Testing Bell Inequalities at the LHC

Alan Barr

Particle Physics Seminar, University of Oxford

$10^{\rm th}$ November 2022

AJB, Phys.Lett.B 825 (2022) 136866 — 2106.01377 [hep-ph]
AJB, P. Caban, J.Rembieliński — 2204.11063 [quant-ph]
R.Ashby-Pickering, AJB, A.Wierzchucka — 2209.13990 [quant-ph]
C.Altomonte, AJB, ORA-2022

Outline

- Motivation
- Bell inequalities
- $H \to W^+W^-$ as a Bell experiment

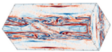
[interlude]

- Some tools from quantum information theory
- Bell inequalities & particle decays
- The LHC as a laboratory for testing quantum foundations

[conclusion]

Motivation

Interesting physics $\neq$ 'new' physics $\neq$ beyond-SM physics



ON THE COVER

Heating of Magnetically Dominated Plasma by Alfvén-Wave Turbulence

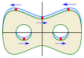
February 14, 2022

Three-dimensional kinetic simulation of the onset of relativistic wave turbulence in the collision of two magnetic shear waves. Selected for a Viewpoint in *Physics*.

Joonas Nättilä and Andrei M. Beloborodov

Phys. Rev. Lett. **128**, 075101 (2022)

Issue 7 Table of Contents | More Covers



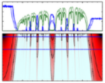
Physics NEWS AND COMMENTARY

A Quantized Surprise from Fermi Surface Topology

February 16, 2022

The quantized conductance of a two-dimensional electron gas can reflect its Fermi surface topology.

Synopsis on:
C. L. Kane
Phys. Rev. Lett. **128**, 076801 (2022)



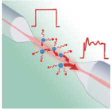
EDITORS' SUGGESTION

Chaotic Diffusion in Delay Systems: Giant Enhancement by Time Lag Modulation

Laminar chaotic diffusion is found in systems with delayed nonlinearity, accompanied by a reduction of the effective dimensionality.

Tony Albers, David Müller-Bender, Lukas Hille, and Günter Radons

Phys. Rev. Lett. **128**, 074101 (2022)



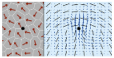
EDITORS' SUGGESTION

Collective Radiative Dynamics of an Ensemble of Cold Atoms Coupled to an Optical Waveguide

An ensemble of cold atoms is coherently coupled in a controlled way to a tapered optical fiber, demonstrating collective effects in this system.

Riccardo Pennetta *et al.*

Phys. Rev. Lett. **128**, 073601 (2022)



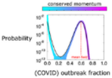
Physics NEWS AND COMMENTARY

Extending and Contracting Cells

February 15, 2022

Cell-substrate interactions explain a difference in behavior between individual cells and tissues on a surface.

Synopsis on:
Andrew Killeen, Thibault Bertrand, and Chiu Fan Lee
Phys. Rev. Lett. **128**, 078001 (2022)



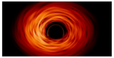
EDITORS' SUGGESTION

Outbreak Size Distribution in Stochastic Epidemic Models

An analytical approach to stochastic epidemic models shows that the statistics of extreme outbreaks depend on an infinite number of minimum-action paths, and that extreme outbreaks define a new class of rare processes for discrete-state stochastic systems.

Jason Hindes, Michael Assaf, and Ira B. Schwartz

Phys. Rev. Lett. **128**, 078301 (2022)



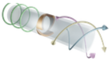
Physics NEWS AND COMMENTARY

Illuminating Black Holes through Turbulent Heating

February 14, 2022

Predictions indicate that it should be possible to directly identify how turbulence heats a given black hole's plasma from the spectrum of that plasma's radiation.

Viewpoint on:
Joonas Nättilä and Andrei M. Beloborodov
Phys. Rev. Lett. **128**, 075101 (2022)



Physics NEWS AND COMMENTARY

Waves in a Solid Imitate Twisted Light

February 11, 2022

Waves of vibration moving through the walls of a pipe can carry orbital angular momentum that could be used for several purposes, according to new theoretical work.

Focus story on:
G. J. Chaplain, J. M. De Ponti, and R. V. Craster
Phys. Rev. Lett. **128**, 064301 (2022)

Some of the old problems are amongst the deepest...

EINSTEIN ATTACKS QUANTUM THEORY

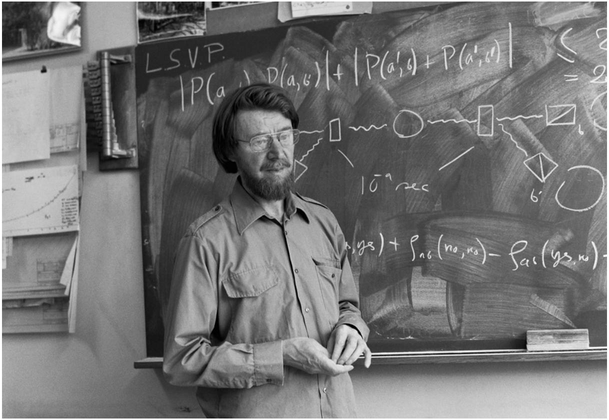
Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

New York Times, May 4 **1935**, reporting on Einstein-Podolsky-Rosen paper,
"*Can Quantum-Mechanical Description of Physical Reality Be Considered Complete*"

. . . and they are experimentally accessible



J.S. Bell *'On the Einstein Podolsky Rosen paradox'* (1964)

Bell inequalities

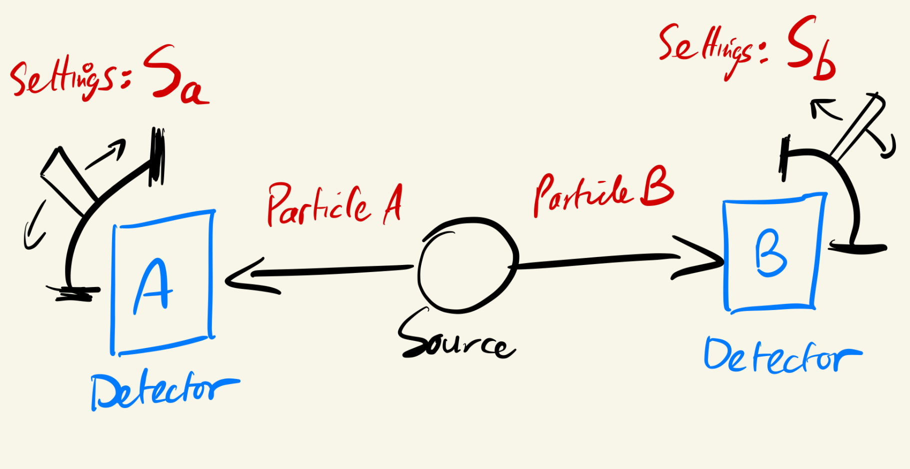
J.S. Bell showed that if we assume:

- **locality**: that there are no **physical influences** traveling faster than the speed of light and
- **realism**: objects have **physical properties** independent of **measurement**

then **correlations** in **measurement outcomes** from two distant observers must necessarily obey an inequality

Rephrasing of Giustina et al 2015

The textbook case – apparatus



(Ensemble of similarly-prepared systems)

Quantum systems – initial thoughts

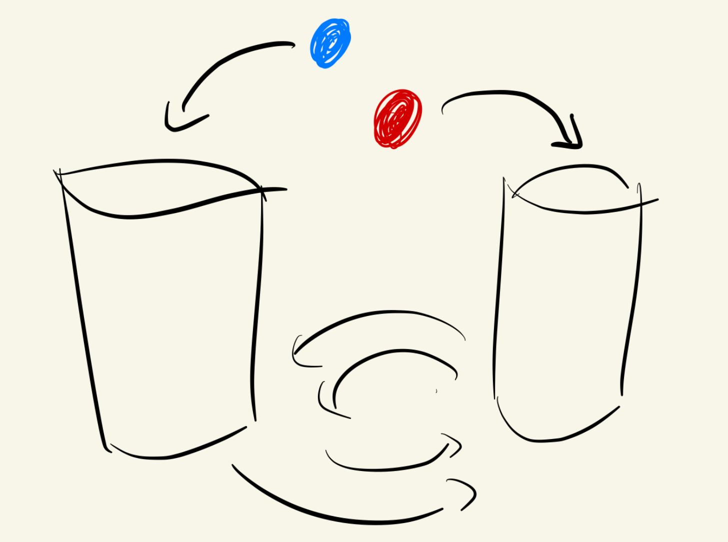
Take a perfectly entangled **Bell state** of two spin-half particles:

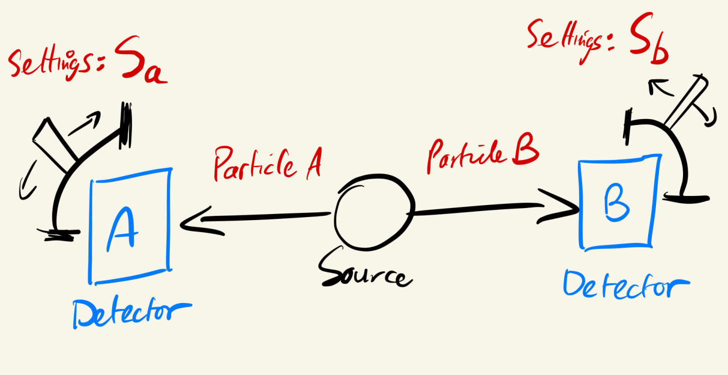
$$|\Psi_{+}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

The measurements of spin for each system separately are uncertain, nevertheless:

- After measuring S_z system A we can tell with certainty about outcome of measuring S_z on system B
- even though A and B may be widely separated

Q: Is this property 'spooky action at a distance'?





We can also change our measurement settings: S_A and S_B

We might expect the probabilities of outcomes at A to depend on:

- the measurement settings S_A at A
- some properties $\vec{\lambda}$ of the AB system

The CHSH Bell inequality

Clauser, Horne, Shimony & Holt (1969)

- The two experiments, A and B, each have two possible outcomes:
  { $+1$ or -1 }
  $E(a, b)$ is the expectation value of the product
- Each experiment has two possible settings :
  { primed or unprimed }
- Calculate the following function of the correlated expectations:

$$\mathcal{I}_2 = E(a,b) - E(a,b') + E(a',b) + E(a',b')$$

The local realism formalism

Assume that there is a well-defined correlation function for the pair of measurement outcomes:

$$P(S_A, S_B) \equiv \int \mathrm{d}\vec{\lambda} \;\; a(S_A, \vec{\lambda}) \;\; b(S_B, \vec{\lambda}) \;\; P(\vec{\lambda})$$

May depend on 'hidden' variables $\vec{\lambda}$ which have a PDF $P(\vec{\lambda})$

Assumptions

- $a(S_A, \vec{\lambda})$ does **not** depend on S_B
- $b(S_B, \vec{\lambda})$ does **not** depend on S_A
- $P(\vec{\lambda})$ does **not** depend on S_A nor on S_B

Demand that marginal probabilities for measurements of A and B are non-negative

The CHSH Bell inequality

$$\mathcal{I}_2 = E(a,b) - E(a,b') + E(a',b) + E(a',b')$$

$$\text{Local realism} \implies |\mathcal{I}_2| \leq 2$$

Quantum Mechanics violates the CHSH inequality

Find CHSH expectation values for the Bell state

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

Quantum mechanics:

- allows values of $\mathcal{I}_2$ larger than two
- up to the Cirel'son bound of $2\sqrt{2}$
- in conflict with local realism

Maximum violation for e.g. $a = 0^\circ$, $a^{'} = 45^\circ$, $b = 22.5^\circ$ and $b^{'} = 67.5^\circ$

Empirical tests of Bell Inequalities

Physical systems

- photons
- ions
- superconducting systems
- nitrogen vacancy centres

Also in pairs of three-outcome measurements using photons

Classic experiments

- Freedman and Clauser (1972)
- Aspect et al.'s experiments (1981 & 1982)
- Zeilinger et al. (1998)
- Three 'loophole-free' tests of 2015:
  Hensen et al., Shalm et al., Giustina (et Zeilinger) et al.

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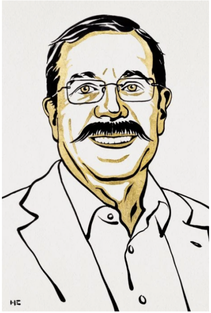
The Nobel Prize in Physics 2022



Ill. Niklas Elmehed © Nobel Prize Outreach

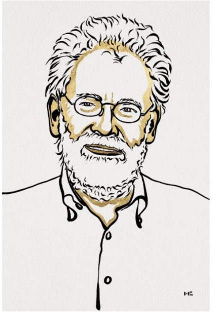
Alain Aspect

Prize share: 1/3

Ill. Niklas Elmehed © Nobel Prize Outreach

John F. Clauser

Prize share: 1/3



Ill. Niklas Elmehed © Nobel Prize Outreach

Anton Zeilinger

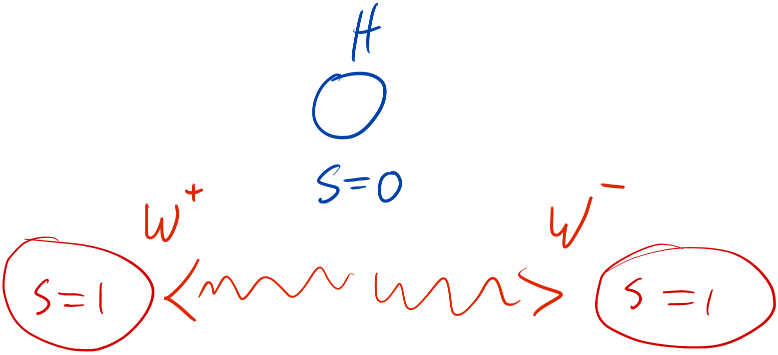
Prize share: 1/3

Results?

Violation of Bell inequalities in each case

In the tested systems and at the tested energies

$H \to W^+W^-$ as a Bell experiment


H
S=0
W+
W−
S=1
S=1

Spin in the $H \to W^+W^-$ decay

The Higgs boson is a scalar, while $W^\pm$ bosons are vector bosons.

- $H \to W^+W^-$ decays produce pairs of W bosons in a singlet spin state
- In the narrow-width and non-relativistic approximations:

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}\Big(\,|+\rangle\,|-\rangle - |0\rangle\,|0\rangle + |-\rangle\,|+\rangle\,\Big)$$

This is also a Bell state

- Bell inequality tests deep in the realm of QFT
- Many orders of magnitude different in energy, length, timescale from existing measurements

W bosons are their own polarimeters

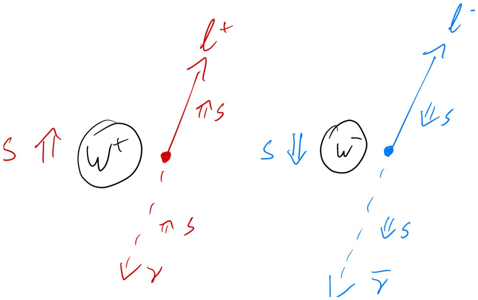
$V - A$ decays

SU(2) weak force is chiral

$$W^+ \to \ell_R^+ + \nu_L$$
$$W^- \to \ell_L^- + \bar{\nu}_R$$

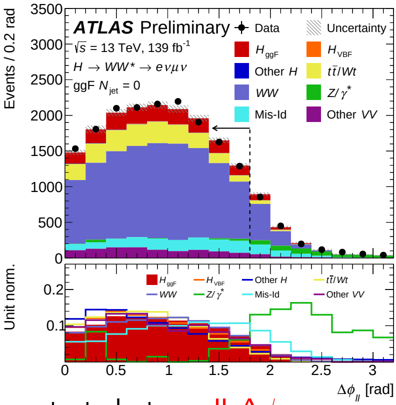
Decay of a $W^\pm$ boson is equivalent to a measurement of its spin along the axis of the emitted lepton

Getting the directions right



- ℓ^+ is emitted preferentially along spin direction (of W^+)
  ℓ^- is emitted preferentially against spin direction (of W^-)
- The $W^\pm$ spins are in different directions
- So the two leptons prefer to go in the same direction as each other

$\ell^+\ell^-$ azimuthal correlations in $H \to W^+W^-$



- Higgs signal concentrated at small $\Delta\phi_{\ell\ell}$
- Used e.g. in discovery searches

Q: Can we measure **Bell inequality** in this system?

\begin{interlude}

discovernorthernireland.com

Belfast City Council has declined to name a street after one of Northern Ireland's most eminent scientists.

Belfast-born, John Stewart Bell who died in 1990, is regarded as one of the 20th Century's greatest physicists.

The council received an application to name a street in Titanic Quarter after Mr Bell.

However, the council rejected the proposal as it has "traditionally avoided using the names of people" when deciding on street names.

Only two streets in the city have been named after individuals since the 1960s: Prince Edward Park in 1962 and Prince Andrew Park in 1987.

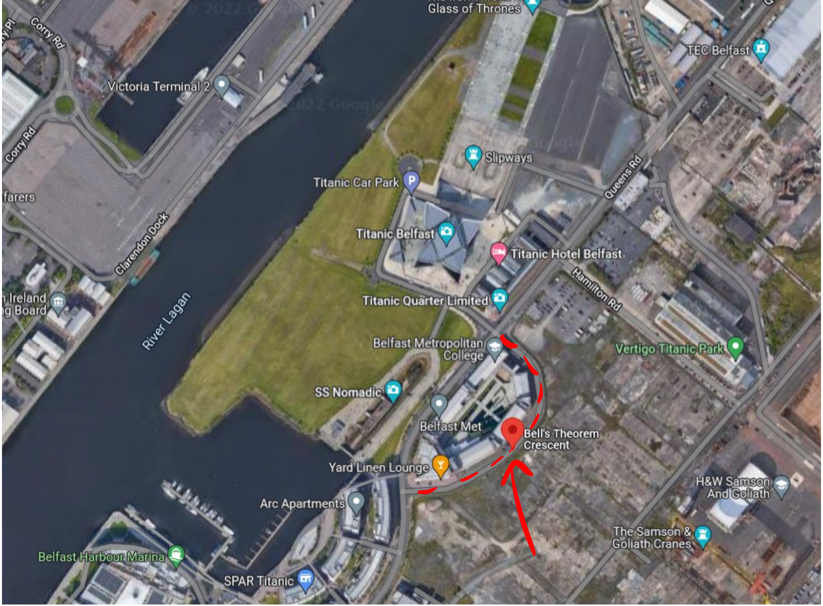
Titanic Quarter Ltd had applied to name a currently unnamed street beside Belfast Metropolitan College as John Bell Crescent.

Bell was born in Belfast in 1928 to a family from a poor background.

He was the only one of his siblings to stay at school over the age of 14, and his family could not afford to send him to one of the city's grammar schools.

Instead he attended Belfast Technical High School, now Belfast Metropolitan College, and then entered Queen's University.

BBC news 19 February 2015


Glass of Thrones
TEC Belfast
Victoria Terminal 2
Corry Rd
Slipways
Titanic Car Park
farers
Queens Rd
Clarendon Dock
Titanic Belfast
Titanic Hotel Belfast
Hamilton Rd
Titanic Quarter Limited
River Lagan
Belfast Metropolitan
College
Vertigo Titanic Park
SS Nomadic
Belfast Met
Bell's Theorem
Crescent
Yard Linen Lounge
H&W Samson
And Goliath
Arc Apartments
The Samson &
Goliath Cranes
Belfast Harbour Marina
SPAR Titanic

Google maps

\end{interlude}

Some tools from quantum information theory

The density matrix ρ

- A fully-characterised quantum system is described by a ket $|\psi\rangle$
- Expectation values of measurement operator $\mathcal{A}$ are given by

$$\langle\psi| \mathcal{A} |\psi\rangle$$

- A more general, not-fully-characterised, quantum system is described by a **density matrix** ρ

$$\rho = \sum_i p_i |\psi\rangle_i \langle\psi|_i$$

  p_i is classical probability
  ρ is a non-negative hermitian operator with unit trace
- Expectation values for operator $\mathcal{A}$ for ρ are given by:

$$\langle\mathcal{A}\rangle = \mathrm{tr}(\rho\mathcal{A})$$

A Bell operator for the CHSH inequality

In QM we can create a Bell operator

$$\mathcal{B}_{\mathrm{CHSH}} = \hat{\mathbf{n}}_1 \cdot \boldsymbol{\sigma} \otimes (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_4) \cdot \boldsymbol{\sigma} + \hat{\mathbf{n}}_3 \cdot \boldsymbol{\sigma} \otimes (\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_4) \cdot \boldsymbol{\sigma}$$

such that:

$$\mathcal{I}_2 = \langle \mathcal{B}_{\mathrm{CHSH}} \rangle = \mathrm{tr}(\rho \mathcal{B}_{\mathrm{CHSH}})$$

- $\mathcal{B}_{\mathrm{CHSH}}$ acts on the two-particle Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- ρ is the two-particle density matrix
- $\boldsymbol{\sigma}$ are the Pauli matrices
- $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3, \hat{\mathbf{n}}_4$ are unit vectors in $\mathbb{R}^3$
  (directions of measurements of a, b, a', b' respectively)

CHSH for spin-1

We can build a generalised CHSH Bell operator for pairs of spin-1 QM systems:

$$\mathcal{B}_{\mathrm{CHSH}} = \hat{\mathbf{n}}_1 \cdot \mathbf{S} \otimes (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_4) \cdot \mathbf{S} + \hat{\mathbf{n}}_3 \cdot \mathbf{S} \otimes (\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_4) \cdot \mathbf{S}$$

where now

$$S_{x,y,z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Local realist expectations

Measurement outcomes: $\in \{+1, 0, -1\}$

The additional 0-outcome can only dilute CHSH expectation value

$\Longrightarrow$ CHSH Bell inequality $|\mathcal{I}_2| \leq 2$ still must be satisfied in LR theory

Finding ρ for two W bosons: 'quantum state tomography'

We need to get the terms in ρ for **two** spin-1 particles

$$\rho \supset \sum_{i,j=1}^{3} \tfrac{1}{4} d_{ij} \, S_i \otimes S_j$$

ρ from data: WW system

We can get the density matrix parameters from the data:

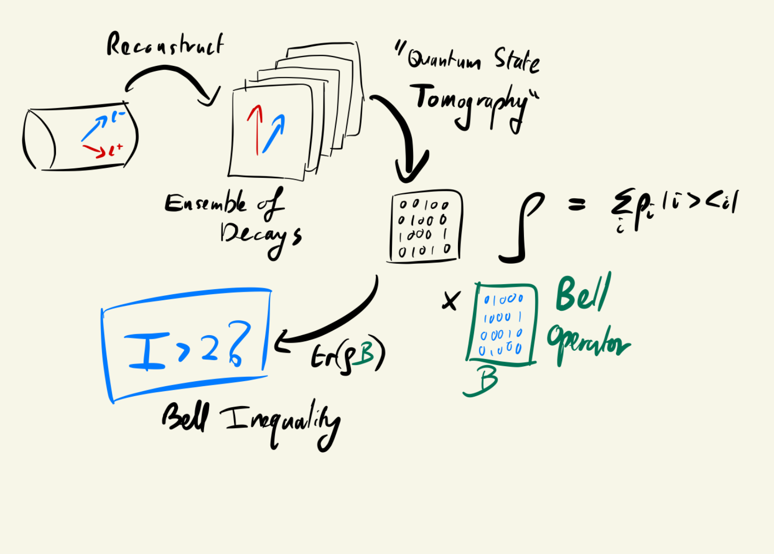
$$d_{ij} = \mathrm{tr}(\rho S_i \otimes S_j) = -4 \left\langle \xi_i^- \xi_j^+ \right\rangle_{\mathrm{av}}$$

$$\xi_i^{\pm} \equiv \cos \theta_{\ell^{\pm}}$$

Then re-write our CHSH Bell inequality as:

$$|\hat{\mathbf{n}}_1 \cdot d \cdot (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_4) + \hat{\mathbf{n}}_3 \cdot d \cdot (\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_4)| \leq 2$$

Summary of technique



Testing Bell inequalities in $H \to W^+W^-$

Simulate $pp \to H \to WW^* \to \ell^+ \nu_\ell \ell^- \bar{\nu}_\ell$

Monte Carlo

Generate $gg \to H \to \ell^+ \nu_\ell \ell^- \bar{\nu}_\ell$ in Madgraph Monte Carlo simulation (10^6 pp events with $\sqrt{s} = 13\,\text{TeV}$)

- Idealise: no detector, assume we can reconstruct $W^\pm$ rest frames
- Cut out the e^+e^- and $\mu^+\mu^-$ events to remove $H \to ZZ^*$
- Place a lower bound $m_W^<$ on the m_W masses
- Optimise over the CHSH measurement directions [1]
- The CHSH Bell inequality is violated iff

$$\mathcal{I}_2 > 2$$

[1] see appendix

. . . drum roll . . .

Results – Part I

$\mathcal{I}_2 > 2$?

$m_W^<$ [GeV]	0	20	30	40	50
$\mathcal{I}_2$	1.78	1.91	1.96	1.94	1.95

No violation! :o(

Why no violation?

No violation of CHSH predicted in Caban et al. (2008)

- For pairs of massive vector bosons $\mathcal{I}_2 \leq 2$
- CHSH is designed for pairs of **qu<u>b</u>its**
- CHSH is **suboptimal** for spin-1 as the 0 outcomes dilute the correlations

Confirmed for relativistic QFT in AJB, P. Caban, J. Rembieliński, 2204.11063

Qutrits

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}\big(|+\rangle |-\rangle - |0\rangle |0\rangle + |-\rangle |+\rangle \big)$$

- This is a maximally entangled state of two **qutrits**
- $|\psi\rangle_{AB} \in (\mathbb{C}^{3})^2$
- Basis for each qutrit $\{0, 1, 2\}$

[On the board: qutrits vs 3-state systems]

The CGLMP Qutrit inequality

Collins Gisin Linden Massar Popescu (2002)

The optimal Bell inequality for pairs of **qutrits**

CGLMP function

$$
\begin{aligned}
\mathcal{I}_3 = P(A_1 = B_1) + P(B_1 = A_2 + 1) \\
+ P(A_2 = B_2) + P(B_2 = A_1) \\
- P(A_1 = B_1 - 1) - P(B_1 = A_2) \\
- P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1).
\end{aligned}
$$

$P(A_i = B_j + k)$ is the probability that A_i and B_j differ by k mod 3

The advantage of being a particle physicist

Use the 8 trace-orthogonal **Gell-Mann** matrices λ_i

The *WW* density matrix (9x9 matrix, 80 free parameters):

$$\rho = \frac{1}{9} I_3 \otimes I_3 + \sum_{i=1}^{8} f_i \lambda_i \otimes I_3 + \sum_{j=1}^{8} g_j I_3 \otimes \lambda_j + \sum_{i,j=1}^{8} h_{ij} \lambda_i \otimes \lambda_j ,$$

CGLMP operator

$$\mathcal{B}^{xy}_{\mathrm{CGLMP}} = -\frac{2}{\sqrt{3}} \left(S_x \otimes S_x + S_y \otimes S_y \right) + \lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5$$

where

$$S_x = \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_6) \quad \text{and} \quad S_y = \frac{1}{\sqrt{2}}(\lambda_2 + \lambda_7).$$

. . . after some matrix algebra, angular integrals and trace relationships . . .

What to measure

CGLMP (qutrit) inequality from data

$$\mathcal{I}_3 = \mathrm{tr}(\rho \mathcal{B}^{xy}_{\mathrm{CGLMP}}) = \frac{8}{\sqrt{3}} \left\langle \xi^+_x \xi^-_x + \xi^+_y \xi^-_y \right\rangle_{\mathrm{av}} + 25 \left\langle \left((\xi^+_x)^2 - (\xi^+_y)^2\right) \left((\xi^-_x)^2 - (\xi^-_y)^2\right) \right\rangle_{\mathrm{av}} + 100 \left\langle \xi^+_x \xi^+_y \xi^-_x \xi^-_y \right\rangle_{\mathrm{av}}$$

So is this Bell inequality violated in (Madgraph simulated) data?

$$\mathcal{I}_3 \leq 2?$$

Results – Part II

$\mathcal{I}_3 > 2 \implies$ violation

$m_W^<$ [GeV]	20	30	40	50
$\mathcal{I}_3^{\rm xyz}$	2.76	2.81	2.82	2.77

YES!

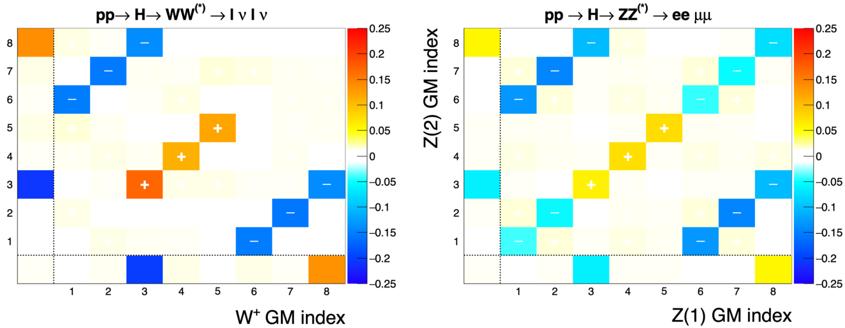
CAVEAT: In the absence of backgrounds, smearings, . . . all under investigation

Confirmed for relativistic QFT in AJB, P. Caban, J. Rembieliński, 2204.11063

The LHC as a laboratory for testing quantum foundations

[. . . as are other (pp, ee, $\mu\mu$, . . .) colliders]

Density matrices from simulated Higgs boson decays



Almost perfect qutrit Bell states

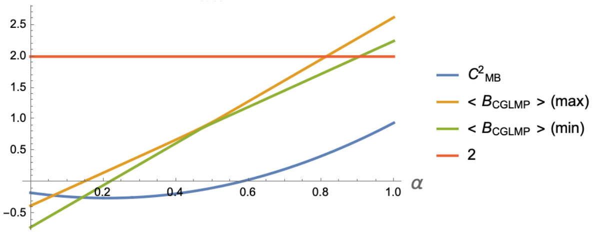
Can perform Bell Tests, entanglement tests, . . .

Sensitive to exchange symmetry of the identical Z bosons

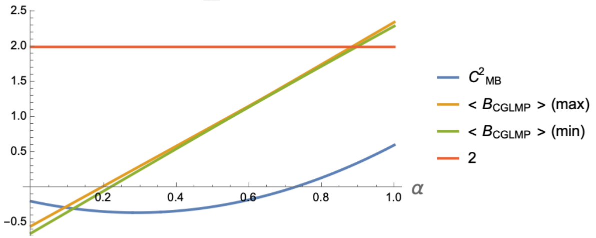
Sensitive to physics beyond the Standard Model

2209.13990

Signal purity needed?



$H \to WW^*$

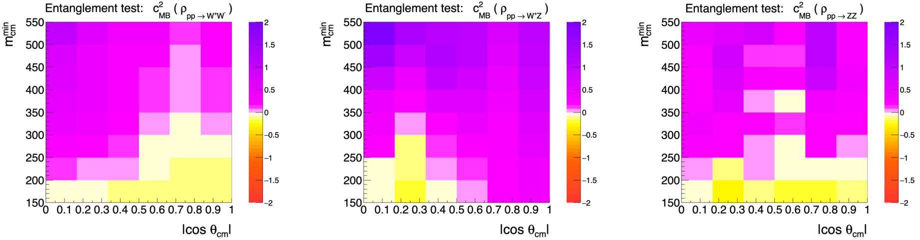


$H \to ZZ^*$

$$\rho = \alpha \rho_{H \to VV^*} + (1 - \alpha) \rho_{pp \to VV^*}$$

2209.13990

Entanglement in diboson continuum?

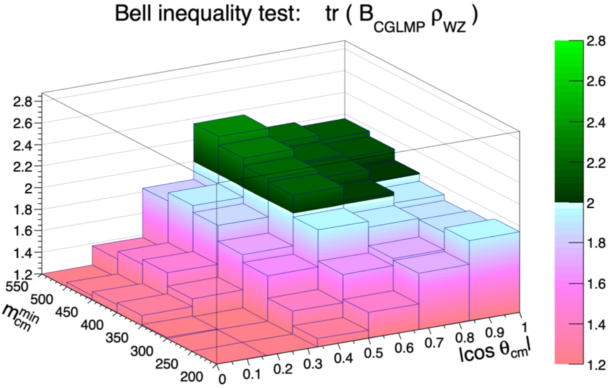


WW, WZ, ZZ, as a function of m_{VV} and $\cos\theta$

Pink/Purple means entangled

2209.13990

Bell violation in WZ continuum?



Green means Bell-inequality violating

2209.13990

So what?

If we observe Bell Inequality violation:

A very different sort of Bell test:

- 12 orders of magnitude higher energy that existing tests (shorter time scale, shorter length scale. . .)
- In 'self-measuring' quantum system
- Deep in the realm of quantum field theory (virtual particles)
- in qutrit rather than qubit systems

If we don't observe Bell Inequality violation (when we expect to):

We have an even more surprising and consequential result . . .

(and yes, it's also a good way to find new fields)

The LHC: a laboratory for probing quantum foundations

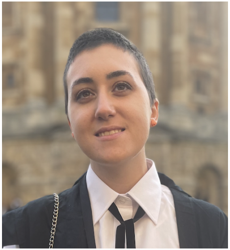
Weak bosons are wonderful quantum probes

- Quantum spin self measurement via chiral weak decays
- Expect entanglement and even Bell inequality violation
- Spin density matrix can be reconstructed from angular distributions ('tomography')

A wide-ranging quantum programme is possible @ LHC

- Local realism tests at $\sim 10^{12}$ higher energy
- Probes of quantum measurement
- Exchange symmetry and distinguishablity
- All in an unexplored region

Co-authors



Clelia Altomonte

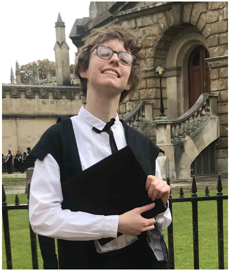
MMathPhys project

$\rightarrow$ *PhD @ Kings*

Rachel Ashby-Pickering

MPhys project

$\rightarrow$ *PhD @ Warwick*



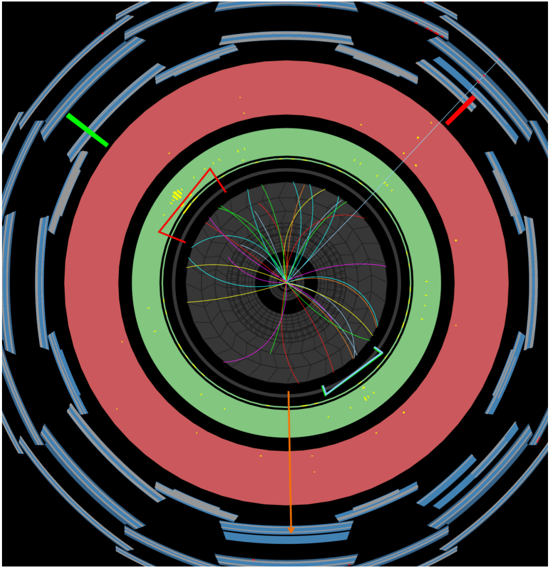
Agnieszka Wierzchucka

Undergrad summer student

(Merton college)

Paweł Caban, *Faculty, Univ. Lodz, Poland*
Jakub Rembieliński, *Faculty, Univ. Lodz, Poland*

EXTRAS



Recent related works by other authors

- Gong, Parida, Tu and Venugopalan, *"Measurement of Bell-type inequalities and quantum entanglement from Λ-hyperon spin correlations at high energy colliders"*, 2107.13007
- Severi, Boschi, Degli Esposti, Maltoni and Sioli, *"Quantum tops at the LHC: from entanglement to Bell inequalities"*, 2110.10112
- Afik, de Nova and Ramón Muñoz, *"Quantum information with top quarks in QCD"*, 2203.05582
- Fabbrichesi, Floreanini and Gabrielli, *"Constraining new physics in entangled two-qubit systems: top-quark, tau-lepton and photon pairs"*, 2208.11723
- Afik, de Nova and Ramón Muñoz, *"Quantum discord and steering in top quarks at the LHC"* 2209.03969
- Aguilar-Saavedra, Bernal, Casas and Moreno, *"Testing entanglement and Bell inequalities in $H \to ZZ$"*, 2209.13441
- Aguilar-Saavedra *"Laboratory-frame tests of quantum entanglement in $H \to WW$"*, 2209.14033

Optimising CHSH inequality over directions

- Find d and its transpose d^{T}
- Find real symmetric positive matrix $M = d^{\mathrm{T}} d$
- Find e-vals $\mu_1 \geq \mu_2 \geq \mu_3$ of M
- Find sum $\Sigma_{\mathrm{CHSH}} = \mu_1 + \mu_2$ of two largest
- Finally the CHSH Bell inequality is violated iff

$$\Sigma_{\mathrm{CHSH}} > 1$$

Maximally entangled qubit pair states

The states for which the Bell inequality violation is maximal are

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B\right) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)
\end{aligned}
$$

These are the Bell states: the maximally entangled states of two qubits

- $|\psi\rangle_{AB} \in (\mathbb{C}^2)^2$
- Basis for each qubit $\{0, 1\}$

QFT calculations

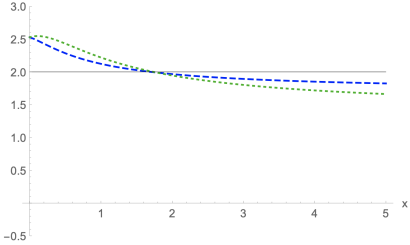


FIG. 3: Comparison of the violation of the CGLMP inequality in the state $|\xi(k, k^\pi)\rangle$ (blue, dashed line) and in the state $|\psi(k, k^\pi)\rangle$ (green, dotted line). The configuration of particles momenta and measurements directions is the following: $\mathbf{n} = (0, 0, 1)$, $\mathbf{w} = (\cos\phi_w \sin\theta_w, \sin\phi_w \sin\theta_w, \cos\theta_w)$, $\mathbf{w} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and $\theta_a = 2.667$, $\phi_a = 4.109$, $\theta_b = 0.924$, $\phi_b = 0.974$, $\theta_c = 2.699$, $\phi_c = 1.005$, $\theta_d = 0$, $\phi_d = 0$.

AJB, P. Caban, J. Rembieliński — 2204.11063 [quant-ph]

Loopholes

Rachel Ashby-Pickering (MMathPhys project)

- **Freedom of Choice**: potential that the violation came from a sort of 'conspiracy' of a locally causal system.
- **Memory**: potential to 'remember' earlier settings of the measurement and so predict the next one, or if the experimental runs aren't independent
- **Efficiency**: potential that the measurements are not representative of the underlying reality.
- **Communication/Locality**: potential that the measurement settings of one of the systems, or the measurement itself could influence the measurement settings or outcome of the other system.

(+other more extreme ways to avoid non-locality: retrocausality, superdeterminism, denial of realism)

Text by Rachel Ashby-Pickering

Three ‘loophole-free’ measurements (2015)

LETTER

doi:10.1038/nature15759

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres

B. Hensen[1,2], H. Bernien[1,2]†, A. E. Dréau[1,2], A. Reiserer[1,2], N. Kalb[1,2], M. S. Blok[1,2], J. Ruitenberg[1,2], R. F. L. Vermeulen[1,2], R. N. Schouten[1,2], C. Abellán[3], W. Amaya[3], V. Pruneri[3,4], M. W. Mitchell[3,4], M. Markham[5], D. J. Twitchen[5], D. Elkouss[1], S. Wehner[1], T. H. Taminiau[1,2] & R. Hanson[1,2]

Selected for a Viewpoint in *Physics*

PRL **115**, 250402 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

Strong Loophole-Free Test of Local Realism*

Lynden K. Shalm,[1,†] Evan Meyer-Scott,[2] Bradley G. Christensen,[3] Peter Bierhorst,[1] Michael A. Wayne,[3,4] Martin J. Stevens,[1] Thomas Gerrits,[1] Scott Glancy,[1] Deny R. Hamel,[5] Michael S. Allman,[1] Kevin J. Coakley,[1] Shellee D. Dyer,[1] Carson Hodge,[1] Adriana E. Lita,[1] Varun B. Verma,[1] Camilla Lambrocco,[1] Edward Tortorici,[1] Alan L. Migdall,[4,6] Yanbao Zhang,[2] Daniel R. Kumor,[3] William H. Farr,[7] Francesco Marsili,[7] Matthew D. Shaw,[7] Jeffrey A. Stern,[7] Carlos Abellán,[8] Waldimar Amaya,[8] Valerio Pruneri,[8,9] Thomas Jennewein,[2,10] Morgan W. Mitchell,[8,9] Paul G. Kwiat,[3] Joshua C. Bienfang,[4,6] Richard P. Mirin,[1] Emanuel Knill,[1] and Sae Woo Nam[1,‡]

[1]*National Institute of Standards and Technology, 325 Broadway, Boulder, Colorado 80305, USA*

[2]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada, N2L 3G1*

Selected for a Viewpoint in *Physics*

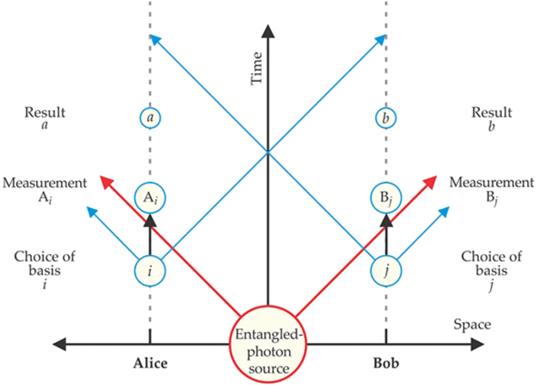
PRL **115**, 250401 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

Significant-Loophole-Free Test of Bell’s Theorem with Entangled Photons

Marissa Giustina,[1,2,*] Marijn A. M. Versteegh,[1,2] Sören Wengerowsky,[1,2] Johannes Handsteiner,[1,2] Armin Hochrainer,[1,2] Kevin Phelan,[1] Fabian Steinlechner,[1] Johannes Kofler,[3] Jan-Åke Larsson,[4] Carlos Abellán,[5] Waldimar Amaya,[5] Valerio Pruneri,[5,6] Morgan W. Mitchell,[5,6] Jörn Beyer,[7] Thomas Gerrits,[8] Adriana E. Lita,[8] Lynden K. Shalm,[8] Sae Woo Nam,[8] Thomas Scheidl,[1,2] Rupert Ursin,[1] Bernhard Wittmann,[1,2] and Anton Zeilinger[1,2,†]

[1]*Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences,*

'Loophole free' measurements

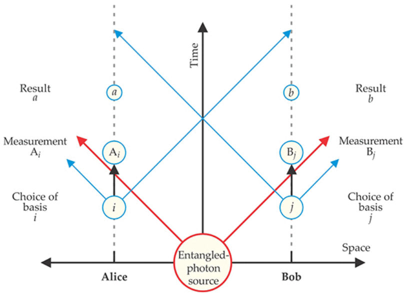


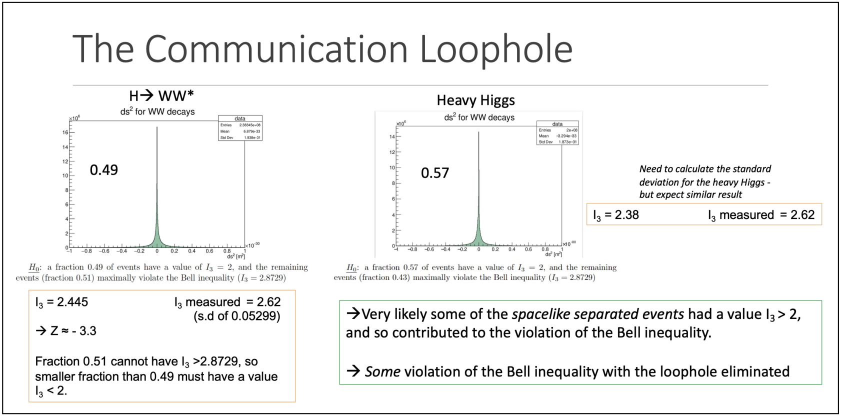
Went to particular trouble to ensure e.g.:

- measurements are space-like separated
- rapid switching of measurements
- basis choice space-like separated from measurement of other system
- measurement settings are 'random'

Communication loophole

- Photon experiments aim for large distances
- Wish to have space-like separation of measurements (& decisions)
- $H \to W^+W^-$ based on QFT calculation
- Mixture of space-like and time-like contributions to amplitude





Results: Can assert space-like separation at least on a statistical basis

Rachel Ashby-Pickering

Freedom of choice loophole

Alice and Bob each have three different sources of random bits that undergo an XOR operation together to produce their random measurement decisions (for more information see Supplemental Material [28]). The first source is based on measuring optical phase diffusion in a gain-switched laser that is driven above and below the lasing threshold. A new bit is produced every 5 ns by comparing adjacent laser pulses [17]. Each bit is then processed through an XOR gate with all past bits that have been produced (for more details see Supplemental Material [28]). The second source is based on sampling the amplitude of an optical pulse at the single-photon level in a short temporal interval. This source produces a bit on demand and is triggered by the synchronization signal. Finally, Alice and Bob each have a different predetermined pseudorandom source that is composed of various popular culture movies and TV shows, as well as the digits of π, processed together through an XOR gate. Suppose that a local-realistic system, with the goal of producing violation of the Bell inequality, was able to manipulate the properties of the photons emitted by the entanglement source before each trial. Provided that the randomness sources correctly extract their bits from the underlying processes of phase diffusion, optical amplitude sampling, and the production of cultural artifacts (such as the movie *Back to the Future*), this powerful local realistic system would be required to predict the outcomes of all of these processes well in advance of the beginning of each trial to achieve its goal. Such a model would have elements of superdeterminism—the fundamentally untestable idea that all events in the Universe are preordained.

Shalm et al 2015

- Many ‘Loophole free’ Bell tests use quantum randomness for $\hat{\mathbf{n}}$ choice

  (amongst other more curious choices)
- So does $H \rightarrow W^+W^-$

Experimental dependence @ LHC?

- Simulate LHC: 140/fb pp @ 13 TeV with Madgraph Monte Carlo simulation
- No backgrounds, some basic selection cuts, Gaussian smearing of each of the W boson momentum components

Expt. Assumptions	Truth	'A'	'B'	'C'
Min $p_T(\ell)$ [GeV]	0	5	20	20
Max $\|\eta(\ell)\|$	—	2.5	2.5	2.5
σ_{smear} [GeV]	0	5	5	10
$\mathcal{I}_3^{\mathrm{xyz}}$	2.62	2.40	2.75	2.16
Signif. $(\mathcal{I}_3^{\mathrm{xyz}} - 2)$	11.7σ	5.2σ	5.3σ	1.0σ

CAVEAT: Indicative only – more realistic version being investigated

In case you're curious

The CGLMP operator is[2]

$$
\mathcal{B}^{xy}_{\mathrm{CGLMP}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\
0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\
0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

[2] after a minor tweak – see 2106.01377

CGLMP limits?

In a local realist theory

$$\mathcal{I}_3 \leq 2$$

In QM

$$\mathcal{I}_3^{\mathrm{QM}} \leq 1 + \sqrt{11/3} \approx 2.9149$$

In QM for a maximally entangled state

$$\mathcal{I}_3^{\mathrm{QM,singlet}} \leq 4/(6\sqrt{3} - 9) \approx 2.8729$$

This is the tightest Bell inequality for pairs of three-outcome experiments

Finding a form for ρ

Parameterise ρ

Spin matrices and their pairwise symmetric products

$$\rho_W = \frac{1}{3} I_3 + \sum_{i=1}^{3} a_i S_i + \sum_{i,j=1}^{3} c_{ij} S_{\{ij\}},$$

where

$$S_{\{ij\}} \equiv S_i S_j + S_j S_i$$

- a_i form a real vector
- c_{ij} form a real symmetric traceless matrix
- $3 + 5 =$ 8 real parameters

Finding ρ - 'quantum state tomography'

$$\rho_W = \tfrac{1}{3} I_3 + \sum_{i=1}^{3} a_i S_i + \sum_{i,j=1}^{3} c_{ij} S_{\{ij\}},$$

Use the trace orthogonality relations

$$\mathrm{tr}(S_i, S_j) = 2\delta_{ij} \quad \text{and} \quad \mathrm{tr}(S_i, S_{\{jk\}}) = 0$$

For an ensemble of $W^{\pm}$ decays we can get the a_i parameter of ρ_W from data

Lepton directions $\rightarrow \rho_W$

$$\langle \xi_i^{\pm} \rangle_{\mathrm{av}} \equiv \langle \hat{\mathbf{n}}_{\ell^{\pm}} \cdot \hat{\mathbf{e}}_i \rangle_{\mathrm{av}} = \mathrm{tr}(\rho_W S_i) = 2a_i$$