LONGITUDINAL BEAM DYNAMICS

JUAS 2023

COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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LESSON 2: SYNCHROTRON DESIGN

MODULE 3: ACCELERATION IN A SYNCHROTRON

→ **Fundamental mode in a pillbox cavity**

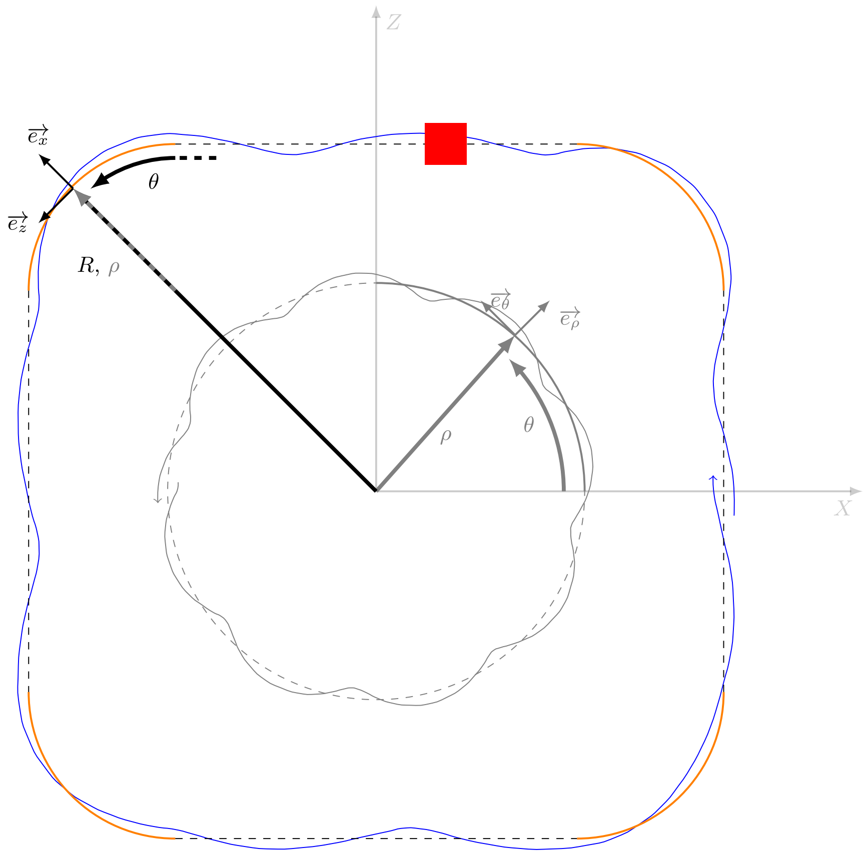
→ **Energy gain**

→ **Transit time factor**

→ **Other sources of energy gain/loss**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



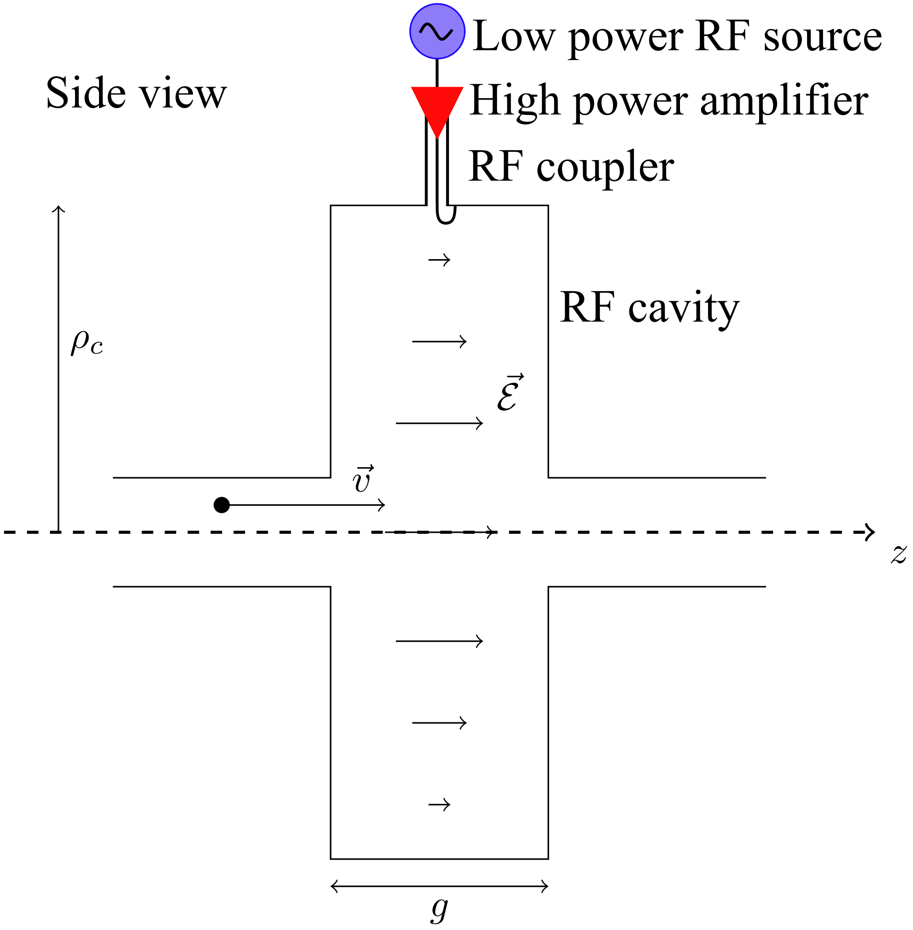
- We start by describing the acceleration of a particle in the RF cavity.

- In the previous lesson we found the expression of the energy gain in the cavity for a single pass

$$\delta E_{\mathrm{rf}} = \int q\mathcal{E}_z\left(\rho, z, t\right) dz$$
$$= qV_{\mathrm{rf}}\left(\rho, \tau\right)$$

- We will find a common expression of $\mathcal{E}_z$ for a simple form of RF cavity.

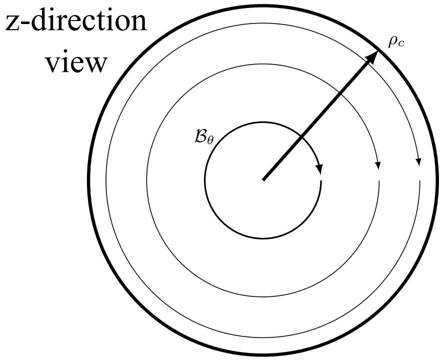
FUNDAMENTAL MODE OF A PILLBOX CAVITY



- A convenient model is the so-called "pillbox cavity", where the fields of the fundamental mode are

$$\mathcal{E}_z(\rho, t) = \mathcal{E}_0 J_0\left(\chi_0 \frac{\rho}{\rho_c}\right)\cos(\omega_r t)$$

$$\mathcal{B}_\theta(\rho, t) = -\frac{\mathcal{E}_0}{c} J_1\left(\chi_0 \frac{\rho}{\rho_c}\right)\sin(\omega_r t)$$

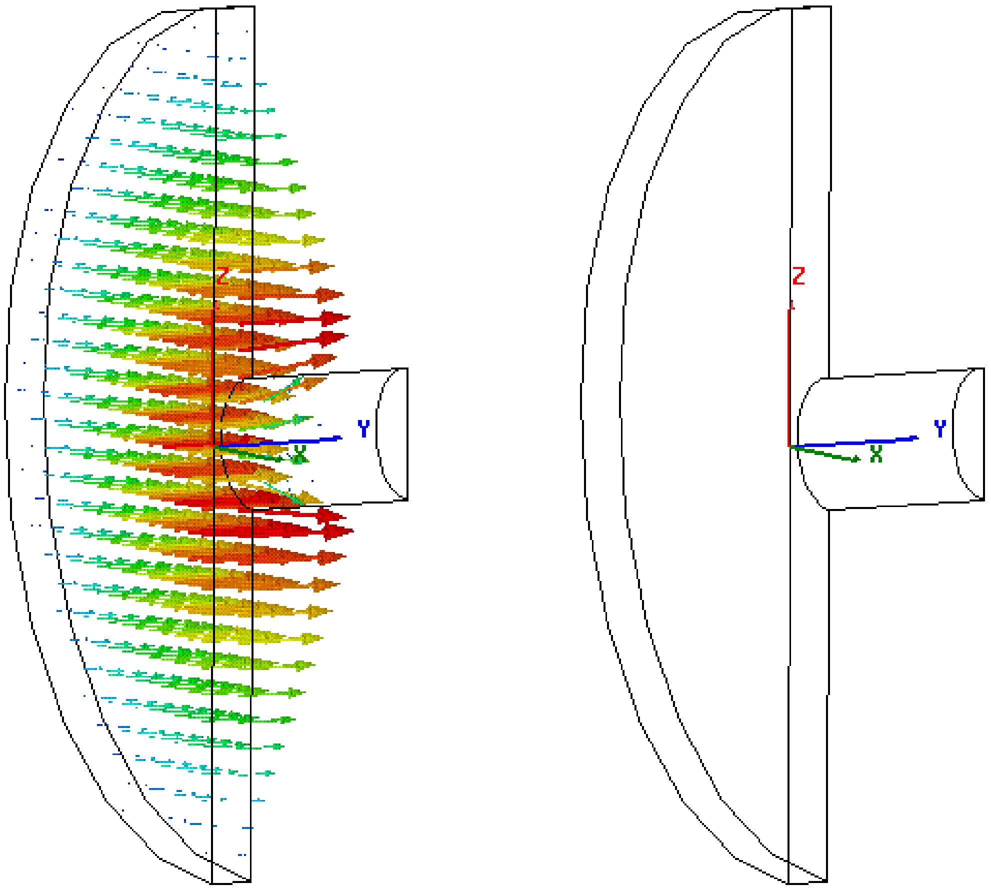


J_n Bessel function

$\chi_0 \approx 2.405$

$\omega_r = \chi_0 c/\rho_c$

FUNDAMENTAL MODE OF A PILLBOX CAVITY



Animation: E. Jensen

- The maximum electric field $\mathcal{E}_0$ achievable in a cavity depends on many parameters including the
  - Cavity material
  - Power amplification
  - Coupling in transmission lines and reflections

- The frequency of a pillbox cavity depends on the radial size, not on the length!

- Dedicated courses in the JUAS Course 2: Introduction and RF Engineering lectures.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

From the wave equation

$$\Delta \overrightarrow{\mathcal{E}} - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{\mathcal{E}}}{\partial t^2} = 0$$

Two conditions on the fields on the boundaries of the cavity (conductor material)

- The electric field is orthogonal to the surface.
- The magnetic field is parallel to the surface.

We neglect the aperture due to the beam pipe and the power coupler.

A large number of modes of oscillation can exist in the cavity, we are interested only in the fundamental mode for which $\overrightarrow{\mathcal{E}} = \mathcal{E}_z \vec{e}_z$ and $\overrightarrow{\mathcal{B}} = \mathcal{B}_\theta \vec{e}_\theta$.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

We will assume a solution of the form $\mathcal{E}_z = \mathcal{E}_0\left(\rho\right)\cos\left(\omega_r t\right)$

Reminder: In cylindrical coordinates

$$\Delta\overrightarrow{\mathcal{E}} = \frac{\partial^2\mathcal{E}_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial\mathcal{E}_z}{\partial\rho} + \frac{\partial^2\mathcal{E}_z}{\partial\rho^2}$$

Reminder: The Bessel differential equation

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + \left(\frac{x}{x_0} - n\right)^2 y = 0$$

$$\rightarrow \quad y = y_0 J_n\left(\frac{x}{x_0}\right)$$

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The wave equation in cylindrical coordinates becomes

$$\frac{\partial^2 \mathcal{E}_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial \mathcal{E}_z}{\partial \rho} + \frac{\partial^2 \mathcal{E}_z}{\partial \rho^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

Assuming a solution of the form $\mathcal{E}_z = \mathcal{E}_{z,\rho}\left(\rho\right)\cos\left(\omega_r t\right)$ lead to

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho}\cos\left(\omega_r t\right) + \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2}\cos\left(\omega_r t\right) - \frac{1}{c^2}\frac{\partial^2 \cos\left(\omega_r t\right)}{\partial t^2}\mathcal{E}_{z,\rho} &= 0 \\
\Longrightarrow \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2}\cos\left(\omega_r t\right) + \frac{1}{\rho}\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho}\cos\left(\omega_r t\right) + \left(\frac{\omega_r}{c}\right)^2 \mathcal{E}_{z,\rho}\cos\left(\omega_r t\right) &= 0
\end{aligned}$$

$$\Longrightarrow \rho^2\frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2} + \rho\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho} + \left(\frac{\rho\omega_r}{c}\right)^2 \mathcal{E}_{z,\rho} = 0$$

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The differential equation

$$\rho^2 \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2} + \rho \frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho} + \left(\frac{\rho \omega_r}{c}\right)^2 \mathcal{E}_{z,\rho} = 0$$

is the Bessel differential equation which has a solution for $\mathcal{E}_{z,\rho}$

$$\mathcal{E}_{z,\rho} = \mathcal{E}_0 J_0 \left(\frac{\rho \omega_r}{c}\right)$$

where $\mathcal{E}_0$ is the amplitude of the field at $\rho = 0$.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The boundary condition for electrical fields implies that $\mathcal{E}_z\left(\rho=\rho_c\right)=0$. We reformulate the electric field

$$\mathcal{E}_{z,\rho}=\mathcal{E}_0 J_0\left(\chi_0\frac{\rho}{\rho_c}\right)$$

where $\chi_0=\rho_c\omega_r/c\approx 2.405$ is the first zero of the Bessel function J_0.

Finally

$$\mathcal{E}_z\left(\rho,t\right)=\mathcal{E}_0 J_0\left(\chi_0\frac{\rho}{\rho_c}\right)\cos\left(\omega_r t\right)$$

with $\omega_r=\chi_0 c/\rho_c\approx 2.405\ c/\rho_c$.

ENERGY GAIN IN AN RF CAVITY

We express the energy gain of a particle passing through a cavity

$$\delta E_{\mathrm{rf}} = \int q\mathcal{E}_z\left(\rho, z, t\right) dz = q\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\omega_r t\right) dz$$

For a particle passing through the center of the cavity ($\rho = 0$), the energy gain becomes

$$\begin{aligned} \delta E_{\mathrm{rf}}\left(\tau\right) &= qV_{\mathrm{rf},0}T_t \cos\left(\omega_r \tau\right) \quad \text{Linac convention} \\ \delta E_{\mathrm{rf}}\left(\tau\right) &= qV_{\mathrm{rf},0}T_t \sin\left(\omega_r \tau\right) \quad \text{Synchrotron convention} \end{aligned}$$

The maximum potential $V_{\mathrm{rf},0}$ (denominator below) and the transit time factor are

$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

ENERGY GAIN IN AN RF CAVITY

DERIVATION

Starting from

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left(\omega_r t\right) dz$$

The longitudinal position of the particle with respect to the cavity is

$$z\left(t\right) = \int_{\tau}^{t} \beta\left(t\right) c \, dt \approx \beta c \left(t - \tau\right)$$

Assumption: the change in velocity of the particle is neglected here. This is not valid for high gradient cavities with non-relativistic particles!

Derive the energy gain and the expression of the transit time factor.

ENERGY GAIN IN AN RF CAVITY

DERIVATION

Starting from

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left[\omega_r \left(\frac{z}{\beta c}\right) - \omega_r \tau\right] dz$$

Using the trigonometric relationship

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) \cos\left(\omega_r \tau\right) dz + \sin\left(\frac{\omega_r z}{\beta c}\right) \sin\left(\omega_r \tau\right) dz$$

The $\sin$ function is odd and cancels in the integral.

ENERGY GAIN IN AN RF CAVITY

DERIVATION

We get

$$\delta E_{\mathrm{rf}} = q\cos\left(\omega_r \tau\right) \int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz$$

We define the maximum possible accelerating potential (no variation with time during particle passage) as

$$V_{\mathrm{rf},0} = \int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz$$

ENERGY GAIN IN AN RF CAVITY

DERIVATION

We define the transit time factor as the ratio between the accelerating potential including the time variation of the field and the maximum potential

$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{V_{\mathrm{rf},0}}$$

The energy gain in the gap finally becomes

$$\delta E_{\mathrm{rf}}\left(\tau\right) = q V_{\mathrm{rf},0} T_t \cos\left(\omega_r \tau\right) = q V_{\mathrm{rf}} \cos\left(\omega_r \tau\right)$$

The $\cos$ which can be interchanged with $\sin$ depending on the convention used (linac vs. synchrotrons).

TRANSIT TIME FACTOR

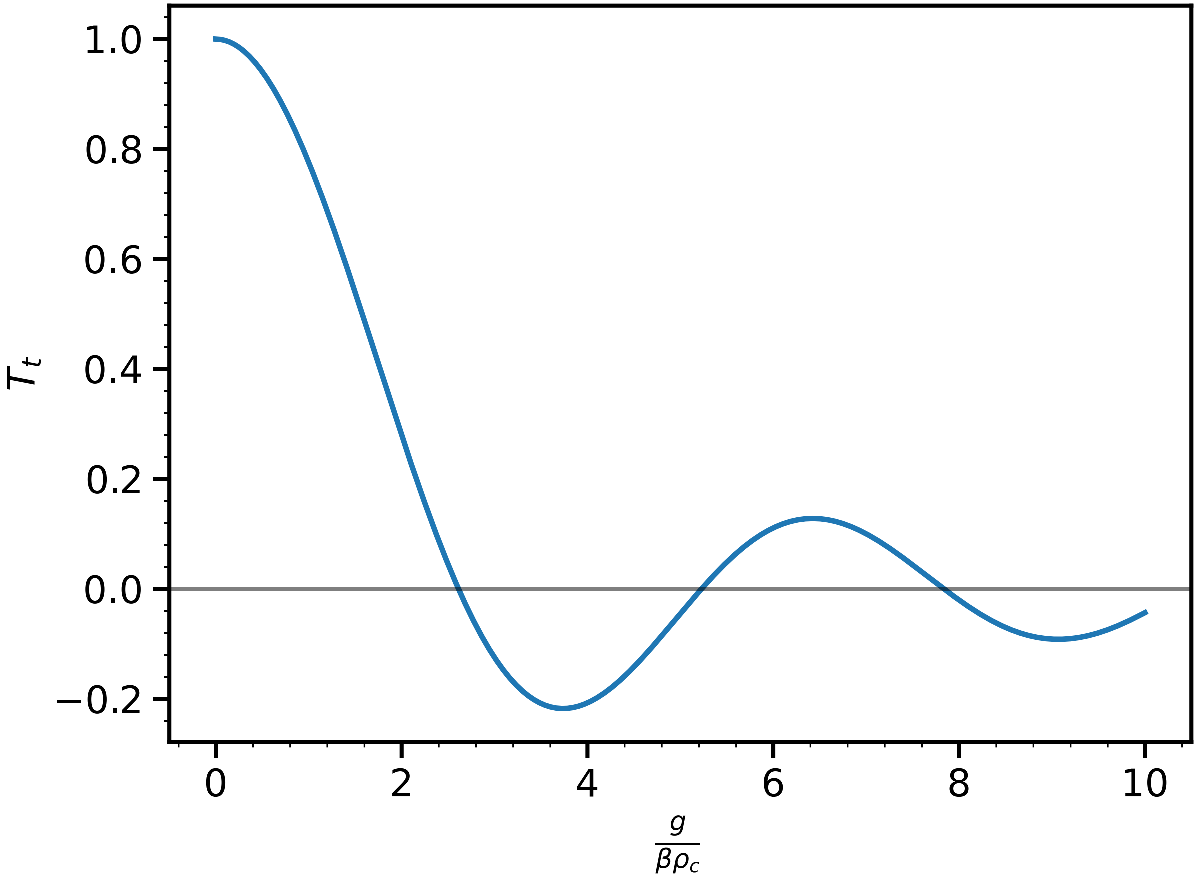
$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

The transit time factor is the ratio between the effective accelerating potential including the time variation of the field (top term) and the maximum potential if a particle would pass instantaneously in the cavity (bottom term, $V_{\mathrm{rf},0}$).

The transit time factor is $T_t \leqslant 1$ and depends on principle on the particle transverse position. For a pillbox cavity, the transit time factor becomes

$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

TRANSIT TIME FACTOR



$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

- The electric field oscillates while the particle goes through the RF cavity
- If the gap is too long, the acceleration potential is effectively reduced.

- A compromise in the design of a cavity is needed to maximize the accelerating potential.

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Derive the transit time factor for a pillbox cavity using the expression of the electric field

$$\mathcal{E}_0 J_0(\rho) \cos\left(\frac{\omega_r z}{\beta c}\right)$$

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Including the expression of the pillbox cavity field in the transit time factor

$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

$$\Longrightarrow T_t = \frac{\mathcal{E}_0 J_0\left(\rho\right) \int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\mathcal{E}_0 J_0\left(\rho\right) \int_{-g/2}^{g/2} dz}$$

$$\Longrightarrow T_t = \frac{1}{g} \int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz$$

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Including the expression of the pillbox cavity field in the transit time factor

$$\Longrightarrow \quad T_t = \frac{1}{g}\int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz$$

$$\Longrightarrow \quad T_t = \frac{\beta c}{\omega_r g}\left[\sin\left(\frac{\omega_r g}{2\beta c}\right) - \sin\left(\frac{-\omega_r g}{2\beta c}\right)\right]$$

$$\Longrightarrow \quad T_t = \frac{\sin\left(\frac{\omega_r g}{2\beta c}\right)}{\left(\frac{\omega_r g}{2\beta c}\right)} = \frac{\sin\left(\frac{\chi_0 g}{2\beta \rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta \rho_c}\right)}$$

ENERGY GAIN IN AN RF CAVITY

SYNCHROTRON CONVENTION

For the rest of the course we will use

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf}}\sin\left(\omega_r \tau\right) \quad \rightarrow \quad \delta E_{\mathrm{rf}}\left(\phi\right) = qV_{\mathrm{rf}}\sin\left(\phi\right)$$

where the transit time factor is included in the definition of V_{rf} (this parameter is often noted $\hat{V}_{\mathrm{rf}}$ in the litterature) and ϕ is the phase of arrival in the cavity.

Assumption: The transit time factor depends on the particle radial position and β. These dependencies will be neglected in the coming derivations.

INDUCTION FORCE IN SYNCHROTRONS

"BETATRONIC" ACCELERATION

During acceleration in a synchrotron the magnetic field is ramped to keep the beam on a constant orbit with

$$\dot{p} = q\dot{\mathcal{B}}_y \rho$$

With the same principle as in the betatron, an azimuthal electric field is induced. This leads to the energy gain (assuming ρ constant)

$$\delta E_\mathrm{b}\left(\rho\right) = q \oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = q \int_0^{2\pi} \int_0^{\rho} \frac{\partial \mathcal{B}_y\left(\rho', \theta, t\right)}{\partial t} \rho' \, d\rho' \, d\theta$$

Assumption: this force is usually negligible in large synchrotrons, although it may not be overlooked to derive precisely longitudinal equations of motion.

SYNCHROTRON RADIATION

ENERGY LOSS IN BENDING MAGNETS

The power of the light emitted by a particle in a curved trajectory is

$$P_{\mathrm{sr}} = \frac{q^2 c}{6\pi\epsilon_0} \frac{(\beta\gamma)^4}{\rho^2}$$

The energy loss over a turn, by multiplying by the time spent in the bending magnet T_b

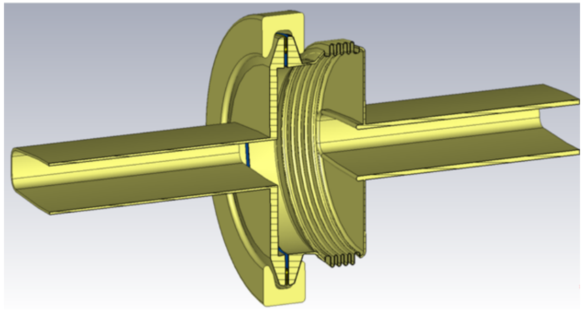
$$\delta E_{\mathrm{sr}}(E,\rho) = \frac{q^2}{3\epsilon_0} \frac{\beta^3 E^4}{\rho\, E_0^4} \quad , \left(T_b = \frac{2\pi\rho}{\beta c} \right)$$

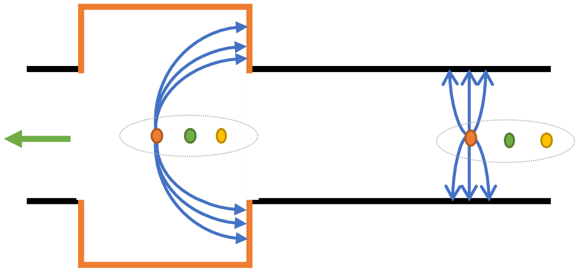
Note the important dependence on E_0^4. Synchrotron radiation is usually neglected for hadron synchrotrons and is predominant for lepton machines.

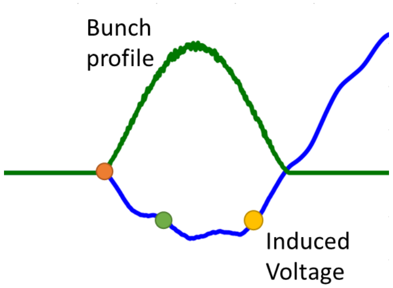
→ Dedicated JUAS course on Synchrotron radiation

SELF INDUCED FIELDS ALONG THE RING

A real accelerator is composed of many equipment, which can lead for example to discontinuities in the beam pipe aperture.







A single particle passing through a cavity-like gap will induce a wakefield $\mathcal{W}\left(\tau\right)$. A bunch with a longitudinal charge density $\lambda\left(\tau\right)$ (number of particles N_b) will induce a voltage $V_{\text{ind}}\left(\tau\right)$, as a convolution product of all the particles single wakes

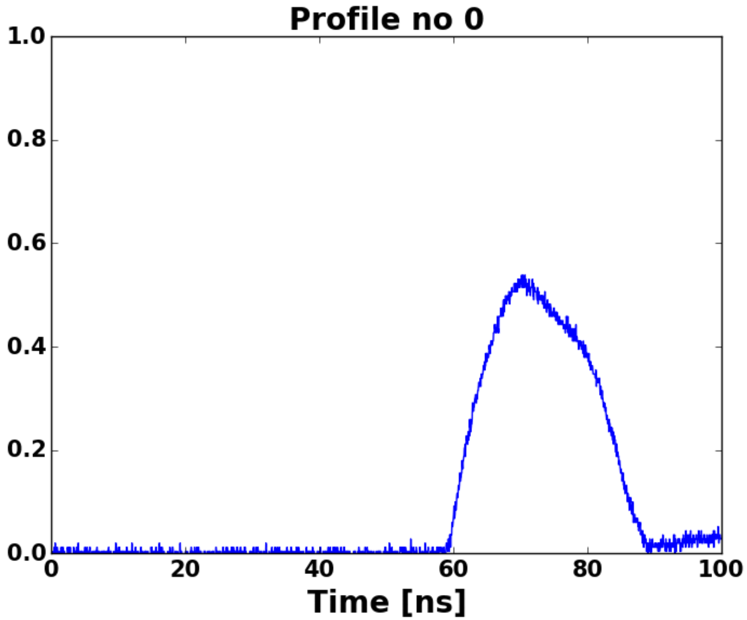
$$\delta E_{\text{ind}}\left(\tau\right) = qV_{\text{ind}}\left(\tau\right) = -qN_b\left(\lambda * \mathcal{W}\right)$$

$\longrightarrow$ Dedicated JUAS course on Collective effects

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity.



- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity (Slide 11)
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\ 80 \cdot 10^6\right) = 1.4\ \mathrm{m}$
- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\ 1.4 \cdot 10^9\right) = 8.2\ \mathrm{cm}$

MODULE 4: THE SYNCHRONOUS PARTICLE

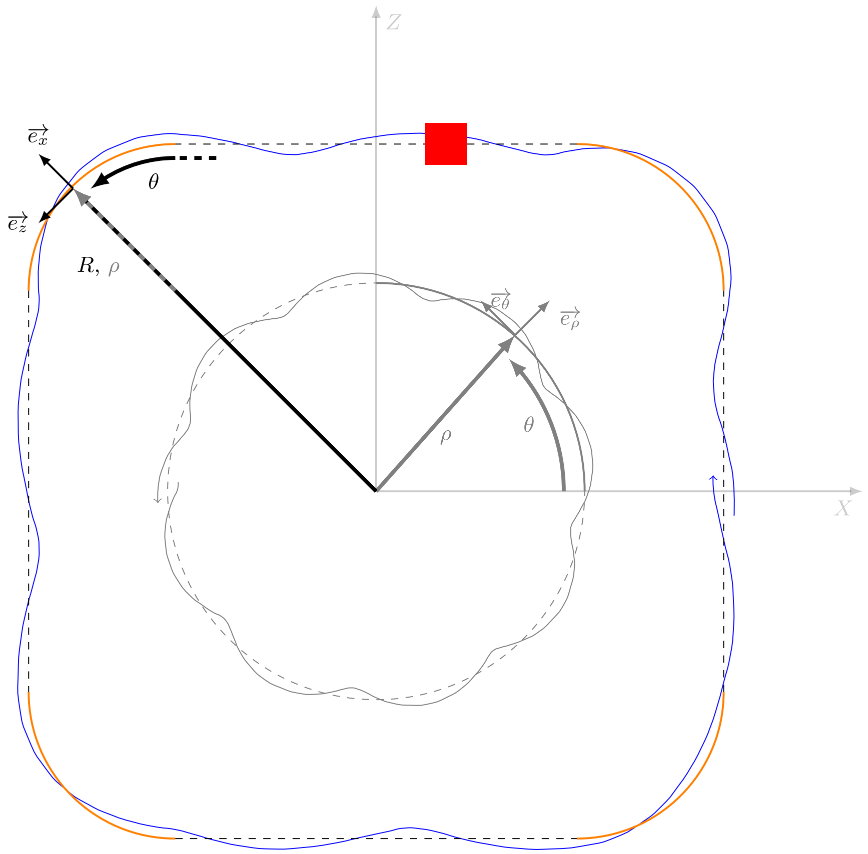
$\longrightarrow$ **Synchronism condition in synchrotrons**

$\longrightarrow$ **Acceleration rate**

$\longrightarrow$ **Magnetic and RF frequency programs**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- The revolution period of an arbitrary particle in a circular machine is

$$T_0 = \frac{C}{v} = \frac{2\pi R}{\beta c}$$

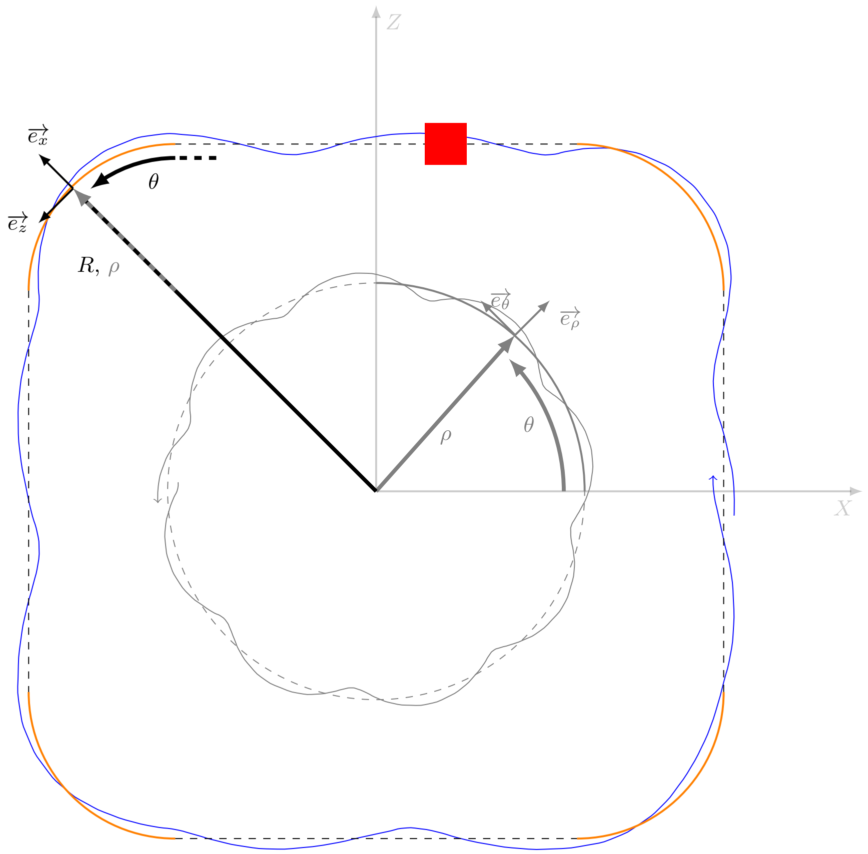
- The corresponding revolution (angular) frequency is

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{\beta c}{R}$$

- We will derive the relationships for the **synchronous** particle (subscript s).

SYNCHRONISM CONDITION

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- A particle is synchronous to the RF if

$$\omega_r = h\,\omega_{0,s} = h\frac{\beta_s c}{R_s}$$

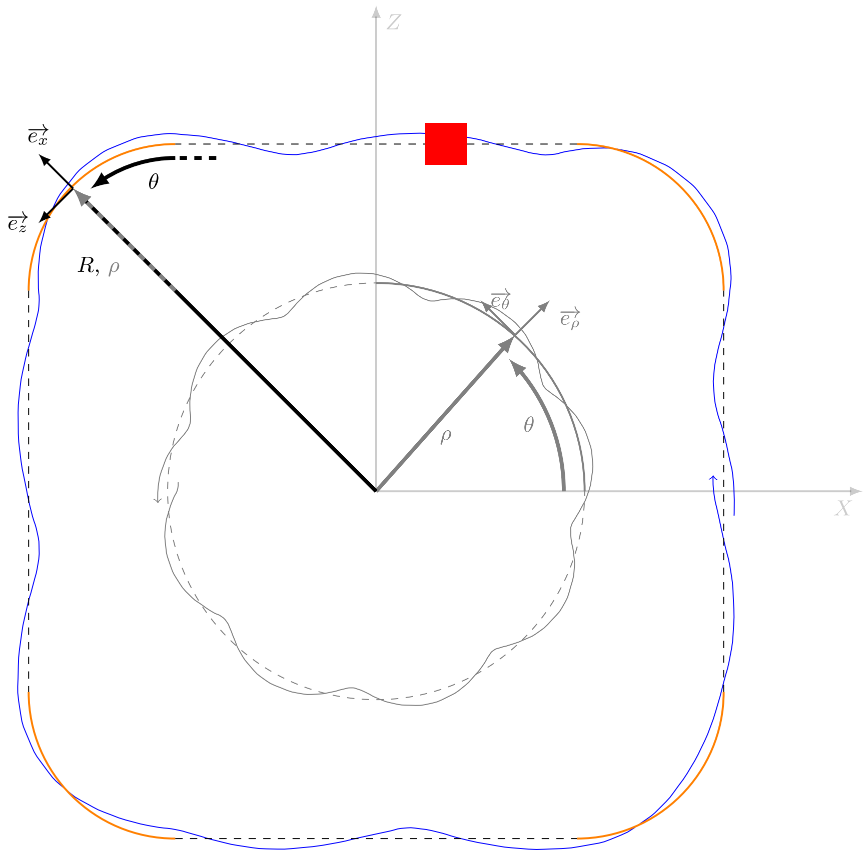
where h is the **RF harmonic number** (integer number).

- There are h different synchronous particles in a synchrotron (and effectively up to h bunches).

- The synchronous particle is fictitious, it is in reality an ideal reference point.

SYNCHRONISM CONDITION

Accelerator seen from above, along the vertical $\vec{Y}$ axis...



- The total energy variation over one turn is

$$\begin{aligned}\delta E_s &= \delta E_{\mathrm{rf},s} + \delta E_{\mathrm{b,s}} \\ &\quad + \delta E_{\mathrm{sr,s}} + \delta E_{\mathrm{ind,s}}\end{aligned}$$

- For the following derivations we will only consider the RF contribution, the energy gain per turn of the synchronous particle is

$$\begin{aligned}\delta E_s &= qV_{\mathrm{rf}} \sin\left(h\omega_{0,s}\tau_s\right) \\ &= qV_{\mathrm{rf}} \sin\left(\phi_s\right)\end{aligned}$$

where ϕ_s is the synchronous phase.

ACCELERATION RATE

Assumption: The acceleration rate with time of the synchronous particle is assumed to be a smooth function with time. The energy gain per turn is usually small (not in Rapid Cycling Synchrotrons!).

The acceleration rate is

$$\dot{E}_s \approx \frac{\delta E_s}{T_{0,s}} \quad \rightarrow \quad \dot{E}_s = \frac{qV_{\mathrm{rf}}}{T_{0,s}} \sin\left(\phi_s\right)$$

The bending field must be increased synchronously, keeping a constant orbit ρ_s, R_s

$$\dot{\mathcal{B}}_y \rho_s = \frac{\dot{p}_s}{q}$$

ACCELERATION RATE

Using the differential relationship

$$\frac{dE}{dp} = \beta c \quad \rightarrow \quad \dot{E} = \beta c \dot{p}$$

and assuming that $\dot{\rho}_s = 0$, we get

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \text{and} \quad \phi_s = \arcsin\left(2\pi \rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\text{rf}}}\right)$$

Reminder: This equation is not related to the induction acceleration. In this case, the acceleration is obtained from the electric field in the RF cavity. The magnetic field in bending magnets is increased so that the synchronous particle keeps fulfilling the synchronism condition.

ACCELERATION RATE

DERIVATION

Derive the acceleration per turn and the synchronous phase assuming $\dot{\rho}_s = 0$

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \text{and} \quad \phi_s = \arcsin\left(2\pi \rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\mathrm{rf}}}\right)$$

ACCELERATION RATE

DERIVATION

$$\frac{d}{dt}\left(\mathcal{B}_y \rho_s\right) = \frac{\dot{p}_s}{q}$$

$$\implies \dot{\mathcal{B}}_y \rho_s + \mathcal{B}_y \dot{\rho}_s = \frac{\dot{E}_s}{q\beta_s c}$$

Assuming $\dot{\rho}_s = 0$

$$\implies \dot{\mathcal{B}}_y \rho_s = \frac{V_{\mathrm{rf}}}{\beta_s c T_{0,s}} \sin\left(\phi_s\right) = \frac{V_{\mathrm{rf}}}{2\pi R_s} \sin\left(\phi_s\right)$$

$$\implies 2\pi q R_s \rho_s \dot{\mathcal{B}}_y = q V_{\mathrm{rf}} \sin\left(\phi_s\right) = \delta E_s$$

RF FREQUENCY SWEEP

To preserve the synchronism condition, the RF frequency must also be adjusted to follow the evolution of β_s during acceleration

$$\omega_r(t) = \frac{hc}{R_s}\beta_s(t)$$

The RF frequency program is linked to the magnetic field

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

The principle of the synchrotron acceleration is to ramp the bending field and the RF frequency synchronously, providing constant R_s and the synchronous phase ϕ_s.

RF FREQUENCY SWEEP

DERIVATION

From

$$\omega_r(t) = \frac{hc}{R_s}\beta_s(t)$$

Express $\beta_s(t)$ as a function of $\mathcal{B}_y(t)$ using the definition of the magnetic rigidity with constant ρ_s (and R_s).

Obtain

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

RF FREQUENCY SWEEP

DERIVATION

Expression of β_s

$$\beta_s\left(t\right) = \frac{pc}{E} = \frac{pc}{\sqrt{\left(pc\right)^2 + E_0^2}}$$

$$= \frac{\mathcal{B}_y \rho_s q c}{\sqrt{\left(\mathcal{B}_y \rho_s q c\right)^2 + \left(m_0 c^2\right)^2}}$$

$$= \frac{\mathcal{B}_y}{\sqrt{\left(\mathcal{B}_y\right)^2 + \left(\frac{m_0 c^2}{\rho_s q c}\right)^2}}$$

RF FREQUENCY SWEEP

DERIVATION

Expression of β_s

$$\beta_s(t) = \sqrt{\frac{\mathcal{B}_y^2}{\mathcal{B}_y^2 + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

Leading to

$$f_r(t) = \frac{\omega_r(t)}{2\pi} = \frac{hc}{2\pi R_s}\beta_s(t)$$
$$= \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

- Revolution period/frequency of the SPS ($\rho_0 = 741.25$ m, $C_0 = 6911.50$ m)
- RF frequency of the SPS ($h = 4620$)
- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
- Smallest RF voltage to accelerate the synchronous particle
- Compute the same parameters with Lead ions $^{208}\mathrm{Pb}^{82+}$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

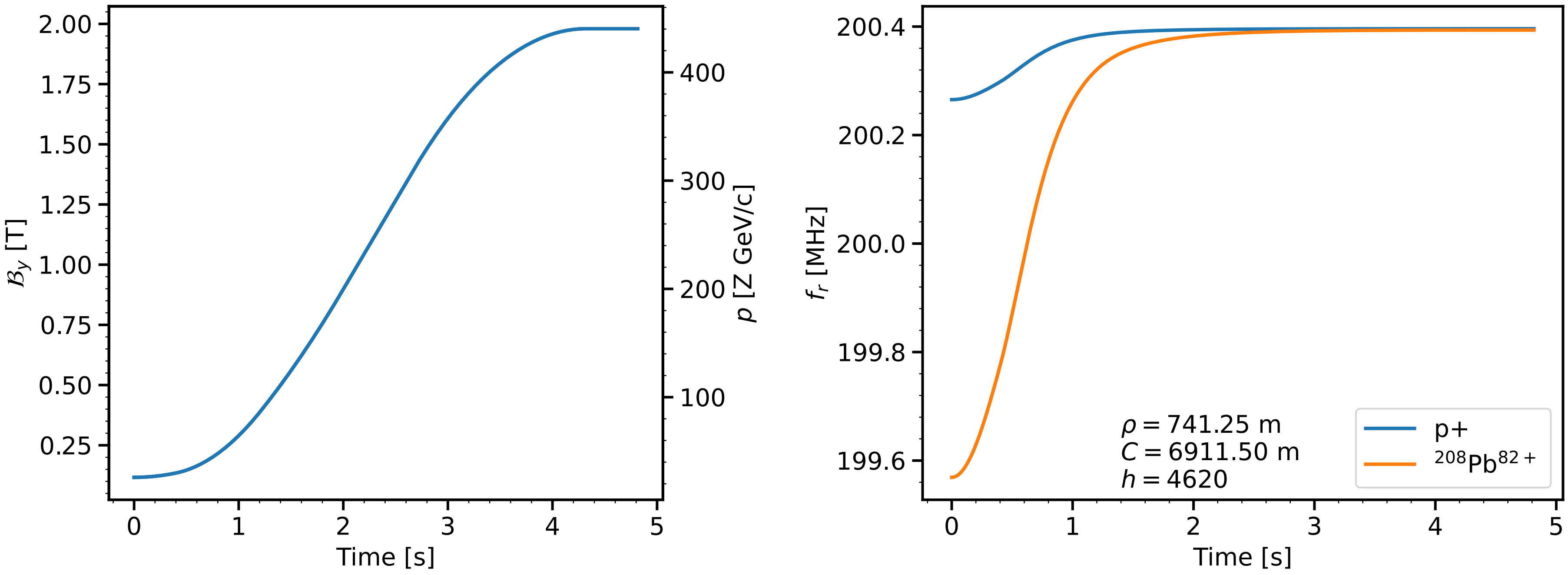
Machine	SPS inj. p+	SPS ext. p+	SPS inj. Pb	SPS ext. Pb
p [GeV/c]	26	450	2132	36900
E [GeV]	26.0169	450.001	2140.786	36900.509
β	0.99935	0.999...	0.9959	0.999..
T_0 [μs]	23.0693	23.0543	23.1493	23.0546
f_r [MHz]	200.266	200.396	199.574	200.394

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

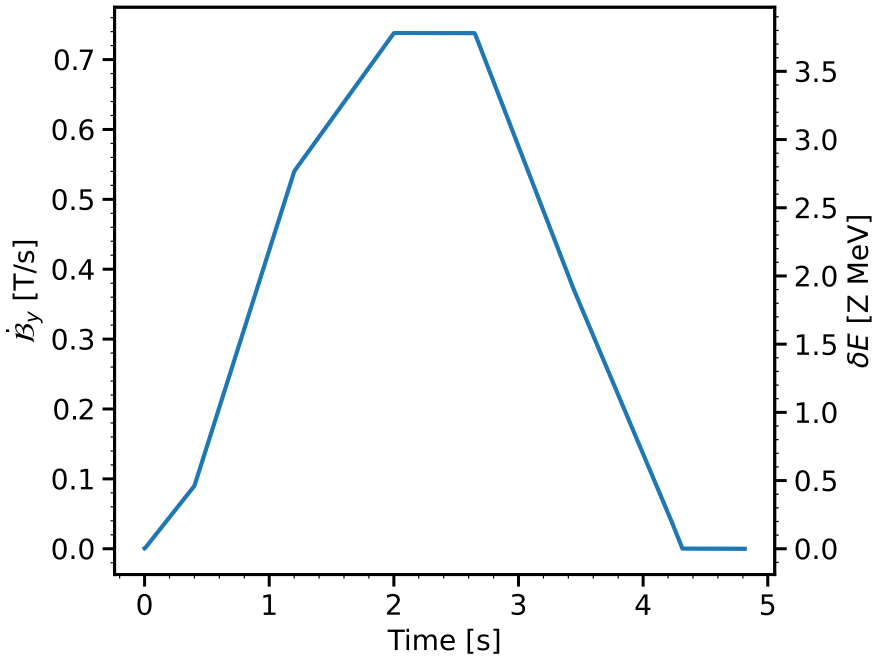
- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
  - $6911.50 \cdot 741.35 \cdot 1 \cdot 0.7 = 3.59\ \mathrm{MeV}$ for p+
  - $6911.50 \cdot 741.35 \cdot 82 \cdot 0.7 = 294\ \mathrm{MeV}$ for Pb
- Smallest RF voltage to accelerate the synchronous particle
  - $3.59\ \mathrm{MV}$ for p+ and Pb ($\delta E_{\mathrm{rf}}/q$)

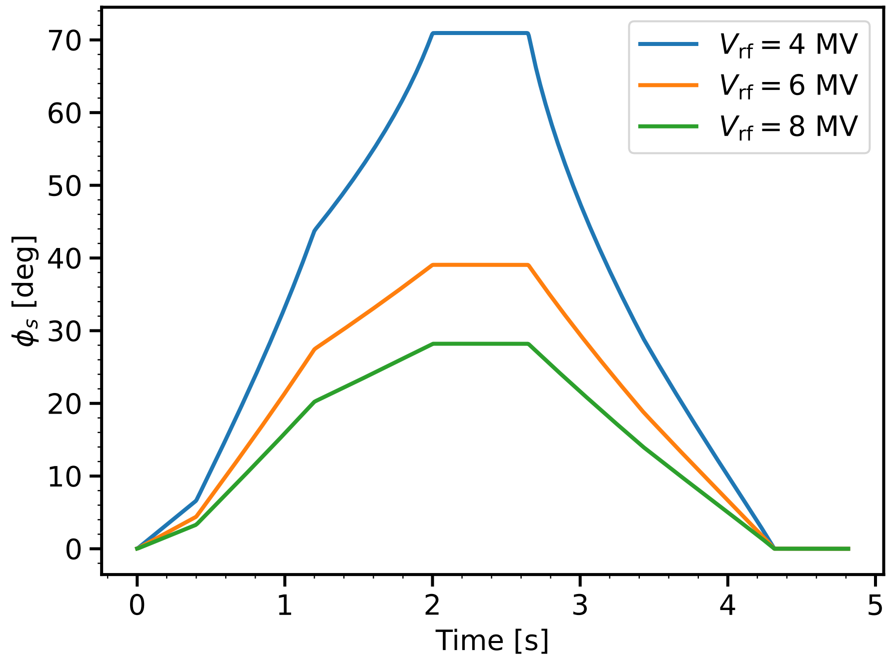
EXAMPLE PROGRAMS IN THE SPS



- The magnetic (and expected momentum) program together with the RF frequency program.
- Due to their larger mass the lead ions have a lower β_s and hence f_r for the same $\mathcal{B}_y \rho_s$

EXAMPLE PROGRAMS IN THE SPS





- The energy gain per turn defines the minimal RF voltage required to accelerate the synchronous particle ($\phi_s < \pi/4$, NB: independent of the charge Z!). Increasing V_{rf} allows to design a shorter cycle in time.

- We will see that there is in reality more considerations to design a V_{rf} program!

MODULE 5: DIFFERENTIAL RELATIONSHIPS IN A SYNCHROTRON

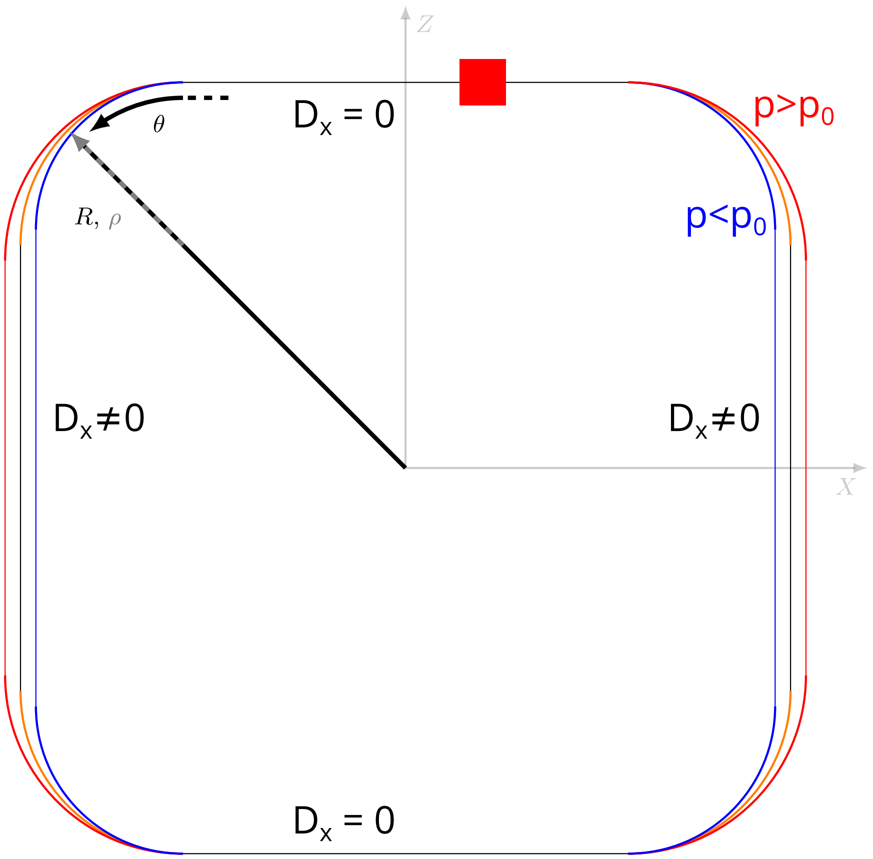
→ **Momentum compaction factor**

→ **Phase slip factor, transition gamma**

→ **Derivation of differential relationships**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



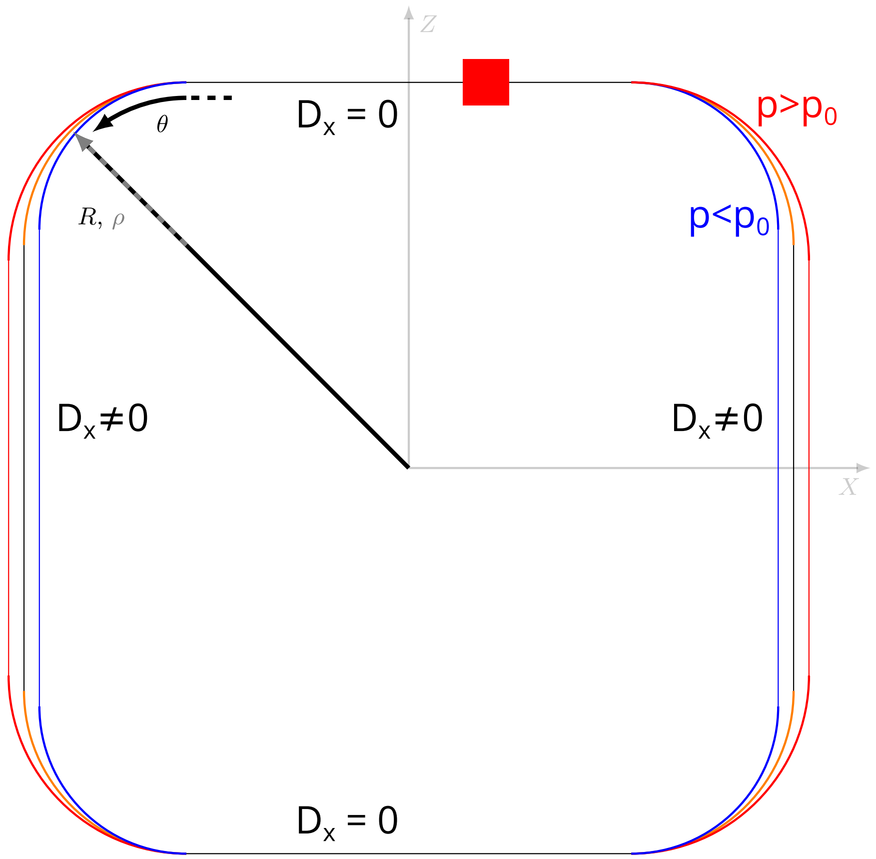
- In the previous module we assumed acceleration with a constant ρ_s.
- Nonetheless, like in cyclotrons, a particle can also be accelerated at fixed magnetic field with

$$\begin{aligned} dp_s &= q\left(d\mathcal{B}_y \rho_s + \mathcal{B}_y d\rho_s\right) \\ &= q\mathcal{B}_y d\rho_s \end{aligned}$$

- The synchronism condition $\omega_r = h\omega_{0,s}$ remains valid, and the RF frequency can be adjusted to accelerate the beam.

ORBIT AND DISPERSION

Accelerator seen from above, along the vertical $\vec{Y}$ axis...

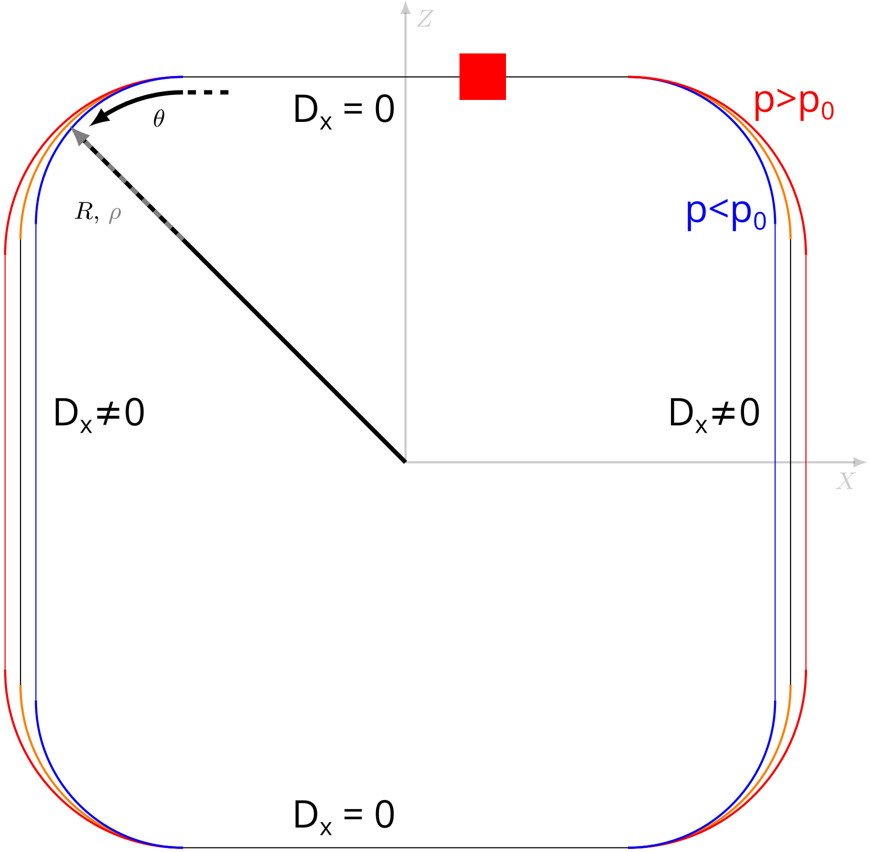


- In synchrotrons, the small beam pipe aperture only allows for small orbit changes.
- Nonetheless, this is used to steer the beam radially or to do very fine adjustments of the beam energy.
- The radial/horizontal offset is given by the transverse dispersion function

$$x_D(z) = D_x(z)\,\frac{dp}{p}$$

MOMENTUM COMPACTION AND PHASE SLIP FACTOR

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



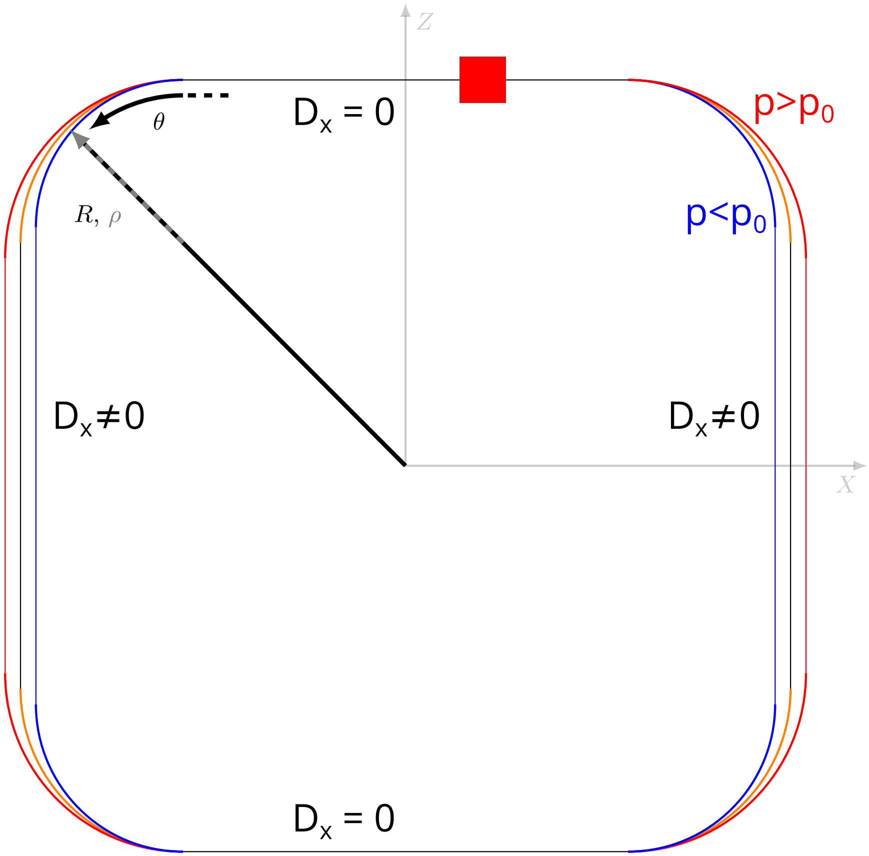
- The relative elongation of the mean radius due to a momentum relative offset is called the "momentum compaction factor"

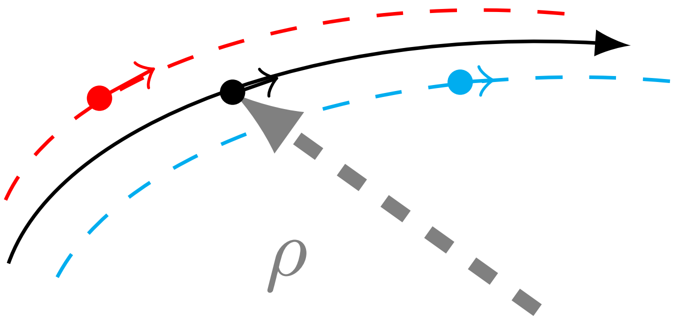
$$\alpha_p = \frac{dR/R}{dp/p}$$

- This parameter is linked to the relative change in revolution frequency, the phase slip factor

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p} = \frac{dT_0/T_0}{dp/p}$$

MOMENTUM COMPACTION AND PHASE SLIP FACTOR





- Is the blue particle arriving before or after the red one after a turn?

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p}$$

- The phase slip factor is fundamental to longitudinal beam dynamics!

MOMENTUM COMPACTION FACTOR

DEFINITION

The momentum is a function of $(\rho, \mathcal{B}_y)$, and consequently of $(R, \mathcal{B}_y)$. It can be differentiated as

$$\frac{dp}{p} = \left(\frac{\partial p}{\partial R}\right)_{\mathcal{B}_y} \frac{R}{p}\frac{dR}{R} + \left(\frac{\partial p}{\partial \mathcal{B}_y}\right)_R \frac{\mathcal{B}_y}{p}\frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

$$\implies \frac{dp}{p} = \frac{1}{\alpha_p}\frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

The momentum compaction factor **assumes a constant** $\mathcal{B}_y$ and is defined as

$$\alpha_p = \left(\frac{\partial p}{\partial R}\right)^{-1}_{\mathcal{B}_y} \frac{p}{R} = \left(\frac{\partial R}{\partial p}\right)_{\mathcal{B}_y} \frac{p}{R}$$

REMINDER DIFFERENTIATION

Differentiating a function $f\left(x,y\right)$

$$df\left(x,y\right)=\left(\frac{\partial f}{\partial x}\right)_y dx+\left(\frac{\partial f}{\partial y}\right)_x dy$$

partial derivatives are taken with all other parameters constant.

Under certain considerations (continuous injective functions), we have

$$\frac{\partial f}{\partial x}=\left(\frac{\partial x}{\partial f}\right)^{-1}$$

MOMENTUM COMPACTION FACTOR

DERIVATION OF GENERAL DEFINITION

Let's remind that $p = \mathcal{B}_y \rho q$, therefore

$$\left(\frac{\partial p}{\partial \mathcal{B}_y}\right)_R \frac{\mathcal{B}_y}{p} = \rho q \frac{\mathcal{B}_y}{p} = 1$$

MOMENTUM COMPACTION FACTOR

COMPUTATION FROM DISPERSION FUNCTION

The horizontal (i.e. radial) offset of a particle closed orbit is obtained from the machine optics (see JUAS Transverse Beam Dynamics Course 3)

$$x_D(z) = D_x(z)\frac{dp}{p}$$

The momentum compaction factor can be computed from

$$\alpha_p = \frac{1}{2\pi R}\int_0^{2\pi R}\frac{D_x(z)}{\rho(z)}dz = \frac{\langle D_x\rangle_\rho}{R}$$

where $\langle D_x\rangle_\rho$ is an averaged dispersion value in the bending magnets (*NB: $\rho \rightarrow \infty$ in straight sections, and $\alpha_p = 0$ in linacs*). For azimuthally symmetric fields, $\alpha_p = 1/Q_x^2$ (which is a reasonable scaling law in general).

MOMENTUM COMPACTION FACTOR

DERIVATION FROM DISPERSION

The total path length increase due to dispersion is

$$dC = 2\pi dR = \int_0^{2\pi} x_D\left(z\right) d\theta$$

Reminder: x_D is a horizontal and i.e. radial offset.

Give an equation to α_p as a function of D_x.

MOMENTUM COMPACTION FACTOR

DERIVATION FROM DISPERSION

The total path length increase due to dispersion is

$$dC = 2\pi dR = \int_0^{2\pi} x_D\left(z\right) d\theta$$

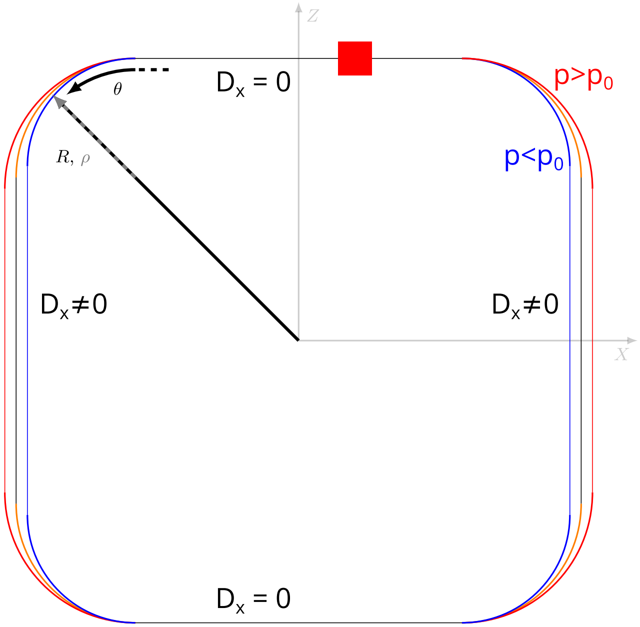
$$\Longrightarrow dR = \frac{1}{2\pi} \int_0^{2\pi R} D_x\left(z\right) \frac{dp}{p} \frac{dz}{\rho\left(z\right)} \quad , \left(dz = \rho\left(z\right) d\theta\right)$$

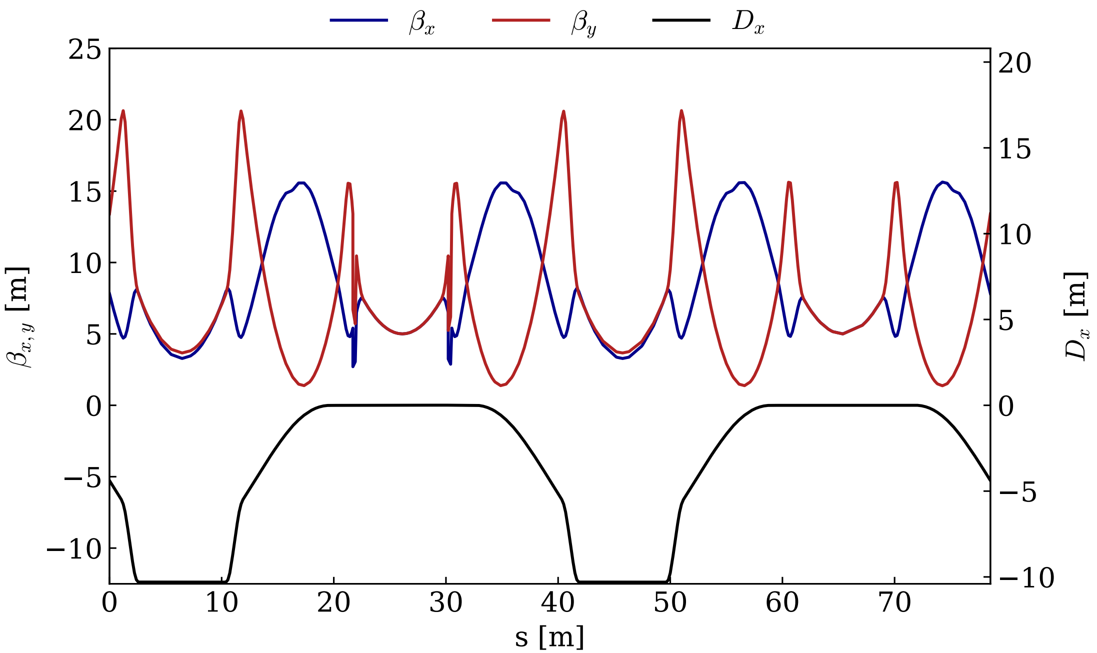
$$\Longrightarrow \frac{dR}{R} \frac{p}{dp} = \frac{1}{2\pi R} \int_0^{2\pi R} \frac{D_x\left(z\right)}{\rho\left(z\right)} dz$$

$$\Longrightarrow \alpha_p = \frac{1}{2\pi R} \int_0^{2\pi R} \frac{D_x\left(z\right)}{\rho\left(z\right)} dz$$

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)

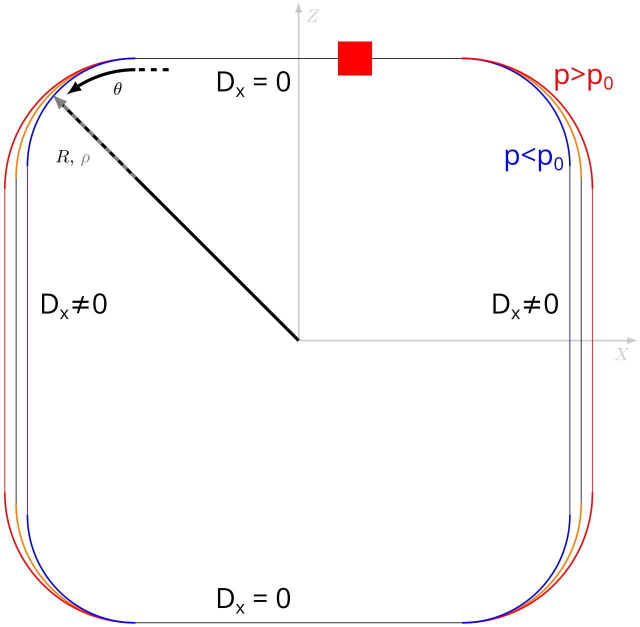


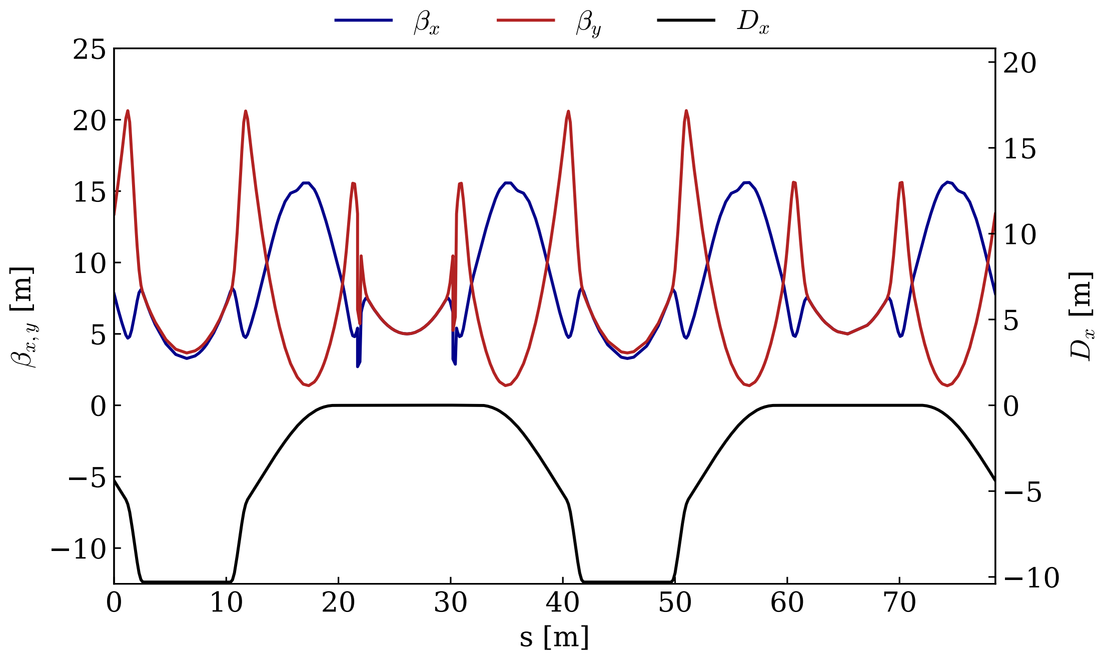


- Dispersion is 0 in two straight sections and large in the other two straight sections.
- NB: Dispersion is displayed as negative but should be taken as positive. Due to convention, integrating from $2\pi R \rightarrow 0$.

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)

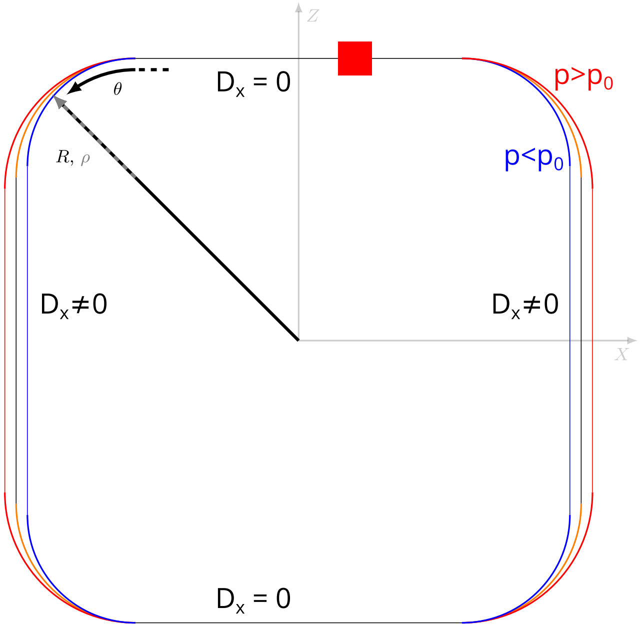


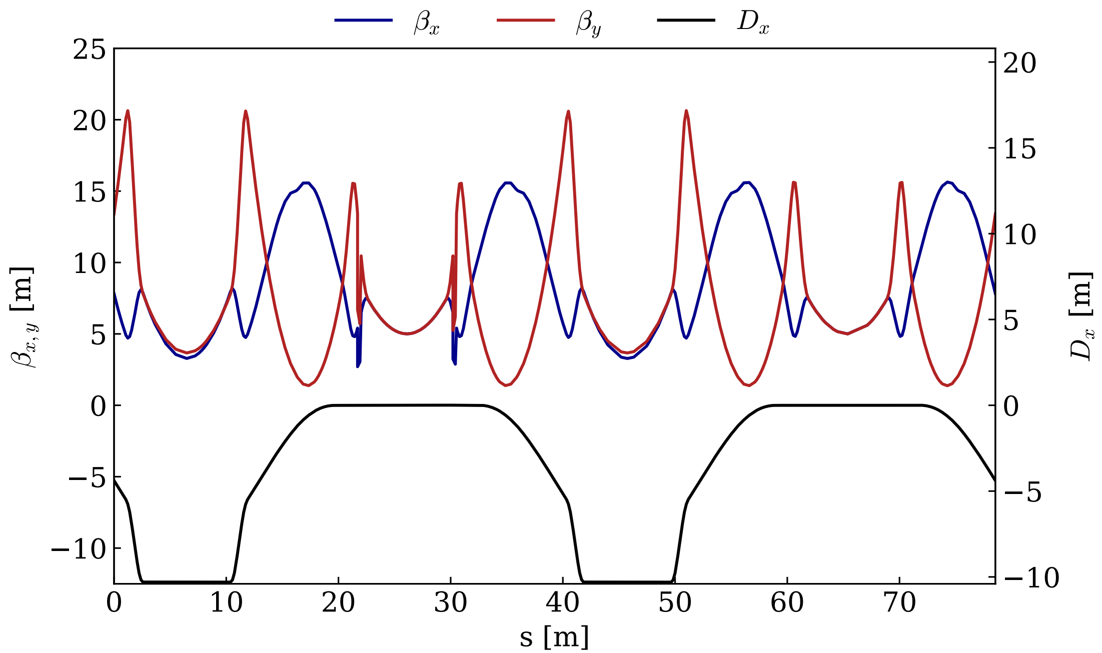


- What is the local radial offset for a particle with $\frac{\Delta p}{p_0} = 10^{-3}$ at large dispersion?

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)





- What is the local radial offset for a particle with $\frac{\Delta p}{p_0} = 10^{-3}$ at large dispersion?
- $x_D \approx 1$ cm.

PHASE SLIP FACTOR

The momentum compaction factor expresses the orbit variation due to a momentum offset. The revolution period/frequency of the particle (and RF frequency via the synchronism condition) is also changed.

Differentiating the revolution frequency

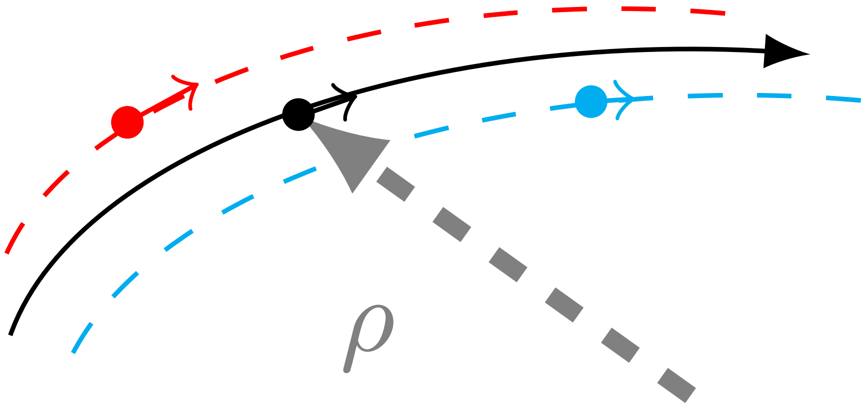
$$\omega_0 = \frac{\beta c}{R} \quad \rightarrow \quad \frac{d\omega_0}{\omega_0} = \frac{d\beta}{\beta} - \frac{dR}{R} \left(= -\frac{dT_0}{T_0} \right)$$

we obtain the phase slip factor

$$\eta = \frac{dT_0/T_0}{dp/p} = -\frac{d\omega_0/\omega_0}{dp/p} = \alpha_p - \frac{1}{\gamma^2}$$

Note: The phase slip is sometimes defined with an opposite sign in the litterature, beware of the used conventions!

TRANSITION ENERGY



- Two regimes

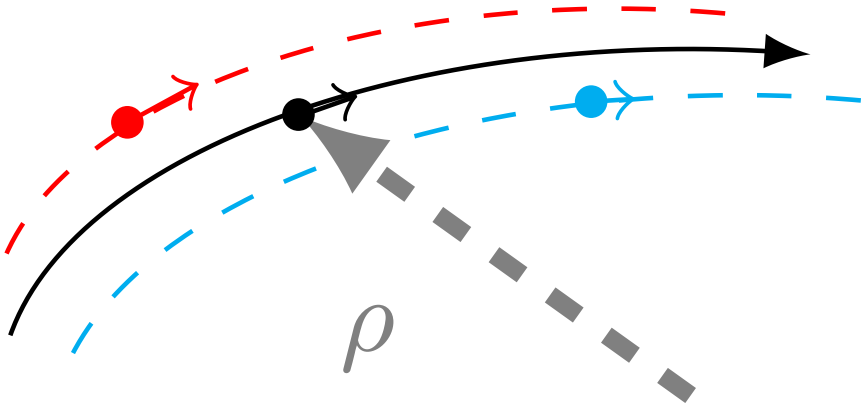
$$\eta > 0 \quad \text{if} \quad \alpha_p > \frac{1}{\gamma^2}, \qquad \gamma > \gamma_t$$

$$\eta < 0 \quad \text{if} \quad \alpha_p < \frac{1}{\gamma^2}, \qquad \gamma < \gamma_t$$

We introduced the transition gamma $\gamma_t = 1/\sqrt{\alpha_p}$.

In the two different regimes, which particle is circulating faster/slower in the machine? What happens for very large γ?

TRANSITION ENERGY



- Two regimes

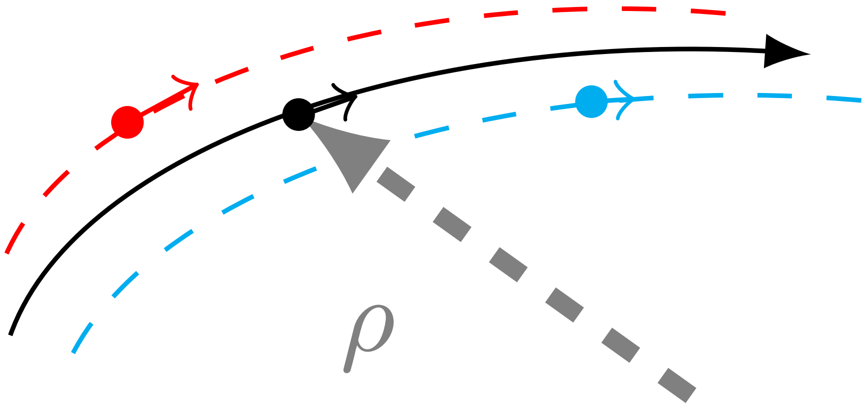
$$\eta > 0 \quad \text{if} \quad \alpha_p > \frac{1}{\gamma^2}, \qquad \gamma > \gamma_t$$

$$\eta < 0 \quad \text{if} \quad \alpha_p < \frac{1}{\gamma^2}, \qquad \gamma < \gamma_t$$

We introduced the transition gamma γ_t.

- For $\gamma < \gamma_t$ (below transition), the particles with **increasing orbit/momentum arrives earlier** as the velocity gain is more important than the increased path length. This happens in particular in **low energy synchrotrons**.
- For $\gamma > \gamma_t$ (above transition), the particles with **increasing orbit/momentum arrives later** as the velocity gain is less important than the increased path length. This happens in particular in **high energy synchrotrons**.

TRANSITION ENERGY



- Special regime

$$\eta = 0 \quad \text{if} \quad \alpha_p = \frac{1}{\gamma^2}, \quad \gamma = \gamma_t$$

- At transition energy, all particles circulate with the same revolution period.
- This change or regime during acceleration requires a special treatment which requires further derivations.
- To avoid transition crossing, special optics with $\langle D_x \rangle < 0 \rightarrow \alpha_p < 0$ can be made, leading mathematically to an imaginary γ_t.

USUAL APPROXIMATIONS

- The momentum compaction factor is computed in synchrotrons with respect to the design orbit R_0 (subscript for design synchrotron parameters 0). Note that the synchronous particle can be offset in orbit with respect to the design trajectory.
- The momentum compaction factor can be expanded in series around p_0 and coefficients computed from the non-linear dispersion function (JUAS Lecture on Transverse Non Linearities)

$$\alpha_{p_0} = \alpha_0 + \alpha_1 \frac{\Delta p}{p_0} + \alpha_2 \left(\frac{\Delta p}{p_0}\right)^2 + ...$$

Assumption: we will assume linear dispersion, momentum compaction factor $\alpha_0 = (\Delta R / R_0) / (\Delta p / p_0)$, phase slip factor $\eta_0 = -(\Delta\omega_0 / \omega_{0,0}) / (\Delta p / p_0)$ for the rest of the course.

AVERAGE MAGNETIC FIELD INDEX

The magnetic rigidity formula can also be written

$$p = q\mathcal{B}_y\rho = q\left\langle\mathcal{B}_y\right\rangle R$$

where we define the average magnetic field along a particle path

$$\left\langle\mathcal{B}_y\right\rangle = \frac{1}{2\pi R}\int_0^{2\pi R}\mathcal{B}_y dz$$

We define the average magnetic field index

$$\left\langle n\right\rangle = -\frac{d\left\langle\mathcal{B}_y\right\rangle / \left\langle\mathcal{B}_y\right\rangle}{dR/R} = 1 - \frac{1}{\alpha_p}$$

These definitions are found in litterature for derivations.

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Show that the definition of the average magnetic field leads to

$$p = q\mathcal{B}_y\rho = q\left\langle \mathcal{B}_y \right\rangle R$$

By differentiating the formula above, demonstrate the relationship between $\langle n \rangle$ and α_p

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Starting from

$$\begin{aligned}
\langle \mathcal{B}_y \rangle &= \frac{1}{2\pi R}\int_0^{2\pi R} \mathcal{B}_y dz \\
&= \frac{1}{2\pi R}\int_0^{2\pi R} \frac{p}{q\rho} dz \\
&= \frac{1}{2\pi R}\frac{p}{q}\int_0^{2\pi R} \frac{dz}{\rho(z)} \\
&= \frac{1}{2\pi R}\frac{p}{q}\frac{2\pi\rho}{\rho} && \rho \rightarrow \infty \text{ in straigt sections} \\
\langle \mathcal{B}_y \rangle R &= \frac{p}{q} && \text{otherwise constant}
\end{aligned}$$

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Differentiating

$$
\begin{aligned}
& p = q \left\langle \mathcal{B}_y \right\rangle R \\
\implies & \frac{dp}{p} = \frac{d\left\langle \mathcal{B}_y \right\rangle}{\left\langle \mathcal{B}_y \right\rangle} + \frac{dR}{R} \\
\implies & \frac{dp/p}{dR/R} = \frac{d\left\langle \mathcal{B}_y \right\rangle / \left\langle \mathcal{B}_y \right\rangle}{dR/R} + 1 \\
\implies & \frac{1}{\alpha_p} = -\left\langle n \right\rangle + 1 \\
\implies & \left\langle n \right\rangle = 1 - \frac{1}{\alpha_p}
\end{aligned}
$$

SYNCHROTRON DIFFERENTIAL EQUATIONS

(1)	$\mathcal{B}_y, p, R$	$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$
(2)	f_0, p, R	$\frac{dp}{p} = \gamma^2 \frac{df_0}{f_0} + \gamma^2 \frac{dR}{R}$
(3)	$\mathcal{B}_y, f_0, p$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma_t^2 \frac{df_0}{f_0} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$
(4)	$\mathcal{B}_y, f_0, R$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma^2 \frac{df_0}{f_0} + \left(\gamma^2 - \gamma_t^2\right) \frac{dR}{R}$

SYNCHROTRON DIFFERENTIAL EQUATIONS

OTHER USEFUL RELATIONSHIPS

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

$$\frac{dp}{p} = \frac{dR}{R} + \frac{d\left\langle \mathcal{B}_y \right\rangle}{\left\langle \mathcal{B}_y \right\rangle}$$

$$\frac{dR}{R} = \frac{1}{\alpha_p}\frac{d\rho}{\rho}$$

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]		
γ		
T_0 [μs]		
α_p [10^{-3}]		
E_t [GeV]		
η [10^{-3}]		

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]	14.03	450
γ	14.95	479.6
T_0 [μs]	23.11	23.05
α_p [10^{-3}]	3.086	3.086
E_t [GeV]	16.89	16.89
η [10^{-3}]	-1.385	3.082

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
  - Transition is crossed since η changes sign
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
  - $\Delta R = 3.086 \cdot 10^{-3} \cdot 6911.50/(2 \cdot 3.14) \cdot \left(-10^{-4}\right) =$-0.34 mm
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?
  - Low E: $\Delta T_0 = -1.385 \cdot 10^{-3} \cdot 23.11 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$ 3.2 ps (late)
  - High E: $\Delta T_0 = 3.082 \cdot 10^{-3} \cdot 23.05 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$-7.1 ps (early)

EXERCISES

- Demonstrate the differential equations (1), (2), (3), (4)

EXERCISES

DERIVATION

(1) Definition in the lecture

(2) Combining

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta} \quad \text{and} \quad \frac{d\beta}{\beta} = \frac{df_0}{f_0} + \frac{dR}{R}$$

(3) Substituting $\frac{dR}{R}$ from (1) in (2)

(4) Substituting $\frac{dp}{p}$ from (1) in (2)

TAKE AWAY MESSAGE

ENERGY GAIN

- RF energy gain

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf},0}T_t \sin\left(\omega_r \tau\right) \quad \rightarrow \delta E_{\mathrm{rf}}\left(\phi\right) = qV_{\mathrm{rf}} \sin\left(\phi\right)$$

- Transit time factor

$$T_t\left(\rho, \beta\right) = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

- Assumptions:
  - β is not changing in the computation of T_t
  - The $\left(\rho, \beta\right)$ dependence of T_t will be neglected

TAKE AWAY MESSAGE

PILLBOX CAVITY (FUNDAMENTAL MODE)

- Pillbox cavity properties

$$\mathcal{E}_z(\rho, t) = \mathcal{E}_0 J_0\left(\chi_0 \frac{\rho}{\rho_c}\right)\cos(\omega_r t)$$

J_n Bessel function, $\chi_0 \approx 2.405, \omega_r = \chi_0 c/\rho_c$

- Transit time factor of pillbox cavity

$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

TAKE AWAY MESSAGE

OTHER ENERGY GAIN/LOSS IN A RING

- Induction acceleration (small in large synchrotrons)

$$\delta E_{\mathrm{b}}\left(\rho\right) = q \int_0^{2\pi} \int_0^{\rho} \frac{\partial \mathcal{B}_y\left(\rho', \theta, t\right)}{\partial t} \rho' \, d\rho' \, d\theta$$

- Synchrotron radiation (relevant for lepton accelerators)

$$\delta E_{\mathrm{sr}}\left(E, \rho\right) = \frac{q^2}{3\epsilon_0} \frac{\beta^3 E^4}{\rho \, E_0^4}$$

- Self induced field

$$\delta E_{\mathrm{ind}}\left(\tau\right) = q V_{\mathrm{ind}}\left(\tau\right) = -q N_b \left(\lambda * \mathcal{W}\right)$$

TAKE AWAY MESSAGE

SYNCHRONISM IN SYNCHROTRON

- The revolution period and frequency

$$T_0 = \frac{C}{v} = \frac{2\pi R}{\beta c} \quad , \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{\beta c}{R}$$

- Synchronism condition with RF frequency

$$\omega_r = h\, \omega_{0,s} = h \frac{\beta_s c}{R_s}$$

TAKE AWAY MESSAGE

ACCELERATION

- Acceleration rate (subscript s for synchronous particle)

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \rightarrow \quad \phi_s = \arcsin\left(2\pi \rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\mathrm{rf}}}\right)$$

- RF frequency program

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

- Assumptions: Acceleration with constant R_s and ρ_s

TAKE AWAY MESSAGE

RADIAL DISPLACEMENT

- Momentum compaction factor (subscript 0 for design orbit/momentum, transition gamma γ_t)

$$\alpha_p = \frac{dR/R}{dp/p} = \frac{\langle D_x \rangle_\rho}{R} = \frac{1}{\gamma_t^2} \approx \frac{\Delta R/R_0}{\Delta p/p_0} \approx \frac{\Delta R/R_s}{\Delta p/p_s}$$

- Phase slip factor

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p} = \frac{dT_0/T_0}{dp/p} = \alpha_p - \frac{1}{\gamma^2} \approx -\frac{\Delta\omega_{0,0}/\omega_{0,0}}{\Delta p/p_0} \approx -\frac{\Delta\omega_{0,s}/\omega_{0,s}}{\Delta p/p_s}$$

- Assumptions: Radial displacement with constant $\mathcal{B}_y$

TAKE AWAY MESSAGE, DIFFERENTIAL EQS.

(1)	$\mathcal{B}_y, p, R$	$$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$
(2)	f_0, p, R	$$\frac{dp}{p} = \gamma^2 \frac{df_0}{f_0} + \gamma^2 \frac{dR}{R}$$
(3)	$\mathcal{B}_y, f_0, p$	$$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma_t^2 \frac{df_0}{f_0} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$$
(4)	$\mathcal{B}_y, f_0, R$	$$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma^2 \frac{df_0}{f_0} + \left(\gamma^2 - \gamma_t^2\right) \frac{dR}{R}$$