New developments in Relativistic Hydrodynamics for Heavy-Ion Collisions: Lecture II

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Outline

LECTURE I

Outline

LECTURE II

3 Glauber model for initial stages

3.1 Various approaches too IS

Various approaches to initial stages

colour glass condensate, AdS/CFT correspondence, Glauber model, ...

Various approaches to initial stages

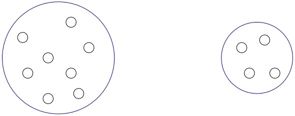
GLAUBER MODEL

Roy Glauber
receiving Nobel Prize
Stockholm, Dec. 2005.

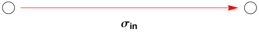
Independent collisions

Glauber model treats a nucleus-nucleus collision as a multiple nucleon-nucleon collision process.

In the Glauber model, **the nucleon distributions in nuclei are random and given by the nuclear density profiles**

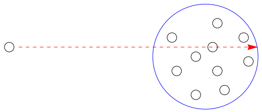


whereas the **elementary nucleon-nucleon collision is characterized by the total inelastic cross section** $\sigma_{\rm in}$.

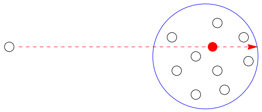


Independent collisions

Initially, the Glauber model was applied only to elastic collisions. In this case a nucleon does not change its properties in the individual collisions, so all nucleon interactions can be well described by the same cross section.



Applying the Glauber model to inelastic collisions, we assume that after a single inelastic collision an excited nucleon-like object is created that interacts basically with the same inelastic cross section with other nucleons.



From NN to AA collisions

Thickness function for the nucleus-nucleus collision

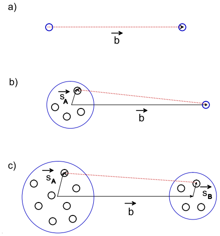
$$T_{AB}(\mathbf{b}) = \int dz_A \int d^2s_A\, \rho_A(\mathbf{s}_A, z_A) \int dz_B \int d^2s_B\, \rho_B(\mathbf{s}_B, z_B)\, t(\mathbf{b} + \mathbf{s}_B - \mathbf{s}_A), \tag{1}$$

with the corresponding normalization condition

$$\int d^2b\, T_{AB}(\mathbf{b}) = 1. \tag{2}$$

The quantity $T_{AB}(\mathbf{b})\, \sigma_{\rm in}$ is the **averaged** probability that a nucleon-nucleon collision takes place in a nucleus-nucleus collision characterized by the impact parameter **b**. In the limit $t(\mathbf{b}) \to \delta^{(2)}(\mathbf{b})$ we may write

$$T_{AB}(\mathbf{b}) = \int d^2s_A\, T_A(\mathbf{s}_A)\, T_B(\mathbf{s}_A - \mathbf{b}). \tag{3}$$



Binary collisions

In a more symmetric form we have

$$T_{AB}(\mathbf{b}) = \int d^2s\, T_A\left(\mathbf{s} + \frac{1}{2}\mathbf{b}\right)\, T_B\left(\mathbf{s} - \frac{1}{2}\mathbf{b}\right). \tag{4}$$

The nucleus-nucleus thickness function $T_{AB}(\mathbf{b})$ can be used to calculate the probability of having n inelastic binary nucleon-nucleon collisions in a nucleus-nucleus collision at the impact parameter $\mathbf{b}$.

$$P(n; AB; \mathbf{b}) = \begin{pmatrix} AB \\ n \end{pmatrix} \left[1 - T_{AB}(\mathbf{b})\,\sigma_{\rm in}\right]^{AB-n} \left[T_{AB}(\mathbf{b})\,\sigma_{\rm in}\right]^n. \tag{5}$$

The result for the average number of the collisions is

$$\boxed{\overline{n}(AB; \mathbf{b}) = AB\, T_{AB}(\mathbf{b})\, \sigma_{\rm in}.} \tag{6}$$

Binary collisions

The total probability of an inelastic nuclear collision is the sum over n from $n = 1$ to $n = AB$

$$P_{\rm in}\left(AB; \mathbf{b}\right) = \sum_{n=1}^{AB} P\left(n; AB; \mathbf{b}\right) = 1 - \left[1 - T_{AB}\left(\mathbf{b}\right)\sigma_{\rm in}\right]^{AB}. \quad (7)$$

From (7), by integrating over the impact parameter space, one may obtain the **total inelastic cross section for the collision of the two nuclei *A* and *B***

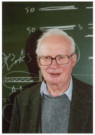
$$\boxed{\sigma_{\rm in}^{AB} = \int d^2b \left(1 - \left[1 - T_{AB}\left(\mathbf{b}\right)\sigma_{\rm in}\right]^{AB}\right).} \quad (8)$$

Using the thickness function for the Au+Au collisions we find $\sigma_{\rm in}^{\rm AuAu}$ = 6.8 b for $\sigma_{\rm in}$ = 30 mb and $\sigma_{\rm in}^{\rm AuAu}$ = 7.0 b for $\sigma_{\rm in}$ = 40 mb. We note that those cross sections are larger than the geometric cross section $\sigma_{\rm geo}^{\rm AuAu} = 4\pi R^2 \approx 5\pi A^{2/3}$ = 5.3 b. This is due to the tails of the Woods-Saxon distribution, which make possible that a nucleon-nucleon collision occurs in the nuclear collision at the impact parameter b larger than $2R$.

Wounded nucleons

A. Bialas, M. Bleszynski, W. Czyz
Multiplicity Distributions in Nucleus-Nucleus Collisions at High-Energies

Nucl.Phys. B111 (1976) 461



The Glauber model can be used also to calculate the number of the participants. To be more precise we distinguish between the **participants which may interact elastically** and the **participants which interact only inelastically**. The latter are called the **wounded nucleons**, and their number is given by the formula

$$\begin{aligned} \overline{w}(A;B;\mathbf{b}) \quad &= \quad A\int d^2s\, T_A(\mathbf{b}-\mathbf{s})\left(1-\left[1-\sigma_{\rm in}\, T_B(\mathbf{s})\right]^B\right) \\ &+ B\int d^2s\, T_B(\mathbf{b}-\mathbf{s})\left(1-\left[1-\sigma_{\rm in}\, T_A(\mathbf{s})\right]^A\right). \qquad (9) \end{aligned}$$

ZEROTH ORDER PARADIGM: **SOFT AND HARD PROCESSES SCALE WITH THE NUMBER OF WOUNDED NUCLEONS AND BINARY COLLISIONS, RESPECTIVELY**

Input for hydrodynamics

Since the final multiplicities are are determined mainly by the number of wounded nucleons, it is reasonable to assume that the initial entropy density of the thermalized system is proportional to the density of wounded nucleons

$$\sigma_{\rm i}(\mathbf{x}_\perp) \propto \overline{w}\,(\mathbf{x}_\perp)$$

short thermalization/equilibration time, $\tau_{\rm therm} < 1$ fm

Perfect-fluid

PERFECT-FLUID MODELS

Hydrodynamic equations

main assumption: system is in local thermal equilibrium – conservation of the baryon number (and other conserved charges), energy and momentum

$$
\begin{aligned}
\partial_\mu N^\mu &= 0 & \qquad N^\mu &\equiv n u^\mu \\
\partial_\mu T^{\mu\nu}_{id} &= 0 & \qquad T^{\mu\nu}_{id} &\equiv \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}\mathcal{P}
\end{aligned}
$$

- 6 independent variables: $(\mathcal{E}, \mathcal{P}, n, u^\mu(3))$
- 6 equations (equation of state $\mathcal{E}(n, \mathcal{P})$)

Equation of state

in ultra relativistic collisions we may neglect the baryon number

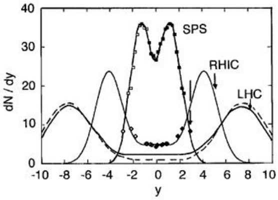
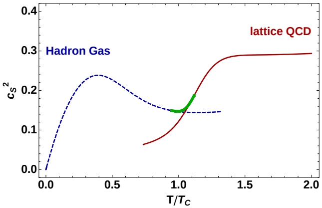


FIG. 1: Net-proton rapidity spectra in the Relativistic Diffusion Model (RDM), solid curves: Transition from the double-humped shape at SPS energies of $\sqrt{s_{NN}}$ = 17.3 GeV to a broad midrapidity valley in the three-sources model at RHIC (200 GeV) and LHC (5.52 TeV). See [11] for details.

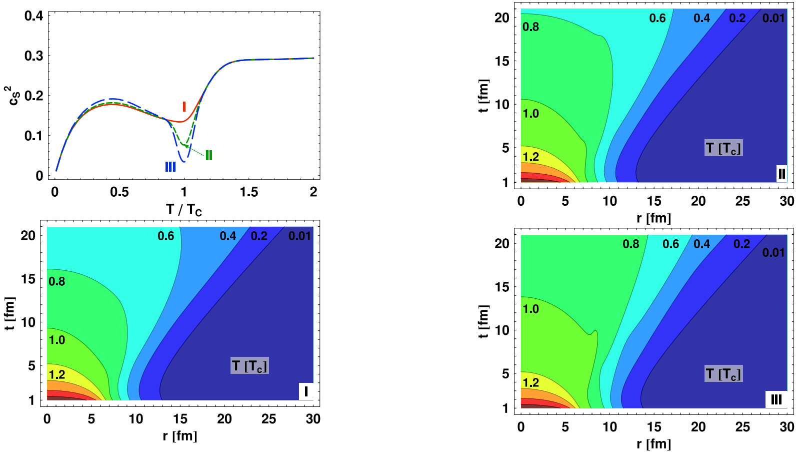
R. Kuiper and G. Wolschin, Annalen Phys. 16, 67 (2007)



M. Chojnacki, WF, Acta Phys.Pol. B38 (2007) 3249

Equation of state

EOS can be checked experimentally by looking at the HBT correlations that give information about the space-time extensions of the system



further evidence from complete hydro simulations
W. Broniowski, M. Chojnacki, WF, A. Kisiel, PRL 101 (2008) 022301

Landau model

E. Fermi, High-energy nuclear events, Prog. Theor. Phys. 5 (1950) 570
I. Y. Pomeranchuk, On the theory of multiple particle production in a single collision Dokl. Akad. Nauk Ser. Fiz. 78 (1951) 889
L. D. Landau, On the multiparticle production in high-energy collisions Izv. Akad. Nauk SSSR 17 (1953) 51

One-dimensional expansion of matter along the collision axis z, the equations of **perfect-fluid relativistic hydrodynamics with zero baryon chemical potential**,

$$\begin{aligned} u^0\partial_0(Tu^0) + u^3\partial_3(Tu^0) &= \partial^0 T, \\ u^0\partial_0(Tu^3) + u^3\partial_3(Tu^3) &= \partial^3 T. \end{aligned} \tag{10}$$

Equations (10) together with the normalization condition for four-velocity give

$$\frac{\partial}{\partial x^0}\left(Tu_3\right) = \frac{\partial}{\partial x^3}\left(Tu_0\right), \tag{11}$$

which means that Tu_0 and Tu_3 may be written as the derivatives of a potential field Φ_{L},

$$Tu_0 = -\partial_0\Phi_{\mathrm{L}}, \quad Tu_3 = -\partial_3\Phi_{\mathrm{L}}. \tag{12}$$

Landau model

The next convenient step is to perform the **Legendre transformation** and switch from the potential Φ_L to the potential χ defined as

$$\chi = \Phi_\mathrm{L} + T u^0 t - T u^3 z. \tag{13}$$

The total differential of χ is (with $u^0 = \cosh\vartheta$, $u^3 = \sinh\vartheta$)

$$\begin{aligned} d\chi &= \left(u^0 t - u^3 z\right) dT + tT\, du_0 - Tz\, du^3 \\ &= (t \cosh\vartheta - z \sinh\vartheta)\, dT + (t \sinh\vartheta - z \cosh\vartheta)\, T d\vartheta. \end{aligned} \tag{14}$$

From the entropy conservation

$$\frac{\partial}{\partial t}(\mathcal{S} \cosh\vartheta) + \frac{\partial}{\partial z}(\mathcal{S} \sinh\vartheta) = 0. \tag{15}$$

we come to the single, partial differential equation for χ,

$$\frac{1}{\mathcal{S}} \frac{d\mathcal{S}}{dT} \left(\frac{\partial \chi}{\partial T} - \frac{1}{T} \frac{\partial^2 \chi}{\partial \vartheta^2} \right) + \frac{\partial^2 \chi}{\partial T^2} = 0. \tag{16}$$

Landau model

Further simplifications may be achieved if we introduce

$$Y = \ln\left(\frac{T}{T_{\rm i}}\right). \tag{17}$$

In this way we obtain the **Khalatnikov equation**

$$\boxed{\frac{\partial^2\chi}{\partial\vartheta^2} - c_{\rm s}^2\frac{\partial^2\chi}{\partial Y^2} + (c_{\rm s}^2 - 1)\frac{\partial\chi}{\partial Y} = 0.} \tag{18}$$

For constant sound velocity it becomes a partial differential equation with constant coefficients. For example, in the case $c_{\rm s}^2 = 1/3$,

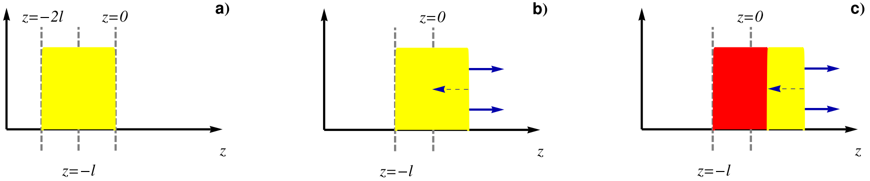
$$3\frac{\partial^2\chi}{\partial\vartheta^2} - \frac{\partial^2\chi}{\partial Y^2} - 2\frac{\partial\chi}{\partial Y} = 0. \tag{19}$$

Landau model

(a) at the beginning, matter forms a highly compressed disk of the width $\Delta = 2l$, because of the reflection symmetry with respect to the plane $z = -l$ we may consider $z \geq -l$,

(b) next, the evolution of matter consists of the expansion into vacuum (indicated by the two solid arrows) and the rarefaction wave entering the system (indicated by the dashed arrow),

(c) after $t_0 = l/c_\mathrm{s}$, when the rarefaction wave hits the plane $z = -l$, the evolution of the central region becomes quite complicated, it consists of the incident rarefaction wave and the reflected waves, in the outer region the simple Riemann solution always holds, and should be matched to the non-trivial solution found by Khalatnikov.



Landau model

Landau finds that in the leading order of magnitude the function χ is given by

$$\chi = -l T_{\rm i} \exp\left[-Y + \sqrt{Y^2 - c_{\rm s}^2 \vartheta^2}\right] \tag{20}$$

where

$$l = \frac{\Delta}{2} = R\, \frac{2 m_{\rm N}}{\sqrt{s_{\rm NN}}}. \tag{21}$$

Landau further argues that the end of one-dimensional motion takes place when

$$t^2 - z^2 = R_{\rm f}^2 \quad \text{(connection between } t \text{ and } z,\ R_{\rm f} \approx 2R\text{)}. \tag{22}$$

New variable L is introduced

$$L = -2Y + \sqrt{Y^2 - \vartheta^2/3} \quad \text{(connection between } T \text{ and } \vartheta\text{)}. \tag{23}$$

$$L = \ln\left(\frac{R_{\rm f}}{\Delta}\right) = \ln\left(\frac{\sqrt{s_{\rm NN}}}{2 m_{\rm N}}\right). \tag{24}$$

Landau model

Rapidity profile of temperature,

$$T = T_\mathrm{i} \exp\left[-\frac{1}{3}\left(2L - \sqrt{L^2 - \vartheta^2}\right)\right]. \tag{25}$$

Similarly, we find the **rapidity profile of the entropy density**,

$$\mathcal{S} = \mathcal{S}_\mathrm{i} \exp\left[-2L + \sqrt{L^2 - \vartheta^2}\right]. \tag{26}$$

In order to calculate the rapidity distribution we first consider a thin slice of the fluid. Since the fluid element moves in the center-of-mass frame with the velocity $v = \tanh\vartheta$, may write

$$dS = \pi R^2\, R_\mathrm{f}\, \mathcal{S}_\mathrm{i} \exp\left[-2L + \sqrt{L^2 - \vartheta^2}\right] d\vartheta. \tag{27}$$

For $\vartheta \ll L$ we find

$$\frac{dS}{d\vartheta} = \pi R^2\, R_\mathrm{f}\, e^{-L}\, \mathcal{S}_\mathrm{i} \exp\left(-\frac{\vartheta^2}{2L}\right). \tag{28}$$

if we identify the entropy density with the particle density

$$\boxed{\frac{dN}{d\mathrm{y}} = \frac{N}{(2\pi L)^{1/2}} \exp\left(-\frac{\mathrm{y}^2}{2L}\right)} \tag{29}$$

GAUSSIAN RAPIDITY DISTRIBUTION!

Bjorken model

J. D. Bjorken, Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, Phys.Rev. D27 (1983) 140

Landau model – initial conditions are specified for a given laboratory time, considered in the center-of-mass frame, when the matter is highly compressed and at rest.

Landau's description looses one aspect of high-energy hadronic collisions – fast particles are produced later and further away from the collision center than the slow ones. It is possible to account for this effect in the hydrodynamic description by imposing special initial conditions. This idea was proposed and studied by Bjorken.

The Bjorken hydrodynamic model was based on the assumption that the rapidity distribution of the charged particles, $dN_{\rm ch}/d{\rm y}$, is constant in the mid-rapidity region. This fact means that the central region is invariant under Lorentz boosts along the beam axis. This in turn implies that the longitudinal flow has the form $v_z = z/t$ and all thermodynamic quantities characterizing the central region depend only on the longitudinal proper time $\tau = \sqrt{t^2 - z^2}$ and the transverse coordinates x and y.

Bjorken model

One-dimensional hydrodynamic model where thermodynamic variables are functions of the longitudinal proper time only,

$$\mathcal{E} = \mathcal{E}(\tau), \qquad \mathcal{P} = \mathcal{P}(\tau), \qquad T = T(\tau), \qquad \text{etc.} \tag{30}$$

The initial conditions for the hydrodynamic expansion are imposed along the hyperbola of the constant proper time

$$\sqrt{t^2 - z^2} = \tau_{\rm i}. \tag{31}$$

In this way one accounts for the time dilation effects characterizing the particle production. The fluid four-velocity field has the form

$$u^\mu = \frac{1}{\tau}(t, 0, 0, z) = \gamma\left(1, 0, 0, \frac{z}{t}\right). \tag{32}$$

This form implies

$$\partial_\mu u^\mu = \frac{1}{\tau}. \tag{33}$$

Bjorken model

We identify all kinds of rapidities

$$\mathrm{y} = \text{arctanh}\left(\mathrm{v}_{\parallel}\right) = \text{arctanh}\left(\frac{z}{t}\right), \tag{34}$$

$$t = \tau \cosh \mathrm{y}, \qquad z = \tau \sinh \mathrm{y} \tag{35}$$

baryon number density in the central region is negligible, entropy conservation gives

$$\partial_\mu \left(\mathcal{S} u^\mu\right) = \frac{d\mathcal{S}\left(\tau\right)}{d\tau} + \frac{\mathcal{S}\left(\tau\right)}{\tau} = 0. \tag{36}$$

the solution of this equation is

$$\mathcal{S}\left(\tau\right) = \mathcal{S}\left(\tau_{\mathrm{i}}\right) \frac{\tau_{\mathrm{i}}}{\tau} \tag{37}$$

for the energy density we find

$$\frac{d\mathcal{E}}{\mathcal{E} + \mathcal{P}} = -\frac{d\tau}{\tau} \tag{38}$$

This equation can be solved if the equation of state is known. For ultra-relativistic particles $\mathcal{P} = \lambda \mathcal{E}$ (with $\lambda = c_s^2 = 1/3$) and we find

$$\mathcal{E}\left(\tau\right) = \mathcal{E}\left(\tau_{\mathrm{i}}\right) \left(\frac{\tau_{\mathrm{i}}}{\tau}\right)^{1+\lambda}. \tag{39}$$

Bjorken model

in the reference frame where the fluid element is at rest we have

$$d^3x = d^2x_\perp \tau \, d\mathrm{y} \tag{40}$$

thus the entropy contained in the interval $d\mathrm{y}$ around $\mathrm{y} = 0$ is

$$dS = \tau \, \mathcal{S}(\tau) \int d^2x_\perp \, d\mathrm{y} \tag{41}$$

$$\frac{d}{d\tau}\left[\frac{dS}{d\mathrm{y}}\right] = \int d^2x_\perp \frac{d}{d\tau}\left[\tau \, \mathcal{S}(\tau)\right] = 0 \tag{42}$$

simple estimate of the energy density achieved in the central region, we first calculate the entropy density $\mathcal{S}(\tau_\mathrm{i})$ at the time τ_i when the hydrodynamic description starts

$$\mathcal{S}(\tau_\mathrm{i}) = \frac{1}{\tau_\mathrm{i}\,\mathcal{A}} \frac{dS}{d\mathrm{y}}(\mathrm{y}=0) \approx \frac{3.6}{\tau_\mathrm{i}\,\mathcal{A}} \frac{dN}{d\mathrm{y}}(\mathrm{y}=0)\,. \tag{43}$$

Here we used the result $\mathcal{S}/n = 2\pi^4/(45\zeta(3)) \approx 3.6$, the transverse overlap area of the two colliding nuclei $\mathcal{A} = \pi\left(3A/(4\pi\rho_0)\right)^{2/3}$, where A is the mass number of the nuclei and ρ_0 is the nuclear saturation density,

$$\mathcal{E}(\tau_\mathrm{i}) = \left[\frac{1215\,\mathcal{S}^4(\tau_\mathrm{i})}{128\,g\,\pi^2}\right]^{\frac{1}{3}} \approx \frac{5.4}{g^{\frac{1}{3}}}\left[\frac{1}{\tau_\mathrm{i}\,\mathcal{A}}\frac{dN}{d\mathrm{y}}\right]^{\frac{4}{3}}. \tag{44}$$

Viscous fluid dynamics

VISCOUS (DISSIPATIVE) HYDRO MODELS

Relativistic Navier-Stokes equations

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

\# of unknowns: 5 + 6 ($\mathcal{E}, \mathcal{P}, u^\mu(3), \Pi, \pi^{\mu\nu}(5)$)
\# of equations: 4 + 1 (equation of state $\mathcal{E}(\mathcal{P})$)
we need 6 extra equations - different methods possible

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta, \qquad \theta = \partial_\mu u^\mu \text{– expansion scalar}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu}, \qquad \sigma^{\mu\nu}\text{– shear flow tensor}$$

T, u^μ are the only hydrodynamic variables, $u^\mu{}_\mu = 1$
kinetic coefficients: $\tau_\Pi \beta_\Pi = \zeta \rightarrow$ bulk viscosity, $\tau_\pi \beta_\pi = \eta \rightarrow$ shear viscosity

Israel-Stewart equations

Israel-Stewart equations — $\Pi, \pi^{\mu\nu}$ promoted to dynamic variables — non-hydrodynamic modes are introduced with the appropriate relaxation times τ_Π, τ_π

W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Annals of Physics 118 (1979) 341

$$\begin{aligned} \dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} \end{aligned}$$

1) HYDRODYNAMIC EQUATIONS DESCRIBE BOTH HYDRODYNAMIC AND NON-HYDRODYNAMIC MODES

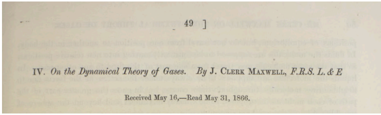
2) HYDRODYNAMIC MODES CORRESPOND TO GENUINE HYDRODYNAMIC BEHAVIOR

3) NON-HYDRODYNAMIC MODES (TERMS) SHOULD BE TREATED AS REGULATORS OF THE THEORY

4) NON-HYDRODYNAMIC MODES GENERATE ENTROPY

Old Maxwell's idea?

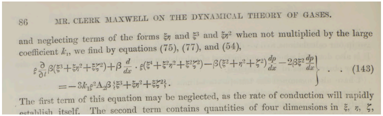
J. Clerk Maxwell, On the Dynamical Theory of Gases, Phil. Trans. R. Soc. Lond. 147 (1867) 49-88, Eq. (143)



[49]

IV. *On the Dynamical Theory of Gases.* By J. CLERK MAXWELL, *F.R.S. L. & E.*

Received May 16,—Read May 31, 1866.



86 MR. CLERK MAXWELL ON THE DYNAMICAL THEORY OF GASES.

and neglecting terms of the forms $\xi\eta$ and ξ^3 and $\xi\eta^2$ when not multiplied by the large coefficient k_1, we find by equations (75), (77), and (54),

$$\left.\begin{array}{l} \rho\frac{\partial}{\partial t}\beta(\xi^3+\xi\eta^2+\xi\zeta^2)+\beta\frac{d}{dx}\cdot\rho(\xi^4+\xi^2\eta^2+\xi^2\zeta^2)-\beta(\xi^2+\eta^2+\zeta^2)\frac{dp}{dx}-2\beta\xi^2\frac{dp}{dx} \\ =-3k_1\rho^2\mathrm{A}_2\beta\{\xi^3+\xi\eta^2+\xi\zeta^2\}. \end{array}\right\} \quad . \quad . \quad (143)$$

The first term of this equation may be neglected, as the rate of conduction will rapidly establish itself. The second term contains quantities of four dimensions in ξ, η, ζ,

C. Cattaneo, Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée

Comptes Rendus 247(4) (1958) 431

What is the real problem with the relativistic Navier-Stokes theory?

P. Kovtun: Existence of gapped modes (non-hydrodynamic modes) with frequencies that have a positive imaginary part. These are unphysical modes. These (UV) modes are outside of the validity regime of the low-energy hydro approximation.

HYDRODYNAMIC VS. NON-HYDRODYNAMICMODES

perturbations $\sim \exp(-i\omega_k t + ikx)$, hydro modes $\omega_k \to 0$ for $k \to 0$, non-hydro modes $\omega_k \to \text{const} \neq 0$ for $k \to 0$

instability for $\mathrm{Im}(\omega_k) > 0$.

Most popular fix is the Israel-Stewart theory: the hydro equations are coupled to extra UV degrees of freedom, which in turn kill the unphysical UV modes.

Analogy to quantum field theory.

Rayski-Pauli-Villars regularization: introduction of physical heavy particles whose presence regulates the UV behavior.

ISRAEL-STEWART ENTROPY PRODUCTION ANALYSIS

First-law of thermodynamics + extensivity of energy, entropy, particle (baryon) number

$$dE = TdS - PdV + \mu dN \quad \longrightarrow \quad d\varepsilon = Td\sigma + \mu dn, \tag{45}$$

$$E + PV = TS + \mu N \quad \longrightarrow \quad \varepsilon + P = T\sigma + \mu n. \tag{46}$$

We can make it looking „covariant"

$$S^\mu_{\rm eq} = \mathcal{S} u^\mu = P\beta^\mu - \xi N^\mu_{\rm eq} + \beta_\lambda T^{\lambda\mu}_{\rm eq}, \tag{47}$$

$$\beta^\mu = \frac{u^\mu}{T}, \quad \beta = \sqrt{\beta^\lambda \beta_\lambda} = \frac{1}{T}, \quad \xi = \frac{\mu}{T}. \tag{48}$$

$$dS^\mu_{\rm eq} = -\xi dN^\mu_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq}, \quad d(P\beta^\mu) = N^\mu_{\rm eq} d\xi - T^{\lambda\mu}_{\rm eq} d\beta_\lambda. \tag{49}$$

Near equilibrium entropy current

$$S^\mu = P\beta^\mu - \xi N^\mu + \beta_\lambda T^{\lambda\mu} + Q^\mu \tag{50}$$

and its growth

$$\partial_\mu S^\mu = -\left(N^\mu - N^\mu_{\rm eq}\right)\partial_\mu \xi + \left(T^{\lambda\mu} - T^{\lambda\mu}_{\rm eq}\right)(\partial_\mu \beta_\lambda) + \partial_\mu Q^\mu. \tag{51}$$

$$T\partial_\mu S^\mu \quad = \quad \frac{nT}{\mathcal{E}+P} q^\lambda \partial_\lambda \left(\frac{\mu}{T}\right) + \pi^{\lambda\mu}\partial_{<\mu}u_{\lambda>} - \Pi\theta + T\partial_\mu Q^\mu . \tag{52}$$

The requirement that the entropy production is positive (with $Q^\mu = 0$) gives:
heat flow (not discussed before)

$$q^\lambda = -\frac{\lambda n T^2}{\mathcal{E}+P}\nabla^\lambda \left(\frac{\mu}{T}\right), \tag{53}$$

bulk pressure

$$\Pi = -\zeta\partial_\mu u^\mu = -\zeta\theta, \tag{54}$$

shear stress tensor

$$\pi_{\lambda\mu} = 2\eta\partial_{<\lambda}u_{\mu>} = 2\eta\sigma_{\lambda\mu}. \tag{55}$$

BEYOND NAVIER-STOKES — a non-zero Q^μ proposed second-order theory

$$S^\mu = S^\mu_{\rm NS} - \left(\beta_0 \Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\lambda}\pi^{\nu\lambda}\right) \frac{u^\mu}{2T} - \alpha_0 \frac{\Pi q^\mu}{T} + \alpha_1 \frac{\pi^{\mu\nu} q_\nu}{T}. \tag{56}$$

$$\begin{aligned} T\partial_\mu S^\mu = & -\Pi \left[\theta + \beta_0 D\Pi + T\Pi \partial_\lambda \left(\frac{\beta_0 u^\lambda}{2T}\right) + \alpha_0 \partial_\nu q^\nu\right] \\ & - q^\mu \left[\nabla_\mu \ln T - Du_\mu - \beta_1 Dq_\mu - Tq_\mu \partial_\lambda \left(\frac{\beta_1 u^\lambda}{2T}\right) - \alpha_1 \partial_\nu \pi^\nu{}_\mu + T\nabla_\mu \left(\frac{\alpha_0 \Pi}{T}\right)\right] \\ & + \pi^{\mu\nu} \left[\sigma_{\mu\nu} - \beta_2 D\pi_{\mu\nu} - T\pi_{\mu\nu}\partial_\lambda \left(\frac{\beta_2 u^\lambda}{2T}\right) + T\nabla_\mu \left(\frac{\alpha_1 q_\nu}{T}\right)\right]. \end{aligned} \tag{57}$$

MIS equations

Müller-Israel-Stewart or Muronga-Israel-Stewart (MIS)

I. Müller, *Zum Paradoxon der Wärmeleitungstheorie*, Zeit. f. Physik 198 (1967) 329

A. Muronga, *Second-order dissipative fluid dynamics for ultra relativistic nuclear collisions*, PRL 88 (2002) 062302

$$
\begin{aligned}
\dot{\Pi} + \frac{\Pi}{\tau_\Pi} &= -\beta_\Pi \theta - \frac{\zeta T}{2\tau_\Pi} \Pi \, \partial_\lambda \left(\frac{\tau_\Pi}{\zeta T} u^\lambda \right) \\
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{\eta T}{2\tau_\pi} \pi^{\mu\nu} \, \partial_\lambda \left(\frac{\tau_\pi}{\eta T} u^\lambda \right)
\end{aligned}
$$

BRSSS equations

Baier, Romatschke, Son, Starinets, Stephanov (BRSSS)
symmetry arguments due to Lorentz and conformal symmetry, ...

R. Baier, P. Romatschke, D.T. Son, A. O. Starinets, M. A. Stephanov,

Relativistic viscous hydrodynamics, conformal invariance, and holography, JHEP 0804 (2008) 100

$$\partial_\mu T^{\mu\nu}_{vis} = 0 \qquad T^{\mu\nu}_{vis} = \mathcal{E} u^\mu u^\nu - \Delta^{\mu\nu}(\mathcal{P} + \Pi) + \pi^{\mu\nu}$$

$$\begin{aligned} \Pi &= 0 \\ \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} &= 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{\lambda_1}{\tau_\pi \eta^2}\pi^{\langle\mu}_{\ \lambda}\pi^{\nu\rangle\lambda} \end{aligned}$$

(+ terms including vorticity and curvature)

DNMR equations

Denicol, Niemi, Molnar, Rischke (DNMR)
simultaneous expansion in the Knudsen number and inverse Reynolds number

approach based on the kinetic theory

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \delta_{\Pi\Pi} \Pi \theta \underline{+ \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu} \omega^{\nu\rangle\gamma} - \delta_{\pi\pi} \pi^{\mu\nu} \theta \underline{- \tau_{\pi\pi} \pi_\gamma^{\langle\mu} \sigma^{\nu\rangle\gamma} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu}}$$

the version of equations shown is for RTA version of the Boltzmann kinetic equation, with neglected vorticity, for standard form of the collision term additional terms (with new kinetic coefficients) appear

shear-bulk coupling $\eta - \zeta$

Review of different viscous-fluid frameworks

Bjorken viscous expansion
$\phi = -\pi^y_y$ component of the shear stress tensor (the only independent one)
energy-momentum conservation

$$\tau \dot{\epsilon} = -\frac{4}{3}\epsilon + \phi$$

BRSSS

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1 \phi^2}{2\eta^2} - \frac{4\tau_\pi \phi}{3\tau} - \phi \tag{58}$$

DNMR with RTA kinetic equation

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{38}{21}\frac{\tau_\pi \phi}{\tau} - \phi \tag{59}$$

MIS with RTA kinetic equation

$$\tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{4\tau_\pi \phi}{3\tau} - \phi \tag{60}$$