Flavour Symmtries (for Leptons)

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CERN Neutrino Platform Pheno Week
Department of Theoretical Physics
CERN, Geneva
March 13, 2023

The Flavour Problem

Understanding the origins of flavour in both quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing and of CP violation in the quark and lepton sector, is one of the most challenging fundamental problems in contemporary particle physics. "Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."

From Model Physicist, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 24 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

In what follows we will discuss a new approach to the flavour problem within the three family framework.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

- **Why** $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$ ($m_{\nu_j} \lesssim 0.5$ **eV**, $m_l \geq 0.511$ **MeV**, $m_q \gtrsim 2$ **MeV**);

- **The origins of the patterns of**
i) neutrino mixing of 2 large and 1 small angles ($\theta^l_{12} = 33.4^\circ$, $\theta^l_{23} = 42.4^\circ$ (49.0°), $\theta^l_{13} = 8.59^\circ$), **and of ii)** Δm^2_{ij}, **i.e., of** $\Delta m^2_{21} \ll |\Delta m^2_{31}|$, $\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30$.

- **The origin of the hierarchical pattern of charged lepton masses:**
$m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The first two added new important aspects to the flavour problem.

$m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$:

seesaw mechanism(s), Weinberg operator, radiative ν mass generation, extra dimensions.
However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

The quark Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

• **The origin(s) of the observed patterns of up- and down-type quark masses characterized by strong hierachies.**

$$m_d \ll m_s \ll m_b\,, \ \ \frac{m_d}{m_s} = 5.02\times10^{-2}\,, \ \ \frac{m_s}{m_b} = 2.22\times10^{-2}\,, \ \ m_b = 4.18 \ \text{GeV};$$

$$m_u \ll m_c \ll m_t\,, \ \ \frac{m_u}{m_c} = 1.7\times10^{-3}\,, \ \ \frac{m_c}{m_t} = 7.3\times10^{-3}\,, \ \ m_t = 172.9 \ \text{GeV};$$

• **The origin of the pattern of the quark mixing: the three quark mixing angles are small and hierarchical,** $\sin\theta^q_{13} \ll \sin\theta^q_{23} \ll \sin\theta^q_{12} \ll 1$, $\sin\theta^q_{12} \cong 0.22$, $\sin\theta^q_{12} \cong 0.22$, $\sin\theta^q_{23} \cong 0.042$, $\sin\theta^q_{13} \cong 0.0038$ **(**$\theta^q_{12} = 12.96^\circ$, $\theta^q_{23} = 2.42^\circ$, $\theta^q_{13} = 0.22^\circ$**).**

• **The origin and magnitude of CP violation in the quark sector.**

My talk: The Lepton Flavour Problem.

Flavour symmetries - principal approach to the Flavour Problem (S. Pakvasa, H. Sugawara, 1978 (S_3); C.D. Frogatt, H.B. Nielsen, 1979 ($U(1)_{\text{FN}}$)).

Vast literature; different varieties (continuos, discrete).

My choice: Non-Abelian discrete symmetry and Modular invariance approaches to the lepton flavour problem (from bottom-up perspective).

The Non-Abelian Discrete Symmetry Approach

With the observed pattern of neutrino mixing Nature is sending us a Message. The Message is encoded in the values of the neutrino mixing angles, leptonic CP violation phases and neutrino masses. In my opinion, Nature gave us also a hint what the content of Nature's Message is.

Neutrino Mixing: New Symmetry?

- $\theta_{12} = \theta_{\odot} \cong \frac{\pi}{5.4}, \quad \theta_{23} = \theta_{\rm atm} \cong \frac{\pi}{4}(?), \quad \theta_{13} \cong \frac{\pi}{20}$

$$U_{\rm PMNS} \cong \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \epsilon \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}}(?) \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}}(?) \end{pmatrix};$$

Very different from the CKM-matrix!

- $\theta_{12} \cong \sin^{-1}\frac{1}{\sqrt{3}} - 0.020$; $\theta_{12} \cong \pi/4 - 0.20$,

$\theta_{13} \cong 0 + \pi/20, \quad \theta_{23} \cong \pi/4 \mp 0.10.$

- $U_{\rm PMNS}$ **due to new approximate symmetry?**

A Natural Possibility (vast literature):

$$U = U_{\text{lep}}^{\dagger}(\theta_{ij}^{\ell}, \delta^{\ell})\; Q(\psi, \omega) U_{\text{TBM,BM,LC,...}}\; \bar{P}(\xi_1, \xi_2)\,,$$

with

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}; \quad U_{\text{BM}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \pm\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \mp\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- $U_{\text{lep}}^{\dagger}(\theta_{ij}^{\ell}, \delta^{\ell})$ - from diagonalization of the l^- mass matrix;
- $U_{\text{TBM,BM,LC,...}}\; \bar{P}(\xi_1, \xi_2)$ - from diagonalization of the ν mass matrix;
- $Q(\psi, \omega)$, - from diagonalization of the l^- and/or ν mass matrices.

P. Frampton, STP, W. Rodejohann, 2003

$U_{\rm LC}$, $U_{\rm GRAM}$, $U_{\rm GRBM}$, $U_{\rm HGM}$:

$$U_{\rm LC} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{c^{\nu}_{23}}{\sqrt{2}} & \frac{c^{\nu}_{23}}{\sqrt{2}} & s^{\nu}_{23} \\ \frac{s^{\nu}_{23}}{\sqrt{2}} & -\frac{s^{\nu}_{23}}{\sqrt{2}} & c^{\nu}_{23} \end{pmatrix} ; \quad \mu - \tau \text{ symmetry : } \quad \theta^{\nu}_{23} = \mp\pi/4 \, ;$$

$$U_{\rm GR} = \begin{pmatrix} c^{\nu}_{12} & s^{\nu}_{12} & 0 \\ -\frac{s^{\nu}_{12}}{\sqrt{2}} & \frac{c^{\nu}_{12}}{\sqrt{2}} & -\sqrt{\frac{1}{2}} \\ -\frac{s^{\nu}_{12}}{\sqrt{2}} & \frac{c^{\nu}_{12}}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} ; \quad U_{\rm HGM} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} , \quad \theta^{\nu}_{12} = \pi/6 \, .$$

$U_{\rm GRAM}$: $\sin^2\theta^{\nu}_{12} = (2 + r)^{-1} \cong 0.276$, $r = (1 + \sqrt{5})/2$
(GR: $r/1$; $a/b = a + b/a$, $a > b$)

$U_{\rm GRBM}$: $\sin^2\theta^{\nu}_{12} = (3 - r)/4 \cong 0.345$.

GRB and HG mixing: W. Rodejohann et al., 2009.

A Natural Possibility (vast literature):

$$U = U^{\dagger}_{\text{lep}}(\theta^{\ell}_{ij}, \delta^{\ell})\ Q(\psi, \omega) U_{\text{TBM,BM,LC,...}}\ \bar{P}(\xi_1, \xi_2)\,,$$

with

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix};\quad U_{\text{BM}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \pm\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \mp\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- $U^{\dagger}_{\text{lep}}(\theta^{\ell}_{ij}, \delta^{\ell})$ - from diagonalization of the l^- mass matrix;
- $U_{\text{TBM,BM,LC,...}}\ \bar{P}(\xi_1, \xi_2)$ - from diagonalization of the ν mass matrix;
- $Q(\psi, \omega)$, - from diagonalization of the l^- and/or ν mass matrices.

P. Frampton, STP, W. Rodejohann, 2003

$U_{\text{TBM(BM)}}$: Groups A_4, T', S_4 (S_4),... (vast literature)

(Reviews: G. Altarelli, F. Feruglio, arXiv:1002.0211; M. Tanimoto et al., arXiv:1003.3552; S. King and Ch. Luhn, arXiv:1301.1340)

- U_{GRA}: Group A_5,...; $s^2_{13} = 0$ and possibly $s^2_{12} = 0.276$ and $s^2_{23} = 1/2$ must be corrected.

L. Everett, A. Stuart, arXiv:0812.1057;...

- U_{LC}: alternatively $U(1)$, $L' = L_e - L_\mu - L_\tau$

S.T.P., 1982

- U_{LC}: $s^2_{12} = 1/2$, $s^2_{13} = 0$, s^ν_{23} - free parameter; $s^2_{13} = 0$ and $s^2_{12} = 1/2$ must be corrected.

• U_{GRB}: Group D_{10},...; $s^2_{13} = 0$ and possibly $s^2_{12} = 0.345$ and $s^2_{23} = 1/2$ must be corrected.

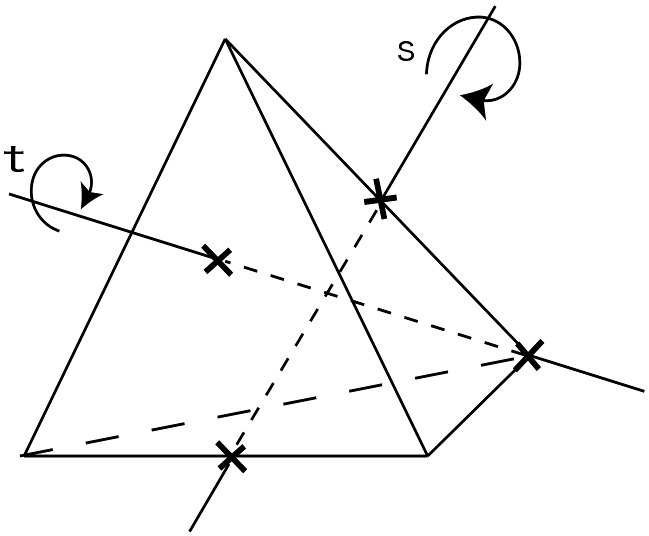
• U_{HG}: Group D_{12},...; $s^2_{13} = 0$, $s^2_{12} = 0.25$ and possibly $s^2_{23} = 1/2$ must be corrected.

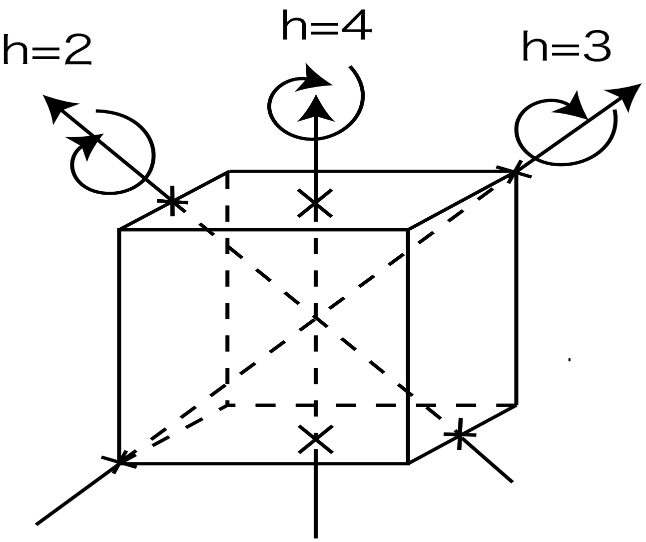
For all symmetry forms considered we have: $\theta^\nu_{13} = 0$, $\theta^\nu_{23} = \mp\pi/4$.
They differ by the value of θ^ν_{12}:
TBM, BM, GRA, GRB and HG forms correspond to $\sin^2\theta^\nu_{12} = 1/3;\ 0.5;\ 0.276;\ 0.345;\ 0.25$.

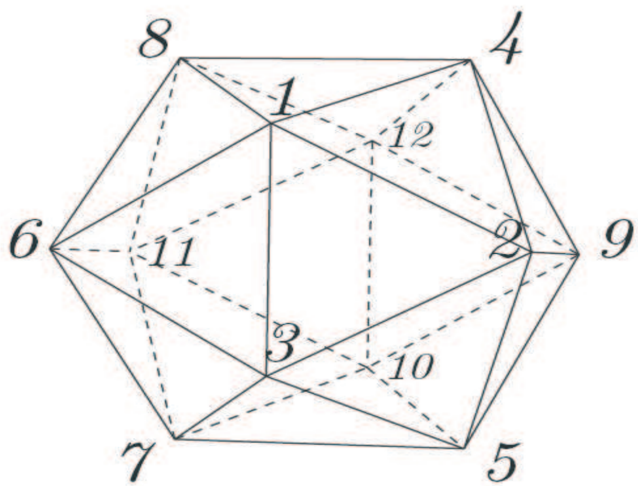
The observed pattern of 3-ν mixing, two large and one small mixing angles,
$\theta_{12} \cong 33^\circ$**,** $\theta_{23} \cong 45^\circ \pm 6^\circ$ **and** $\theta_{13} \cong 8.4^\circ$**,**
can most naturally be explained by extending the Standard Model (SM) with a flavour symmetry corresponding to a non-Abelian discrete (finite) group G_f**.**

$G_f = A_4$, T', S_4, A_5, D_{10}, D_{12},...

Vast literature; reviews: G. Altarelli, F. Feruglio, 1002.0211; H. Ishimori et al., 1003.3552; M. Tanimoto, AIP Conf.Proc. 1666 (2015) 120002; S. King and Ch. Luhn, 1301.1340; D. Meloni, 1709.02662; STP, 1711.10806







Examples of symmetries: A_4, S_4, A_5

From M. Tanimoto et al., arXiv:1003.3552

Group	Number of elements	Generators	Irreducible representations
S_4	24	S, T (U)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3}'$
S'_4	48	S, T (R)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3}'$, $\hat{\mathbf{1}}$, $\hat{\mathbf{1}}'$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$, $\hat{\mathbf{3}}'$
A_4	12	S, T	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{3}$
T'	24	S, T (R)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{2}$, $\mathbf{2}'$, $\mathbf{2}''$, $\mathbf{3}$
A_5	60	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$, $\mathbf{5}$
A'_5	120	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$, $\mathbf{5}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{2}}'$, $\hat{\mathbf{4}}$, $\hat{\mathbf{6}}$.

Number of elements, generators and irreducible representations of S_4, S'_4, A_4, $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.

Predictions and Correlations

$U_\nu = U_{\rm TBM,BM,GRA,GRB,HG}\, \bar{P}(\xi_1,\xi_2)$; θ^ν_{12};

$U_\ell^\dagger = R_{12}(\theta^\ell_{12})\, Q$, $Q = {\rm diag}(e^{i\varphi}, 1, 1)$
(the "minimal" = simplest case ($SU(5)\times T'$,...)

$U_\ell^\dagger = R_{12}(\theta^\ell_{12})\, R_{23}(\theta^\ell_{23}) Q$, $Q = {\rm diag}(1, e^{-i\psi}, e^{-i\omega})$
(next-to-minimal case)

$\cos\delta = \cos\delta(\theta_{12},\theta_{23},\theta_{13};\theta^\nu_{12},...)$,

$J_{CP} = J_{CP}(\theta_{12},\theta_{23},\theta_{13},\delta) = J_{CP}(\theta_{12},\theta_{23},\theta_{13};\theta^\nu_{12},...)$,

θ^ν_{12},... - **known (fixed) parameters, depend on the underlying symmetry.**

For arbitrary fixed θ^{ν}_{12} and any θ_{23} ("minimal" and "next-to-minimal" cases):

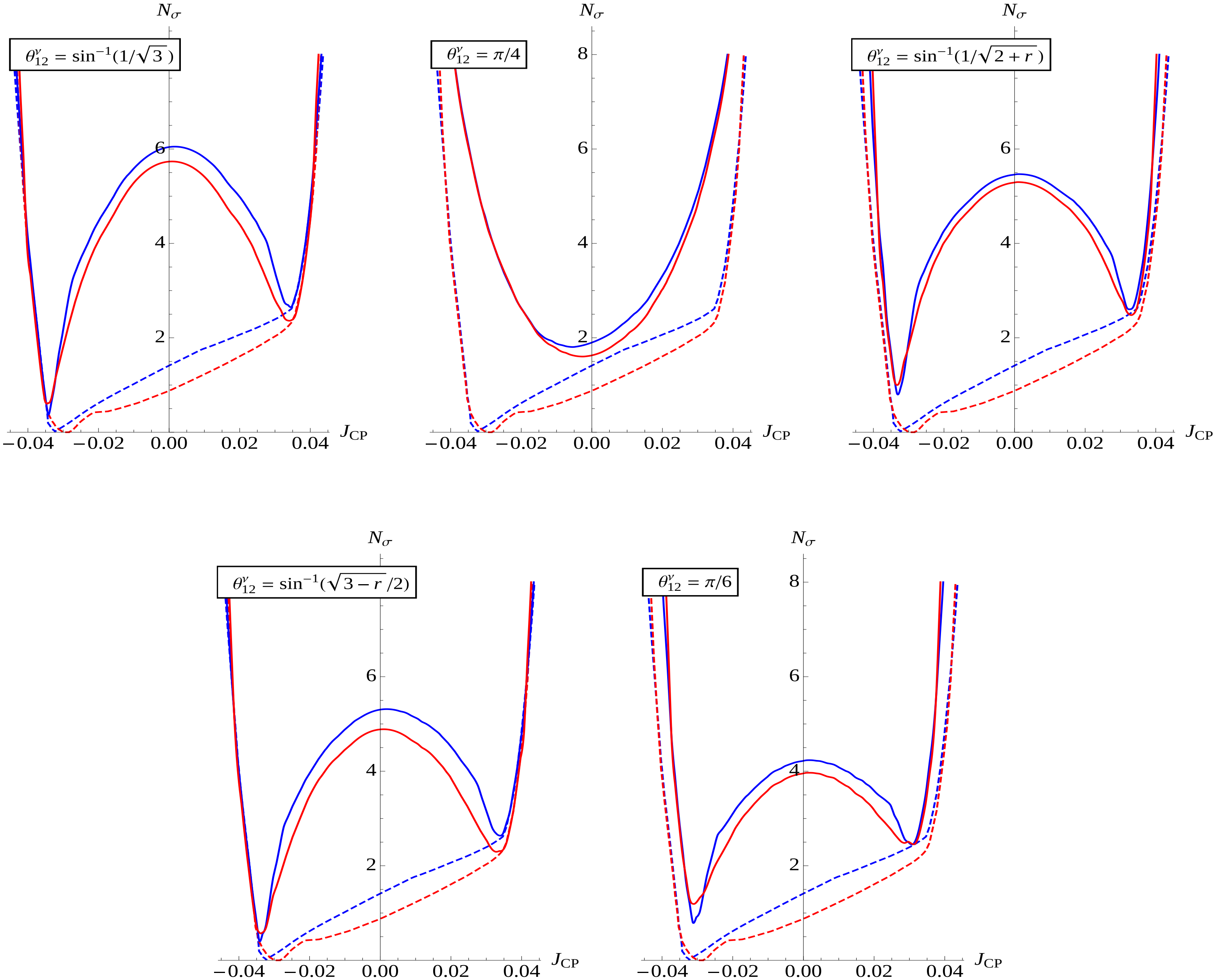
$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\cos 2\theta^{\nu}_{12}\right.$$

$$\left. + \left(\sin^2\theta_{12} - \cos^2\theta^{\nu}_{12}\right)\left(1 - \cot^2\theta_{23}\,\sin^2\theta_{13}\right)\right].$$

S.T.P., arXiv:1405.6006

This results is exact.

"Minimal" case: $\sin^2\theta_{23} = \frac{1}{2}\,\frac{1-2\sin^2\theta_{13}}{1-\sin^2\theta_{13}}$.



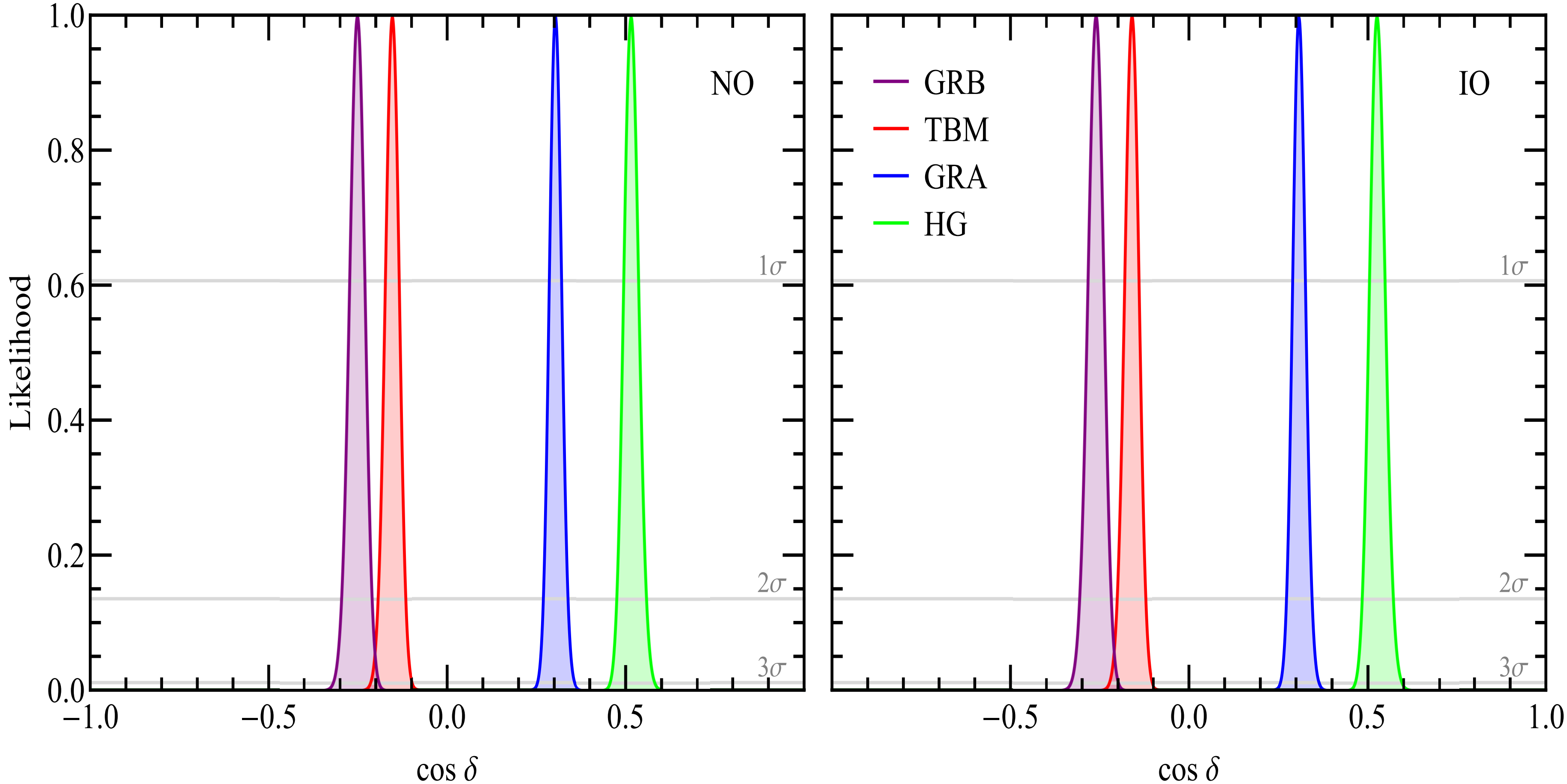
I. Girardi, S.T.P., A. Titov, arXiv:1410.8056

Prospective precision:

$\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO),**

$\delta(\sin^2\theta_{13}) = 3\%$ **(Daya Bay),**

$\delta(\sin^2\theta_{23}) = 5\%$ **(T2K, NOνA combined).**



Updated from I. Girardi, S.T.P., A. Titov, arXiv:1410.8056

b.f.v. of $\sin^2\theta_{ij}$ (Esteban et al., Jan., 2018) + the prospective precision used.

$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\cos 2\theta^{\nu}_{12} + \left(\sin^2\theta_{12} - \cos^2\theta^{\nu}_{12}\right)\left(1 - \cot^2\theta_{23}\sin^2\theta_{13}\right)\right] .$$

$\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE).**

How does it Work.

Choose G_f**.**

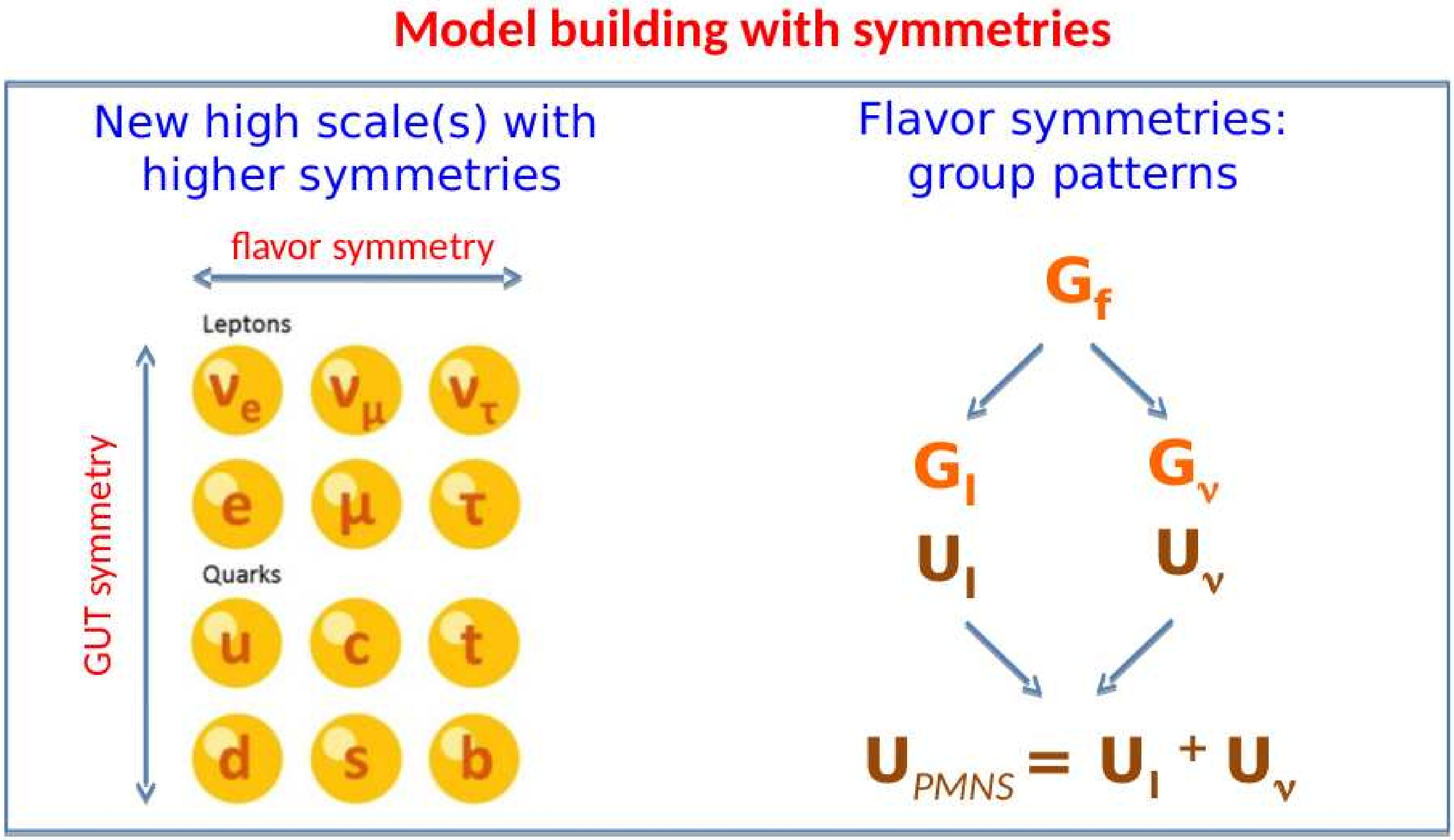
$\nu_{eL}(x), \nu_{\mu L}(x), \nu_{\tau L}(x)$**: assigned to** $\rho^{(\nu)}(g_f)$ **- irreducible representation of** G_f**, where** g_f **is an element of** G_f**.**

$e_L(x), \mu_L(x), \tau_L(x)$**: assigned to** $\rho^{(e)}(g_f)$ **- IRREP of** G_f**.**

$G_f = S_4$**,** A_4**,** T'**,** A_5**:** $\rho^{(\nu)}(g_f)$**,** $\rho^{(e)}(g_f)$ **- triplet IRREP.**

$e_R(x), \mu_R(x), \tau_R(x)$**: singlets of** G_f**.**

How Does it Work



E. Lisi, TAUP 2019

ν_j, **Majorana mass term,** $m_j \neq m_k$, $j \neq k = 1,2,3$: $G_\nu = Z_2 \times Z_2$, Z_2

$G_e = Z_2$; Z_n, $n > 2$; $Z_n \times Z_m$, $n,m \geq 2$

M_e **- charged lepton mass matrix (L-R convention).**

U_e**:** $U_e^\dagger M_e M_e^\dagger U_e = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$.

G_e **- residual symmetry group of** $M_e M_e^\dagger$**:**

$\rho^{(e)}(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger$,

$\rho^{(e)}(g_e)$ **generator(s) of** G_e **in the triplet rep.**

$\rho^{(e)}(g_e)$ **and** $M_e M_e^\dagger$ **commute: both are diagonalised by** U_e**.**

$\rho^{(e)}(g_e)$ **- known! Thus,** U_e **- fixed!**

M_ν - neutrino Majorana mass matrix (R-L convention).

U_ν: $U_\nu^T M_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3)$.

G_ν - residual symmetry group of M_ν:

$$\rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu,$$

g_ν: an element of G_ν,
$\rho(g_\nu)$ generator of G_ν in the triplet repr.

$\rho(g_\nu)$ and $M_\nu^\dagger M_\nu$ commute: both are diagonalised by U_ν.
$\rho(g_\nu)$ - known! Thus, U_ν-fixed.

$$U_{\mathrm{PMNS}} = U_e^\dagger U_\nu$$

A_4: $G_e = Z_3^T = \{1, T, T^2\}$, $G_\nu = Z_2^S = \{1, S\}$

$(S^2 = T^3 = (ST)^3 = \mathbf{I})$

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \omega = e^{i2\pi\tau/3} \text{ (A} - \text{F)}.$$

$U_e = \mathbf{I}$, $U_{\text{PMNS}} = U_e^\dagger U_\nu = U_{\text{TBM}} U_{13}(\theta^\nu_{13}, \alpha)$, θ^ν_{13}, α - free.

W. Grimus, L. Lavoura, 2008

$\sin^2\theta_{12} = \frac{1}{3(1-\sin^2\theta_{13})} \cong 0.34$;

$\cos\delta = \frac{\cos 2\theta_{23} \cos 2\theta_{13}}{\sin 2\theta_{23} \sin\theta_{13}(2-3\sin^2\theta_{13})^{\frac{1}{2}}}$; **if** $\theta_{23} = \frac{\pi}{4}$, $\delta = \pm\frac{\pi}{2}$.

Examples of Predictions and Correlations II.

- $\sin^2\theta_{23} = \frac{1}{2}$.
- $\sin^2\theta_{23} \cong \frac{1}{2}(1 \mp \sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong \frac{1}{2}(1 \mp 0.022)$.
- $\sin^2\theta_{23} = 0.455;\ 0.463;\ 0.537;\ 0.545;\ 0.604$.
- $\sin^2\theta_{12} \cong \frac{1}{3}(1 + \sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong 0.340$.
- $\sin^2\theta_{12} \cong \frac{1}{3}(1 - 2\sin^2\theta_{13}) + O(\sin^4\theta_{13}) \cong 0.319$.
- **and/or** $\cos\delta = \cos\delta(\theta_{12}, \theta_{23}, \theta_{13}; \theta^{\nu}_{12}, ...)$,

$J_{CP} = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}, \delta) = J_{CP}(\theta_{12}, \theta_{23}, \theta_{13}; \theta^{\nu}_{12}, ...)$,

$\theta^{\nu}_{12}, ...$ - **known (fixed) parameters, depend on the underlying symmetry**.

The Approach is testable/falsifyable experimentally!

The measurement of the Dirac phase in the PMNS mixing matrix, together with an improvement of the precision on the mixing angles θ_{12}, θ_{13} and θ_{23}, can provide unique information about the possible existence of new fundamental symmetry in the lepton sector.

Prospective (useful/requested) precision:

$\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO),**

$\delta(\sin^2\theta_{13}) = 3\%$ **(Daya Bay),**

$\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE; T2K+NOνA(?)).**

$\delta(\delta) = 10^\circ$ **at** $\delta = 3\pi/2$ **(THKK?)**

The Power of Data

Systematic analysis (I. Girardi *et al.*, 2016): all possible combinations of residual symmetries G_e and G_ν of the lepton flavour symmetry groups $G_f = S_4$, A_4, T' and A_5, leading to correlations between some of the three neutrino mixing angles and/or between the neutrino mixing angles and the Dirac CPV phase δ, were considered.

(A) $G_e = Z_2$ **and** $G_\nu = Z_k$, $k > 2$ **or** $Z_m \times Z_n$, $m, n \geq 2$;

(B) $G_e = Z_k$, $k > 2$ **or** $Z_m \times Z_n$, $m, n \geq 2$ **and** $G_\nu = Z_2$;

(C) $G_e = Z_2$ **and** $G_\nu = Z_2$.

In these cases $U_e^\dagger$ and/or U_ν of $U = U_e^\dagger U_\nu = (\tilde{U}_e)^\dagger \Psi \tilde{U}_\nu Q_0$, are partially (or fully) determined by residual discrete symmetries of $G_f = S_4$, A_4, T' and A_5.

More specifically:

A. $G_e = Z_2$, $G_\nu = Z_n$, $n > 2$ **or** $Z_n \times Z_m$, $n, m \geq 2$;
U_ν **fixed; A1, A2 (A3):** θ_{23}, $\cos\delta$ **(**θ_{12}, θ_{13}**) predicted.**

B. $G_e = Z_n$, $n > 2$ **or** $G_e = Z_n \times Z_m$, $n, m \geq 2$, $G_\nu = Z_2$;
U_e **fixed; B1,B2 (B3):** θ_{12}, $\cos\delta$ **(**θ_{23}, θ_{13}**) predicted.**

C. $G_e = Z_2$ **and** $G_\nu = Z_2$: θ_{12} **or** θ_{23} **or** $\cos\delta$ **predicted.**

$G_f = A_4$, S_4, T', A_5.

A_4: **3** Z_2, **4** Z_3, **1** $Z_2 \times Z_2$ **subgroups (total 8).**

T': **similar to** A_4.

S_4: **9** Z_2, **4** Z_3, **3** Z_4, **4** $Z_2 \times Z_2$ **subgroups (total 20).**

A_5: **has 15** Z_2, **10** Z_3, **6** Z_5, **5** $Z_2 \times Z_2$ **subgroups (36).**

In the case of A_4 (T') symmetry only there are 64 models (up to permutation of rows and columns).

A_4:
$(G_e, G_\nu) = (Z_2, Z_3)$, **A1 - A3;**
$(G_e, G_\nu) = (Z_2, Z_2)$, **A1 - A3;**
$(G_e, G_\nu) = (Z_3, Z_2)$, **B1 - B3;**
$(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$, **B1 - B3;**
$(G_e, G_\nu) = (Z_2, Z_2)$, **C1 - C9.**

For A_4, S_4 and A_5 the total number of models to be analysed is extremely large. However, a total of only 14 models survive the 3σ constraints on $\sin^2\theta_{ij}$ from the current data and the requirement $|\cos\delta| \leq 1$.

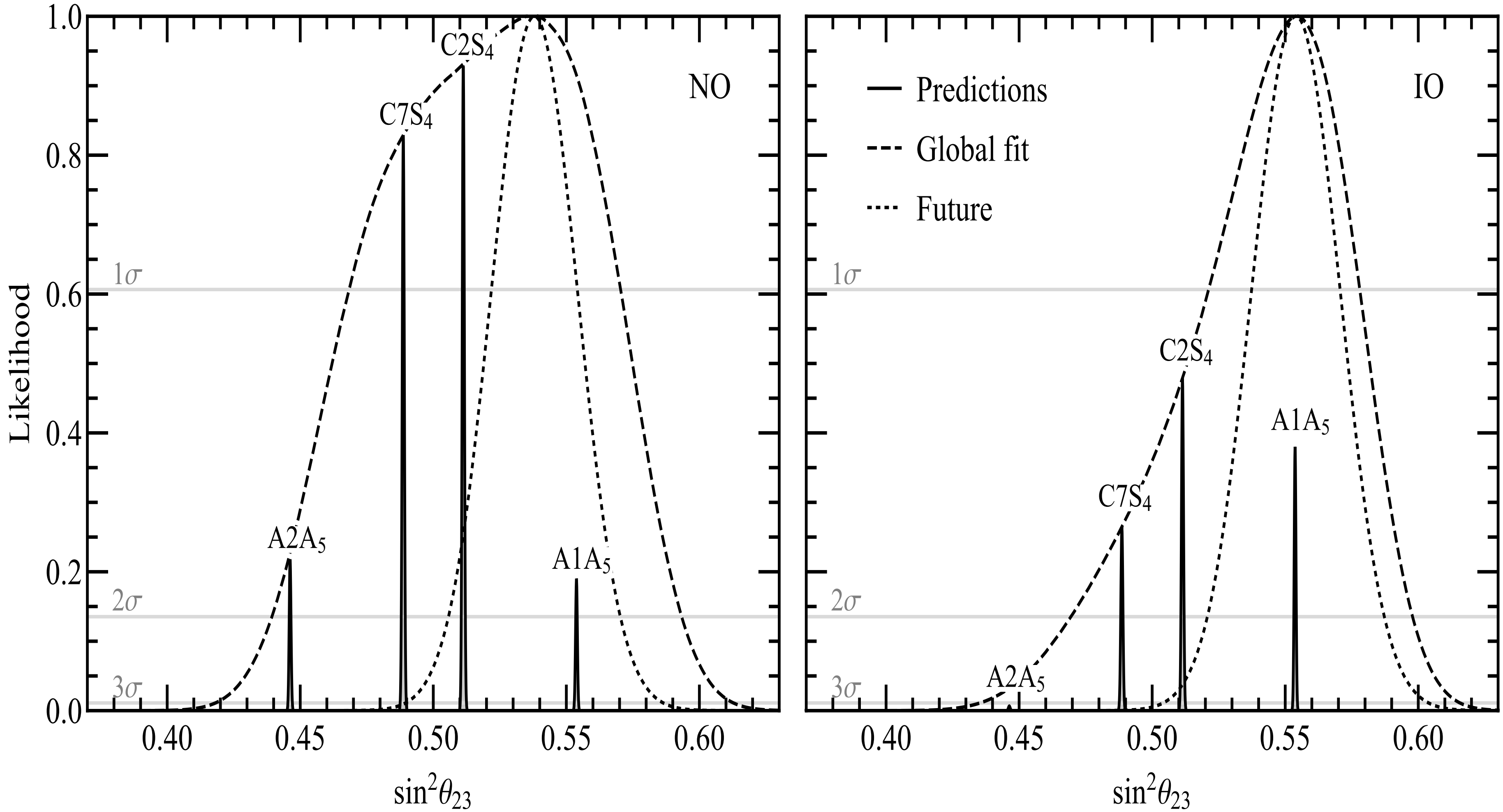
Phenomenologically Viable Predictions

A1 (A2), A_5 **(**$G_e = Z_2$**,** $G_\nu = Z_3$ **(Dirac** ν_j**)):**
$\sin^2\theta_{23} \cong 0.553\ (0.447)$; $\cos\delta \cong 0.716\ (-0.716)$.

A1, S_4**:** $\sin^2\theta_{23} \cong 0.5(1 - \sin^2\theta_{13}) \cong 0.489$;
$\cos\delta \cong -1$ **requires** $\sin^2\theta_{12} \cong 0.348$ **(!)**

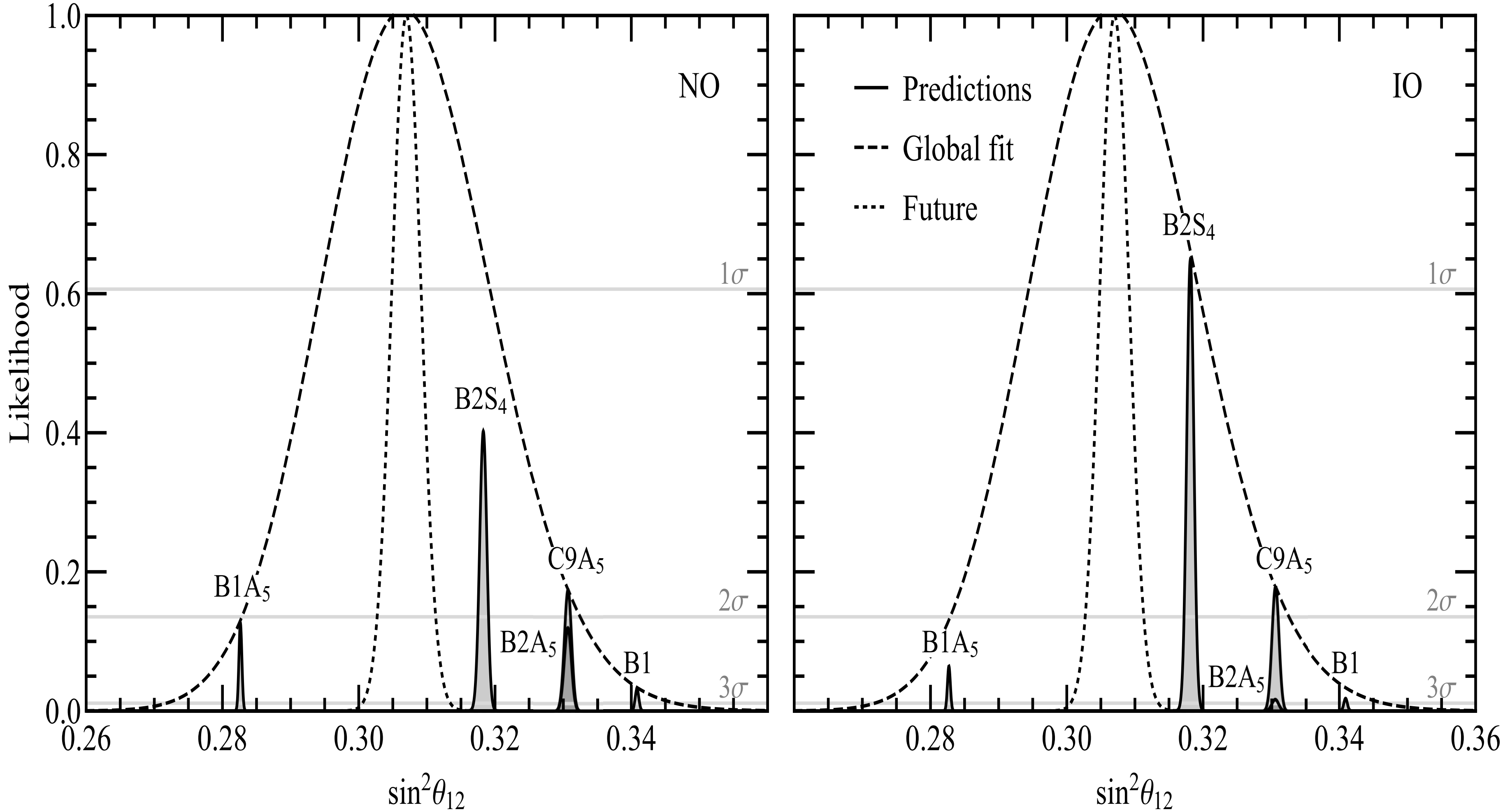
B1, A_4 **(**T'**,** S_4**,** A_5**) (**$G_e = Z_3^T$**,** $G_\nu = Z_2^S$**):**
$U_{\text{PMNS}} = U_{\text{TBM}} U_{13}(\theta^\nu_{13}, \delta_{13}) Q_0$;
$\sin^2\theta_{12} = 1/(3\cos^2\theta_{13}) \cong 0.340$; $\cos\delta \cong 0.570$.

B2, S_4 **(**$G_e = Z_3^T$**,** $G_\nu = Z_2^{SU}$**):**
$\sin^2\theta_{12} \cong (1 - 2\sin^2\theta_{13})/3 = 0.319$; $\cos\delta \cong -0.269$.



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2\theta_{23}) = 3\%$ **(T2HK, DUNE).**



S.T.P., A. Titov, arXiv:1804.00182

Future: $\delta(\sin^2\theta_{12}) = 0.7\%$ **(JUNO).**

A total of 6 models would survive out of the currently viable 14 (of the extremely large number) considered if $\delta(\sin^2\theta_{23}) = 3\%$, $\delta(\sin^2\theta_{12}) = 0.7\%$ and the current b.f.v. would not change:

$A1A_5$, $C2S_4$, C3, $C3A_5$, $C4A_5$, C8.

Will be constrained further by the data on δ.

The Problem

The correct lepton mixing pattern in a model with non-Abelian discrete symmetry G_f is determined by the appropriate choice of residual symmetries G_e and G_ν and is not directly related to the charged lepton and neutrino mass generation.
The breaking of G_f has to ensure the correct generation of the fermion masses and keep G_e and G_ν intact.

The symmetry breaking in the lepton and quark flavour models based on non-Abelian discrete symmetries is impressively cumbersome: it requires the introduction of a plethora of "flavon" scalar fields having elaborate potentials, which in turn require large shaping symmetries to ensure the requisite breaking of the symmetry leading to correct mass and mixing patterns.

To give an exmaple, in the theoretical lepton flavour model based on the T' symmetry constructed in I. Girardi et al., arXiv:1312.1966, there are 4 triplets, 3 doublets and 7 singlets of flavon fields and the scalar potential has a $Z_8 \times Z_4 \times Z_4 \times Z_3 \times Z_3 \times Z_2$ shaping symmetry.

The Flavour Problem: Modular Invariance Approach

Modular invariance approach to the flavour problem was proposed in F. Feruglio, arXiv:1706.08749 and has been intensively developed in the last four years.

In this approach the flavour (modular) symmetry is broken by the vacuum expectation value (VEV) of a single scalar field - the modulus τ. The VEV of τ can also be the only source of violation of the CP symmetry.

Many (if not all) of the drawbacks of the widely studied alternative approaches are absent in the modular invariance approach to the flavour problem.

The first phenomenologically viable "minimal" (in terms of fields, i.e., without flavons) lepton flavour model based on modular symmetry appeared in June of 2018 (J.T. Penedo, STP, arXiv:1806.11040). Since then various aspects of this approach were and continue to be extensively studied – the number of publications on the topic exceeds 160.

Matter Fields and Modular Forms

The matter(super)fields (charged lepton, neutrino, quark) transform under $\overline{\Gamma} \simeq PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{I, -I\}$ $\quad$ ($\Gamma \simeq SL(2,\mathbb{Z})$) as "weighted" multiplets:

$$\psi_i \xrightarrow{\gamma} (c\tau + d)^{-k_\psi} \rho_{ij}(\tilde{\gamma})\psi_j \,, \gamma \in \overline{\Gamma} \;\; (\gamma \in \Gamma)\,,$$

$$\left(\gamma\tau = \frac{a\tau+b}{c\tau+d}\,,\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,,\, a,b,c,d \in \mathbb{Z}\,,\, ad - bc = 1\,,\, \mathrm{Im}\tau > 0\right)$$

k_ψ **is the weight of** ψ; $k_\psi \in \mathbb{Z}$ **(or rational number).**
$\Gamma(N)$ **- principal congruence (normal) subgroup of** $SL(2,\mathbb{Z})$**.**
$\rho(\tilde{\gamma})$ **is a unitary representation of the** *inhomogeneous (homogeneous)* **finite modular group** $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ **(**$\Gamma'_N = \Gamma/\Gamma(N)$**),** $\tilde{\gamma}$ **− representation of** γ **in** Γ_N **(**Γ'_N**)**

F. Feruglio, arXiv:1706.08749; S. Ferrara et al., Phys.Lett. B233 (1989) 147, B225 (1989) 363

As we have indicated in brackets, one can consider also the case of Γ **and** $\gamma \in \Gamma(N)$**. Then** $\rho(\gamma)$ **will be a unitary representation of the** *homogeneous* **finite modular group** Γ'_N**.**

Remarkably, for $N \leq 5$, the inhomogeneous finite modular groups Γ_N are isomorphic to non-Abelian discrete groups widely used in flavour model building:
$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$.
Γ_N is presented by two generators S and T satisfying:

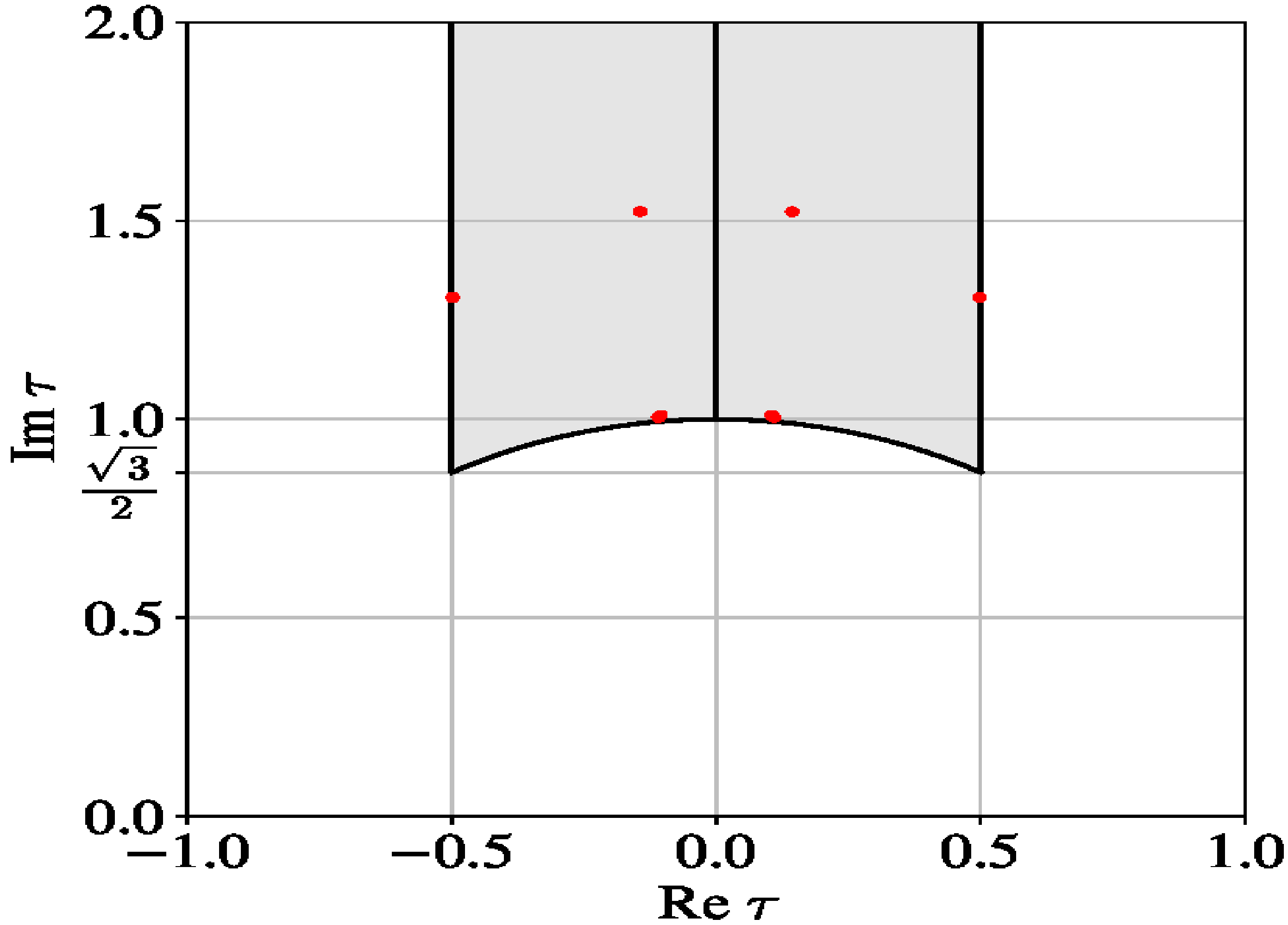
$$S^2 = (ST)^3 = T^N = I\,.$$

The group theory of $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$ is summarized, e.g., in P.P. Novichkov *et al.*, JHEP 07 (2019) 165, arXiv:1905.11970.

$\Gamma \simeq SL(2,\mathbb{Z})$ – *homogeneous* **modular group, $\Gamma(N)$ and the quotient groups $\Gamma'_N \equiv \Gamma/\Gamma(N)$ – homogeneous finite modular groups. For $N = 3, 4, 5$, Γ'_N are isomorphic to the double covers of the corresponding non-Abelian discrete groups:**
$\Gamma'_3 \simeq A'_4 \equiv T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$.
Γ'_N is presented by two generators S and T satisfying:

$$S^4 = (ST)^3 = T^N = I\,,\;\; S^2\,T = T\,S^2 \quad (S^2 = R)\,.$$

The group theory of $\Gamma'_3 \simeq A'_4$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$ for flavour model building was developed in X.-G. Liu, G.-J. Ding, arXiv:1907.01488 (A'_4); P.P. Novichkov et al., arXiv:2006.03058 (S'_4); C.-Y. Yao et al., arXiv:2011.03501 (A'_5).



The Fundamental Domain of $\overline{\Gamma}$ (Γ) shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Relevant sub-groups of Γ_N and Γ'_N: :

$\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$

$\mathbb{Z}_N^{T} = \{I, T, (T)^2, ..., T^{N-1}\}$

Γ_N: $\mathbb{Z}_2^{S} = \{I, S\}$

Γ'_N: $\mathbb{Z}_4^{S} = \{I, S, S^2, S^3\}$ **(** $R^2 = I$, $\mathbb{Z}_2^{R} = \{I, R\}$, $R\tau = \tau$ **)**

Group	Number of elements	Generators	Irreducible representations
S_4	24	S, T (U)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3}'$
S'_4	48	S, T (R)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3}'$, $\hat{\mathbf{1}}$, $\hat{\mathbf{1}}'$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$, $\hat{\mathbf{3}}'$
A_4	12	S, T	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{3}$
T'	24	S, T (R)	$\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{2}$, $\mathbf{2}'$, $\mathbf{2}''$, $\mathbf{3}$
A_5	60	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$, $\mathbf{5}$
A'_5	120	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3}'$, $\mathbf{4}$, $\mathbf{5}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{2}}'$, $\hat{\mathbf{4}}$, $\hat{\mathbf{6}}$.

Number of elements, generators and irreducible representations of S_4, S'_4, A_4, $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.

Modular Forms

Within the considered framework the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are expressed in terms of modular forms of a certain level N and weight k_f.

The modular forms are functions of a single complex scalar field – the modulus τ – and have specific transformation properties under the action of the modular group.

Both the modular forms of given level N and weight k_f and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$.

Once τ acquires a VEV, the modular forms and thus the Yukawa couplings and the form of the mass matrices get fixed, and a certain flavour structure arises.

Quantitatively and barring fine-tuning, the magnitude of the values of the non-zero elements of the fermion mass matrices and therefore the fermion mass ratios are determined by the modular form values (which in turn are functions of the τ's VEV).

Modular Forms (contd.)

The key elements of the considered framework are modular forms $f(\tau)$ of *weight k_f and level N* **− holomorphic functions of τ, which transform under $\overline{\Gamma}$ (Γ) as follows:**

$$F\,(\gamma\tau) = (c\tau + d)^{k_F}\,\rho_{\mathbf{r}}(\tilde{\gamma})F(\tau)\,, \quad \gamma \in \overline{\Gamma} \quad (\gamma \in \Gamma)\,,$$

F. Feruglio, arXiv:1706.08749

ρ_{r} is a unitary representation of the finite modular group Γ_N (Γ'_N).
In the case of $\overline{\Gamma}$ (Γ) *non-trivial* **modular forms exist only for positive even integer (positive integer) weight k_F.**
For given k, N (N is a natural number), the modular forms span a linear space of finite dimension:
of weight k and level 3, $\mathcal{M}_k(\Gamma_3^{(\prime)} \simeq A_4^{(\prime)})$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4^{(\prime)} \simeq S_4^{(\prime)})$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5^{(\prime)} \simeq A_5^{(\prime)})$, is $5k+1$.
Thus, $\dim\mathcal{M}_1(\Gamma'_3 \simeq A'_4) = 2$, $\dim\mathcal{M}_1(\Gamma'_4 \simeq S'_4) = 3$, $\dim\mathcal{M}_1(\Gamma'_5 \simeq A'_5) = 6$.

Multiplets of Γ_N (Γ'_N) of higher weight modular forms can be constructed from tensor products of the lowest weight 2 (weigh 1) multiplets (they represent homogeneous polynomials of the lowest weight modular forms).

Following arXiv:1706.08749, it was of highest priority and of crucial importance for model building to find the basis of modular forms of the lowest weight 2 (weight 1) transforming in irreps of Γ_N (Γ'_N).

F. Feruglio, 1706.08749 ($\Gamma_3 \simeq A_4$, $k_f = 2$: the 3 mod.forms form a 3 of A_4);
T. Kobayashi et al., 1803.10391 ($\Gamma_2 \simeq S_3$, $k_f = 2$: the 2 mod. forms form a **2** of S_3);
J. Penedo, STP, 1806.11040 ($\Gamma_4 \simeq S_4$, $k_f = 2$: the 5 mod. forms form a **2** and $3'$ of S_4);
P.P. Novichkov et al., 1812.02158; G.-J. Ding et al., 1903.12588 (($\Gamma_5 \simeq A_5$), $k_f = 2$: the 11 basis modular forms were shown to form a **3**, a $3'$ and a **5** of A_5).

More elegant constuction: modular forms for A'_4, S'_4, A'_5, and A_4, S_4, A_5.
In each of three cases of A'_4, S'_4 and A'_5 the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the lowest weight 1 multiplets, were shown (respectively in X.-G. Liu, G.-J. Ding, 1907.01488, P.P. Novichkov et al., 2006.03058 and C.-Y. Yao et al., 2011.03501) to be expressed in terms of only two independent functions of τ.

These pairs of functions are different for the three different groups; but they all are related (in different ways) to the Dedekind η-function (in the case of A'_5 (A_5) - to two Jacobi theta constants also) and have similar (fastly converging) $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

Thus, in the case of a flavour symmetry described by a finite modular group $\Gamma_N^{(\prime)}$, $N = 2, 3, 4, 5$, the elements of the matices of the Yukawa couplings in the considered approach represent homogeneous polynomials of various degree of only two (holomorphic) functions of τ. They include also a limited (relatively small) number of constant parameters.

The modular forms of level $N = 2,3,4,5$ for $\Gamma^{(\prime)}_{2,3,4,5} \simeq S_3, A^{(\prime)}_4, S^{(\prime)}_4, A^{(\prime)}_5$ have been constructed by use of the of Dedekind eta function, $\eta(\tau)$:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}} \, , q = e^{i2\pi\tau} \, .$$

In the cases of $\Gamma^{(\prime)}_5 \simeq A^{(\prime)}_5$ two "Jacobi theta constants" are also used.
Modular forms of level $N = 4$ for $\Gamma'_4 \simeq S'_4$ ($\Gamma_4 \simeq S_4$) − in terms of $\theta(\tau)$, $\varepsilon(\tau)$:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau) \, , \; \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau) \, .$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1,2,4$, is the Dedekind eta.
Modular forms of level $N = 3$ for $\Gamma'_3 \simeq A'_4$ ($\Gamma_3 \simeq A_4$) − in terms of $\hat{e}_1$ and $\hat{e}_2$:

$$\hat{e}_1 = \frac{\eta^3(3\tau)}{\eta(\tau)}, \quad \hat{e}_2 = \frac{\eta^3(\tau/3)}{\eta(\tau)} \, .$$

Modular forms of level $N = 5$ for $\Gamma'_3 \simeq A'_5$ ($\Gamma_3 \simeq A_4$) − in terms of $\theta_5(\tau)$ and $\varepsilon_5(\tau)$: $\theta_5(\tau) = \exp(-\,i\pi/10)\,\Theta_{\frac{1}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$, $\varepsilon_5(\tau) = \exp(-\,i\,3\pi/10)\,\Theta_{\frac{3}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$.

Example: S_4'

P.P. Novichkov, J.T. Penedo. S.T.P., arXiv:2006.03058

Weight 1 modular forms furnishing a $\hat{3}$ of S_4':

$$Y_{\hat{\mathbf{3}}}^{(1)}(\tau) = \begin{pmatrix} \sqrt{2}\,\varepsilon\,\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix}$$

Modular S_4 lowest-weight 2 multiplets furnish a 2 and a $3'$ irreducible representations of S_4 (S_4') and are given by: :

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y_{\mathbf{3}'}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 - \varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

At weight $k = 3$, a non-trivial singlet and two triplets exclusive to S_4' arise:

$$Y_{\hat{\mathbf{1}}'}^{(3)}(\tau) = \sqrt{3}\left(\varepsilon\,\theta^5 - \varepsilon^5\,\theta\right),$$

$$Y_{\hat{\mathbf{3}}}^{(3)}(\tau) = \begin{pmatrix} \varepsilon^5\,\theta + \varepsilon\,\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\,\varepsilon^2\,\theta^4 - \varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6 - 5\,\varepsilon^4\,\theta^2\right) \end{pmatrix}, \quad Y_{\hat{\mathbf{3}}'}^{(3)}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\,\theta^3 \\ \theta^6 + 3\,\varepsilon^4\,\theta^2 \\ -3\,\varepsilon^2\,\theta^4 - \varepsilon^6 \end{pmatrix}.$$

At weight $k = 4$ one again recovers the S_4 result: the modular forms furnish a 1, 2, 3 and $3'$ irreducible representations of S_4 (S_4').

$$Y_{\mathbf{1}}^{(4)}(\tau) = \frac{1}{2\sqrt{3}}\left(\theta^8 + 14\,\varepsilon^4\,\theta^4 + \varepsilon^8\right)\,, \qquad Y_{\mathbf{2}}^{(4)}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - 10\,\varepsilon^4\,\theta^4 + \varepsilon^8\right) \\ \sqrt{3}\left(\varepsilon^2\,\theta^6 + \varepsilon^6\,\theta^2\right) \end{pmatrix}\,,$$

$$Y_{\mathbf{3}}^{(4)}(\tau) = \frac{3}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\left(\varepsilon^2\,\theta^6 - \varepsilon^6\,\theta^2\right) \\ \varepsilon^3\,\theta^5 - \varepsilon^7\,\theta \\ -\varepsilon\,\theta^7 + \varepsilon^5\,\theta^3 \end{pmatrix}\,, \qquad Y_{\mathbf{3'}}^{(4)}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - \varepsilon^8\right) \\ \frac{1}{2\sqrt{2}}\left(\varepsilon\,\theta^7 + 7\,\varepsilon^5\,\theta^3\right) \\ \frac{1}{2\sqrt{2}}\left(7\,\varepsilon^3\,\theta^5 + \varepsilon^7\,\theta\right) \end{pmatrix}\,,$$

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ are given by:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,, \qquad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1,2,4$, is the Dedekind eta function.

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions - power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$ ($\mathrm{Im}(\tau) \geq \sqrt{3}/2$, $|q_4| \lesssim 0.26$) :

$$\theta(\tau) = 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2\,q_4^4 + 2\,q_4^{16} + \ldots\,,$$

$$\varepsilon(\tau) = 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2\,q_4 + 2\,q_4^9 + 2\,q_4^{25} + \ldots\,.$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, $\theta \to 1$, $\varepsilon \to 0$.

In this limit $\varepsilon \sim 2\,q_4$ and ε can be used as an expansion parameter instead of q_4.

Due to quadratic dependence in the exponents of q_4, the $q-$expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_4| \leq \exp(-\pi\sqrt{3}/4) \simeq 0.26$.

Similar conclusions are valid for the pair of functions in terms of which the lowest weight 1 modular forms, and thus all higher weight modular forms of A_4' and A_5' are expressed.

The Framework

$\mathcal{N}=1$ **rigid (global) SUSY, the matter action** $\mathcal{S}$ **reads:**

$$\mathcal{S}=\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\overline{\theta}\;\; K(\tau,\overline{\tau},\psi,\overline{\psi})+\left(\int \mathrm{d}^4x\,\mathrm{d}^2\theta\;\; W(\tau,\psi)+\mathrm{h.c.}\right)\,,$$

K **is the Kähler potential,** W **is the superpotential,** ψ **denotes a set of chiral supermultiplets** ψ_i**,** θ **and** $\overline{\theta}$ **are Grassmann variables;**
τ **is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation** τ **for the lowest complex scalar component of the modulus superfield and call this component also "modulus").**
τ **and** ψ_i **transform under the action of** $\overline{\Gamma}$ **(**Γ**) in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that** $\psi_i=\psi_i(x)$ **transform in a certain irrep** $\mathbf{r}_i$ **of** Γ_N **(**Γ'_N**), the transformations read:**

$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\overline{\Gamma}\;(\Gamma):\qquad \begin{cases}\tau\to\dfrac{a\tau+b}{c\tau+d}\,,\\ \psi_i\to(c\tau+d)^{-k_i}\,\rho_{\mathbf{r}_i}(\gamma)\,\psi_i\,.\end{cases}$$

ψ_i **is not a modular form multiplet, the integer** $(-k_i)$ **can be** >0**,** <0**, 0.**
Invariance of $\mathcal{S}$ **under these transformations implies (global SUSY):**

$$\begin{cases} W(\tau,\psi) \to W(\tau,\psi)\,, \\ K(\tau,\overline{\tau},\psi,\overline{\psi}) \to K(\tau,\overline{\tau},\psi,\overline{\psi}) + f_K(\tau,\psi) + \overline{f_K}(\overline{\tau},\overline{\psi})\,. \end{cases}$$

The second line represents a Kähler transformation.
An example Kähler potential that is widely used in model building reads:

$$K(\tau,\overline{\tau},\psi,\overline{\psi}) = -\Lambda_0^2 \log(-i\tau + i\overline{\tau}) + \sum_i \frac{|\psi_i|^2}{(-i\tau + i\overline{\tau})^{k_i}}\,,$$

$\Lambda_0 > 0$ **having mass dimension one.**
More general $K(\tau,\overline{\tau},\psi,\overline{\psi})$ **and the possible consequences they can have for flavour model building are discussed in**
Mu-Chun Chen et al., arXiv:1909.06910 and 2108.02240; Y. Almumin et al., arXiv:2102.11286.

$$W(\tau,\psi) \rightarrow W(\tau,\psi)\,,$$

The superpotential can be expanded in powers of ψ_i:

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_s g_{i_1\,\dots\,i_n,s}\left(Y_{i_1\,\dots\,i_n,s}(\tau)\,\psi_{i_1}\dots\psi_{i_n}\right)_{\mathbf{1},s}\,,$$

$\mathbf{1}$ **stands for an invariant singlet of Γ_N (Γ'_N). For each set of n fields $\{\psi_{i_1},\dots,\psi_{i_n}\}$, the index s labels the independent singlets. Each of these is accompanied by a coupling constant $g_{i_1\,\dots\,i_n,s}$ and is obtained using a modular multiplet $Y_{i_1\,\dots\,i_n,s}$ of the requisite weight. To ensure invariance of W under Γ_N (Γ'_N), $Y_{i_1\,\dots\,i_n,s}(\tau)$ must transform as:**

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)\,Y(\tau)\,,$$

$\mathbf{r}_Y$ is a representation of Γ_N (Γ'_N), and k_Y and $\mathbf{r}_Y$ are such that

$$k_Y = k_{i_1} + \dots + k_{i_n}\,, \tag{1}$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \dots \otimes \mathbf{r}_{i_n} \supset \mathbf{1}\,. \tag{2}$$

Thus, $Y_{i_1\,\dots\,i_n,s}(\tau)$ represents a multiplet of weight k_Y and level N modular forms transforming in the representation $\mathbf{r}_Y$ of Γ_N (Γ'_N).

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \, ,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma) \, \psi \quad (\rho(\gamma) \, , \, \Gamma_N^{(\prime)} \, , \, N = 2, 3, 4, 5) \, ,$$
$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho_{r^c}^c(\gamma) \, \psi^c \, , \quad (\rho^c(\gamma) \, , \, \Gamma_N^{(\prime)}) \, .$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \, .$$

Inputs in the Analyses

Lepton sector: reference 3-ν mixing scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

ν_j, $m_j \neq 0$: **Majorana particles (assumed).**

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ **eV;**
the value of $\min(m_j)$ **and the "ordering" unknown.**

Δm^2_{21}, $|\Delta m^2_{31}|$ - **known.**

The PMNS matrix U - 3×3 **unitary:** θ_{12}, θ_{13}, θ_{23} - **known; CPV phases** δ, α_{21}, α_{31} - **unknown.**

Thus, 5 known + 4 unknown parameters + MO.

"Known" = measured; "unknown" = not measured.

m_e, m_μ, m_τ **also known - used as input.**

Example: Lepton Flavour Models Based on S_4 (Seesaw Models without Flavons)

P.P. Novichkov et al., arXiv:1811.04933

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism.

The fields involved:

- two Higgs doublets H_u and H_d; transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, $k_u = k_d = 0$;
- three lepton $SU(2)$ doublets L_1, L_2, L_3; furnish a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_L = 2$;
- three neutral lepton gauge singlets N_1^c, N_2^c, N_3^c; transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_N = 0$;
- three charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c; transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1}', \mathbf{1}, \mathbf{1}'$ and carry weights $k_{1,2.3} = 0, 2, 2$.

With these assumptions, the superpotential has the form:

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c\, L\, f_{E_i}(Y)\right)_{\mathbf{1}} H_d + g \left(N^c\, L\, f_N(Y)\right)_{\mathbf{1}} H_u + \Lambda \left(N^c\, N^c\, f_M(Y)\right)_{\mathbf{1}}$$

$\alpha_{1,2,3}$, g, g', Λ are constants.

We work in a basis in which the S_4 generators S and T are represented by symmetric matrices for all irreducible representations $\mathbf{r}$. In this basis the triplet irreps of S and T to be used read:

$$S = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \;\; T = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$**. The plus (minus) corresponds to the irrep $\mathbf{3}$ ($\mathbf{3'}$) of S_4.**

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

By specifying the weights of the matter fields one obtains the weights of the relevent modular forms.
After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings, λ and $\mathcal{Y}$, as well as the Majorana mass matrix M for heavy neutrinos, are generated:

$$W = \lambda_{ij}\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}\, N_i^c\, L_j\, H_u + \frac{1}{2}\, M_{ij}\, N_i^c\, N_j^c\,,$$

a sum over $i, j = 1, 2, 3$ is assumed. After integrating out N^c and after EWS breaking, the charged lepton mass matrix M_e and the light neutrino Majorana mass matrix M_ν are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$M_e = v_d\, \lambda^\dagger\,, \quad v_d \equiv \mathrm{vev}(\mathrm{H_d^0})\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \quad v_u \equiv \mathrm{vev}(\mathrm{H_u^0})\,.$$

The Majorana mass term for heavy neutrinos

Assume $k_\Lambda = 0$, i.e., no non-trivial modular forms are present in $\Lambda\,(N^c\,N^c\,f_M\,(Y))_1$, $k_N = 0$, and for both $\rho_N \sim \mathbf{3}$ or $\rho_N \sim \mathbf{3}'$

$$(N^c\,N^c)_1 = N^c_1\,N^c_1 + N^c_2\,N^c_3 + N^c_3\,N^c_2\,,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\,\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \textbf{for} \quad k_\Lambda = 0\,.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale Λ, which can be rendered real. The Majorana mass term conserves a "non-standard" lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

The neutrino Yukawa couplings

The lowest non-trivial weight, $k_L = 2$, leads to

$$g\left(N^c\, L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g'\left(N^c\, L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u\,.$$

There are 4 possible assignments of ρ_N and ρ_L we consider. Two of them, namely $\rho_N = \rho_L \sim \mathbf{3}$ and $\rho_N = \rho_L \sim \mathbf{3'}$ give the following form of $\mathcal{Y}$:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix}\right], \quad \textbf{for} \quad k_L + K_N = 2 \quad \textbf{and} \quad \rho_N = \rho_L\,.$$

The two remaining combinations, $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3'})$ and $(\mathbf{3'}, \mathbf{3})$, lead to:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}\right], \quad \textbf{for} \quad k_L + k_N = 2 \quad \textbf{and} \quad \rho_N \neq \rho_L\,.$$

In both cases, up to an overall factor, the matrix $\mathcal{Y}$ depends on one complex parameter g'/g and the VEV of τ, vev(τ).

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 - \varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

The charged lepton Yukawa couplings

In the minimal (in terms of weights) viable possibility for $L_{1,2,3}$ furnishing a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3'}$, and carrying a weight $k_L = 2$, and $E^c_{1,2,3}$ transforming as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1'}\,,\mathbf{1}\,,\mathbf{1'}$ (up to permutations) and carrying weights $k_{1,2.3} = 0, 2, 2$, the relevant part of W, W_e, can take 6 different forms which lead to the same matrix U_e diagonalising $M_e M_e^\dagger = v_d^2\, \lambda^\dagger \lambda$, and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to $\rho_L = \mathbf{3}$, $\rho_1 = \mathbf{1'}$, $\rho_2 = \mathbf{1}$, $\rho_3 = \mathbf{1'}$:

$$\alpha \left(E_1^c\, L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c\, L\, Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c\, L\, Y^{(4)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d \,.$$

This leads to

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1 Y_4 - Y_2 Y_5) & \beta\,(Y_1 Y_3 - Y_2 Y_4) & \beta\,(Y_1 Y_5 - Y_2 Y_3) \\ \gamma\,(Y_1 Y_4 + Y_2 Y_5) & \gamma\,(Y_1 Y_3 + Y_2 Y_4) & \gamma\,(Y_1 Y_5 + Y_2 Y_3) \end{pmatrix} \,,$$

In this “minimal” example the matrix λ depends on 3 free parameters, α, β and γ, which can be rendered real by re-phasing of the charged lepton fields.

We recall that

$$M_e = v_d\, \lambda^\dagger \,, \qquad v_d \equiv \mathrm{vev}(\mathrm{H_d^0}) \,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y} \,, \qquad v_u \equiv \mathrm{vev}(\mathrm{H_u^0}) \,.$$

Parameters of the model: α, β, γ, g^2/Λ − real; g' and VEV of τ − complex, i.e., 6 real parameters + 2 (1) phases for description of 12 observables (3 charged lepton masses, 3 neutrino masses, 3 mixing angles and 3 CPV phases). Excellent description of the data is obtained also for real g' (i.e., 6 real parameters + 1 phase, employing gCP).

The 3 real parameters $v_d\alpha$, β/α, γ/α − fixed by fitting m_e, m_μ and m_τ. The remaining 3 real parameters and 2 (1) phases − $v_u^2 g^2/\Lambda$, $|g'/g|$, $|\tau|$ and $\arg(g'/g)$, $\arg\tau$ ($\arg\tau$) − describe the 5 ν measured observables - 3 mixing angles, 2 Δm_{ij}^2.

The model considered leads to testable predictions for $\min(m_j)$ ($\sum_i m_i$), type of the ν mass spectrum (NO or IO), the 3 CPV Dirac and Majorana phases; predicted are also $|\langle m\rangle|$, the range of θ_{23}, as well as of correlations between different observables.

Seven real parameters (5 real couplings + the complex VEV of τ) − is the minimal number of parameters in the constructed so far phenomenologically viable lepton flavour models with massive Majorana neutrinos based on modular invariance.

Numerical Analysis

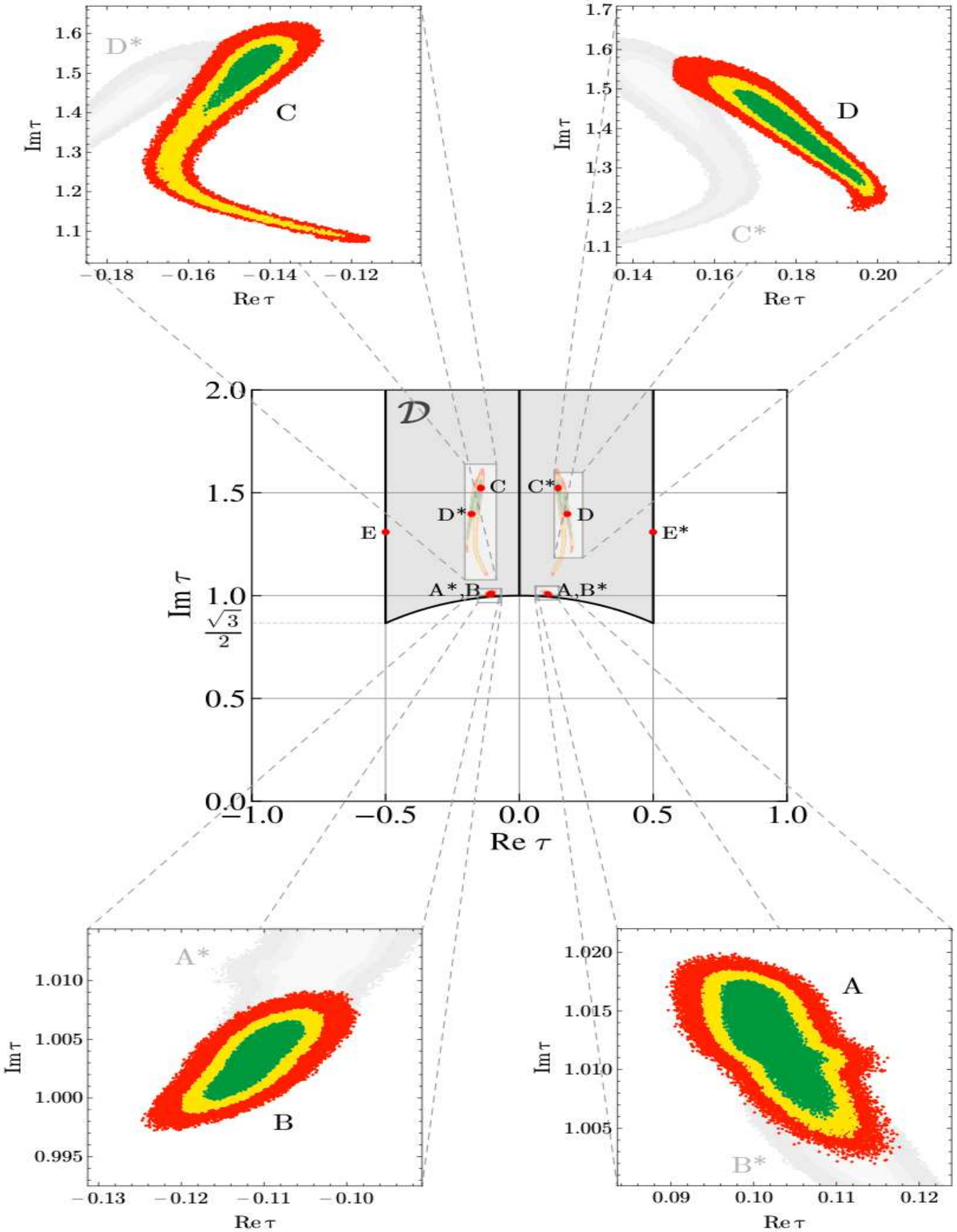
Each model depends on a set of dimensionless parameters

$$p_i = (\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots)\,,$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: $v_d\,\alpha$ for M_e and $v_u^2\,g^2/\Lambda$ for M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarized in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \mathrm{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\lvert\Delta m^2\rvert/(10^{-3}\ \mathrm{eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/\lvert\Delta m^2\rvert$	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2\theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

Best fit values and 1σ ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best fit value and 1σ range of δ did not drive the numerical searches here reported.



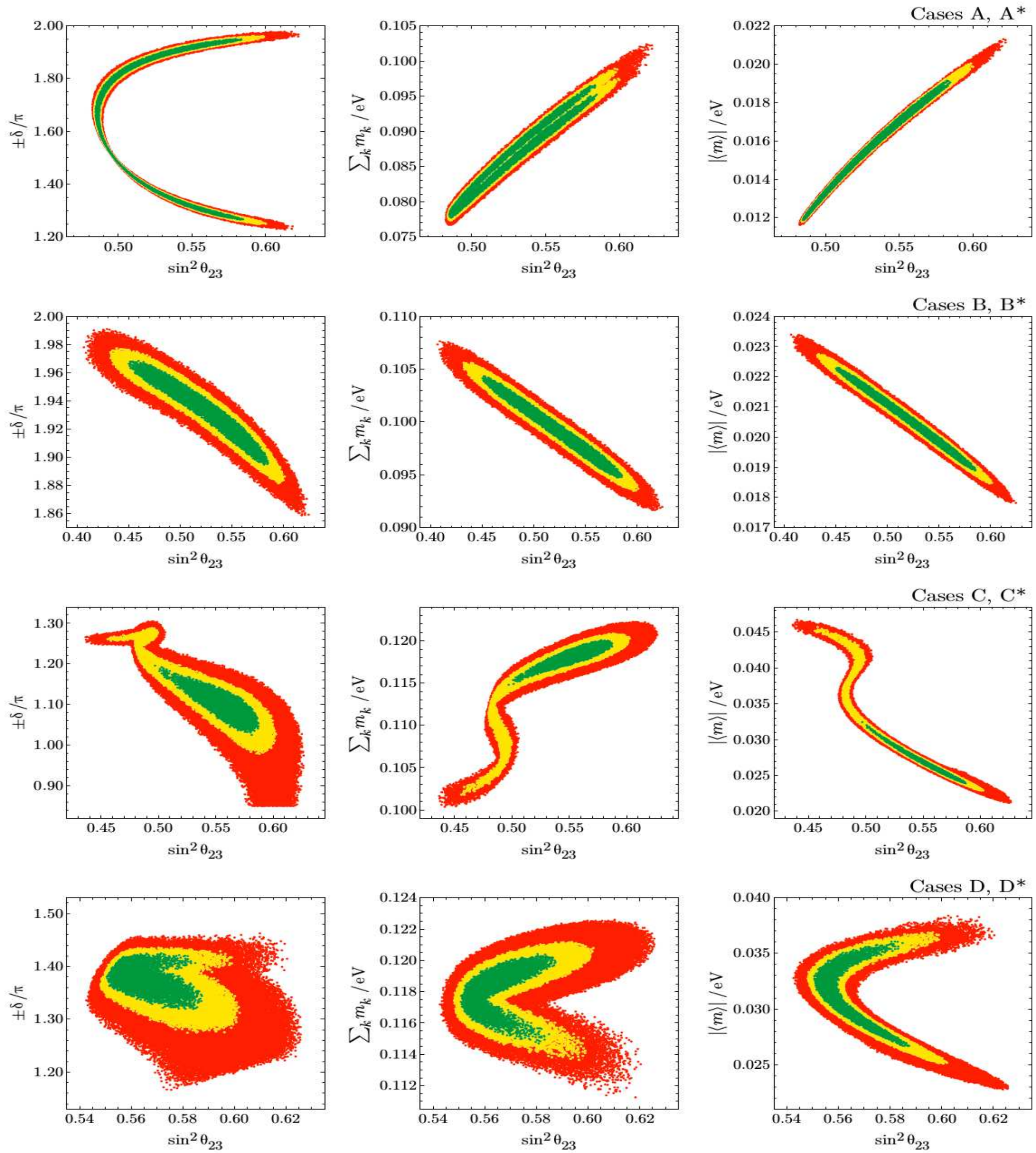
P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	2σ range	3σ range
$\mathrm{Re}\,\tau$	± 0.1045	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
$\mathrm{Im}\,\tau$	1.01	1.006 – 1.018	1.004 – 1.018
β/α	9.465	8.247 – 11.14	7.693 – 12.39
γ/α	0.002205	0.002032 – 0.002382	0.001941 – 0.002472
$\mathrm{Re}\,g'/g$	0.233	$-0.02383 - 0.387$	$-0.02544 - 0.4417$
$\mathrm{Im}\,g'/g$	± 0.4924	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d\,\alpha$ [MeV]	53.19		
$v_u^2\,g^2/\Lambda$ [eV]	0.00933		
m_e/m_μ	0.004802	0.004418 – 0.005178	0.00422 – 0.005383
m_μ/m_τ	0.0565	0.048 – 0.06494	0.04317 – 0.06961
r	0.02989	0.02836 – 0.03148	0.02759 – 0.03224
δm^2 [10^{-5} eV^2]	7.339	7.074 – 7.596	6.935 – 7.712
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	2.413 – 2.494	2.392 – 2.513
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3313	0.2656 – 0.3449
$\sin^2\theta_{13}$	0.02125	0.01988 – 0.02298	0.01912 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4846 – 0.5846	0.4838 – 0.5999
Ordering	NO		
m_1 [eV]	0.01746	0.01196 – 0.02045	0.01185 – 0.02143
m_2 [eV]	0.01945	0.01477 – 0.02216	0.01473 – 0.02307
m_3 [eV]	0.05288	0.05099 – 0.05405	0.05075 – 0.05452
$\sum_i m_i$ [eV]	0.0898	0.07774 – 0.09661	0.07735 – 0.09887
$\lvert\langle m\rangle\rvert$ [eV]	0.01699	0.01188 – 0.01917	0.01177 – 0.02002
δ/π	± 1.314	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
α_{21}/π	± 0.302	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
α_{31}/π	± 0.8716	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases A and A*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.1045 + i\,1.01$).

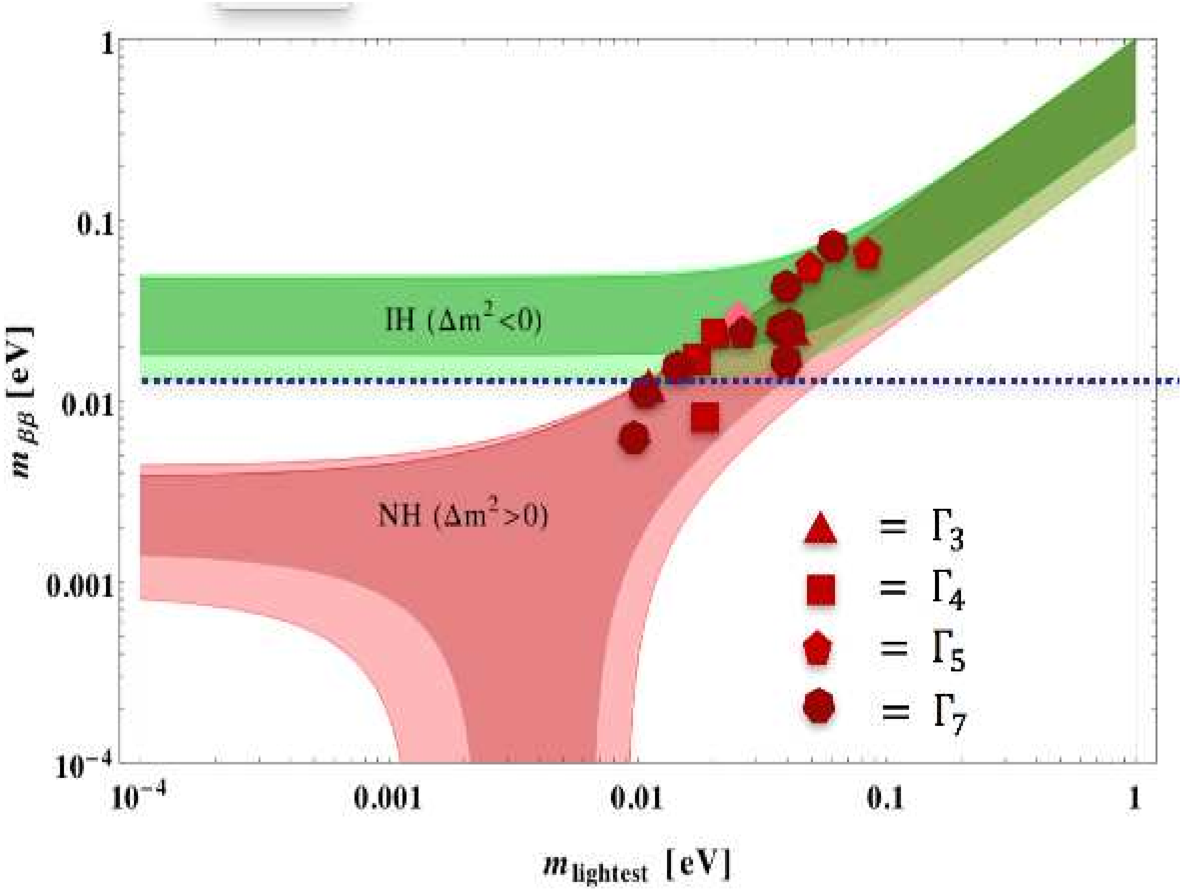
	Best fit value	2σ range	3σ range
$\mathrm{Re}\,\tau$	∓ 0.109	$\mp(0.1051 - 0.1172)$	$\mp(0.103 - 0.1197)$
$\mathrm{Im}\,\tau$	1.005	0.9998 – 1.007	0.9988 – 1.008
β/α	0.03306	0.02799 – 0.03811	0.02529 – 0.04074
γ/α	0.0001307	0.0001091 – 0.0001538	0.0000982 – 0.0001663
$\mathrm{Re}\,g'/g$	0.4097	0.3513 – 0.5714	0.3241 – 0.5989
$\mathrm{Im}\,g'/g$	∓ 0.5745	$\mp(0.5557 - 0.5932)$	$\mp(0.5436 - 0.5944)$
$v_d\,\alpha$ [MeV]	893.2		
$v_u^2\,g^2/\Lambda$ [eV]	0.008028		
m_e/m_μ	0.004802	0.004425 – 0.005175	0.004211 – 0.005384
m_μ/m_τ	0.05649	0.04785 – 0.06506	0.04318 – 0.06962
r	0.0299	0.02838 – 0.03144	0.02757 – 0.03223
δm^2 [10^{-5} eV^2]	7.34	7.078 – 7.59	6.932 – 7.71
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	2.414 – 2.494	2.393 – 2.514
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3314	0.2662 – 0.3455
$\sin^2\theta_{13}$	0.02125	0.0199 – 0.02302	0.01914 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4503 – 0.5852	0.4322 – 0.601
Ordering	NO		
m_1 [eV]	0.02074	0.01969 – 0.02374	0.01918 – 0.02428
m_2 [eV]	0.02244	0.02148 – 0.02522	0.02101 – 0.02574
m_3 [eV]	0.05406	0.05345 – 0.05541	0.05314 – 0.05577
$\sum_i m_i$ [eV]	0.09724	0.09473 – 0.1043	0.0935 – 0.1056
$\lvert\langle m\rangle\rvert$ [eV]	0.01983	0.01889 – 0.02229	0.01847 – 0.02275
δ/π	± 1.919	$\pm(1.895 - 1.968)$	$\pm(1.882 - 1.977)$
α_{21}/π	± 1.704	$\pm(1.689 - 1.716)$	$\pm(1.681 - 1.722)$
α_{31}/π	± 1.539	$\pm(1.502 - 1.605)$	$\pm(1.484 - 1.618)$
$N\sigma$	0.02435		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases B and B*, (which refer to $(k_\Lambda, k_g) = (0,2)$ and $\tau = \pm 0.109 + i\,1.005$).



P.P. Novichkov et al., arXiv:1811.04933

Predictions for the neutrinoless double beta decay effective Majorana mass.



F. Feruglio, talk at Bethe Colloquium, 18/06/2020

Predictions of modular invariant models of lepton flavoour for the neutrinoless double beta decay effective Majorana mass. The predictions are in the range of sensitivity of some of the current and upcoming neutrinoless double beta decay experiments (LEGEND, nEXO, KamLAND-Zen II, NEXT).

CP Symmetry in Modular Invariant Flavour Models

The formalism of combined finite modular and generalised CP (gCP) symmetries for theories of flavour was developed in P.P. Novichkov et al., arXiv:1905.11970.

gCP invariance was shown to imply that the constants g, which accompany each invariant singlet in the superpotential, must be real (in a symmetric basis of S and T and at least for $\Gamma_N^{(\prime)}$, $N \leq 5$). Thus, the number of free parameters in modular-invariant models which also enjoy a gCP symmetry gets reduced, leading to "minimal" models which have higher predictive power.

In these models, the only source of both modular symmetry breaking and CP violation is the VEV of the modulus τ.

The "minimal" phenomenologically viable modular-invariant flavour models with gCP symmetry constructed so far
– of the lepton sector with massive Majorana neutrinos (12 observables) contain $\geq$ 7 real parameters – 5 real couplings + the complex τ (6 real constants + 1 phase);
– of the quark sector contain $\geq$ 9 real parameters – 7 real coulplings + the complex τ;
– while the models of lepton and quark flavours (22 observables) have $\geq$ 15 real parameters - 13 real couplings + the complex τ.

See, e.g., B.-Y. Qu et al., arXiv:2106.11659

Under the CP transformatoion,

$$\tau \xrightarrow{\text{CP}} -\tau^* \,.$$

P.P. Novichkov et al., 1905.11970; A. Baur et al., 1901.03251 and 1908.00805

It was further demonstrated that CP is conserved for

$$\text{Re}\tau = \pm 1/2\,; \;\; \tau = e^{i\theta}\,,\; \theta = [\pi/3, 2\pi/3]\,; \text{Re}\tau = 0\,,\; \text{Im}\tau \geq 1\,.$$

i.e., for the values of τ's VEV at the boundary of the fundamental domain and on the imaginary axis.

Further Developments

Progress on the problem of having the charged lepton (and quark) mass hierarchies determined by the properties of the modular forms (without fine tuning). The problem studied in H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al., 2101.08718 (see also G-J. Ding et al., 1910.03460).

A possible solution was proposed in P.P. Novichkov, J.T. Penedo, STP, arXiv:2102.07488, where also a non-fine-tuned model of lepton flavour was constructed. Makes use of the existence of values of the VEV of τ (fixed points) which break the modular symmetry only partially to certain residual discrete symmetries.

Interesting results were obtained in studies of the modulus stabilisation (i.e., finding the VEV of the modulus as minima of modulus potential derived from "first principles").

The "top-down" approach was also actively being developed based on ultraviolet complete theories
(some of the latest publications include (the list is far from complete): Chen:2019ewa (M.-C. Chen et al.), Nilles:2020nnc,Kobayashi:2020hoc, Kobayashi:2020uaj,Nilles:2020kgo,Kikuchi:2020frp,Nilles:2020tdp, Kikuchi:2020nxn,Baur:2020jwc,Ishiguro:2020nuf,Nilles:2020gvu, Ishiguro:2020tmo,Hoshiya:2020hki,Baur:2020yjl,Almumin:2021fbk).

Recently in A. Baur et al., arXiv:2207.10677, the first string-derived flavor model with realistic phenomenology, based on "eclectic" flavour symmetry, one "component" of which is the T' modular symmetry, was constructed.

Before concluding: concise review of the literature on modular invariance approach to the flavour problem.

Bottom-up modular invariance approaches to the lepton flavour problem have been exploited first using the finite modular groups

$\Gamma_3 \simeq A_4$ (F. Feruglio, 1706.08479; J.C. Criado, F. Feruglio, 1807.01125);

$\Gamma_2 \simeq S_3$ (T. Kobayashi et al., 1803.10391);

$\Gamma_4 \simeq S_4$ (J.T. Penedo, S.T. Petcov, 1806.11040, minimal, no flavons).

After these first studies, the interest in the approach grew significantly and models based on these and othere groups have been constructed and extensively studied:

$\Gamma_4 \simeq S_4$
(Novichkov:2018ovf,Kobayashi:2019mna,Okada:2019lzv,Kobayashi:2019xvz,Gui-JunDing:2019wap,Wang:2019ovr,Wang:2020dbp,Gehrlein:2020jnr);

$\Gamma_5 \simeq A_5$
(P.P. Novichkov et al., 1812.02158; Ding:2019xna,Gehrlein:2020jnr);

$\Gamma_3 \simeq A_4$
(Kobayashi:2018scp, Novichkov:2018yse, Nomura:2019jxj, Nomura:2019yft, Ding:2019zxk, Okada:2019mjf, Nomura:2019lnr, Asaka:2019vev, Gui-JunDing:2019wap, Zhang:2019ngf,

Nomura:2019xsb, Kobayashi:2019gtp, Wang:2019xbo, Abbas:2020vuy, Okada:2020dmb, Ding:2020yen, Behera:2020sfe, Nomura:2020opk, Nomura:2020cog, Behera:2020lpd, Asaka:2020tmo, Nagao:2020snm, Hutauruk:2020xtk);

$\Gamma_2 \simeq S_3$ (Okada:2019xqk,Mishra:2020gxg);

$\Gamma_7 \simeq PSL(2, \mathbb{Z}_7)$ (G.-J. Ding et al., 2004.12662).

Similarly, attempts have been made to construct viable models of quark flavour and of quark-lepton unification (including based on GUTs): (H.Okada, M. Tanimoto, 1812.09677, 1905.13421; T. Kobayashi et al., 1906.10341; Kobayashi:2018wkl,Lu:2019vgm,Abbas:2020qzc, Okada:2020rjb,Du:2020ylx,Zhao:2021jxg,Chen:2021zty,Ding:202

The formalism of the interplay of modular and gCP symmetries has been developed and first applications made
(P.P. Novichkov et al., 1905.11970);
it was further extensively explored
(Kobayashi:2019uyt,Okada:2020brs,Yao:2020qyy,Wang:2021mkw,Qu:2021jdy),
as was the possibility of coexistence of multiple moduli
(P.P. Novichkov et al., 1811.04933 and 1812.11289 (pheno); deMedeiros-Varzielas:2019cyj,King:2019vhv,deMedeirosVarzielas:2020kji, Ding:2020zxw).

Modular invariant theories of flavour with more than one modulus, based on simplectic modular groups were also developed
(G.-J. Ding et al., 2010.07952 and 2102.06716).

The formalism of double covers Γ'_N has been developed and viable flavour models constructed for the cases of
$\Gamma'_3 \simeq T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$
(X.-G. Liu, G.-J. Ding, 1907.01488 (T'); P.P. Novichkov et al., 2006.03058 (S'_4): X. Wang et al., 2010.10159 (A'_5); Liu:2020akv, Yao:2020zml);
the formalism of metaplectic (two-fold) cover group of the modular group $SL(2,\mathbb{Z})$, involving half-integral (rational) weigth modular forms, has also been developed (X.-G. Liu et al., 2007.13706).

It was also realised that there esist three fixed (symmetry) points of the VEV of τ, $\tau_{\text{sym}} = \omega(= -1/2 + i\sqrt{3}/2)$, $i\infty$, i (in the mod. group fund. domain), at which the flavour (=finite modular) symmetry Γ_N (Γ'_N) is broken to non-trivial residual symmetries, $\mathbb{Z}_3^{ST}$, $\mathbb{Z}_N^T$ and $\mathbb{Z}_2^S$ ($\mathbb{Z}_4^S \times \mathbb{Z}_2^R$)
(P.P. Novichkov et al., 1811.04933 (2006.03058));
this fact was further exploited in flavour model building
(Novichkov:2018yse,Novichkov:2018nkm,Okada:2020brs)
and especially in connection with the posibility to build viable flavour models with observed charged lepton (quark) mass hirarchies in the vicinity of the symmetry points
(H. Okada, M. Tanimoto, 2009.14242, 2012.0188; F. Feruglio et al.,

2101.08718)
even without fine-tuning (P.P. Novichkov et al., 2102.07488).

It was shown also that one can have successful leptogeneis in theories with modular flavour symmetries
(T. Asaka et al., 1909.06520; X. Wnag, S. Zhou, 1910.09473; H. Okada et al., 2105.14292).

The bottom-up analyses are expected to eventually connect with the results of the top-down approach based on ultraviolet-complete theories (Kobayashi:2018rad,Kobayashi:2018bff,deAnda:2018ecu,Baur:2019kwi, Kariyazono:2019ehj,Baur:2019iai,Nilles:2020nnc,Kobayashi:2020hoc, Abe:2020vmv,Ohki:2020bpo,Kobayashi:2020uaj,Nilles:2020kgo,Kikuchi:2020frp, Nilles:2020tdp,Kikuchi:2020nxn,Baur:2020jwc,Ishiguro:2020nuf,Nilles:2020gvu, Ishiguro:2020tmo,Hoshiya:2020hki,Baur:2020yjl,Kikuchi:2021ogn).

The presented list of publications is not exhaustive.

Instead of Conclusions

To summarise, there is still very important work to be done in the field of the modular invariance approach to the flavour problem. The stakes are high and worth the efforts: we are trying to develop The Theory of Flavour using the power and the beauty of the modular invariance.