LONGITUDINAL BEAM DYNAMICS

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COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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LESSON 4: SYNCHROTRON MOTION

MODULE 8: LINEAR SYNCHROTRON MOTION

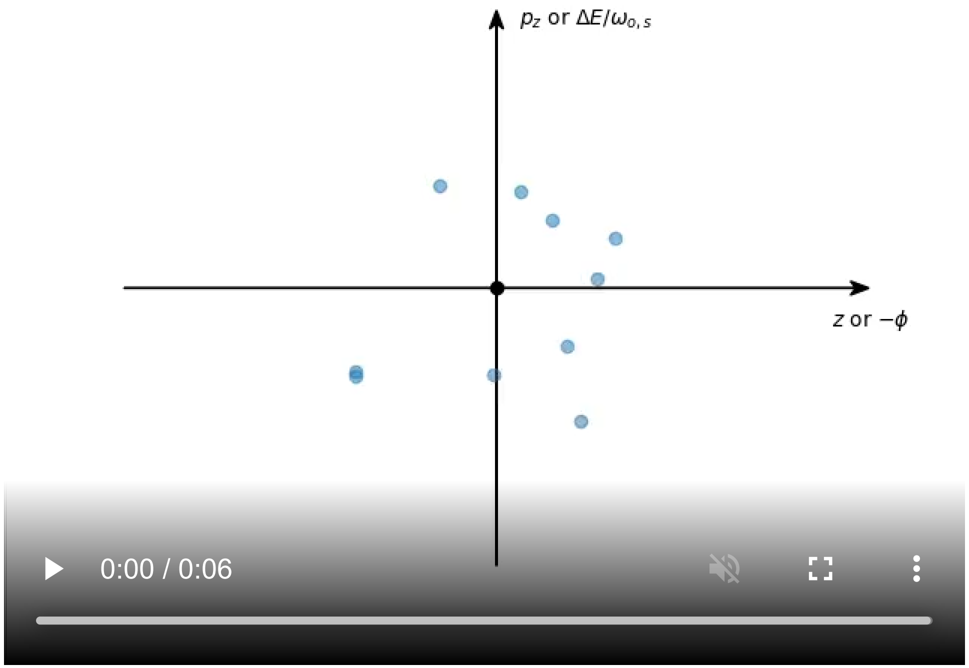
→ **Combined linear equations of motion**

→ **Linear synchrotron frequency, tune**

→ **Phase stability, transition crossing**

→ **Emittance, adiabaticity**

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

- The longitudinal equations of motion describe the evolution of the phase ϕ and energy ΔE of an arbitrary particle compared to the synchronous particle.

COMBINING THE EQUATIONS OF MOTION

The two equations of motion are inter-dependant and can be combined (*note that we replaced* $\omega_{0,s}$ *by* ω_r / h*, which is equivalent and will become relevant later*)

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_r}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right] \quad (1)$$

$$\frac{d\phi}{dt} = \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right) \quad (2)$$

By incorporating (2) in (1), we get

$$\frac{d}{dt}\left(\frac{d\phi}{dt}\frac{\beta_s^2 E_s}{\eta\omega_r^2}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

COMBINING THE EQUATIONS OF MOTION

We will first make two important approximations:

- The machine and beam parameters s are changing slowly with time (only ϕ and ΔE are functions of time)
- We consider small phase oscillations $\Delta\phi = \phi - \phi_s$ (reminder: $\dot{\phi}_s = 0$ by definition)

The $\sin$ functions on the right hand side are linearized

$$\begin{aligned}\sin(\phi) - \sin(\phi_s) &= \sin(\phi_s + \Delta\phi) - \sin(\phi_s) \\ &= \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi - \sin\phi_s \\ &\approx \cos\phi_s \Delta\phi\end{aligned}$$

COMBINING THE EQUATIONS OF MOTION

The approximations lead to

$$\frac{d^2\Delta\phi}{dt^2} = \frac{qV_{\mathrm{rf}}\eta\omega_r^2}{2\pi h\beta_s^2 E_s}\cos\phi_s\Delta\phi$$
$$\implies \frac{d^2\Delta\phi}{dt^2} + \omega_{s0}^2\Delta\phi = 0$$

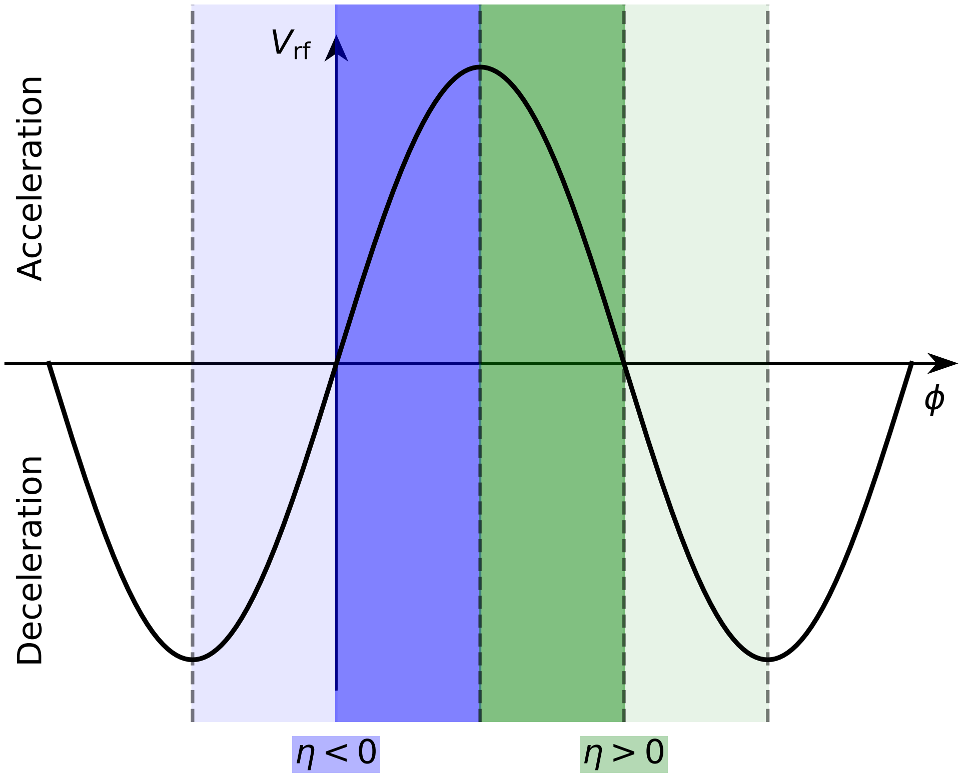
where the linear synchrotron (angular) frequency is defined as (**beware $\boldsymbol{\omega_{s0} \neq \omega_{0,s}}$**)

$$\omega_{s0} = 2\pi f_{s_0} = \sqrt{-\frac{qV_{\mathrm{rf}}\omega_r^2\eta\cos\phi_s}{2\pi h\beta_s^2 E_s}}$$

The motion of the particles in the longitudinal phase space (synchrotron motion) is a harmonic oscillator for small $\Delta\phi$, under the condition that $\boldsymbol{\eta\cos\phi_s < 0}$.

PHASE STABILITY

EXAMPLE FOR POSITIVELY CHARGED PARTICLES



- The phase stability condition

$$\eta \cos \phi_s < 0$$

imposes that the synchronous phase is

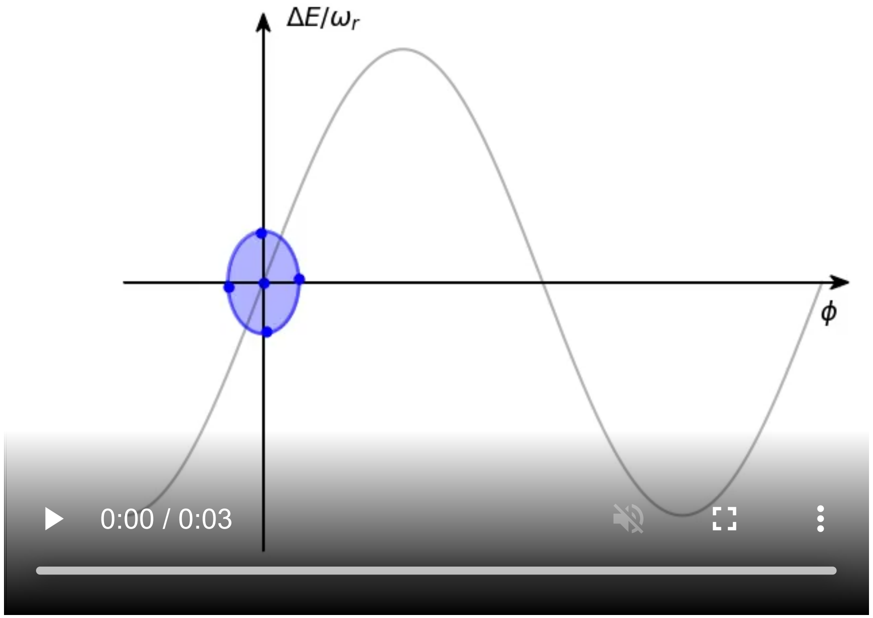
$$\eta < 0 \rightarrow \phi_s \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

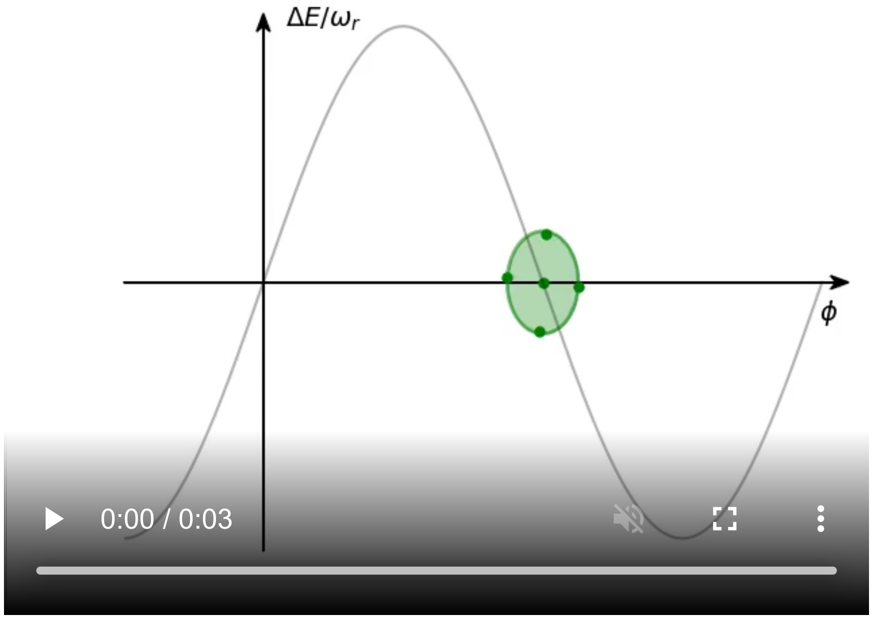
$$\eta > 0 \rightarrow \phi_s \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$$

- A non-synchronous particle rotates around the synchronous particle only if the phase stability condition is fulfilled.

PHASE STABILITY

EXAMPLE FOR POSITIVELY CHARGED PARTICLES, INTUITIVELY

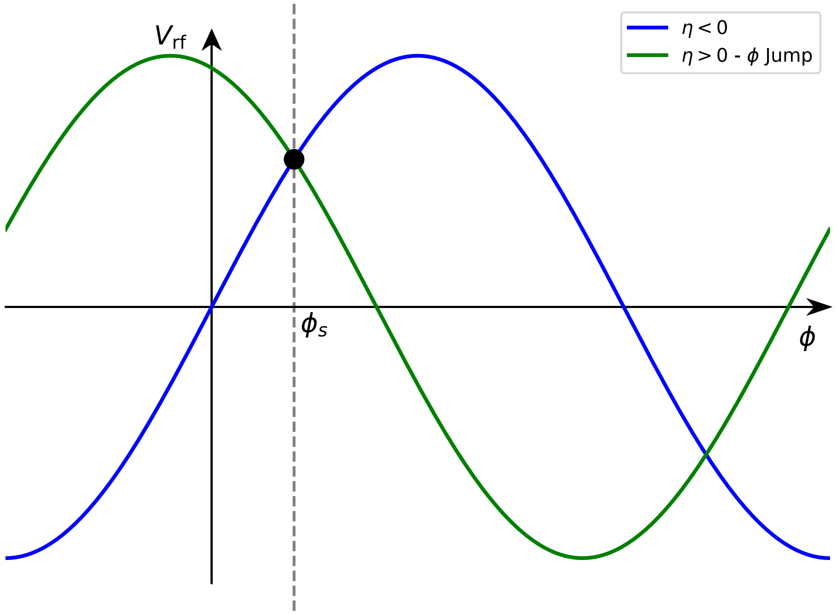




- Below transition $\eta < 0$, early particles should lose energy (velocity) and be delayed.
- Above transition $\eta > 0$, early particles should gain energy to travel a longer orbit and be delayed.

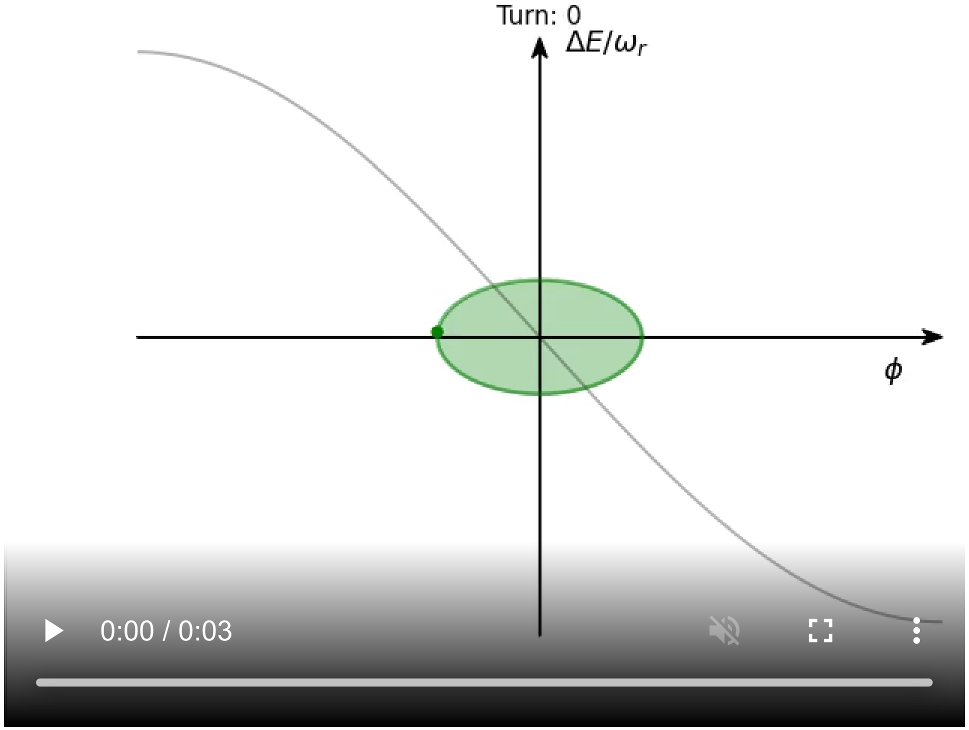
TRANSITION CROSSING

During acceleration, transition energy is crossed and η changes sign. As a reminder, $\eta\left(t\right) = \alpha_p - 1/\gamma_s^2\left(t\right)$.



In that occurence, the phase of the RF phase after transition must be rapidly shifted by $\pi - 2\phi_s$ to preserve the phase stability.

LINEAR SYNCHROTRON TUNE



- The linear synchrotron tune is defined as the ratio of the synchrotron frequency to the revolution frequency

$$Q_{s0} = \frac{\omega_{s0}}{\omega_{0,s}} = \sqrt{-\frac{qV_{\mathrm{rf}} h\eta \cos\phi_s}{2\pi\beta_s^2 E_s}}$$

- The inverse of the synchrotron tune gives the number of machine turns needed to perform one full period in longitudinal phase space.

- The synchrotron tune (longitudinal, $\mathcal{O}\left(10^{-3} - 10^{-2}\right)$) is usually much smaller than the betatron tune (transverse, $\mathcal{O}\left(1 - 10^{2}\right)$).

AMPLITUDE OF OSCILLATIONS

The solutions for the evolution of the parameters of the non-synchronous particle are

$$\Delta\phi\left(t\right) = \Delta\phi_m \sin\left(\omega_{s0} t\right)$$
$$\left(\frac{\Delta E}{\omega_r}\right)\left(t\right) = \left(\frac{\Delta E}{\omega_r}\right)_m \cos\left(\omega_{s0} t\right)$$

where the maximum amplitudes of oscillations in phase and energy are noted with the subscript m. The synchrotron angle is noted $\psi = \omega_{s0} t$.

The ratio in the amplitudes of oscillation is

$$\frac{\left(\Delta E/\omega_r\right)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{\left|\eta\right| \omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{\left|\eta\right| h^2 \omega_{0,s}} Q_{s0}$$

AMPLITUDE OF OSCILLATIONS

DERIVATION

Demonstrate the ratio of maximum amplitudes in phase/energy

$$\frac{(\Delta E/\omega_r)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{|\eta|\,\omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{|\eta|\,h^2\omega_{0,s}}Q_{s0}$$

Hint: Include the solution for $\Delta\phi$ in the equation of motion as a start.

AMPLITUDE OF OSCILLATIONS

DERIVATION

Starting from

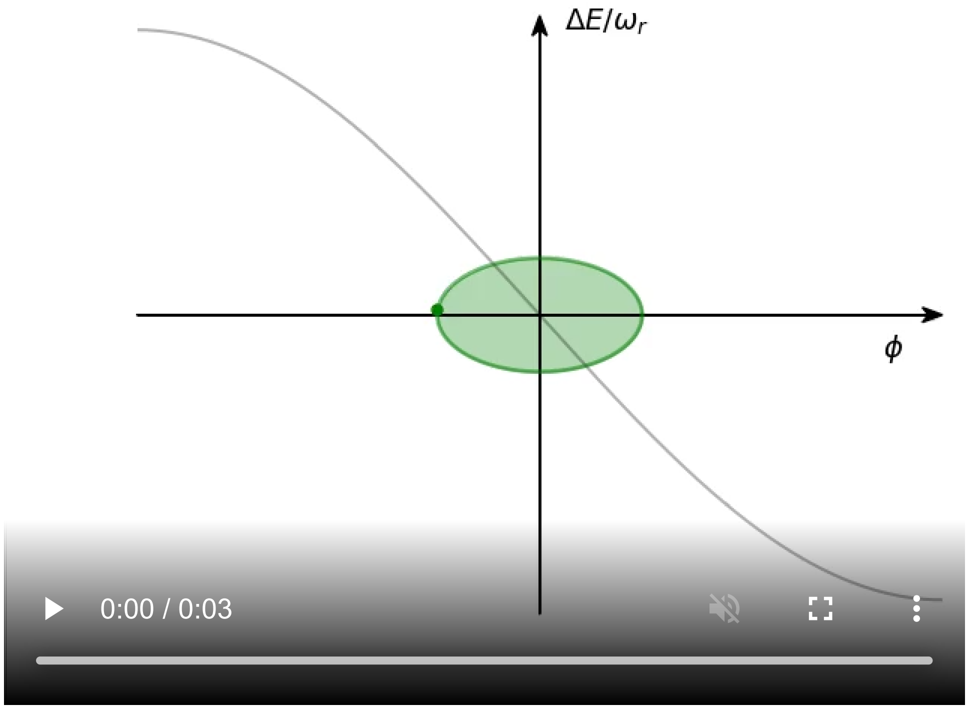
$$\begin{aligned} \Delta\phi\left(t\right) &= \Delta\phi_m \sin\left(\omega_{s0} t\right) \\ \implies \Delta\dot{\phi} &= \Delta\phi_m \omega_{s0} \cos\left(\omega_{s0} t\right) \\ \implies \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)_m \cos\left(\omega_{s0} t\right) &= \Delta\phi_m \omega_{s0} \cos\left(\omega_{s0} t\right) \end{aligned}$$

We obtain

$$\frac{\left(\Delta E/\omega_r\right)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{\left|\eta\right| \omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{\left|\eta\right| h^2 \omega_{0,s}} Q_{s0}$$

We took $\left|\eta\right|$ to obtain positive phase/energy maximum amplitudes.

LINEAR LONGITUDINAL EMITTANCE



- The trajectory of the particles in phase space is an ellipse of the form

$$\left(\frac{\Delta\phi}{\Delta\phi_m}\right)^2 + \left(\frac{\Delta E/\omega_r}{[\Delta E/\omega_r]_m}\right)^2 = 1$$

- The surface of the ellipse corresponds to the longitudinal emittance of a particle. A linear approximation is

$$\varepsilon_{l,0} = \frac{\pi}{\omega_r}\Delta E_m \Delta\phi_m = \pi \Delta E_m \Delta\tau_m$$

- The longitudinal emittance is expressed in the $[\mathrm{eV}\cdot\mathrm{s}]$ unit and is constant for a particle as long as machine parameters are changed slowly (adiabatically).

LINEAR LONGITUDINAL EMITTANCE

EXPRESSION

We note the bunch length $\tau_l = 2\Delta\tau_m$ corresponding to the diameter of the particle with the largest amplitude.

The linear longitudinal emittance becomes

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \frac{\pi \beta_s^2 E_s}{4\left|\eta\right|} \omega_{s0} \tau_l^2$$
$$= \frac{\pi \left|\eta\right|}{\beta_s^2 E_s} \frac{1}{\omega_{s0}} \Delta E_m^2$$

In practice, the longitudinal emittance of a bunch is estimated from the measured bunch length together with the machine parameters.

The bunch momentum spread is $\delta_p = 2\Delta p_m / p_s = 2\Delta E_m / \left(\beta_s^2 E_s\right)$.

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Demonstrate that

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \frac{\pi \beta_s^2 E_s}{4\left|\eta\right|} \omega_{s0} \tau_l^2$$
$$= \frac{\pi \left|\eta\right|}{\beta_s^2 E_s} \frac{1}{\omega_{s0}} \Delta E_m^2$$

Hint: replace the phase or energy deviation with the one obtained from the energy/phase amplitude ratios.

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Replacing ΔE_m and using $\Delta\phi_m = \omega_r \tau_l / 2$

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \pi \omega_r \Delta\phi_m \frac{\beta_s^2 E_s}{|\eta|\, \omega_r^2} \omega_{s0} \frac{\tau_l}{2} = \pi \frac{\omega_r \tau_l}{2} \frac{\beta_s^2 E_s}{|\eta|\, \omega_r} \omega_{s0} \frac{\tau_l}{2}$$

$$\varepsilon_{l,0} = \frac{\pi \beta_s^2 E_s}{4\,|\eta|} \omega_{s0} \tau_l^2$$

$$= \frac{\pi \beta_s^2 E_s}{4\,|\eta|} \sqrt{-\frac{q V_{\mathrm{rf}} \omega_r^2 \eta \cos\phi_s}{2\pi h \beta_s^2 E_s}} \tau_l^2$$

$$= \tau_l^2 \sqrt{-\frac{\pi}{32} \frac{\omega_{0,s}^2 \beta_s^2 E_s}{\eta} q V_{\mathrm{rf}} h \cos\phi_s}$$

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Replacing $\Delta\phi_m$

$$\varepsilon_{l,0} = \frac{\pi}{\omega_r}\Delta E_m \Delta\phi_m = \frac{\pi}{\omega_r}\Delta E_m \frac{1}{\omega_{s0}}\frac{|\eta|\,\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)_m$$

$$\varepsilon_{l,0} = \frac{\pi\,|\eta|}{\beta_s^2 E_s}\frac{1}{\omega_{s0}}\Delta E_m^2$$

$$= \frac{\pi\,|\eta|}{\beta_s^2 E_s}\sqrt{-\frac{2\pi h \beta_s^2 E_s}{q V_{\mathrm{rf}} \omega_r^2 \eta \cos\phi_s}}\Delta E_m^2$$

$$= \Delta E_m^2 \sqrt{-2\pi^3 \frac{\eta}{\omega_{0,s}^2 \beta_s^2 E_s}\frac{1}{q V_{\mathrm{rf}} h \cos\phi_s}}$$

ADIABATICITY

- The longitudinal emittance of a bunch is preserved as long as the machine parameters are changed "adiabatically". The relative variation of the synchrotron frequency with time should be small compared to the synchrotron frequency

$$\left|\frac{\dot{\omega}_{s0}}{\omega_{s0}}\right| \ll \omega_{s0}$$

- The adiabaticity parameter is then

$$\alpha_{\mathrm{ad}} = \left|\frac{1}{\omega_{s0}^2}\frac{d\omega_{s0}}{dt}\right| \ll 1$$

- Intuitively, the parameters of the machine (e.g. energy, RF voltage, RF phase...) must be changed much slower than the synchrotron motion for the bunch to adapt to its new trajectory in phase space.

SCALING LAWS

The following scaling laws allow to evaluate the change in bunch length and energy spread from relative variations in emittance and machine parameters (NB: E_s and η are interdependent).

BUNCH LENGTH

$$\tau_l \propto \varepsilon_{l,0}^{1/2}\ V_{\mathrm{rf}}^{-1/4}\ h^{-1/4}\ E_s^{-1/4}\ \eta^{1/4}$$

ENERGY DEVIATION

$$\Delta E_m \propto \varepsilon_{l,0}^{1/2}\ V_{\mathrm{rf}}^{1/4}\ h^{1/4}\ E_s^{1/4}\ \eta^{-1/4}$$

- During acceleration, with all parameters constants except E_s, the bunch length reduces with $\tau_l \propto E_s^{-1/4}$. This is adiabatic "damping" of phase oscillations.
- The energy spread scales inversely with $\Delta E_m \propto E_s^{1/4}$, $\varepsilon_{l,0}$ is constant.

EXERCISES

- Compute the linear synchrotron frequency and tune in the SPS at $p = 14$ GeV/c and $p = 450$ GeV/c, with an RF harmonic $h = 4620$ and voltage $V_{\mathrm{rf}} = 4.5$ MV (find the other SPS parameters obtained in the exercises from Module 5). The beam is not accelerated.
- Compute the approximate emittance and momentum spread at $p = 14$ GeV/c for a bunch length $\tau_l = 3$ ns.
- What would be the bunch length at $p = 450$ GeV/c if the emittance is preserved?
- What would be the bunch length and energy spread at transition energy?
- Evaluate required increase in rf voltage to shorten the bunch length by a factor 2.

EXERCISES

- Linear synchrotron frequency and tune
  - Low energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14} \sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14/23.11 \cdot 10^6)^2 \cdot 1.385 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}}$$

$$\approx 784 \text{ Hz}$$

$$Q_{s0} = 784 \cdot 23.11 \cdot 10^{-6} \approx 1.81 \cdot 10^{-2}$$

EXERCISES

- Linear synchrotron frequency and tune
  - High energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14}\sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6)^2 \cdot 3.082 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot 1 \cdot 450 \cdot 10^9}} \approx 206\ \mathrm{Hz}$$

$$Q_{s0} = 206 \cdot 23.05 \cdot 10^{-6} \approx 4.76 \cdot 10^{-3}$$

EXERCISES

- Linear emittance

$$\varepsilon_{l,0} = \frac{3.14 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{4 \cdot 1.385 \cdot 10^{-3}} 2 \cdot 3.14 \cdot 784 \cdot (3 \cdot 10^{-9})^2 \approx 0.35 \text{ eVs}$$

- Energy spread

$$\delta_p = 2\frac{\Delta E_m}{\beta_s^2 E_s} = 4\frac{\varepsilon_{l,0}}{\pi \tau_l \beta_s^2 E_s} = \frac{4 \cdot 0.35}{3.14 \cdot 3 \cdot 10^{-9} (14/14.03)^2 \cdot 14.03 \cdot 10^9}$$
$$\approx 1.06 \times 10^{-2}$$

- Adiabatic damping

$$\tau_{l,\text{high}} = \tau_{l,\text{low}} \left(\frac{E_\text{high}}{E_\text{low}}\right)^{-1/4} = 3 \cdot \left(\frac{450}{14.03}\right)^{-1/4} \approx 1.26 \text{ ns}$$

EXERCISES

- Transition

The bunch length would tend to zero while the energy spread diverge to infinity! Non-adiabatic theory needed to better evaluate bunch parameters at transition crossing.

- Adiabatic bunch shortening

$$\tau_{l,\mathrm{high}} = \tau_{l,\mathrm{low}} \left(\frac{V_{\mathrm{high}}}{V_{\mathrm{low}}}\right)^{-1/4}$$

$$\implies V_{\mathrm{high}} = V_{\mathrm{low}} \left(\frac{\tau_{l,\mathrm{high}}}{\tau_{l,\mathrm{low}}}\right)^{-4} = V_{\mathrm{low}} \times 16$$

The required voltage increase is a factor 16! Not very efficient shortening.

MODULE 9: NON-LINEAR SYNCHROTRON MOTION

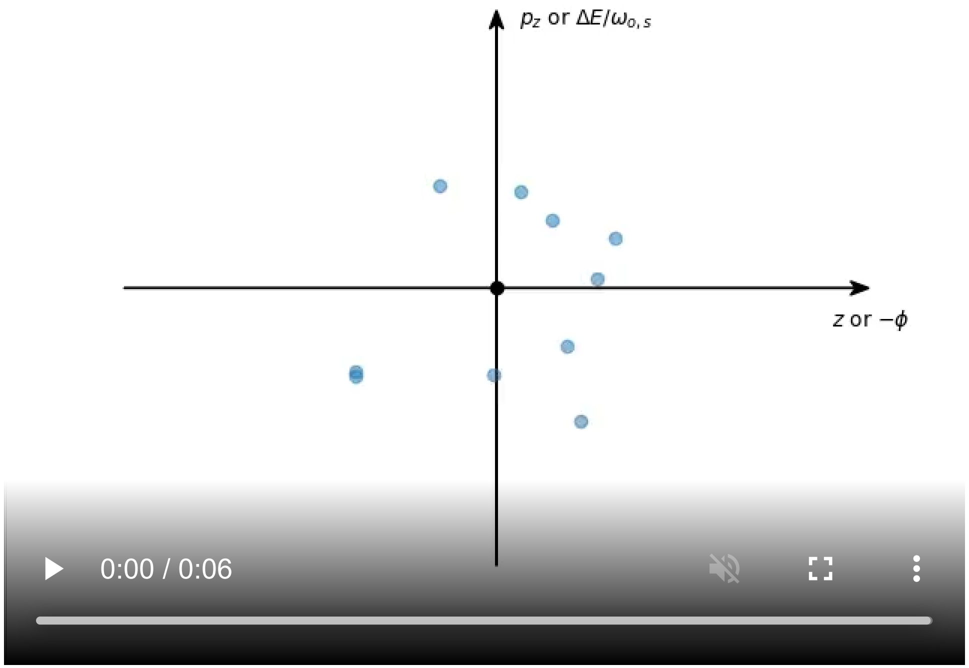
→ **Combined non-linear equations of motion (Hamiltonian)**

→ **RF bucket parameters and bunch emittance**

→ **Non-linear synchrotron frequency**

→ **Matching**

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_r}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)$$

- Starting from the same equations of motion as in Module 8.

COMBINING THE EQUATIONS OF MOTION

Starting over from the same combined equation of motion

$$\frac{d}{dt}\left(\frac{d\phi}{dt}\frac{\beta_s^2 E_s}{\eta\omega_r^2}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left(\sin\phi - \sin\phi_s\right)$$

We assume again that the change of machine parameters with time is negligible (adiabaticity), hence

$$\frac{d^2\phi}{dt^2} + \frac{\omega_{s0}^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

We can solve for $\dot{\phi}$. By multiplying by $\dot{\phi}$ and integrating with time, we get

$$\frac{\dot{\phi}^2}{2\omega_{s0}^2} - \frac{\cos\phi + \phi\sin\phi_s}{\cos\phi_s} = \mathcal{H}$$

COMBINING THE EQUATIONS OF MOTION

DERIVATION

For the first term, we use the differential identity

$$d\left(x^2\right) = 2\,x\,dx \quad \rightarrow \quad \dot{\phi}\ddot{\phi} = \frac{1}{2}\frac{1}{dt}\frac{d\left(\phi^2\right)}{dt^2}$$

For the second term, we integrate

$$\begin{aligned} &\frac{\omega_{s0}^2}{\cos\phi_s}\int\left(\sin\phi - \sin\phi_s\right)\frac{d\phi}{dt}dt \\ =&\frac{\omega_{s0}^2}{\cos\phi_s}\left[\int \sin\phi\, d\phi - \int \sin\phi_s d\phi\right] \\ =&-\frac{\omega_{s0}^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right)\end{aligned}$$

COMBINING THE EQUATIONS OF MOTION

The integration constant $\mathcal{H}$ can be offset so that its value is zero for the synchronous particle. Since $\dot{\phi}_s = 0$, we get

$$\mathcal{H} = \frac{\dot{\phi}^2}{2\omega_{s0}^2} - \frac{\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s}{\cos\phi_s}$$

Replacing $\dot{\phi}$ using the phase differential equation definition and ω_{s0}, we finally obtain

$$\mathcal{H} = \frac{\eta\omega_r^2}{2\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)^2 + \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

Linearized for small $\Delta\phi$, the equation describes the same ellipse as obtained in the previous module.

HAMILTONIAN OF SYNCHROTRON MOTION

The same result can be obtained from the expression of the Hamiltonian using

$$\frac{d\phi}{dt} = \frac{\partial \mathcal{H}}{\partial \left(\Delta E/\omega_r\right)} \quad \text{and} \quad \frac{d\left(\Delta E/\omega_r\right)}{dt} = -\frac{\partial \mathcal{H}}{\partial \phi}$$

In the previous equations, $\mathcal{H}$ effectively represents the Hamiltonian of a particle in our system, corresponding to the **energy of synchrotron oscillations** (beware: this is not the actual particle energy!)

The Hamiltonian is composed of two parts

$$\mathcal{H} = \mathcal{T}\left(\frac{\Delta E}{\omega_r}\right) + \mathcal{U}\left(\phi\right)$$

where $\mathcal{T}$ is the "kinetic" energy of synchrotron oscillations and $\mathcal{U}$ the "potential" energy.

HAMILTONIAN MECHANICS

APARTÉ

The Hamiltonian of a particle can be obtained from the canonical Hamilton equations

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}$$

where p and q are the conjugate momentum and coordinate.

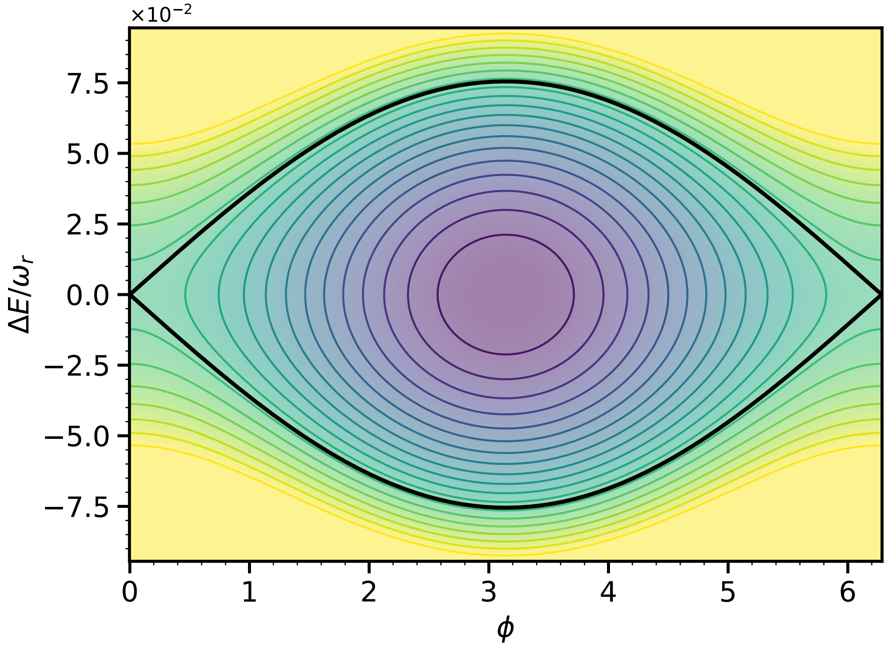
A time invariant Hamiltonian is then expressed

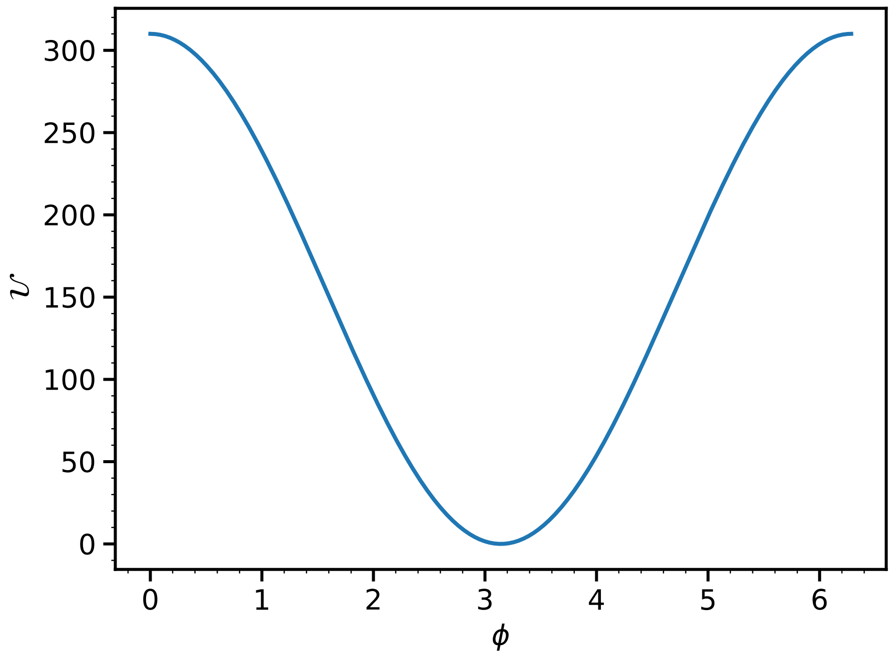
$$\mathcal{H} = \int \frac{\partial \mathcal{H}}{\partial p} dp + \int \frac{\partial \mathcal{H}}{\partial q} dq$$

A time invariant $\mathcal{H}$ is a constant of motion.

HAMILTONIAN OF SYNCHROTRON MOTION

STATIONNARY BUNCH, $\Phi_S = 0$

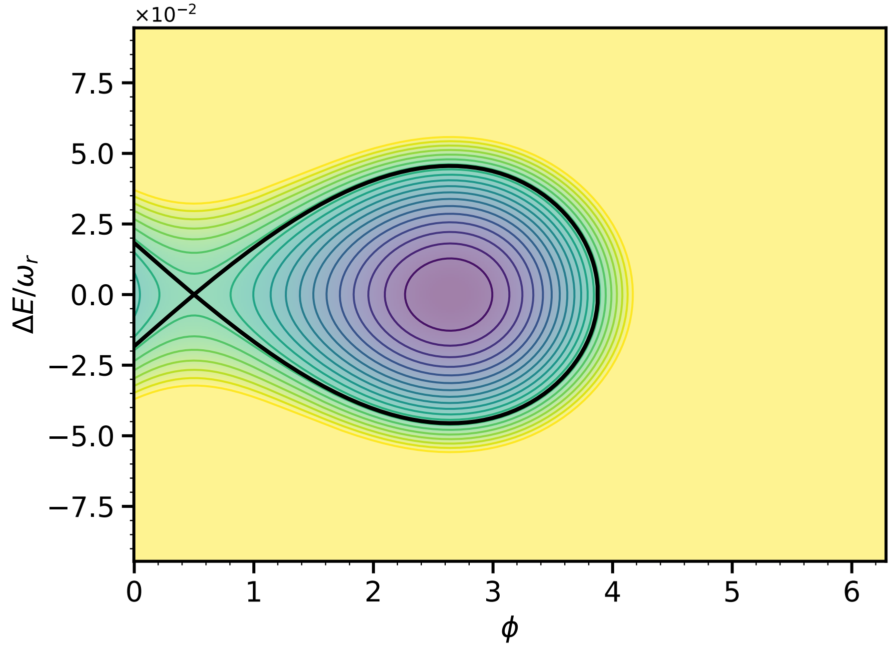


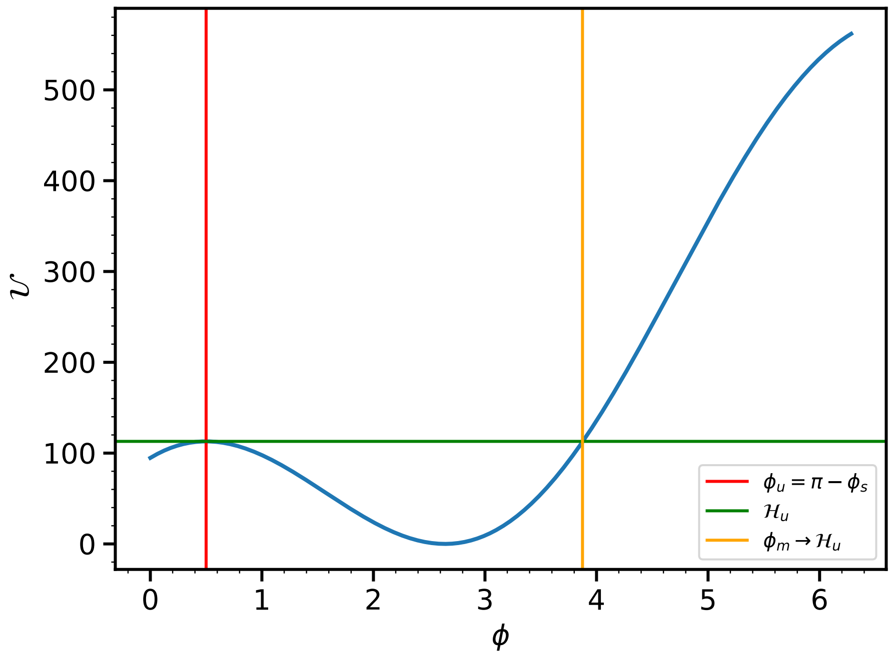


- The Hamiltonian gives the trajectory of the particle in phase space (ellipse at low $\Delta\phi$).
- A particle oscillates in phase space and performs a bounded motion if its energy $\mathcal{H}$ is lower than the maximum of the potential well.

HAMILTONIAN OF SYNCHROTRON MOTION

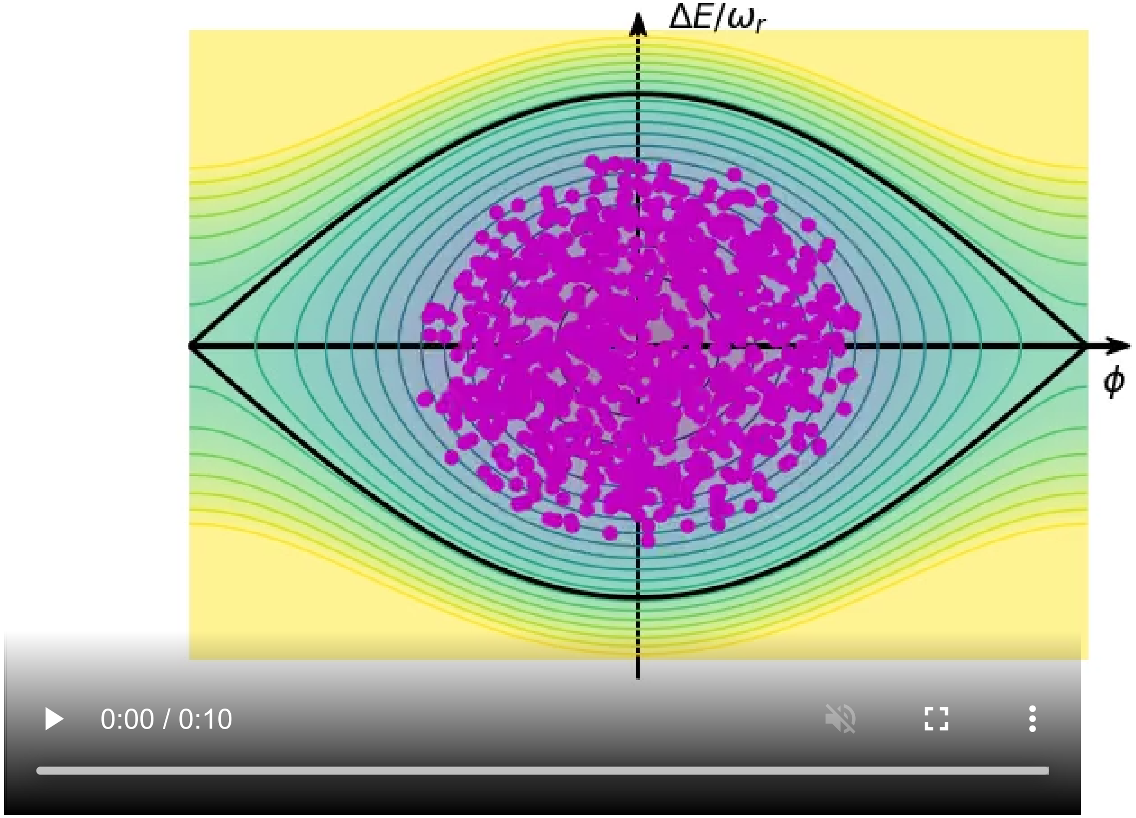
ACCELERATION Φ_S





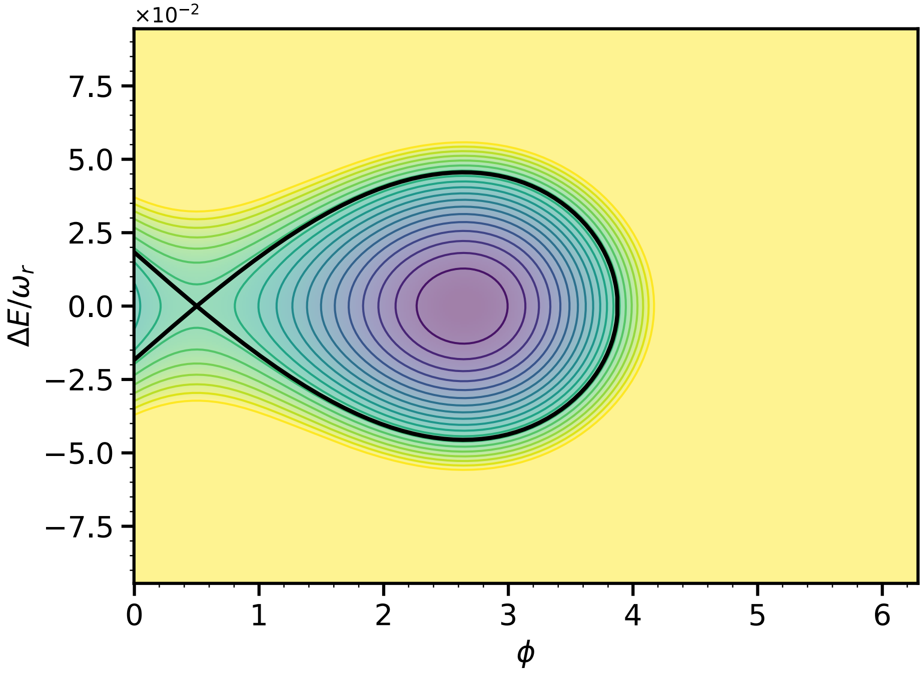
- During acceleration, the potential well is modified with ϕ_s.
- The stable fixed point (center of the RF bucket) is shifted to ϕ_s, the trajectories of non-synchronous particles are asymetric.

MOTION OF PARTICLES IN THE RF BUCKET



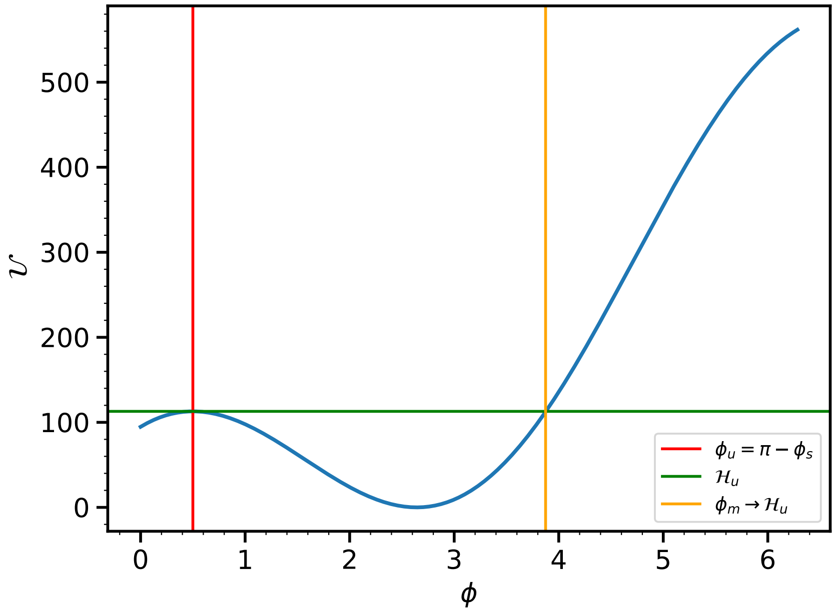
- Example of particles rotating in longitudinal phase space, with non-linear synchrotron motion.

SEPARATRIX



- The maximum contour in which the particles have a bounded motion around the synchronous phase is the **separatrix**.
- The separatrix is the limit of the **RF bucket**, where particles can be captured in a bunch.

SEPARATRIX



- The limit of the separatrix is given by the unstable fixed point $\phi_u = \pi - \phi_s$ (obtained from $d\mathcal{U}/d\phi = 0$) on one side.
- On the other side, the phase ϕ_m corresponds to the turning point where $\mathcal{U}_m = \mathcal{U}(\pi - \phi_s) = \mathcal{H}_u(\pi - \phi_s, \Delta E = 0)$

SEPARATRIX

EXPRESSION

The expression for the separatrix is obtained from the Hamiltonian

$$\mathcal{H} = \frac{\eta \omega_r^2}{2\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)^2 + \mathcal{U}(\phi)$$

$$\mathcal{H}_u = \mathcal{U}(\pi - \phi_s)$$

The maximum trajectory in energy is

$$\Delta E_{\text{sep}} = \pm\sqrt{\frac{2\beta_s^2 E_s}{|\eta|}}\sqrt{\mathcal{U}(\pi - \phi_s) - \mathcal{U}(\phi)}$$

RF BUCKET HEIGHT

The RF bucket height is obtained from the maximum height of the separatrix at $\Delta E_{\text{sep}}\left(\phi_s\right)$

The RF bucket height in energy is (*NB: this is the half size from 0 to* $\Delta E_{\text{sep,m}}$*, and should be* $\times 2$ *for the full bucket height*)

$$\Delta E_{\text{sep,m}} = \sqrt{\frac{2qV_{\text{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}} Y\left(\phi_s\right)$$

where

$$Y\left(\phi_s\right) = \left|-\cos\phi_s + \frac{\left(\pi - 2\phi_s\right)}{2}\sin\phi_s\right|^{1/2}$$

is the reduction of the bucket height during acceleration $Y \leqslant 1$.

RF BUCKET HEIGHT

DERIVATION

The potential well at $\pi - \phi_s$, using the trigonometric identity, is

$$
\begin{aligned}
\mathcal{U}(\pi - \phi_s) &= \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\cos(\pi - \phi_s) - \cos\phi_s + (\pi - 2\phi_s)\sin\phi_s\right] \\
&= \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\cos\pi\cos\phi_s + \sin\pi\sin\phi_s - \cos\phi_s + (\pi - 2\phi_s)\sin\phi_s\right] \\
&= \frac{qV_{\mathrm{rf}}}{2\pi h}\left[-2\cos\phi_s + (\pi - 2\phi_s)\sin\phi_s\right] \\
&= \frac{qV_{\mathrm{rf}}}{\pi h}\left[-\cos\phi_s + \frac{(\pi - 2\phi_s)}{2}\sin\phi_s\right]
\end{aligned}
$$

RF BUCKET AREA (ACCEPTANCE)

The bucket area (acceptance) is obtained by integrating within the separatrix contour

$$\mathcal{A}_{\mathrm{bk}} = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\mathcal{U}\left(\pi-\phi_s\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

The bucket area can be reformulated as

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|}}\Gamma\left(\phi_s\right)$$

where

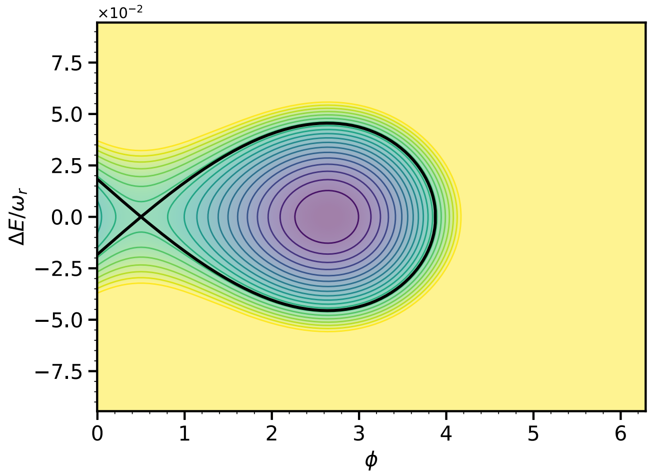
$$\Gamma\left(\phi_s\right) = \frac{1}{4\sqrt{2}} \int_{\phi_u}^{\phi_m} \sqrt{-\cos\phi_s - \cos\phi + \left(\pi - \phi - \phi_s\right)\sin\phi_s}\, d\phi$$

RF BUCKET AREA (ACCEPTANCE)

The function $\Gamma\left(\phi_s\right)$ is the reduction of the bucket area during acceleration $\Gamma \leqslant 1$ and can be approximated to give the formula

$$\mathcal{A}_{\mathrm{bk}} \approx \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}}\frac{1-\sin\phi_s}{1+\sin\phi_s}$$

For the stationary RF bucket $\mathcal{A}_{\mathrm{bk}} = 8\Delta E_{\mathrm{sep,m}}/\omega_r$



RF BUCKET AREA (ACCEPTANCE)

DERIVATION

$$\mathcal{A}_{\mathrm{bk}} = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\mathcal{U}\left(\pi - \phi_s\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

$$= 2\sqrt{\frac{qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\ldots}\, d\phi$$

$$\begin{aligned} \ldots &= \left[\cos\left(\pi - \phi_s\right) - \cos\phi_s + \left(\pi - 2\phi_s\right)\sin\phi_s\right] \\ &\quad - \left[\cos\phi - \cos\phi_s + \left(\phi - \phi_s\right)\sin\phi_s\right] \\ &= \left[-2\cos\phi_s + \left(\pi - 2\phi_s\right)\sin\phi_s\right] \\ &\quad - \left[\cos\phi - \cos\phi_s + \left(\phi - \phi_s\right)\sin\phi_s\right] \\ &= -\cos\phi_s - \cos\phi + \left(\pi - \phi - \phi_s\right)\sin\phi_s \end{aligned}$$

RF BUCKET AREA (ACCEPTANCE)

DERIVATION

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}}\frac{1}{4\sqrt{2}}\int_{\phi_u}^{\phi_m}\sqrt{\dots}\,d\phi$$

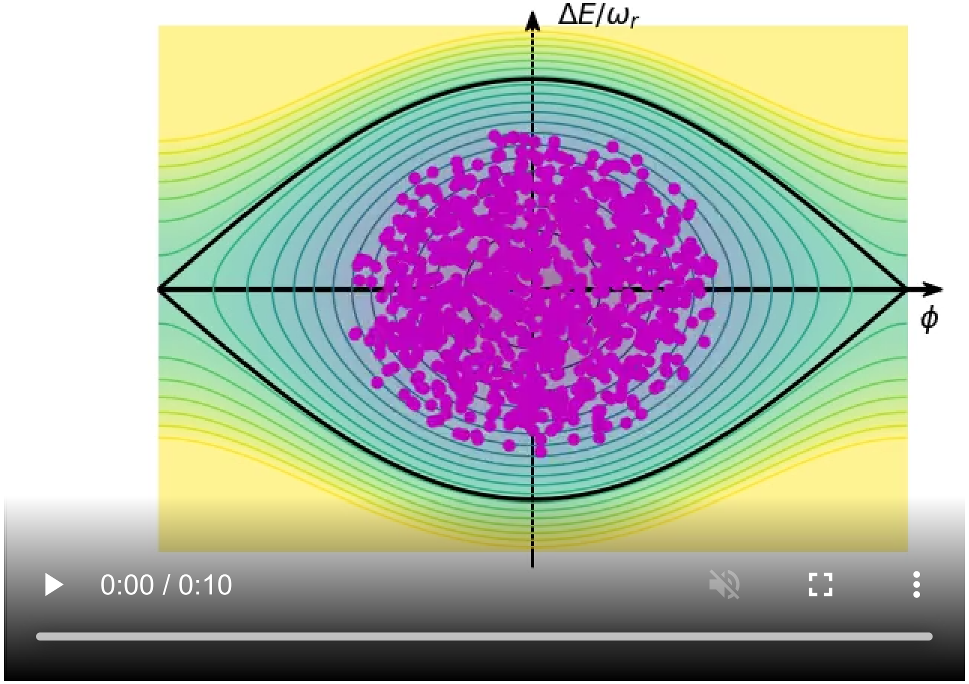
$$= \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}}\Gamma\left(\phi_s\right)$$

where

$$\Gamma\left(\phi_s\right) = \frac{1}{4\sqrt{2}}\int_{\phi_u}^{\phi_m}\sqrt{-\cos\phi_s - \cos\phi + \left(\pi - \phi - \phi_s\right)\sin\phi_s}\,d\phi$$

$$\approx \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

LONGITUDINAL EMITTANCE, FILLING FACTOR



- The longitudinal emittance can be calculated accounting for the non-linearities of the RF bucket

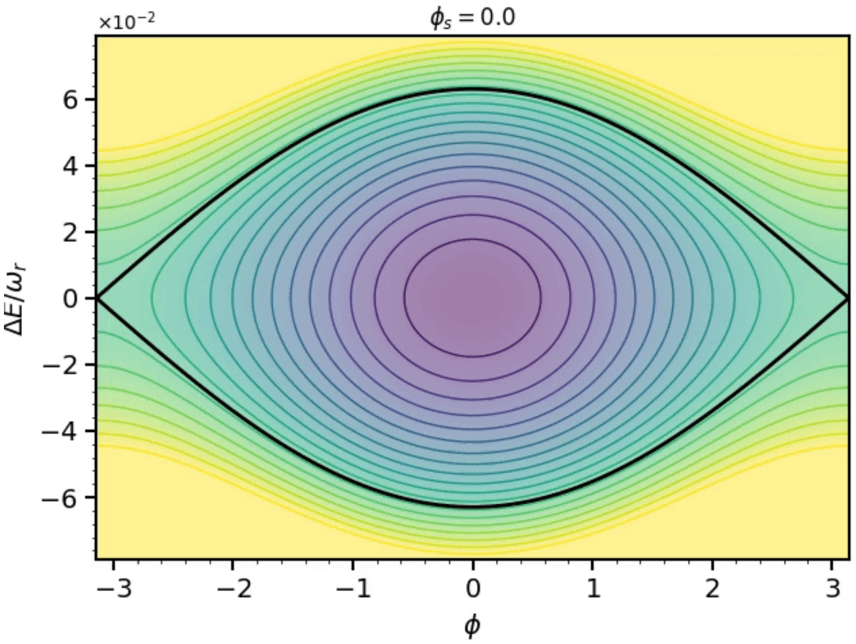
$$\varepsilon_l = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \cdot \int_{\phi_{b,l}}^{\phi_{b,r}} \sqrt{\mathcal{U}\left(\phi_{b,lr}\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

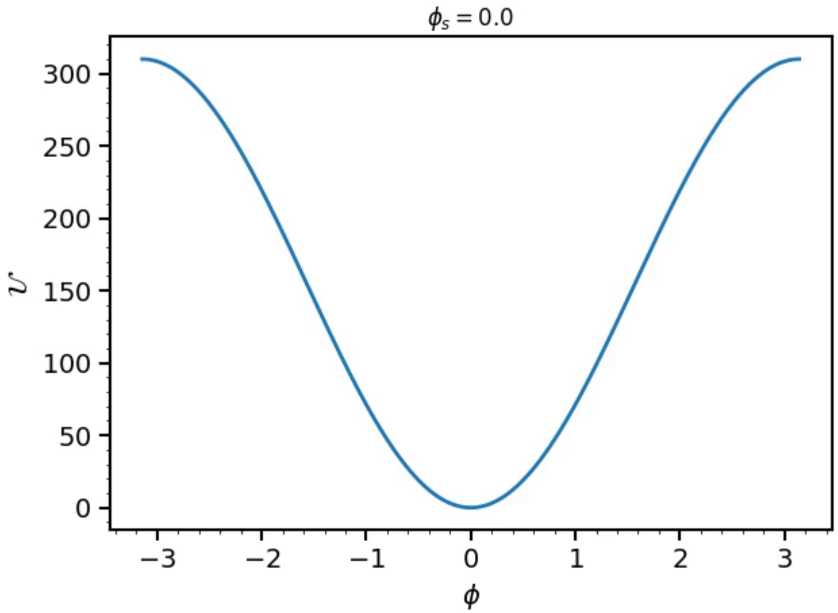
where b,l and b,r stands for the left/right edge of the bunch in phase (amplitude ϕ_b and full length $2\phi_b$).

The filling factor is commonly defined in emittance: $\varepsilon_l/\mathcal{A}_{\mathrm{bk}}$ or in energy: $\Delta E_{b,m}/\Delta E_{\mathrm{sep},m}$

ACCELERATION

BELOW TRANSITION

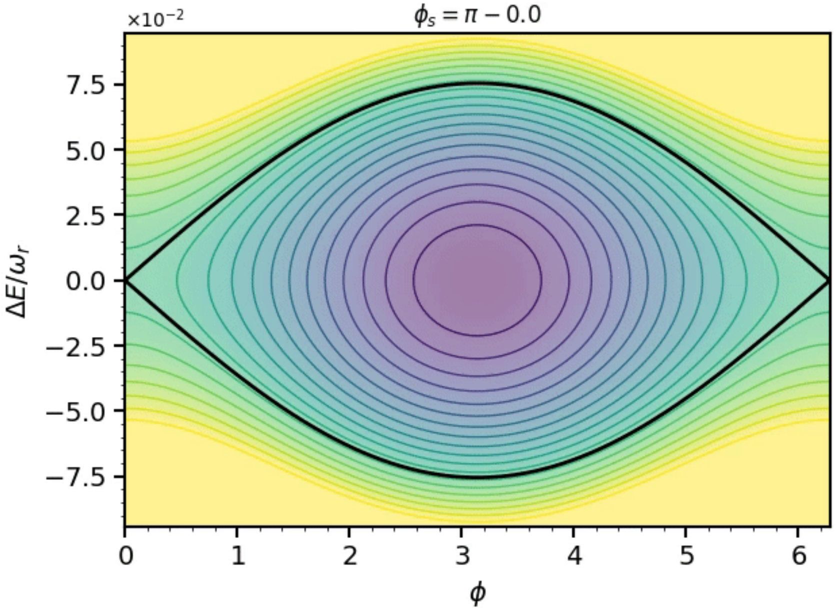


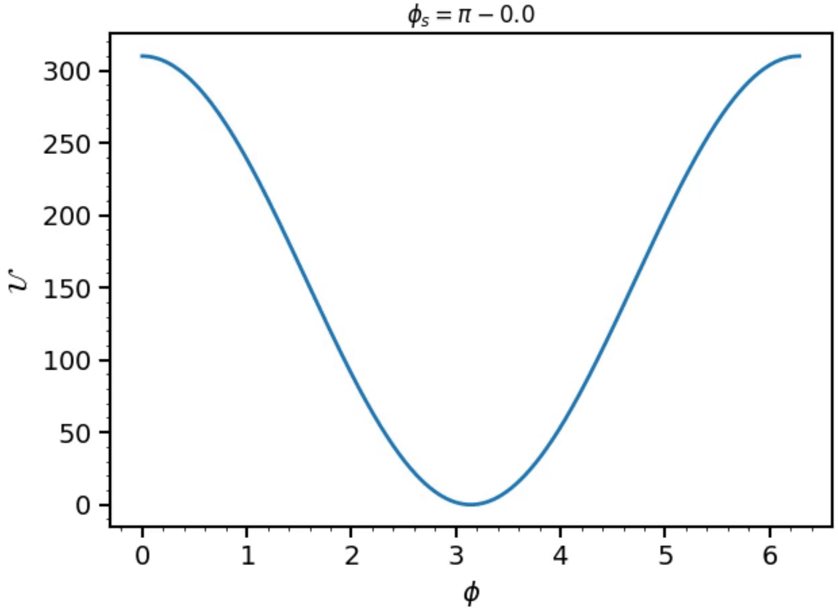


- Notice the shape of the bucket: below transition, pointing towards positive ϕ.
- The bucket shrinks if ϕ_s is too large! The acceleration should be limited or RF voltage increased to keep a reasonable filling factor (losses otherwise).

ACCELERATION

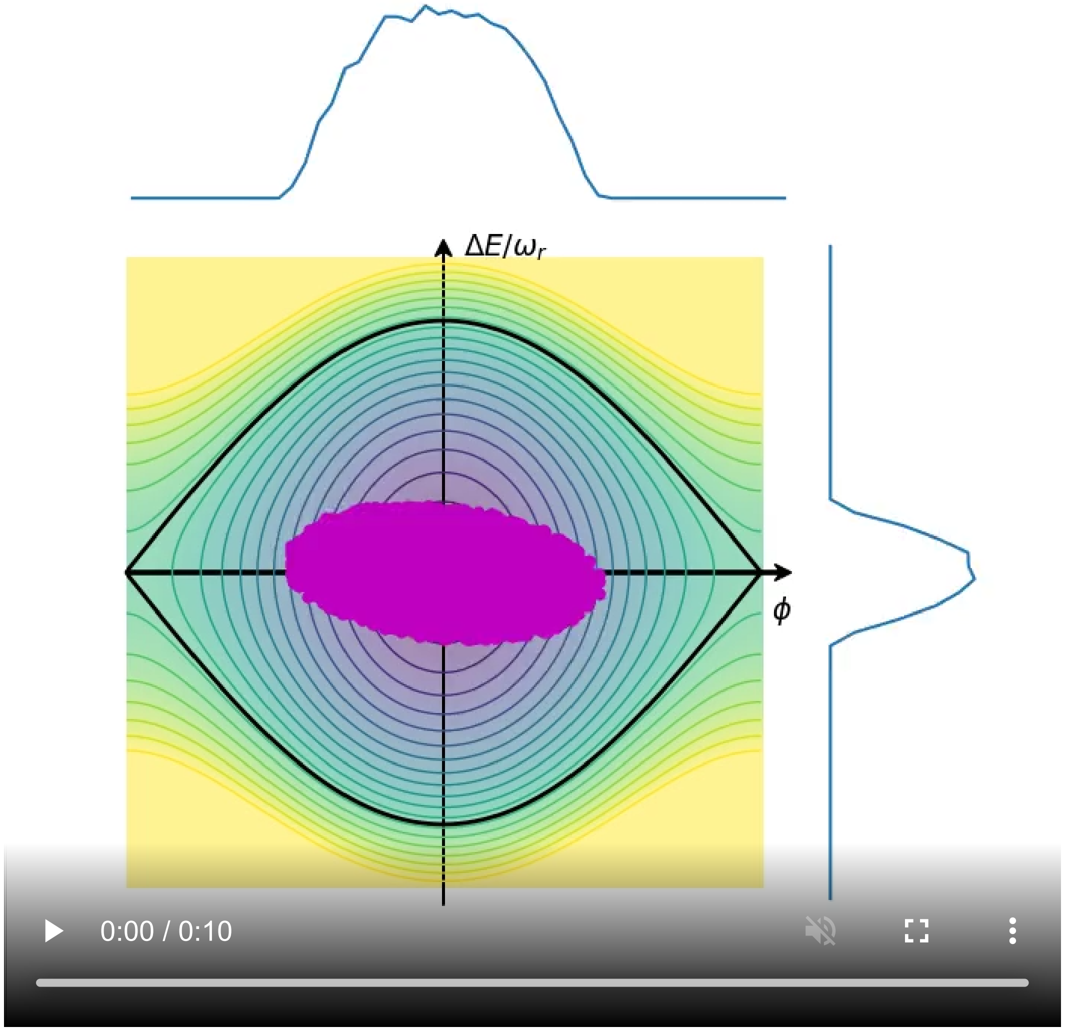
ABOVE TRANSITION





- Notice the shape of the bucket: above transition, pointing towards negative ϕ.
- The bucket shrinks if ϕ_s is too large! The acceleration should be limited or RF voltage increased to keep a reasonable filling factor (losses otherwise).

NON-LINEAR SYNCHROTRON FREQUENCY



- The non-linear frequency is obtained by integrating

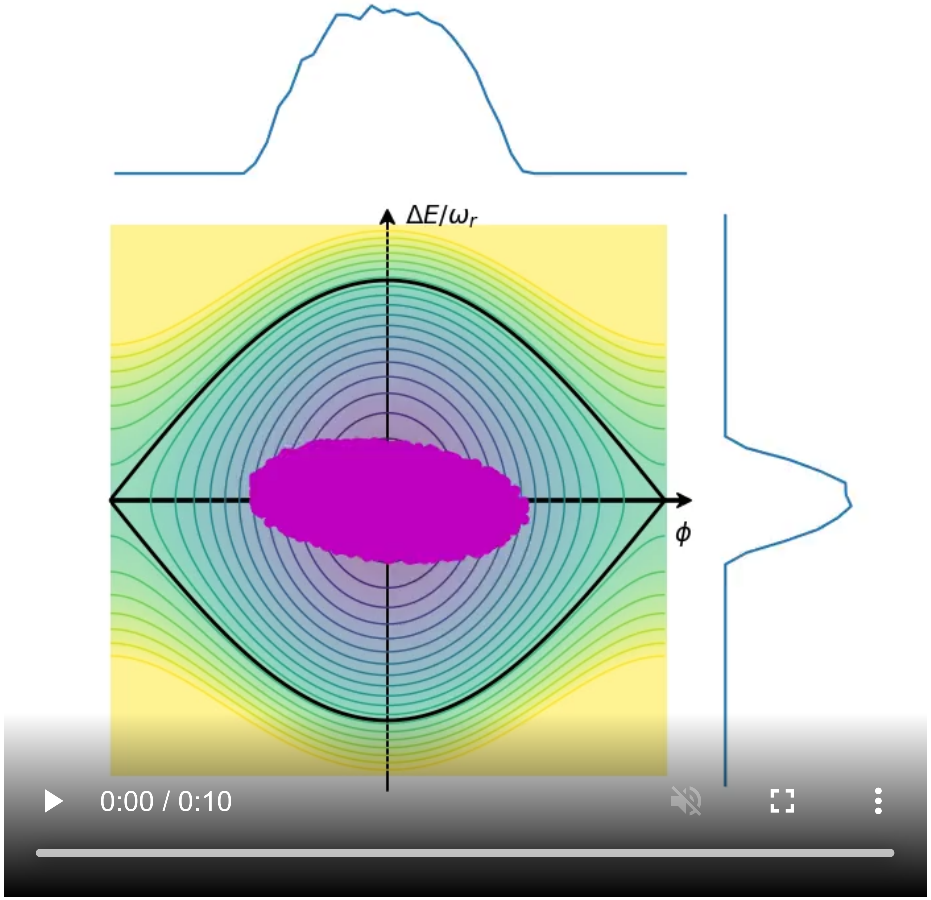
$$T_s = \int_{\phi_{b,l}}^{\phi_{b,r}} \frac{d\phi}{\dot{\phi}}$$

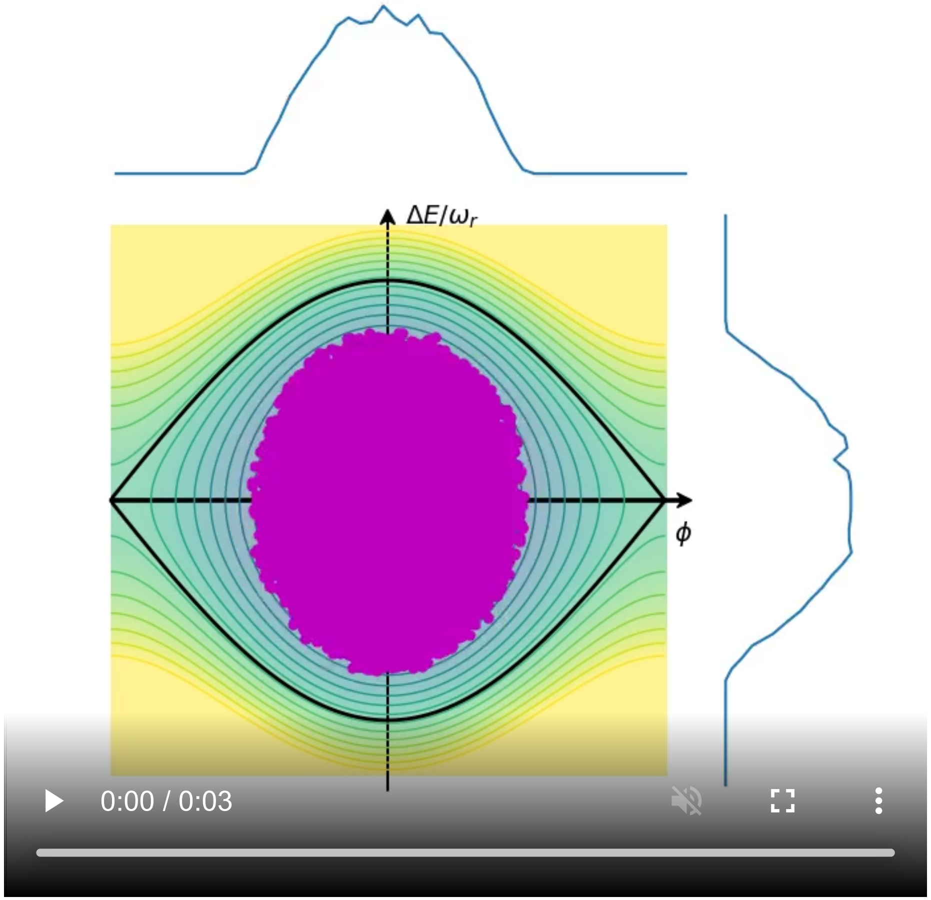
- The integration leads to the relationship

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2K\left(\sin\frac{\phi_b}{2}\right)}$$

$$\approx 1 - \frac{\phi_b^2}{16}$$

MATCHING AND FILAMENTATION





- The bunch is matched if the density along a iso-Hamiltonian line is constant. If mismatched, the bunch filaments and the statistical emittance increases.

EXERCISES

- Compute the RF bucket area (or acceptance) using the SPS parameters from Module 5 and 8.
- Compute the bucket height.
- Compute the filling factor for 3 ns bunch at 14 GeV/c (use the linear approximation for the emittance calculation)
- The bunch length oscillations at injection indicate that the energy spread is too small by 10%. How much should the RF voltage be reduced to improve the matching?

EXERCISES

- Low energy 14 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.11 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 0.50 \text{ eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 79.1 \text{ MeV}$$

EXERCISES

- High energy 450 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 1.91\ \mathrm{eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 301\ \mathrm{MeV}$$

EXERCISES

- Filling factor

From the previous module exercise, the longitudinal emitatnce is 0.35 eVs. The filling factor in area is $0.35/0.50 \approx 70$ %.

- Matching

The bunch length and energy spread are fixed at injection. In orderto match the bunch, the bucket height should be reduced by 10%. The RF voltage can be reduced to reduce the bucket height, with a scaling $\sqrt{V_{\mathrm{rf}}}$

$$\frac{\Delta E_{\mathrm{sep},m,2}}{\Delta E_{\mathrm{sep},m,1}} = 0.9 = \sqrt{\frac{V_{\mathrm{rf},2}}{V_{\mathrm{rf},1}}} \quad \rightarrow \quad V_{\mathrm{rf},2} = 0.9^2 V_{\mathrm{rf},1} \approx 0.81 V_{\mathrm{rf},1}$$

The RF voltage should be reduced by 20% (useful tip: $(1-\epsilon)^n \approx 1 - n\epsilon \rightarrow (1-0.1)^2 \approx 1 - 2 \cdot 0.1$)

TAKE AWAY MESSAGE

LINEAR SYNCHROTRON MOTION

- Linear synchrotron frequency

$$\omega_{s0} = 2\pi f_{s_0} = \sqrt{-\frac{qV_{\mathrm{rf}} h \omega_{0,s}^2 \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Linear synchrotron tune

$$Q_{s0} = \frac{\omega_{s0}}{\omega_{0,s}} = \sqrt{-\frac{qV_{\mathrm{rf}} h \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Phase stability condition

$$\eta \cos\phi_s < 0$$

TAKE AWAY MESSAGE

LINEAR OSCILLATION AMPLITUDE AND EMITTANCE

- Oscillation amplitude ratio

$$\frac{(\Delta E/\omega_r)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{|\eta|\,\omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{|\eta|\,h^2\omega_{0,s}} Q_{s0}$$

- Approximate longitudinal emittance

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \frac{\pi\beta_s^2 E_s}{4\,|\eta|}\omega_{s0}\tau_l^2 = \frac{\pi\,|\eta|}{\beta_s^2 E_s}\frac{1}{\omega_{s0}}\Delta E_m^2$$

TAKE AWAY MESSAGE

BUNCH PARAMETERS LINEAR SCALING LAWS

- Bunch length

$$\tau_l \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{-1/4}\, h^{-1/4}\, E_s^{-1/4}\, \eta^{1/4}$$

- Energy deviation

$$\Delta E_m \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{1/4}\, h^{1/4}\, E_s^{1/4}\, \eta^{-1/4}$$

TAKE AWAY MESSAGE

HAMILTONIAN

$$\mathcal{H} = \frac{\eta \omega_r^2}{2\beta_s^2 E_s} \left(\frac{\Delta E}{\omega_r}\right)^2 + \frac{q V_{\mathrm{rf}}}{2\pi h} \left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

TAKE AWAY MESSAGE

RF BUCKET PARAMETERS

- RF bucket height

$$\Delta E_{\text{sep,m}} = \sqrt{\frac{2qV_{\text{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}} \left| -\cos\phi_s + \frac{(\pi - 2\phi_s)}{2} \sin\phi_s \right|^{1/2}$$

- RF bucket area (acceptance)

$$\mathcal{A}_{\text{bk}} \approx \frac{8}{\omega_r} \sqrt{\frac{2qV_{\text{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}} \frac{1 - \sin\phi_s}{1 + \sin\phi_s}$$

- For the stationary RF bucket, the RF bucket length is 2π and $\mathcal{A}_{\text{bk}} = 8\Delta E_{\text{sep,m}} / \omega_r$

TAKE AWAY MESSAGE

NON-LINEAR SYNCHROTRON FREQUENCY

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2K\left(\sin\frac{\phi_b}{2}\right)} \approx 1 - \frac{\phi_b^2}{16}$$