Solutions for the Tutorial on Transverse Beam Dynamics

Questions to relax

1.) *Can you explain in your own words the meaning of …*

phase advance
beam emittance
β-function

… what does it mean, if you are told that the phase advance per cell is 90 degrees in the horizontal and 60 degrees in the vetical plane ?

Concerning the two parameters β-function and beam emittance: they both determine the beam envelope. Can you explain the difference ?

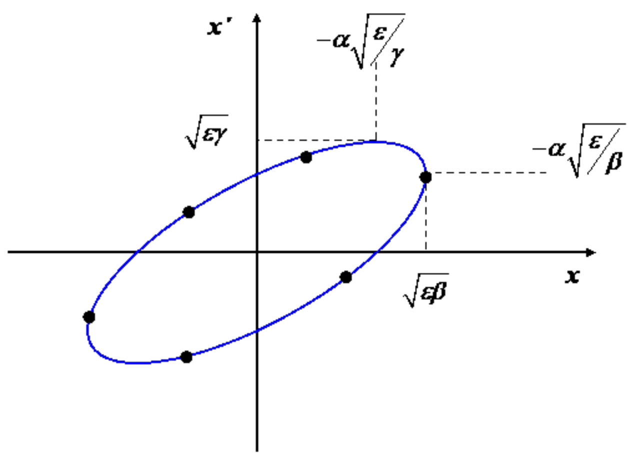
How will the phase space ellipse look like in general ?

Depending on the beta (maximum or minimum) and the α and γ the ellipse will be more or less flat and shaoped & tilted.

General case:

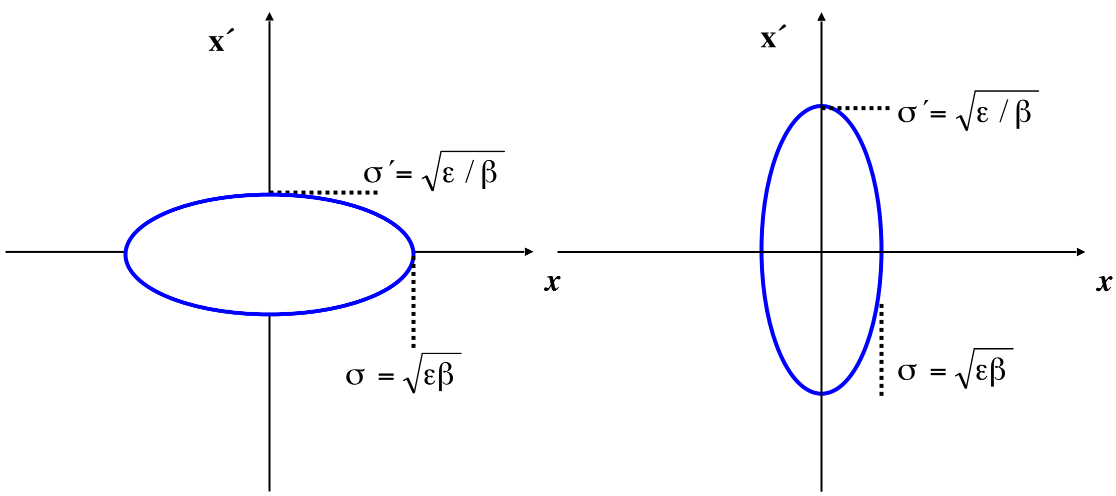
remember that

$$\gamma = \frac{1+\alpha^2}{\beta}$$



And how does it look like at an Interaction point … and why ?

At an IP we have usually $\alpha = 0$ and so we get the following picture: The beam will need the largest aperture if the ellipse is flat, or it will have the largest divergence.



2.) About a real storage ring: LHC: particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN will collide proton beams with a maximum momentum of p=7 TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9m$	
particle momentum	$p = 7\ TeV/c$	
main dipoles	$B = 8.392\ T$	$l_B = 14.2m$
main quadrupoles	$G = 235\ T/m$	$l_q = 5.5m$

Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.

The beam rigidity is obtained in the usual way by the golden rule:

$$B * \rho = \frac{p}{e} = \frac{1}{0.299792} * p[GeV/c]$$

$$= \frac{1}{0.299792} * 7000\ Tm = 23350\ Tm$$

and knowing the magnetic dipole field we get

$$\rho = \frac{7000\ Tm}{8.392T * 0.299792} = 2780m$$

... and determine the number of dipoles that is needed in the machine.

The bending angle for one LHC dipole magnet:

$$\alpha = \frac{l}{\rho} = \frac{14.2m}{2780m} = 5.108\ mrad$$

and as we want to have a closed storage ring we require an overall bending angle of 2π:

$$N = \frac{2\pi}{\alpha} = \frac{6.28rad}{5.108mrad} = 1230\ Magnets$$

Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

We can use the beam rigidity (or the particle momentum) to calculate the normalized quadrupole strength:

$$k = \frac{g}{B * \rho} = \frac{g}{p/e} = 0.299792 * \frac{g}{p[GeV/c]}$$

$$k = 0.299792 * \frac{235\ T/m}{7000\ [GeV/c]} = 0.01 \frac{1}{m^2}$$

$$f = \frac{1}{k * l_q} = \frac{1}{0.01/m^2 * 5.5m} = 18.2m \quad > \quad l_q$$

The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

How does the matrix for such a (foc.) magnet look like?
How would you establish a description of this magnet in thin lens approximation?
Compare the matrix elements.
Nota bene: in our notation a foc. magnet has a negative k-value.

The matrix of a focusing quadrupole is given by

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{|k|} * l) & \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|} * l) \\ -\sqrt{|k|} * \sin(\sqrt{|k|} * l) & \cos(\sqrt{|k|} * l) \end{pmatrix}$$

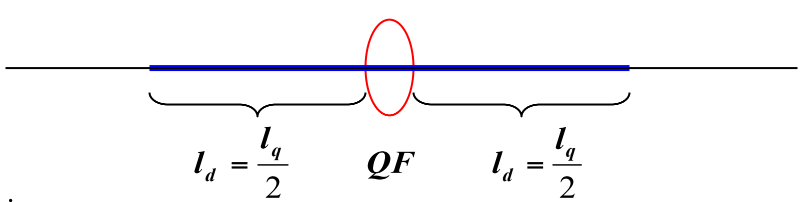
$$M_{QF} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{|f|} & 1 \end{pmatrix}$$

with the focal length $$f = \frac{1}{|kl_q|} = \frac{1}{0.01/m^2 * 5.5m} = 18.2m$$

But we should not forget the overall length of the beast: The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



So we write:

$$M_{thinlens} = \begin{pmatrix} 1 & l_q/2 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_q/2 \\ 0 & 1 \end{pmatrix}$$

Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2} * k * l_q & \frac{l_q}{2} * (2 + k * l_q * \frac{l_q}{2}) \\ k * l_q & 1 + k * l_q \end{pmatrix}$$

With the parameters in the example we get finally

$$\left. \begin{aligned} &\frac{l_q}{2} = 2.75\, m, \\ &k * l_q = \frac{-1}{|f|} = \frac{-1}{18.2 m} = -0.0549 \frac{1}{m} \end{aligned} \right\} \quad M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

which is still quite close to the result of the exact calculation above.

3.) Beam rigidity & particle momentum *(court. Ted Wilson)*

… or the stupid question: after all we have to deal with a relativistic beam!

A synchrotron of 25m radius accelerates protons from a kinetic energy of 50 MeV to 1000 MeV. What is the maximum energy of a deuteron beam (Z=1, A=2) that could be accelerated in the machine ?

The beam rigidity relates the magnetic field to the particle momentum:

$$B * \rho = \frac{p}{e} = 3.333 * p(GeV/c)$$

The momentum is given just by the magnetic dipole field and the bending radius – independent of the particle that is stored in the machine !
Calculation of the momentum of a 1000 MeV proton beam:

$$p^2 c^2 = E^2 - E_0^2$$

kinetic energy: $T = 1000 MeV$

rest energy of a proton: $E_0 = m_0 c^2 = 938 MeV$

overall energy = rest energy + kin. energy: $E = m_0c^2 + T$

$$p^2c^2 = (m_0c^2 + T)^2 - (m_0c^2)^2$$

$$p^2c^2 = (0.938GeV + 1.0GeV)^2 - (0.938GeV)^2$$
$$pc = \sqrt{2.876}\ GeV$$

This is the maximum momentum that can be carried by the machine.

To calculate the kinetic energy for the deuteron we set:

$$p^2c^2 = (T_{deut} + 0.938GeV + 0.939GeV)^2 - (0.938GeV + 0.939GeV)^2 = 2.876GeV^2$$

and solve for T_{deut}

$$T_{deut} = 0.653\ GeV$$

Nota bene: *We do not need the bending radius to obtain this result. But we could use it to calculate the magnetic field B that we need in this machine:*

$$B * \rho = 3.33 * p[GeV/c]$$
$$= 3.33 * \sqrt{2.876}$$

And as $\rho = 25$ m we get $B = 0.22T$

4.) Apertures and Beam Envelopes:

The LHC magnet structure in the arcs consists of a symmetric FoDo with 90° phase advance per cell and an aperture radius of r_0 *=20mm.*
a.) Given the value of β_{max} *= 500m in a QF quadrupole lens, what beam emittance would just touch the vacuum chamber ? (We call this value the "acceptance" of the machine).*

$$\sigma_{\max} = \sqrt{\varepsilon_{\max}\beta} = r_0 = 20\,mm$$

$$\varepsilon_{\max} = \frac{r_0^2}{\beta} = \frac{400\,mm^2}{500\,m} = 8*10^{-7}\,rad\ m$$

b.) If the typical emittance of a stored beam at 450 GeV injection energy is $\varepsilon \approx 7*10^{-9}$ *rad m, how many* σ *of beam envelope fit into the vacuum chamber for* β*=500m ?*

$$\sigma_{beam} = \sqrt{\varepsilon_{typical} * \beta} = \sqrt{7*10^{-9}\,radm * 500\,m}$$

$$\sigma_{beam} = 1.9\,mm$$

$$\rightarrow \quad r_0 \approx 10\sigma$$

the vacuum aperture corresponds to 10 sigma of the stored beam ... indeed there is not much space for errors and distortions.

c) what will happen if – keeping the beam optics constant – you accelerate the beam to an energy of E = 7000 GeV ?

The beam emittance will shrink as a function of the relativistic parameters $\beta*\gamma$. In the given energy range $\beta \approx 1$ and so we can simplify the scaling and set

$$\varepsilon_{7000} = \varepsilon_{450} * \frac{\gamma_{450}}{\gamma_{7000}} = 4.5*10^{-10}$$

The beam dimension will shrink (in both planes) and the beam lifetime and background rates will improve.

During luminosity operation at this energy we require at least 14 sigma aperture due to background and quench safety reasons. What is the maximum beta function that can be accepted if the aperture of our mini beta quadrupoles is 20mm?

$$r_0 = 14*\sqrt{\varepsilon\beta} \quad \rightarrow \frac{r_0^2}{196} = 4.5*10^{-10} * \beta_{\max}$$

$$\beta_{\max} = \frac{400*10^{-6}\,m^2}{196*4.5*10^{-10}\,m} = 4.5\,km$$

5.) Question foir the fun of it:

Let's build a real cheap storage ring. *Just put it to the North Pole and use the magnetic field of the earth whose field lines are perpendicular to the surface at that nice place. Forget about focusing … that's for nitpickers. What will be the size of the ring for a 10 keV electron beam if the earth magnetic field is about 0.5 Gauß? (1 Gauß = 10 $^{-4}$ Tesla).*
And in the end, in which direction do you have to circulate the electrons to get stored beam?

As we all know it is the momentum that defines the magnetic field:

$$\boldsymbol{B} * \rho = \frac{\boldsymbol{p}}{\boldsymbol{e}}$$

So we have to calculate the momentum of the electron beam first and – as it is neither ultra relativistic nor in the classical energy regime – we have to apply the full relativistic stuff.

Overall energy of a particle: $\boldsymbol{E} = \sqrt{\boldsymbol{p}^2\boldsymbol{c}^2 + \boldsymbol{m}^2\boldsymbol{c}^4}$

rest energy of an electron: $\boldsymbol{E}_0 = \boldsymbol{mc}^2 = 511\ \boldsymbol{keV}$

kinetic energy of the beam: $\boldsymbol{E}_{kin} = 10\ \boldsymbol{keV}$

$\Rightarrow E_{total} = 5.21 \cdot 10^5 eV$

calculation of the momentum:

$$p^2c^2 = E_{total}^2 - m^2c^4$$

$$p = \frac{\sqrt{E_{total}^2 - m^2c^4}}{c} = \frac{\sqrt{(5.21 \cdot 10^5 eV)^2 - (5.11 \cdot 10^5 eV)^2}}{2.99792 \cdot 10^8 m/s}$$

$$\rightarrow \quad \frac{\boldsymbol{p}}{\boldsymbol{e}} = \frac{1.02 * 10^5\,\boldsymbol{eV}}{2.99792 * 10^8\,\boldsymbol{m/s} * \boldsymbol{e}} = 3.4 * 10^{-4}\underbrace{\frac{\boldsymbol{Vs}}{\boldsymbol{m}}}_{\boldsymbol{Tm}}$$

bending radius:

$$\rho = \frac{p/e}{B} = \frac{3.4*10^{-4}\ Tm}{5*10^{-5}\ T}$$

$$\rightarrow \rho = 6.8\ m$$

It is astonishing: The storage ring is very small, or in other words: the magnetic field of the earth is quite strong. Indeed, even in HERA we had to compensate for it !!

And now for the beer:

10 keV is similar to the energy in our conventional TV screens - those big and heavy things, based on a electron beam scaning a fluorescent screen. So - if all these considerations are true - given a length of 30cm (for the distance between the TV gun and the screen) the displacement at the screen due to the earth magnetic field is a few millimetres.
So … turning the TV screen around the colours should change !
Do they ?? Or is that all nonsense ???

And finally … in which direction do you have to circulate the electrons to get stored beam?

6.) Dispersion and Chromaticity

Can you explain in your own words the meaning of …
dispersion
chromaticity

7.) Consider a linear collider: *The general structure of such a machine does not differ too much from the arc of a storage ring. Clearly – it is not a circular machine but concerning the optics it is quite similar …*

Does such a Linac have a chromaticity?
And if so how would you correct it ?

8.) … if not explained already in the lecture.
The beta function in the quadrupoles of a symmetric FoDo cell *(i.e. $f_{qd} = -f_{qf}$)*
depends only on the cell length L and the phase advance μ of the cell:

$$\beta_{QF} = L * \frac{1 + \sin\frac{\mu}{2}}{\sin\mu}$$

$$\beta_{QD} = L * \frac{1 - \sin\frac{\mu}{2}}{\sin\mu}$$

a) Establish the matrix of a FoDo in thin lens approximation and proof these relations.
Hint: remember the trigonometric magic tricks $1 - \cos\mu = 2\sin^2\frac{\mu}{2}$

b) Consider a proton beam where in general the emittances are equal in both planes: $\varepsilon_x = \varepsilon_y$
Find the optimum phase advance that will give the smallest value for the radius of a particle beam in both planes

$$r^2 = \sigma_x^2 + \sigma_y^2$$

.

9.) it is a wild world …
During the construction phase of a heavy ion storage ring one quadrupole magnet turned out to be too short by 1 mm: The one meter long yoke was stapled by steel plates, 1mm in thickness each, and one of them just was forgotten (this is no joke !).

Calculate the tune change in both planes if this error is not compensated and the beta functions at the location of the quadrupole are in the hor. and vert. plane respectively

$$\beta_x = 80m, \quad \beta_y = 20m$$

The quadrupole strength is . $k = 2 * 10^{-2}$

The tune shift due to a change in the integrated quadrupole field is given by

$$\Delta Q = \frac{1}{4\pi} * \int_0^{l_q} \Delta k * \beta(s) ds$$

which is in our case approximately

$$\Delta Q \approx \frac{1}{4\pi} * \Delta l_Q * \beta_Q * k_Q$$

As the quadrupole magnet was to short we expect a lower tune in the hor. plane of the quadrupole (and a higher tune in the vertical one).

$$\Delta Q_{hor} \approx -\frac{1}{4\pi} * 1 * 10^{-3} m * 80 m * 2 * 10^{-2} \, {}^{1}\!/_{m^2}$$

$$\Delta Q_{hor} \approx -0.00013$$

In the vertical plane using $\beta = 20m$ instead we get

$$\Delta Q_{vert.} \approx +0.000032$$

Lets assume that the beam will survive this error. (Clearly we corrected the error nevertheless).

Now lets have a look at a typical mini beta insertion:
Given the following parameters:

$$k = 3.4 * 10^{-2} \frac{1}{m^2}, \quad lq = 1.9 m, \quad \beta = 1{,}6 km$$

$$\Delta Q \approx \frac{1}{4\pi} * \Delta k_Q * \beta_Q * l_Q = 0.00013$$

$$\Delta k_Q = \frac{4\pi * 0.00013}{1600 m * 1.9 m} = 5.4 * 10^{-7}$$

which corresponds to a relative error of

$$\frac{\Delta k_Q}{k_Q} = \frac{5.4 * 10^{-7}}{3.4 * 10^{-2}} = 1.6 * 10^{-5}$$

A small error in an arc quadrupole (be it the length of the magnet or the gradient) turns out to be a severe problem at places where the beta function is high.
Message: At the mini beta insertions you should use the best hard ware that you can get.

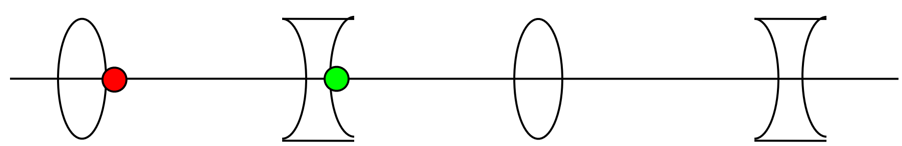
10.) Tuning Quadrupoles:

The main dipole and quadrupole magnets in a storage ring are often powered in series by one power supply. While such a set up facilitates the tracking of the main dipole and quadrupole magnets during acceleration it requires special "tuning" quadrupole circuits for tune adjustments. Assume your machine has one tuning quadrupole per plane, placed at a location where β_x = 180m, β_y =40m.

If the overall tune is Q=64.28, what is the maximum tuning range of this system if the maximum acceptable beta-beat in the machine due to the tuning is limited to10% ?

How can the system described above be improved, to obtain a larger tuning range with a beta-beat that is still smaller than 10% ?

Consider the following schematic lattice:



The position of the tuning quadrupole in the hor. and vertical plane is indicated by the red and green spot.

If the working point of the machine is changed we will obtain – due to the corresponding change of the quadrupole strength – a beta beat of

$$\frac{\Delta\beta}{\beta} = \frac{1}{2sin(2\pi Q)}\int \beta_s \Delta k \; cos \; 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q)ds$$

For one single tuning quadrupole per plane we get approximately:

$$\frac{\Delta\beta}{\beta} = \frac{1}{2sin(2\pi Q)}\beta_s \Delta k l_q \underbrace{cos \; 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q)}_{=1 \text{ for worst case}} ds$$

Using Q=0.28 and β=180m

we require: $\frac{\Delta\beta}{\beta} = 0.509 * 180m\Delta kl \le 10\%$

and the resulting limit for the integrated quadrupole strength is

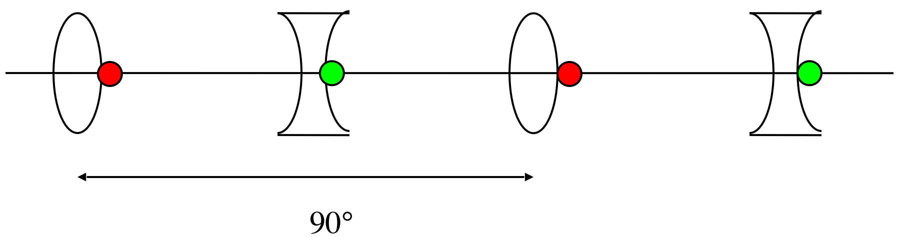
$$\Delta kl \leq 1.1 * 10^{-3}$$

The tune change that can be achieved with that quadrupole strength is

$$\Delta Q = \frac{1}{4\pi} \Delta kl * \beta = 0.015$$

… indeed not very much.

b.) improved lattice for tuning quadrupoles



Assume a phase advance of Φ=90° … as usual.
Consider one cell, starting from the middle of a tuning quadrupole to the next equivalent one. If both tuning quadrupoles are powered in series, the beta beat resulting from this pair is

$$\frac{\Delta\beta}{\beta} = \frac{1}{2sin(2\pi Q)} \int \beta_{\tilde{s}} \Delta k \ cos \ 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q) + \beta_{\tilde{s}} \Delta k \ cos \ 2(\Phi_{\tilde{s}} + 90^o - \Phi_s - \pi Q) ds$$

$$\frac{\Delta\beta}{\beta} = \frac{1}{2sin(2\pi Q)} \int \beta_{\tilde{s}} \Delta k \ cos \ 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q) + \beta_{\tilde{s}} \Delta k \ cos \ 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q + 90^o) ds$$

$$\frac{\Delta\beta}{\beta} \approx \frac{\beta_{\tilde{s}} \Delta k l_q}{2sin(2\pi Q)} \{ \ cos \ 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q) + cos 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q) \cdot \underbrace{cos(2 \cdot 90^o)}_{=-1} - sin 2(\Phi_{\tilde{s}} - \Phi_s - \pi Q) \cdot \underbrace{sin(2 \cdot 90^o)}_{=0}) \}$$

$$\frac{\Delta\beta}{\beta} \approx 0$$

The remaining beta beat from two tuning quadrupoles being 90° apart from each other is – very close to – zero. In a real machine in general we will therefore install a large number of tuning quadrupoles (e.g. one at each main quad) and arrange this scheme in a way that the phase advance between these tuning quadrupole magnets is modulo 90°.