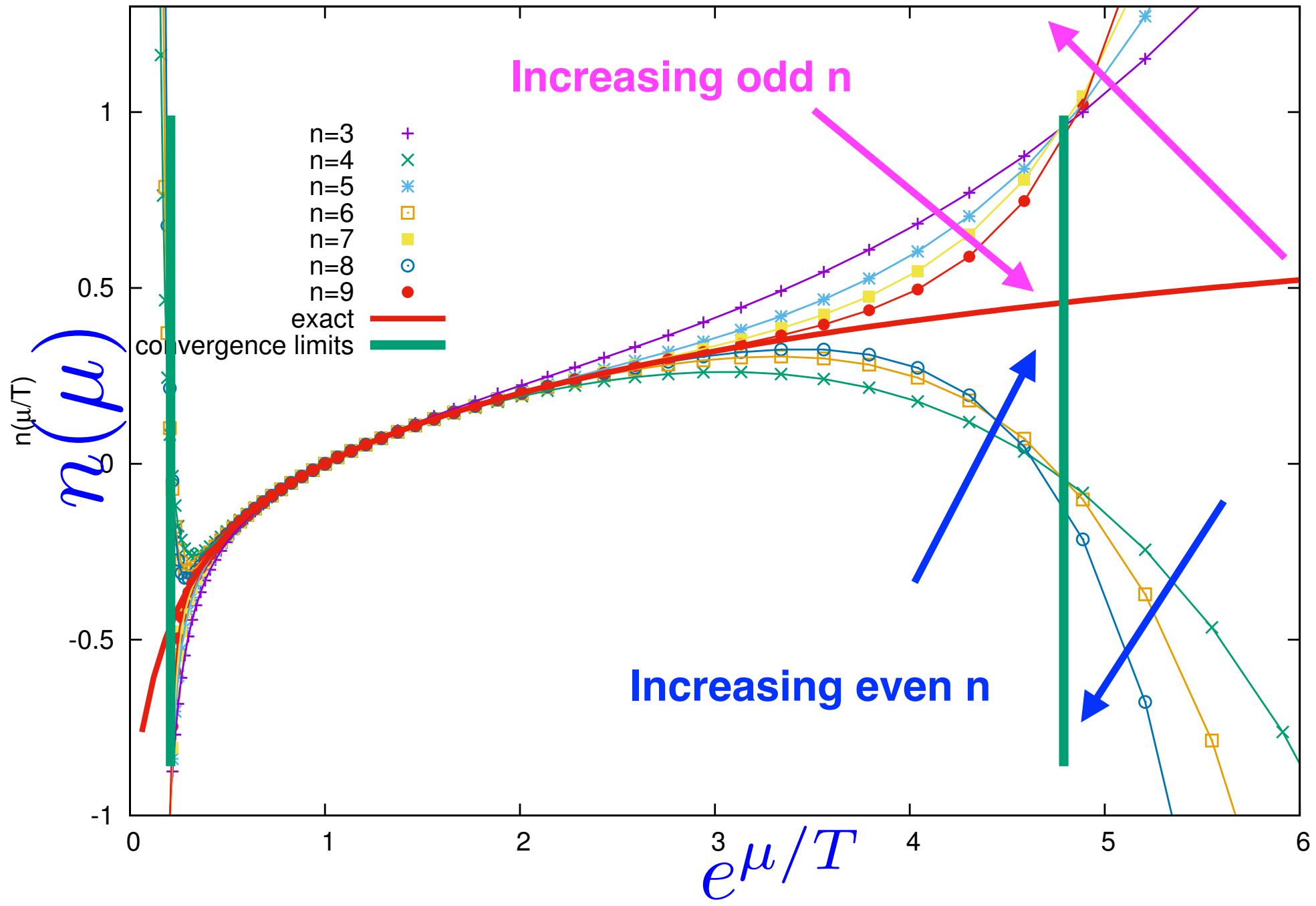
## Partial sums reveal the radius of convergence

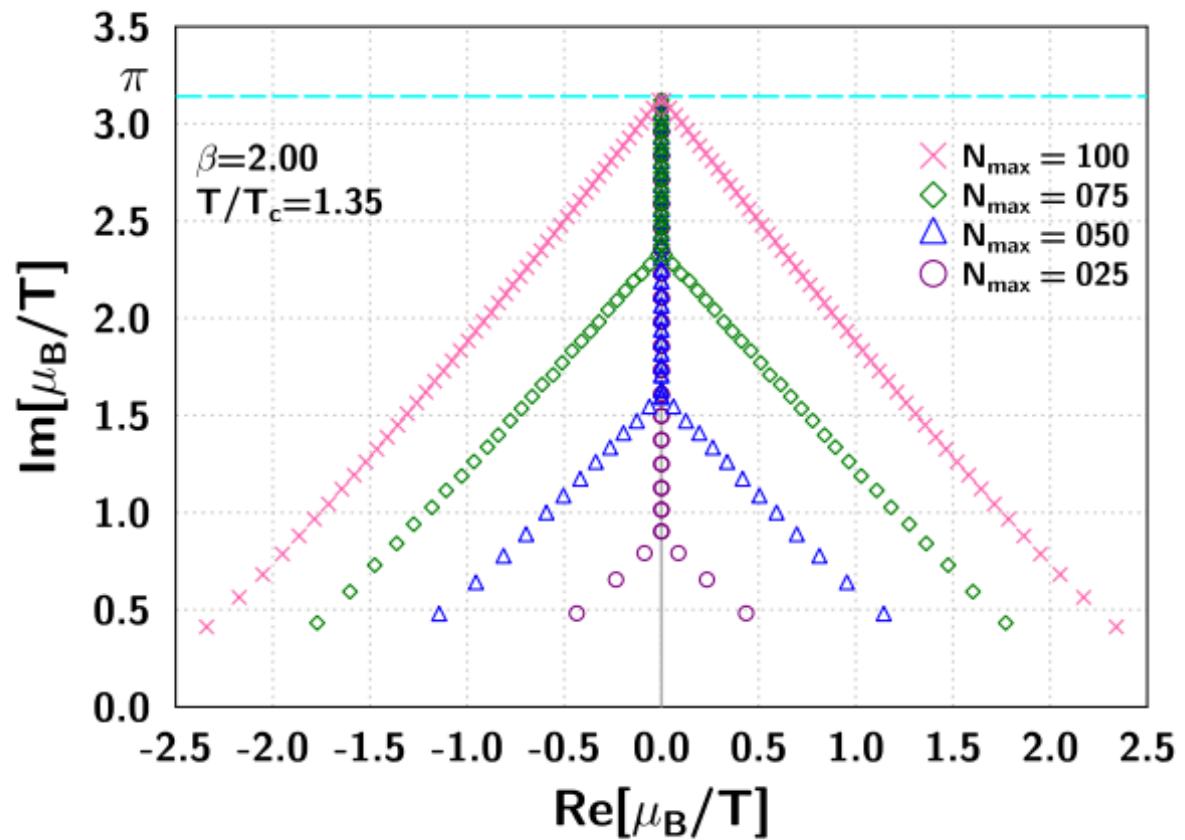
$$n^n(\mu_I) = \sum_1^n a_k \sin(k\mu_I/T)$$

↓

Analytic continuation

$$n^n(\mu) = \sum_1^n a_k \sinh(\mu/Tk)$$



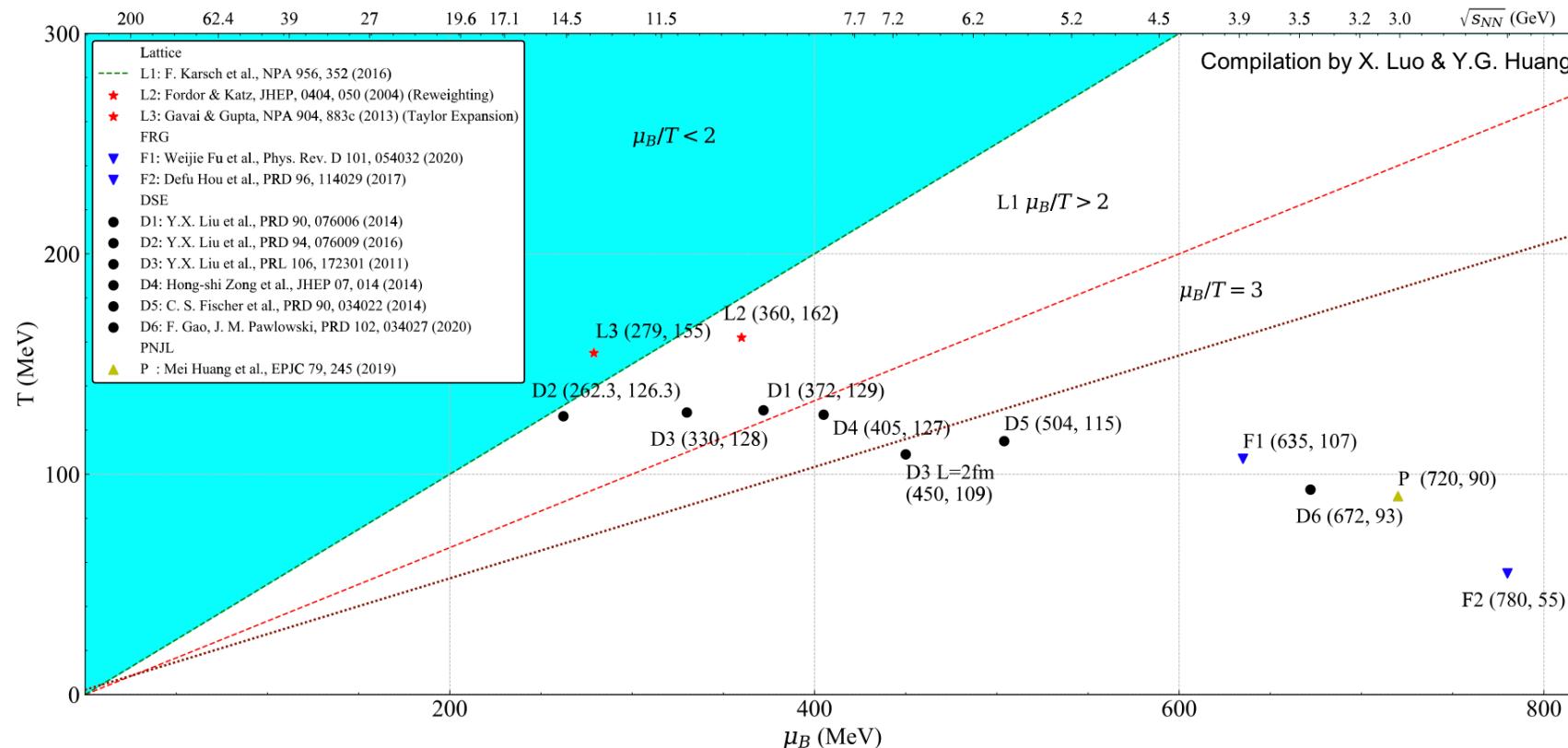


Nakamura et al.



# Location of CP : Theoretical Prediction

*Preliminary collection from Lattice, DSE, FRG and PNJL (2004-2020)*



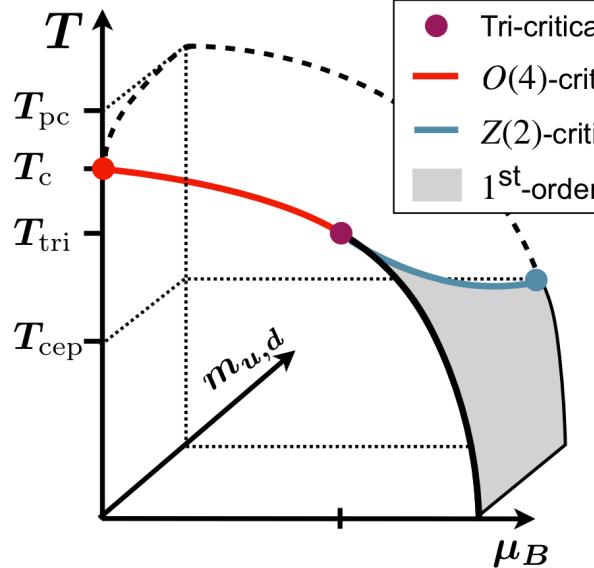
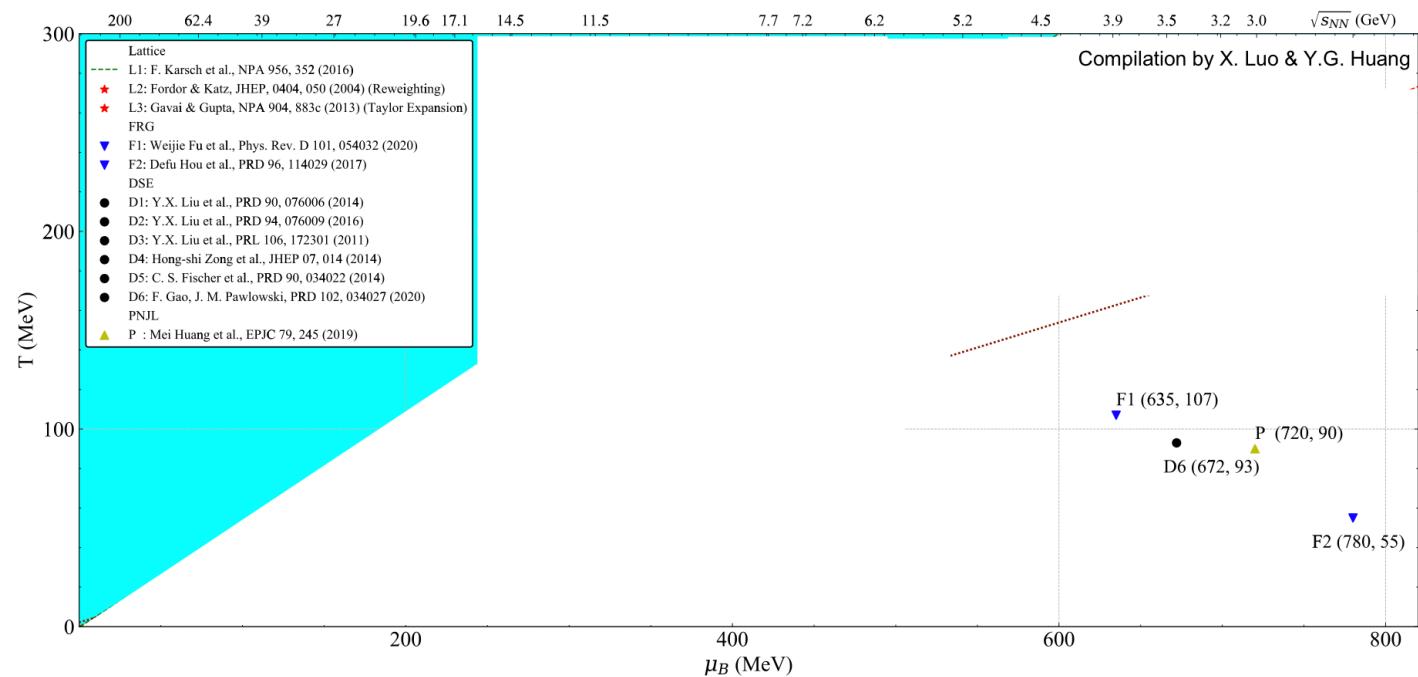
Large uncertainties for the estimation of CP location.



RHIC-BES Seminar Oct. 6th 2020

## Location of CP : Theoretical Prediction

Preliminary collection from Lattice, DSE, FRG and PNJL (2004-2020)



Large uncertainties for the estimation of CP location.

# ‘Essential’ issues in Lattice Field Theory for Extreme QCD

- Euclidean formulation : no real time (still, a sign problem..)

- Monte Carlo evolution : no positive determinant for

$$\begin{aligned}\theta &\neq 0 \\ \mu_q &\neq 0\end{aligned}$$

# ‘Essential’ issues in Lattice Field Theory for Extreme QCD

- Euclidean formulation : no real time (still, a sign problem..)

- Monte Carlo evolution : no positive determinant for

$$\begin{aligned}\theta &\neq 0 \\ \mu_q &\neq 0\end{aligned}$$

Plenty of room for improvements,  
dialogue with other approaches  
and new ideas -

Enjoy Extreme QCD2023!

## References

Axions and Lattice QCD

Guy Moore <https://arxiv.org/abs/1709.09466>

MpL and Anton ~Trunin <https://arxiv.org/pdf/2005.06547.pdf>

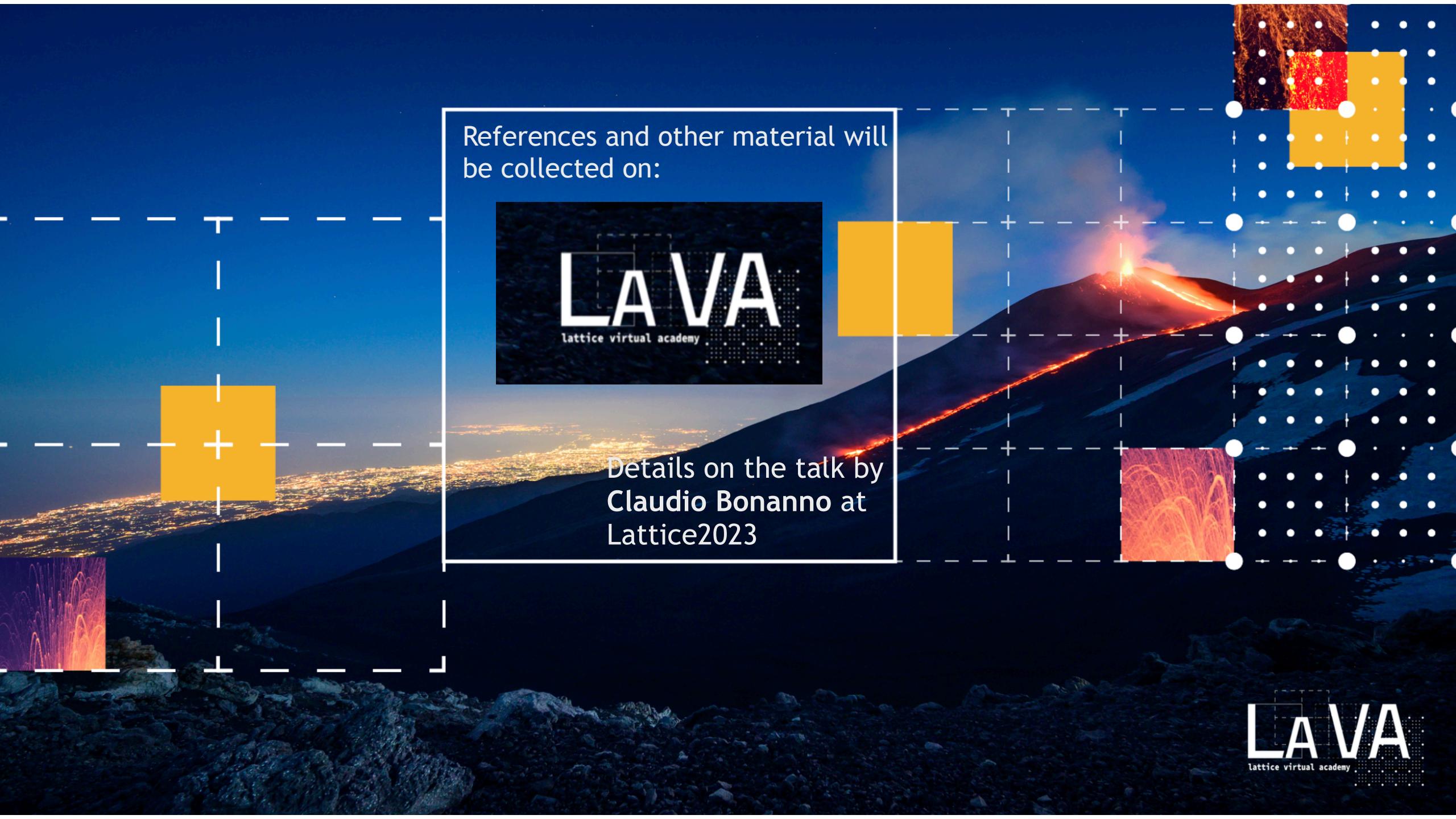
Lattice Finite Density

MpL <https://arxiv.org/pdf/hep-lat/0401021.pdf>

G. Aarts <https://arxiv.org/pdf/1512.05145.pdf>

An overview of Lattice QCD for Phase Transitions (community paper of Strong-2020 network)

G. Aarts et al. <https://arxiv.org/pdf/2301.04382.pdf>



References and other material will  
be collected on:



Details on the talk by  
**Claudio Bonanno** at  
Lattice2023