LONGITUDINAL BEAM DYNAMICS

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COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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LESSON 3: LONGITUDINAL EQUATIONS OF MOTION

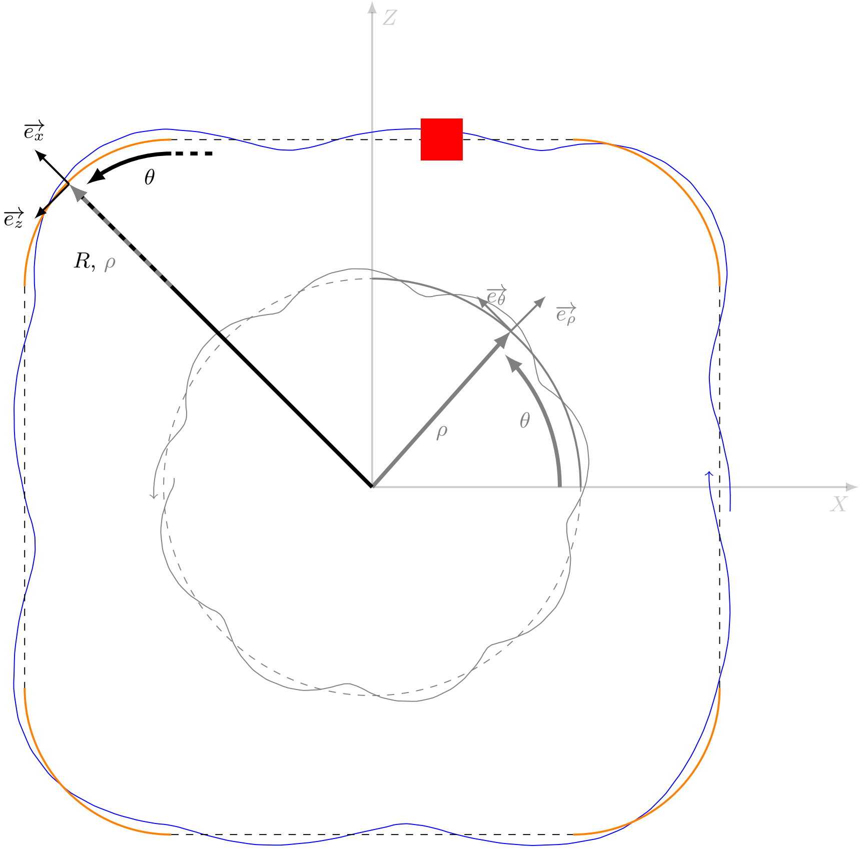
MODULE 6: THE NON-SYNCHRONOUS PARTICLES

$\longrightarrow$ **Energy gain equation of motion**

$\longrightarrow$ **Phase slippage equation of motion**

NON-SYNCHRONOUS PARTICLE

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



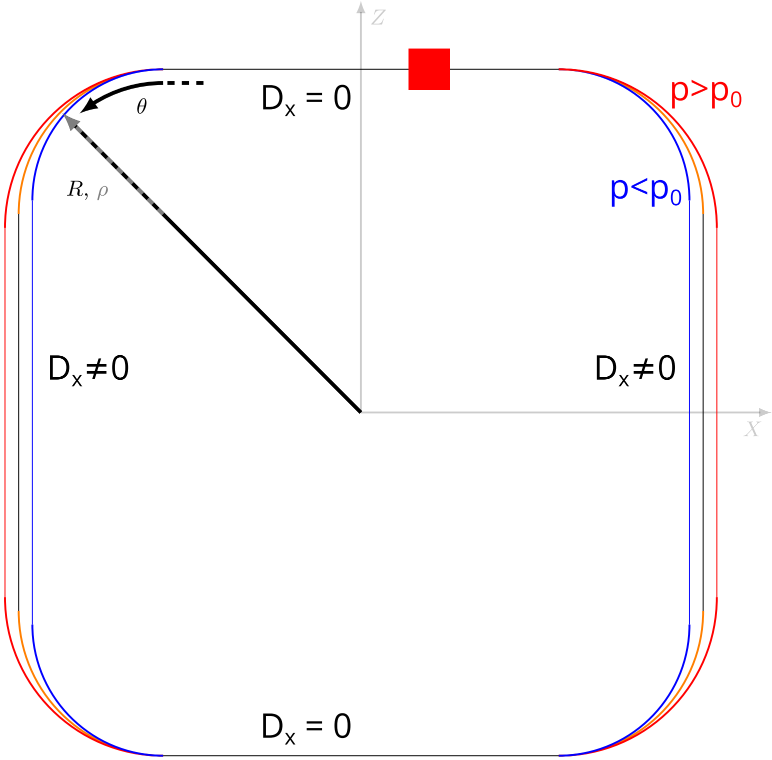
- In the last module we considered only the idealized synchronous particle with subscript s.

- We will consider from now on the equations for a non-synchronous particle with

$$E = E_s + \Delta E$$
$$\omega_0 = \omega_{0,s} + \Delta\omega_0$$
$$\theta = \theta_s + \Delta\theta$$
$$\rho(z) = \rho_s(z) + x_D(z)\ \text{(Dispersion)}$$
$$\ldots$$

EQUATIONS OF MOTION

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



f *and g arbitrary mathematical functions*

- We will describe the evolution of the energy gain of an arbitrary particle arriving at a phase ϕ in the cavity, compared to the synchronous particle.

$$\frac{d\left(\Delta E\right)}{dt} = f\left(\phi\right)$$

- We will then derive the evolution of the phase of an arbitrary particle with different energy ΔE with respect to the synchronous particle.

$$\frac{d\left(\phi\right)}{dt} = g\left(\Delta E\right)$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

In the previous lesson we derived the acceleration rate of the synchronous particle. The acceleration rate is **first approximated to consider only the RF contribution** (no induction force, synchrotron radiation, wakefields...). For the synchronous particle

$$\dot{E}_s \approx \frac{\delta E_s}{T_{0,s}} \quad \rightarrow \quad \dot{E}_s = \frac{qV_{\mathrm{rf}}}{T_{0,s}} \sin\left(\phi_s\right) = \omega_{0,s} \frac{qV_{\mathrm{rf}}}{2\pi} \sin\left(\phi_s\right)$$

The acceleration rate for an arbitrary particle is

$$\dot{E} = \omega_0 \frac{qV_{\mathrm{rf}}}{2\pi} \sin\left(\phi\right)$$

The difference in acceleration rate is

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_{\mathrm{rf}}}{2\pi} \left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

ALL FORCES EXCEPT RF NEGLECTED

From

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_\mathrm{rf}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

Re-organizing the term on the left hand side provides us with the following equation of motion

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_\mathrm{rf}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

The second term on the left hand side can be obtained after thourough derivations.

Including the induction forces, the equation of motion can actually be made simpler!

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Demonstrate we can write

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\Delta E}{\omega_{0,s}} \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}$$

NB: Remember that ω_0 *and* $\omega_{0,s}$ *are functions of time.*

Hint: It may easier to handle T_0 *than* ω_0 *for the initial derivations.*

Hint 2: Keep linear orders in Δ*.*

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Starting from

$$\begin{aligned}\dot{E}\,T_0 - \dot{E}_s\,T_{0,s} &= \dot{E}_s\,T_{0,s} + \Delta\dot{E}\,T_{0,s} + \dot{E}_s\,\Delta T + \Delta\dot{E}\,\Delta T - \dot{E}_s\,T_{0,s} \\ &= \Delta\dot{E}\,T_{0,s} + \dot{E}_s\,\Delta T + \text{2nd order}\end{aligned}$$

replacing T_0 by ω_0 and removing second order terms we get

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} \approx \frac{\Delta\dot{E}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\,\Delta T$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Including $\omega_{0,s}$ inside the derivative with time

$$\begin{aligned}\frac{\Delta \dot{E}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\,\Delta T &= \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) - \Delta E\frac{d}{dt}\left(\frac{1}{\omega_{0,s}}\right) + \frac{1}{2\pi}\dot{E}_s\Delta T \\ &= \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \frac{\Delta E}{\omega_{0,s}}\frac{\dot{\omega}_{0,s}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\,\Delta T \\ &= (1) + (2) + (3)\end{aligned}$$

Expressing (2), using $dE = v\,dp = \omega R\,dp$ and differentiating $\omega = \beta c/R$

$$\frac{\Delta E}{\omega_{0,s}}\frac{\dot{\omega}_{0,s}}{\omega_{0,s}} = R_s \Delta p\left(\frac{\dot{\beta}_s}{\beta_s} - \frac{\dot{R}_s}{R_s}\right)$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Expressing (3), using $dE = v\ dp = \omega R\ dp$ and $\eta = \alpha_p - \gamma^{-2} = (dT/T)\ /\ (dp/p)$

$$\begin{aligned} \frac{1}{2\pi}\dot{E}_s\ \Delta T &= \frac{1}{2\pi}\omega_{0,s} R_s\ \dot{p}_s \eta \frac{\Delta p}{p_s} T_s \\ &= R_s \Delta p \left(\alpha_p - \frac{1}{\gamma_s^2}\right) \frac{\dot{p}_s}{p_s} \end{aligned}$$

Summing (2) + (3) and with $d\beta/\beta = \gamma^{-2} dp/p$

$$(2) + (3) = R_s \Delta p \left(\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right)$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Using the synchrotron differential equation (1) from Module 5

$$R_s \Delta p \left(\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s} \right) = R_s \Delta p \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}$$

From the relativistic differential relationship $dE = \beta c dp, \beta = P/E$ and the definition of the angular revolution frequency $\omega = \beta c / R$

$$\begin{aligned} \ldots &= R_s \frac{\Delta E}{\beta_s c} \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}} \\ &= \left(\frac{\Delta E}{\omega_{0,s}} \right) \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}} \end{aligned}$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Finally

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

The first equation of motion with only the RF contribution is

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

CONTRIBUTION OF INDUCTION FORCES

WHEN THE BETATRON MEETS THE SYNCHROTRON

- Induction forces were presumed to be negligible in the previous lesson.
- The acceleration of the synchronous particle thanks to induction is indeed very small compared to the acceleration obtained from the RF cavity.
- The difference in acceleration for an **arbitrary particle with respect to synchronous particle** to express ΔE is relevant!

CONTRIBUTION OF INDUCTION FORCES

We add the induction contribution on the previous equation of motion to the right hand side

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right] \qquad (1)$$

$$- \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) \qquad (2)$$

$$+ \frac{q}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta \qquad (3)$$

It can be demonstrated that (3) is equal to (2) to the first order!

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

Taking into account the induction force, the acceleration rate for the synchronous particle is

$$\dot{E}_s \approx \frac{\delta E_{\mathrm{rf},s} + \delta E_{\mathrm{b},s}}{T_{0,s}}$$

$$\dot{E}_s = \frac{\omega_{0,s}}{2\pi}\left[qV_{\mathrm{rf}}\sin\left(\phi_s\right) + q\int_0^{2\pi}\int_0^{\rho_s}\frac{\partial \mathcal{B}_y}{\partial t}\rho'\,d\rho'\,d\theta\right]$$

The acceleration rate for an arbitary particle is

$$\dot{E} = \frac{\omega_0}{2\pi}\left[qV_{\mathrm{rf}}\sin\left(\phi\right) + q\int_0^{2\pi}\int_0^{\rho}\frac{\partial \mathcal{B}_y}{\partial t}\rho'\,d\rho'\,d\theta\right]$$

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

The difference in the magnetic flux in the surface between the paths of an arbitrary particle and the synchronous one is

$$
\begin{aligned}
\int_0^{2\pi}\int_{\rho_s}^{\rho} \frac{\partial \mathcal{B}_y}{\partial t}\rho'\, d\rho'\, d\theta &\approx \int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\int_{\rho_s}^{\rho} \rho'\, d\rho'\, d\theta \\
&= \int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\frac{\rho^2-\rho_s^2}{2}\, d\theta \\
&= \int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\frac{\left(\rho_s+x_D\right)^2-\rho_s^2}{2}\, d\theta \\
(\text{2nd order in } \rho \text{ neglected}) &\approx \int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s\, x_D\, d\theta
\end{aligned}
$$

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

The difference in acceleration rate between an arbitrary and the synchronous particle becomes to the first order

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right] + \frac{q}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta$$

We now demonstrate that the induction term on the right hand side is equal to

$$\frac{q}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta = \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

and hence compensate with the term (2) here.

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

Using the definition of the momentum compaction factor (from dispersion)

$$
\begin{aligned}
\frac{q}{2\pi}\int_0^{2\pi}\frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta &= \frac{q}{2\pi}\dot{\mathcal{B}}_{y,s}\rho_s\int_0^{2\pi} D_x\frac{\Delta p}{p}d\theta \\
&= \frac{q}{2\pi}\dot{\mathcal{B}}_{y,s}\rho_s\ 2\pi R_s\ \alpha_p\frac{\Delta p}{p_s} \\
&= \alpha_p\ \frac{q\rho_s}{p_s}\dot{\mathcal{B}}_{y,s}R_s\ \Delta p \\
&= \alpha_p\ \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}R_s\ \frac{\Delta E}{\beta_s c} \\
&= \alpha_p\ \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\ \left(\frac{\Delta E}{\omega_{0,s}}\right) \qquad = \text{-}\ (2)
\end{aligned}
$$

FIRST LONGITUDINAL EQUATION OF MOTION

EVOLUTION OF THE RELATIVE ENERGY OF AN ARBITRARY PARTICLE

We obtain finally

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\text{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

This is the first fundamental longitudinal equation of motion

Note that $\omega_{0,s}$ is inside of the time derivative!

PHASE SLIPPAGE

In the previous lesson, we considered the synchronous particle which is by definition always synchronous to the RF and arrives at the phase ϕ_s.

The evolution of the azimuth of an arbitrary particle with time is

$$\theta\left(t\right) = \int \omega_0 dt$$

The phase of an arbitrary particle with respect to the RF at all time is

$$\phi\left(t\right) = \int \omega_r dt - h\theta\left(t\right) = -h \int \Delta\omega_0 dt$$

Remember that $\omega_r = h\omega_{s,0}$. Notice the $-$ sign, a particle in front in azimuth (higher θ) will arrive earlier in the cavity (lower ϕ)

FIRST LONGITUDINAL EQUATION OF MOTION

EVOLUTION OF THE PHASE WITH RESPECT TO THE RF OF AN ARBITRARY PARTICLE

By differentiating with time and including the phase slip factor η

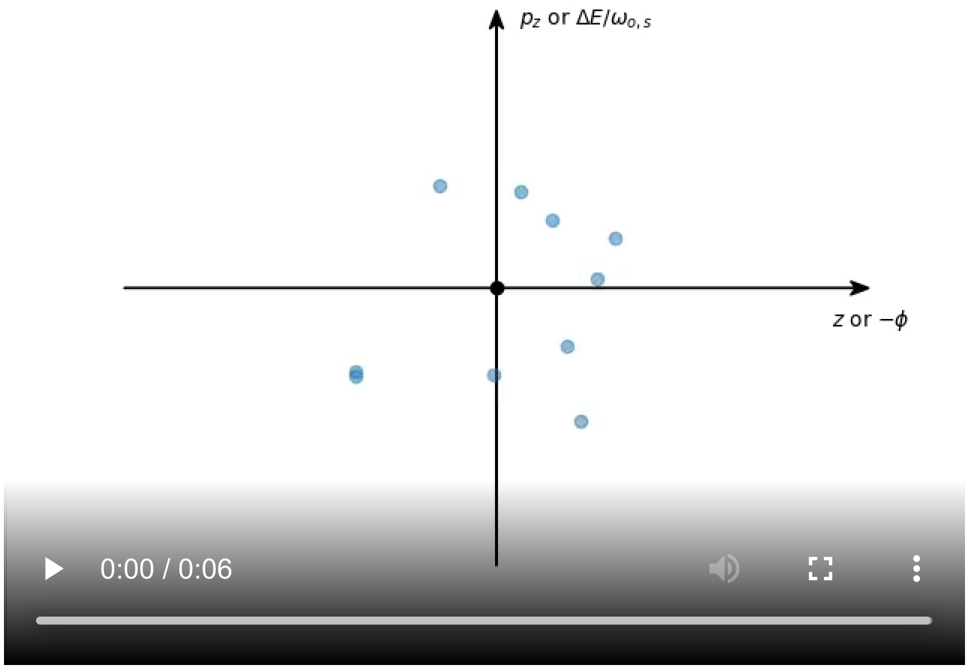
$$\begin{aligned}\frac{d\phi}{dt} &= -h\Delta\omega_0 \\ &= h\eta\omega_{0,s}\frac{\Delta p}{p_s}\end{aligned}$$

Using the differential relationship $dE/E = \beta^2 dp/p$

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

This is the second fundamental longitudinal equation of motion

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

- From now on we will describe the motion in the longitudinal phase space $\left(\phi, \frac{\Delta E}{\omega_{0,s}}\right)$, instead of $\left(z, p_z\right)$

MODULE 7: INTRODUCTION TO PARTICLE TRACKING

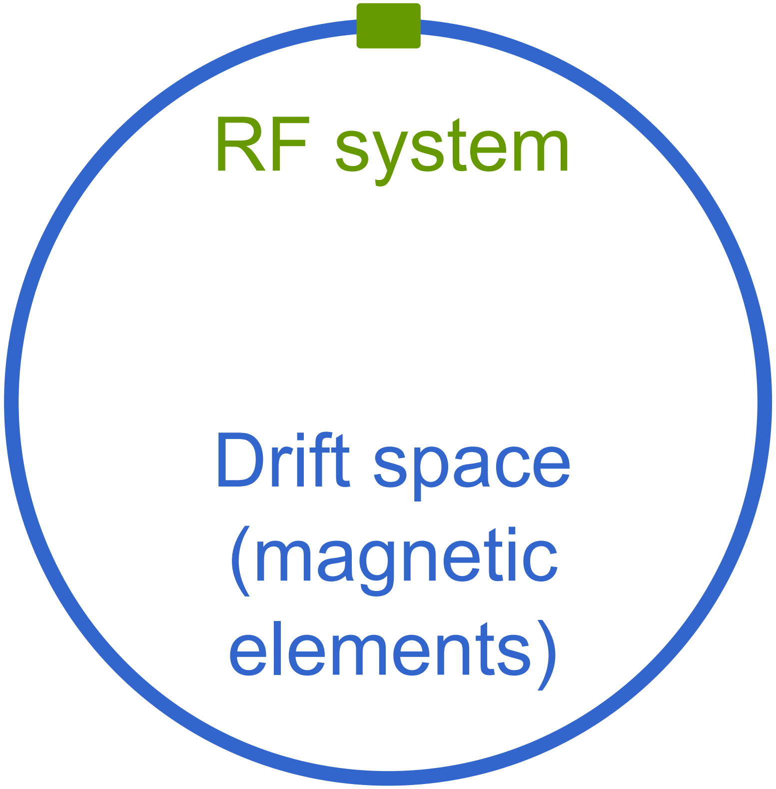
→ **Defining the accelerator**

→ **Implementing and iterating the equations of motion**

→ **Examples**

SYNCHROTRON DEFINITION

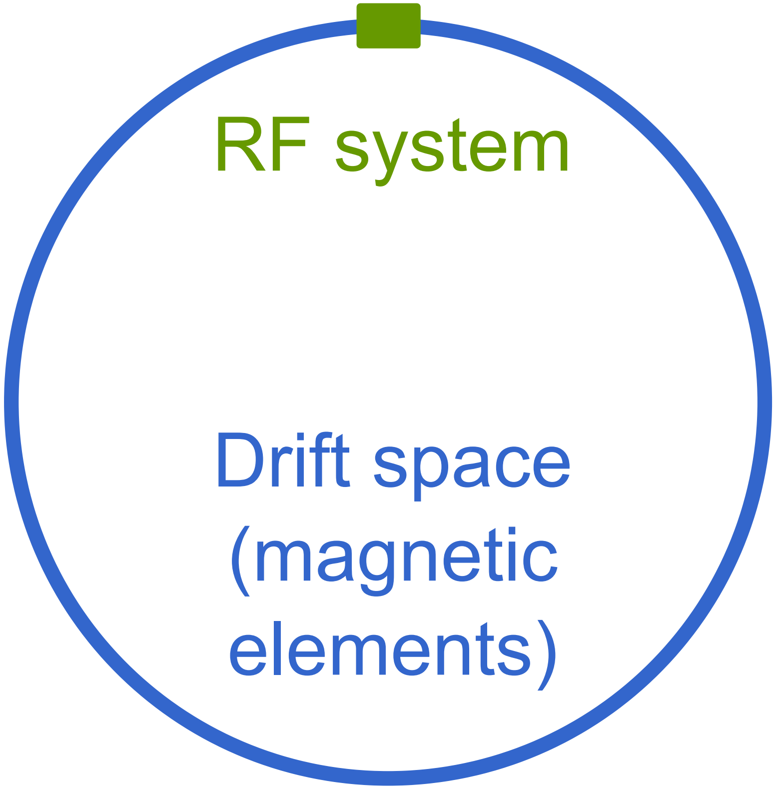
Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- In this part we implement numerically the equations of motion.
- The synchrotron motion can be complex to grasp, a tracking code can help to visualize easily the longitudinal motion in phase space.
- A rather simple tracking code can allow to do very accurate simulations, in a rather small number of code lines!

SYNCHROTRON DEFINITION

Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- Definition of "design" energy (momentum)

$$p_d = q\mathcal{B}_y \rho_d$$

- Definition of "design" revolution period

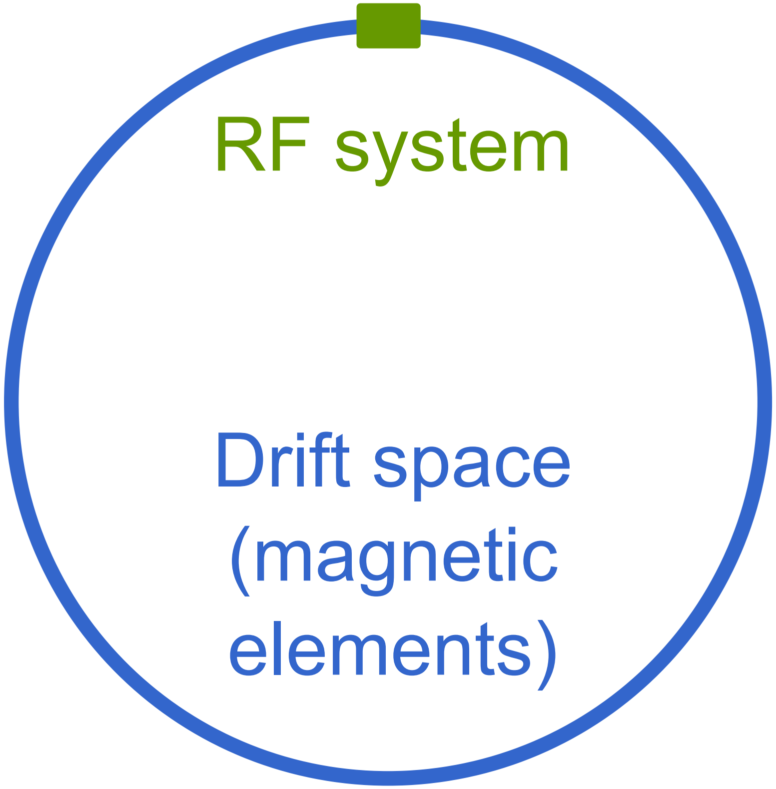
$$\mathcal{B}_y, p_d \rightarrow T_{0,d} = \frac{2\pi}{\omega_{0,d}} = \frac{C_d}{\beta_d c}$$

- Definition of RF parameters

$$\omega_r = h\omega_{0,d}$$

SYNCHROTRON DEFINITION

Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- In the context of the tracking code, we will use the $(\phi, \Delta E)$ coordinate system
- Relative energy of an arbitrary particle

$$\Delta E = E - E_d$$

- Phase of the particle relative to the RF

$$\phi$$

- NB: $(\phi, \Delta E)$ are not canonical variables, $(\tau, \Delta E)$ or $(\phi, \Delta E/\omega_{0,s})$ are canonical.

ENERGY GAIN IN RF CAVITY

- The longitudinal equation of motion (continuous in t) is commonly called the "kick" equation (NB: we neglect $\dot{\omega}_{0,d}$ for simplicity)

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,d}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_d\right)\right] \approx \frac{\Delta E^{n+1} - \Delta E^{n}}{\omega_{0,d} T_{0,d}}$$

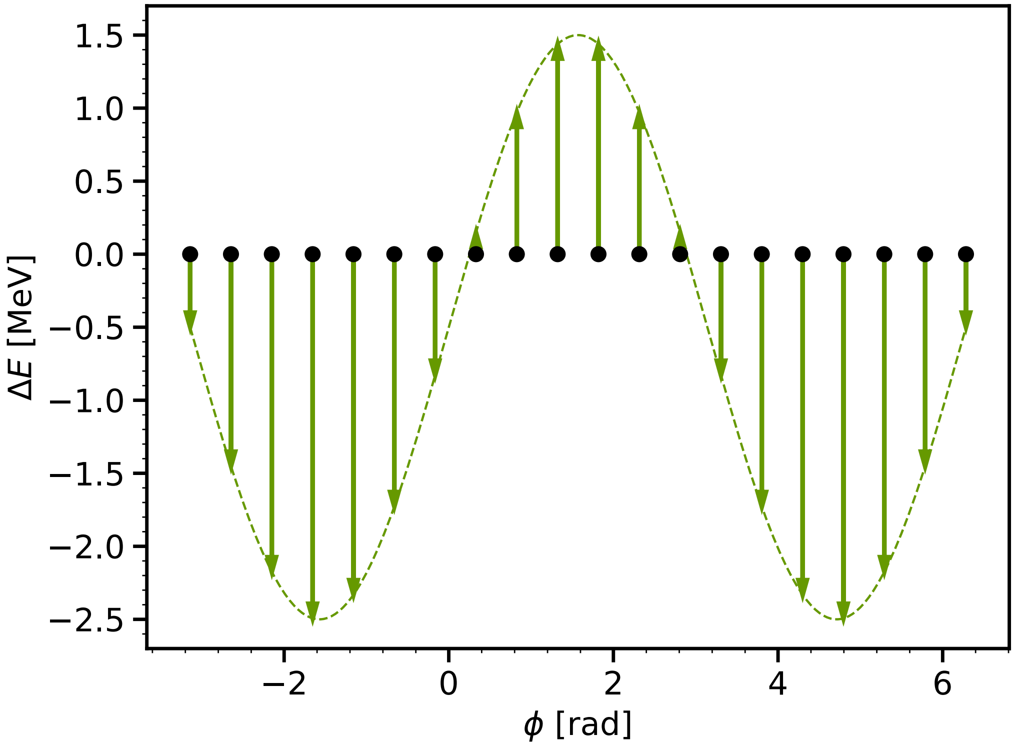
- The equation of motion is actually discrete by nature, the relative energy gain every turn $T_{0,d}$ is

$$\Delta E^{n+1} = \Delta E^{n} + qV\sin\left(\phi\right) - \delta E_d^{n\rightarrow n+1}$$

- The acceleration per turn is

$$\delta E_d^{n\rightarrow n+1} = 2\pi q \rho_d R_d \frac{\mathcal{B}_y^{n+1} - \mathcal{B}_y^{n}}{T_{0,d}}$$

ENERGY GAIN IN RF CAVITY



- Example for protons passing in RF system with $V = 2\mathrm{MV}$, $f_{\mathrm{rf}} = 200\mathrm{MHz}$, $\delta E_d = 0.5\mathrm{MeV}$

DRIFT

- The phase slip equation is commonly called the "drift" equation, neglectic any source of change in ΔE along the magnetic elements

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{\phi^{n+1}-\phi^n}{T_{0,d}}$$

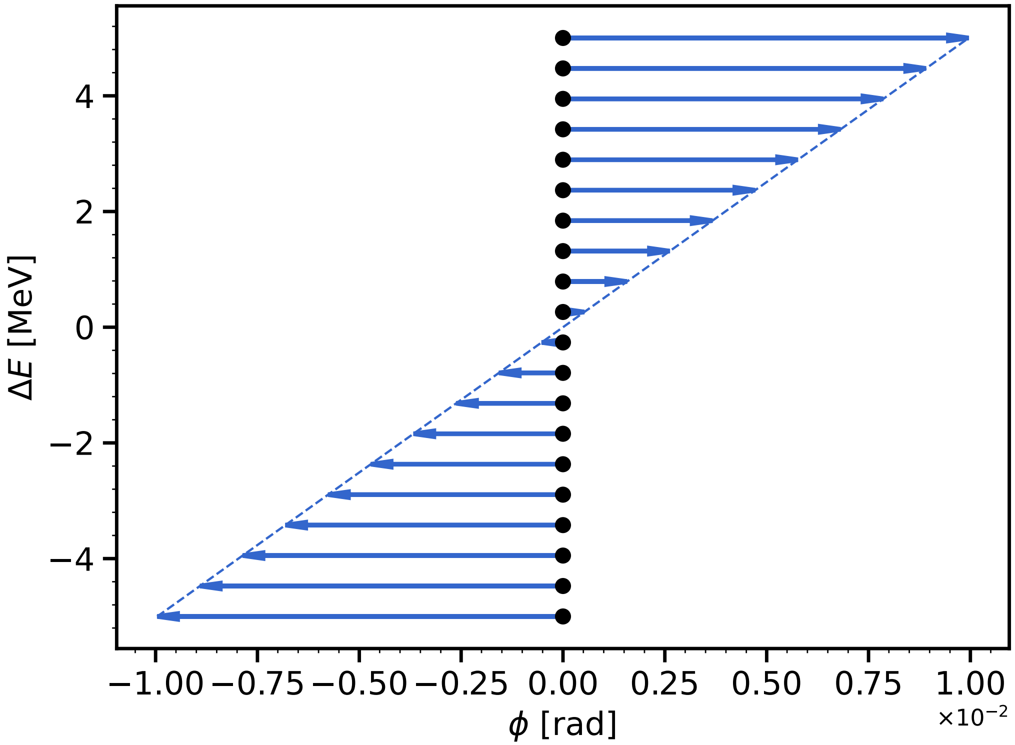
- Drift in time of an arbitrary particle with respect to the design particle after $T_{0,d}$

$$\phi^{n+1} = \phi^n + \left(\frac{2\pi h\eta_0}{\beta^2 E}\right)_d \Delta E$$

with

$$\eta_{0,d} = \frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma_d^2} = \alpha_0 - \frac{1}{\gamma_d^2}$$

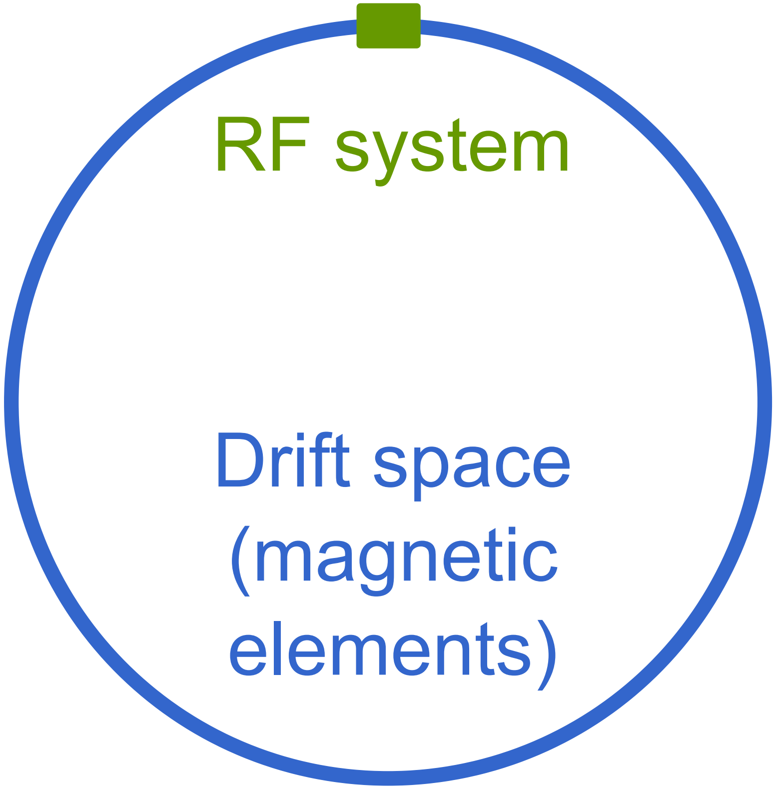
DRIFT



- Example for protons passing along a ring with $C_d = 6911.50\mathrm{m}$, $E_d = 26\mathrm{GeV}$, $\gamma_{\mathrm{tr}} = 18$

TRACKING

The two equations of motion are sufficient to build a simple **(yet very useful)** tracking code



- The tracking can be coded as

```
for n_turns:
    for n_particles:
        dE += rf_kick(phi)
        phi += drift(dE)
```

- Where

```
def rf_kick(phi):
    return q*Vrf*sin(phi) - q*Vrf*sin(phi_s)
```

```
 def drift(dE):
    return 2*pi*h*eta_0 / (beta**2 * E) * dE
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON

The present example is using Python, but any language could be used following the pseudo-code layout from the previous slide.

- We import useful libraries

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.constants import m_p, c, e
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON

- Define our machine parameters

```
Ekin = 26e9  # eV

charge = 1
E0 = m_p * c**2. / e
circumference = 6911.5  # m
energy = Ekin + E0
momentum = np.sqrt(energy**2. - E0**2.)
beta = momentum / energy
gamma = energy / E0

t_rev = circumference / (beta * c)
f_rev = 1 / t_rev

harmonic = 4620
voltage = 4.5e6  # V
f_rf = harmonic * f_rev
t_rf = 1 / f_rf

gamma_t = 18
alpha_c = 1 / gamma_t**2.
eta = alpha_c - 1 / gamma**2.
```

- Print the parameters of the machine

```
print("Beta: " +
      str(beta))
print("Gamma: " +
      str(gamma))
print("Revolution period: " +
      str(t_rev * 1e6) + " mus")
print("RF frequency: " +
      str(f_rf / 1e6) + " MHz")
print("RF period: " +
      str(t_rf * 1e9) + " ns")
print("Momentum compaction factor: " +
      str(alpha_c))
print("Phase slippage factor: " +
      str(eta))
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- Define our machine parameters

```
Ekin = 26e9  # eV

charge = 1
E0 = m_p * c**2. / e
circumference = 6911.5  # m
energy = Ekin + E0
momentum = np.sqrt(energy**2. - E0**2.)
beta = momentum / energy
gamma = energy / E0

t_rev = circumference / (beta * c)
f_rev = 1 / t_rev

harmonic = 4620
voltage = 4.5e6  # V
f_rf = harmonic * f_rev
t_rf = 1 / f_rf

gamma_t = 18
alpha_c = 1 / gamma_t**2.
eta = alpha_c - 1 / gamma**2.
```

- Define your tracking functions

```
def drift(dE, harmonic, eta, beta, energy):

    return 2 * np.pi * harmonic * \
        eta * dE / (beta**2 * energy)

def rf_kick(phi, charge, voltage, phi_s=0):

    return charge * voltage * (
        np.sin(phi) - np.sin(phi_s))
```

- Define your initial particle positions (test example)

```
n_particles = 10  # or millions ? :)

phase_coordinates = np.linspace(
    0, 2 * np.pi, n_particles)

dE_coordinates = np.zeros(n_particles)
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- Track!!!

```
n_turns = 10  # or millions? :)

for idx_turn in range(n_turns):
    dE_coordinates += rf_kick(
        phase_coordinates, charge, voltage)

    phase_coordinates += drift(
        dE_coordinates, harmonic, eta, beta, energy)
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- The particle coordinates can then be monitored at each turn during the tracking.

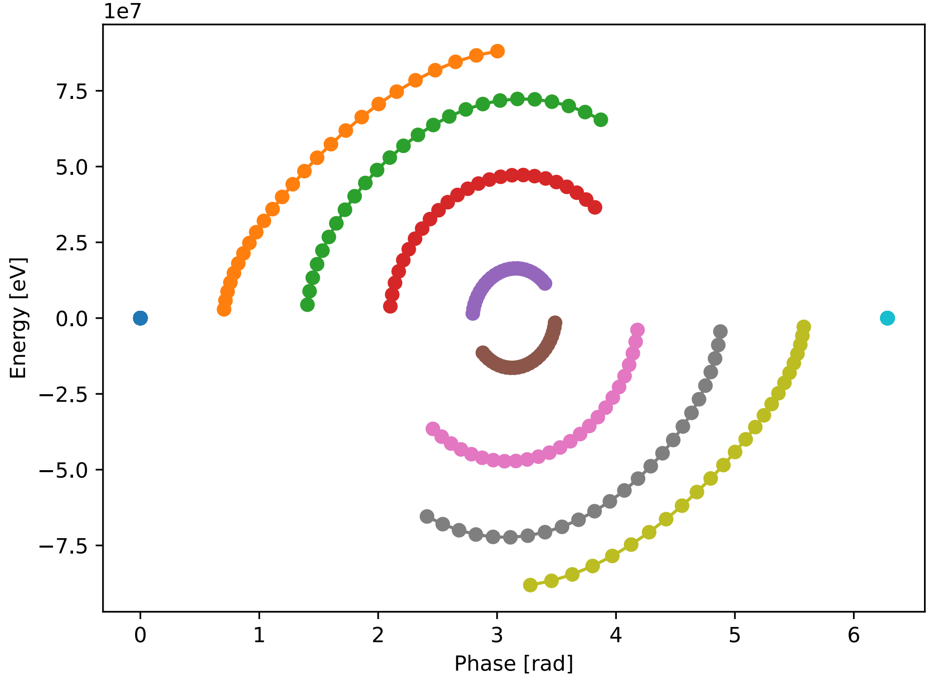
```
n_turns = 25  # or millions? :)

saved_positions_phi = np.zeros((n_particles, n_turns))
saved_positions_dE = np.zeros((n_particles, n_turns))

for idx_turn in range(n_turns):
    dE_coordinates += rf_kick(
        phase_coordinates, charge, voltage)

    phase_coordinates += drift(
        dE_coordinates, harmonic, eta, beta, energy)

    saved_positions_dE[:, idx_turn] = dE_coordinates
    saved_positions_phi[:, idx_turn] = phase_coordinates
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- The trajectory of the particles can be visualized

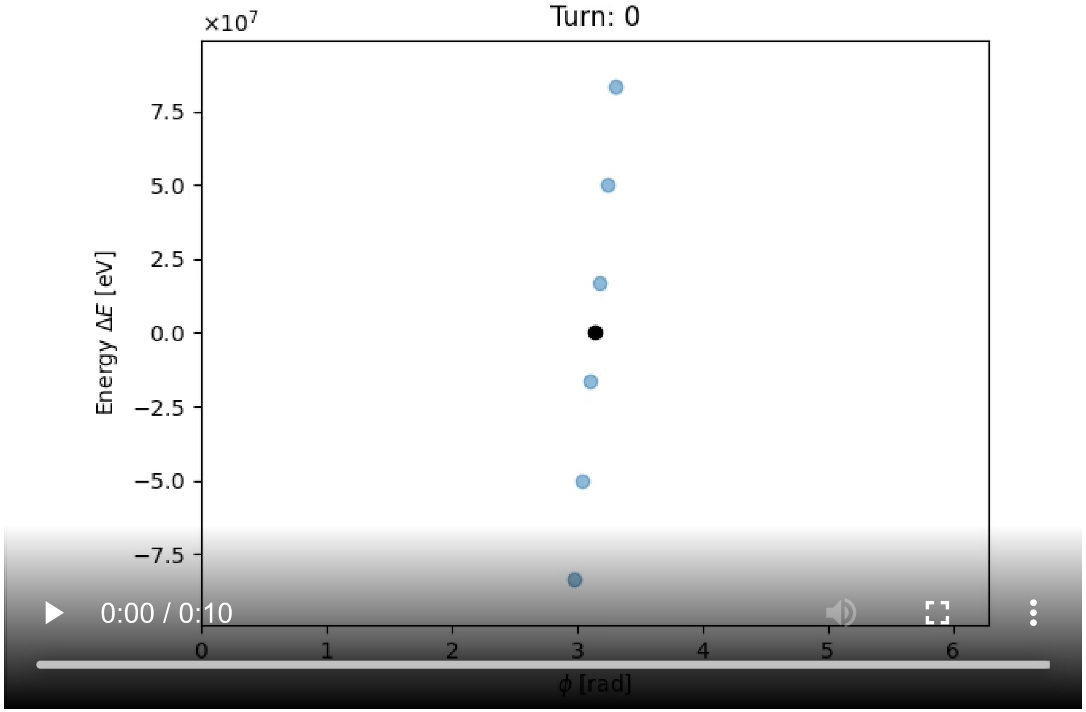
```
plt.figure('phase space')
plt.clf()
for idx_particle in range(n_particles):
    plt.plot(
        saved_positions_phi[idx_particle, :],
        saved_positions_dE[idx_particle, :],
        '-o')
plt.xlabel('Phase [rad]')
plt.ylabel('Energy [eV]')
```



- The final script is less than 100 lines long!

EXAMPLES

DRIFT ONLY

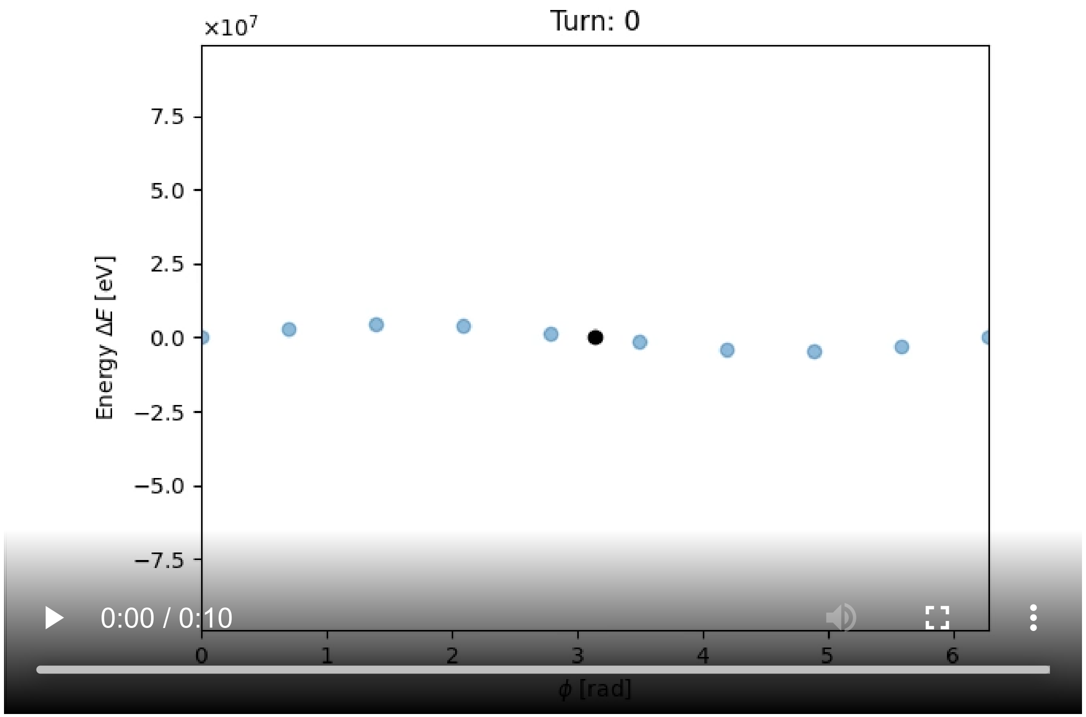


Only with the drift equation of motion, no RF.

Particles get distant from each other.

EXAMPLES

RF KICK ONLY

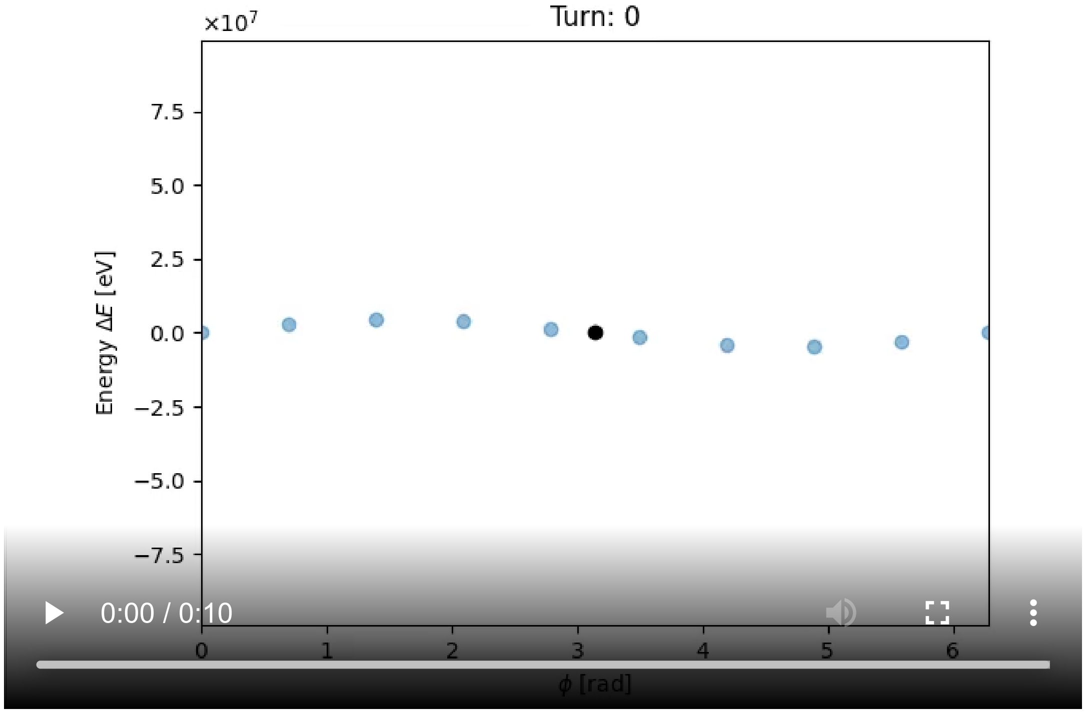


Only RF, no drift.

Particles all get accelerated/decelerated with respect to the synchronous particle.

EXAMPLES

RF KICK AND DRIFT

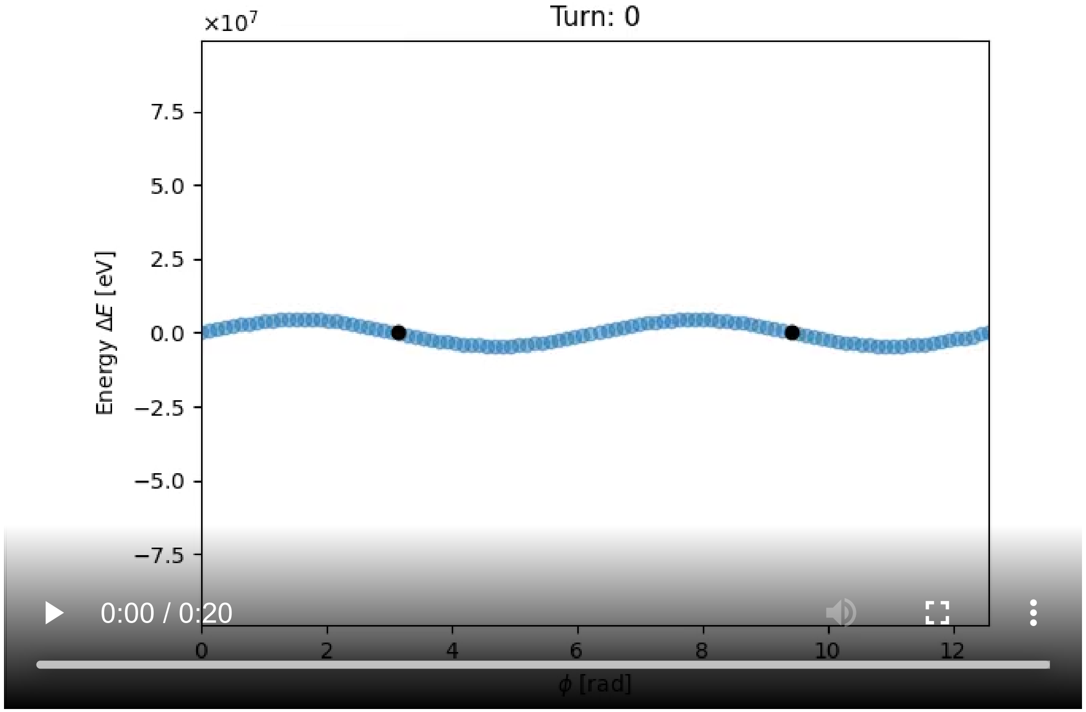


Combining RF kick and drift, particles start to oscillate around the synchronous particle.

Linear close to the synchronous particle, non-linear at large amplitude.

EXAMPLES

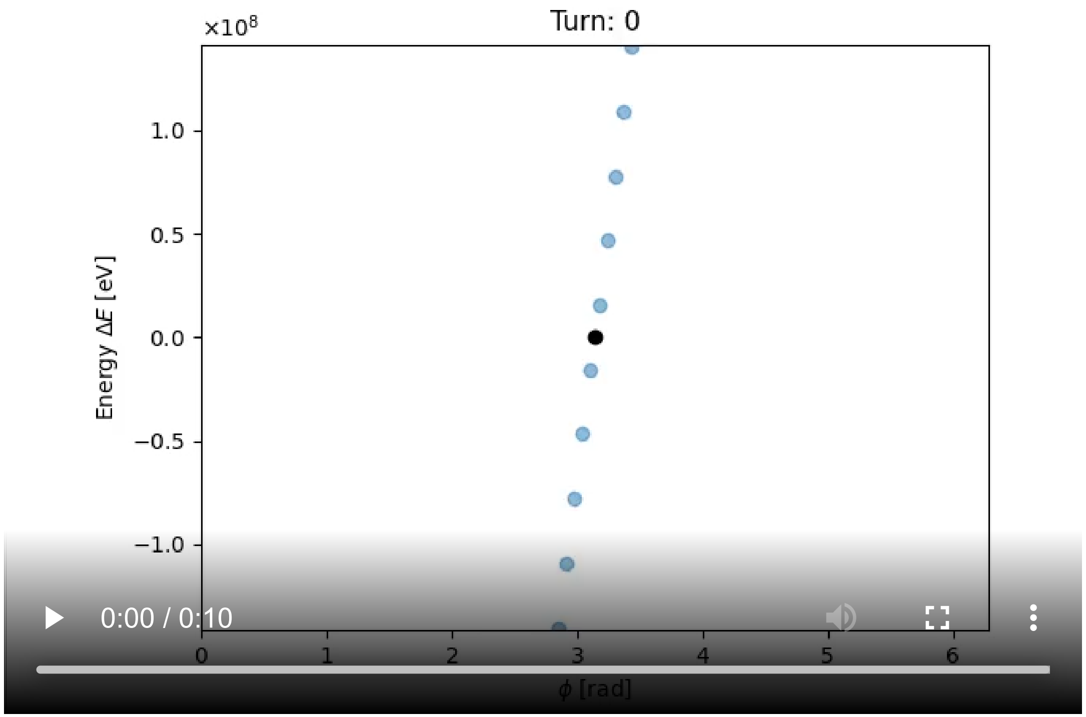
AROUND OTHER SYNCHRONOUS PARTICLES



Particles oscillate around one of the h possible synchronous particles.

EXAMPLES

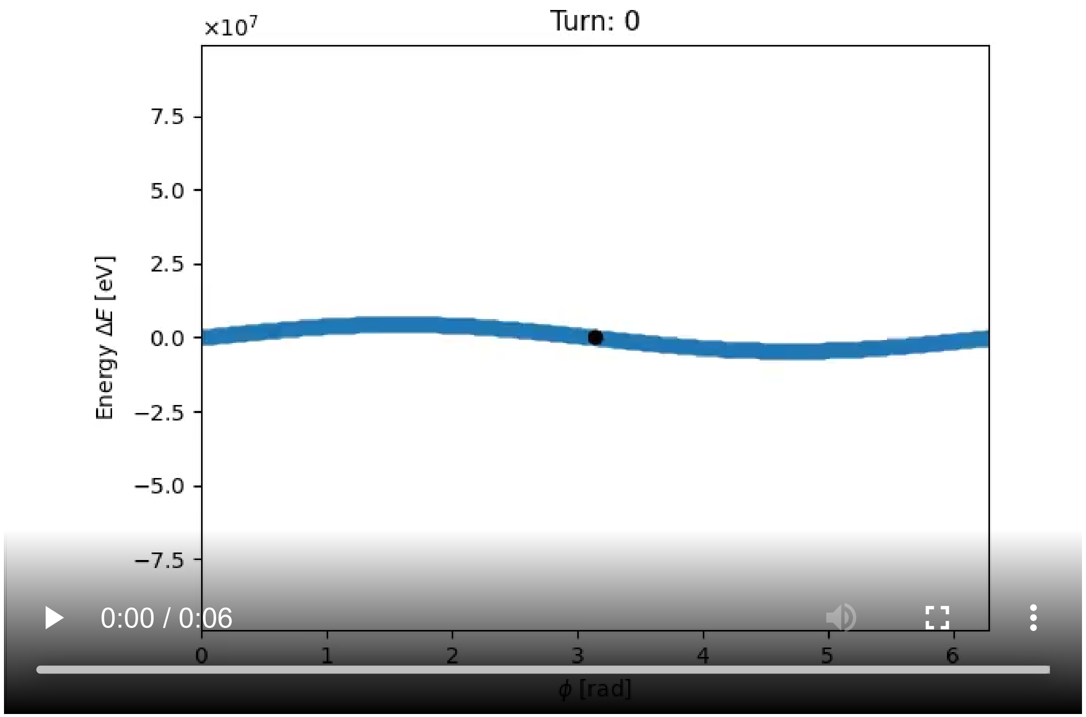
VERY LARGE AMPLITUDES



If particles have very different energies than the synchronous particle, they don't oscillate around a stable point anymore.

EXAMPLES

LIMIT OF PHASE STABILITY

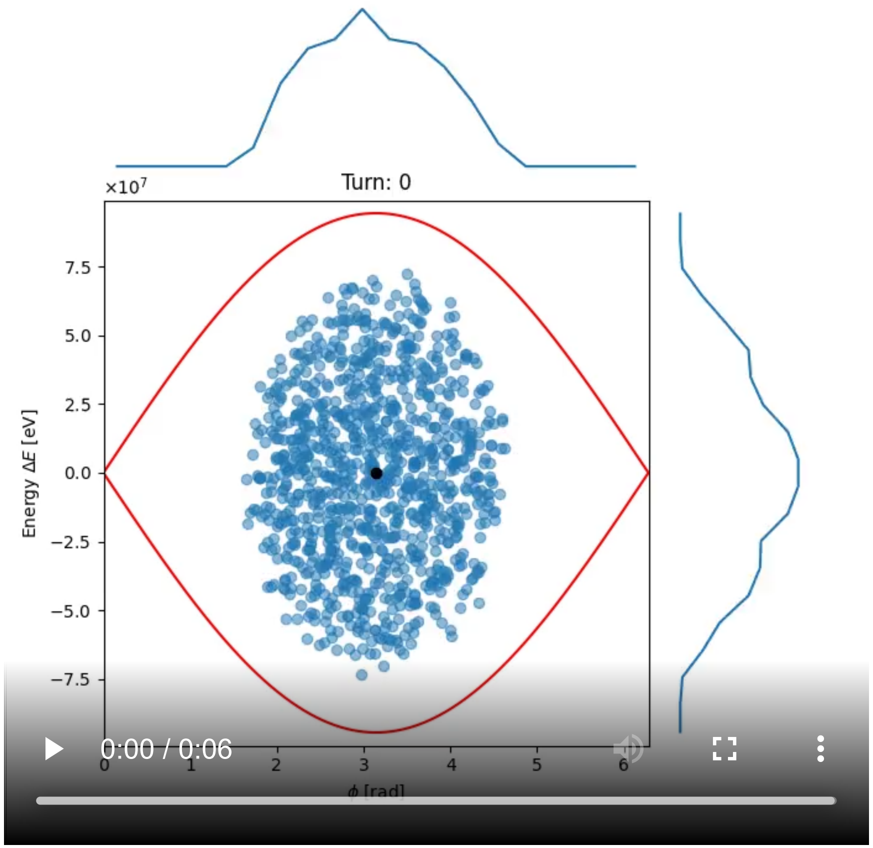


A contour of the limit of phase stability is easily obtained with tracking.

It is called the RF bucket.

EXAMPLES

REALISTIC BUNCH IN THE RF BUCKET



- The only measurable in reality is the line density (top line), corresponding to the histogram in ϕ.
- Tracking is an essential tool to compare computations with measurements!

TAKE AWAY MESSAGE

LONGITUDINAL EQUATIONS OF MOTION

- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\text{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$