LONGITUDINAL BEAM DYNAMICS

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COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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EXERCISES

EXERCISES ON THE EV

- **An accelerator has a potential of 20 MV, what is the corresponding energy gain of a particle in Joules?**
- **What is the total energy of the beam stored in the LHC?** (The beam is composed of 2808 bunches of 1.15×10^{11} protons each at 7 TeV)
- **What is the equivalent speed of a high speed train?** (Assume a 400 tons (200 m long) TGV train)
- **What is the beam power delivered to the LHC beam?** (Consider an acceleration from 450 GeV to 7 TeV in 30 minutes)

EXERCISE ON THE EV

CORRECTION

- **An accelerator has a potential of 20 MV, what is the corresponding energy gain of the beam in Joules?**
  - $20 \cdot 10^{6} \cdot 1.609 \cdot 10^{-19} = 3.2 \cdot 10^{-12}\ \mathrm{J}$
- **What is the total energy of the beam stored in the LHC**
  - $2808 \cdot 1.15 \cdot 10^{11} \cdot 7 \cdot 10^{12} \cdot 1.609 \cdot 10^{-19} = 364\ \mathrm{MJ}$
- **What is the equivalent speed of a high speed train ($E_{\mathrm{LHC}} = E_{\mathrm{kin,train}}$)**
  - $v_{\mathrm{train}} = \sqrt{2E_{\mathrm{LHC}}/m_{\mathrm{train}}} = \sqrt{2 \cdot 364 \cdot 10^{6} / \left(400 \cdot 10^{3}\right)} = 154\ \mathrm{km/h}$
- **What is the power delivered to the LHC beam (1800 s)**
  - $2808 \cdot 1.15 \cdot 10^{11} \cdot (7 - 0.450) \cdot 10^{12} \cdot 1.609 \cdot 10^{-19} / 1800 = 189\ \mathrm{kW}$

EXERCISES

- **Show that**

$$p\,[\mathrm{GeV/c}] \approx 0.3\ Z\ \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

- **Compute the relativistic parameters for the following CERN machines**

Machine	E_0 [MeV]	E_{kin} [GeV]	E [GeV]	γ	β	p [GeV/c]	$\mathcal{B}_y\rho$ [Tm]
PSB inj (p+)		0.160					
PSB ext (p+)		2					
SPS ($^{208}\mathrm{Pb}^{82+}$)							86.4
LHC (p+)			7000				
LEP (e+/e-)			100				

$$m_p = 1.6726\times10^{-27}\ \mathrm{kg},\ m_e = 9.1094\times10^{-31}\ \mathrm{kg},\ u = 1.661\times10^{-27}\ \mathrm{kg}$$

EXERCISES

MACHINE PARAMETERS

Machine	E_0 [MeV]	E_{kin} [GeV]	E [GeV]	γ	β	p [GeV/c]	$\mathcal{B}_y \rho$ [Tm]
PSB inj (p+)	938	0.160	1.098	1.17	0.52	0.57	1.90
PSB ext (p+)	938	2	2.938	3.13	0.95	2.78	9.30
SPS ($^{208}\mathrm{Pb}^{82+}$)	193751	1940.50	2134.25	11.0	0.996	2125.44	86.4
LHC (p+)	938	6999	7000	7460	0.999..	6999.99..	23333
LEP (e+/e-)	0.511	99.99	100	195695	0.999..	99.99..	333.33

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity.
- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity (Slide 11)
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\, 80 \cdot 10^6\right) = 1.4\ \mathrm{m}$
- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\, 1.4 \cdot 10^9\right) = 8.2\ \mathrm{cm}$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

- Revolution period/frequency of the SPS ($\rho_0 = 741.25$ m, $C_0 = 6911.50$ m)
- RF frequency of the SPS ($h = 4620$)
- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
- Smallest RF voltage to accelerate the synchronous particle
- Compute the same parameters with Lead ions $^{208}\mathrm{Pb}^{82+}$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

Machine	SPS inj. p+	SPS ext. p+	SPS inj. Pb	SPS ext. Pb
p [GeV/c]	26	450	2132	36900
E [GeV]	26.0169	450.001	2140.786	36900.509
β	0.99935	0.999...	0.9959	0.999..
T_0 [μs]	23.0693	23.0543	23.1493	23.0546
f_r [MHz]	200.266	200.396	199.574	200.394

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
  - $6911.50 \cdot 741.35 \cdot 1 \cdot 0.7 = 3.59\ \mathrm{MeV}$ for p+
  - $6911.50 \cdot 741.35 \cdot 82 \cdot 0.7 = 294\ \mathrm{MeV}$ for Pb
- Smallest RF voltage to accelerate the synchronous particle
  - $3.59\ \mathrm{MeV}$ for p+ and Pb ($\delta E_{\mathrm{rf}}/q$)

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]		
γ		
T_0 [μs]		
α_p [10^{-3}]		
E_t [GeV]		
η [10^{-3}]		

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]	14.03	450
γ	14.95	479.6
T_0 [μs]	23.11	23.05
α_p [10^{-3}]	3.086	3.086
E_t [GeV]	16.89	16.89
η [10^{-3}]	-1.385	3.082

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
  - Transition is crossed since η changes sign
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
  - $\Delta R = 3.086 \cdot 10^{-3} \cdot 6911.50/(2 \cdot 3.14) \cdot \left(-10^{-4}\right) =$-0.34 mm
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?
  - Low E: $\Delta T_0 = -1.385 \cdot 10^{-3} \cdot 23.11 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$ 3.2 ps (late)
  - High E: $\Delta T_0 = 3.082 \cdot 10^{-3} \cdot 23.05 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$-7.1 ps (early)

EXERCISES

- Compute the linear synchrotron frequency and tune in the SPS at $p = 14$ GeV/c and $p = 450$ GeV/c, with an RF harmonic $h = 4620$ and voltage $V_{\mathrm{rf}} = 4.5$ MV (find the other SPS parameters obtained in the exercises from Module 5). The beam is not accelerated.
- Compute the approximate emittance and momentum spread at $p = 14$ GeV/c for a bunch length $\tau_l = 3$ ns.
- What would be the bunch length at $p = 450$ GeV/c if the emittance is preserved?
- What would be the bunch length and energy spread at transition energy?
- Evaluate required increase in rf voltage to shorten the bunch length by a factor 2.

EXERCISES

- Linear synchrotron frequency and tune
  - Low energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14}\sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14/23.11 \cdot 10^6)^2 \cdot 1.385 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}}$$

$$\approx 784 \text{ Hz}$$

$$Q_{s0} = 784 \cdot 23.11 \cdot 10^{-6} \approx 1.81 \cdot 10^{-2}$$

EXERCISES

- Linear synchrotron frequency and tune
  - High energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14} \sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6)^2 \cdot 3.082 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot 1 \cdot 450 \cdot 10^9}}$$
$$\approx 206\ \mathrm{Hz}$$

$$Q_{s0} = 206 \cdot 23.05 \cdot 10^{-6} \approx 4.76 \cdot 10^{-3}$$

EXERCISES

- Linear emittance

$$\varepsilon_{l,0} = \frac{3.14 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{4 \cdot 1.385 \cdot 10^{-3}} 2 \cdot 3.14 \cdot 784 \cdot (3 \cdot 10^{-9})^2 \approx 0.35 \text{ eVs}$$

- Energy spread

$$\delta_p = 2\frac{\Delta E_m}{\beta_s^2 E_s} = 4\frac{\varepsilon_{l,0}}{\pi \tau_l \beta_s^2 E_s} = \frac{4 \cdot 0.35}{3.14 \cdot 3 \cdot 10^{-9}(14/14.03)^2 \cdot 14.03 \cdot 10^9}$$
$$\approx 1.06 \times 10^{-2}$$

- Adiabatic damping

$$\tau_{l,\text{high}} = \tau_{l,\text{low}} \left(\frac{E_\text{high}}{E_\text{low}}\right)^{-1/4} = 3 \cdot \left(\frac{450}{14.03}\right)^{-1/4} \approx 1.26 \text{ ns}$$

EXERCISES

- Transition

The bunch length would tend to zero while the energy spread diverge to infinity! Non-adiabatic theory needed to better evaluate bunch parameters at transition crossing.

- Adiabatic bunch shortening

$$\tau_{l,\text{high}} = \tau_{l,\text{low}} \left(\frac{V_\text{high}}{V_\text{low}}\right)^{-1/4}$$

$$\implies V_\text{high} = V_\text{low} \left(\frac{\tau_{l,\text{high}}}{\tau_{l,\text{low}}}\right)^{-4} = V_\text{low} \times 16$$

The required voltage increase is a factor 16! Not very efficient shortening.

EXERCISES

- Compute the RF bucket area (or acceptance) using the SPS parameters from Module 5 and 8.
- Compute the bucket height.
- Compute the filling factor for 3 ns bunch at 14 GeV/c (use the linear approximation for the emittance calculation)
- The bunch length oscillations at injection indicate that the energy spread is too small by 10%. How much should the RF voltage be reduced to improve the matching?

EXERCISES

- Low energy 14 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.11 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 0.50\ \mathrm{eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 79.1\ \mathrm{MeV}$$

EXERCISES

- High energy 450 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 1.91 \text{ eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 301 \text{ MeV}$$

EXERCISES

- Filling factor

From the previous module exercise, the longitudinal emitatnce is 0.35 eVs. The filling factor in area is $0.35/0.50 \approx 70\,\%$.

- Matching

The bunch length and energy spread are fixed at injection. In orderto match the bunch, the bucket height should be reduced by 10%. The RF voltage can be reduced to reduce the bucket height, with a scaling $\sqrt{V_{\mathrm{rf}}}$

$$\frac{\Delta E_{\mathrm{sep},m,2}}{\Delta E_{\mathrm{sep},m,1}} = 0.9 = \sqrt{\frac{V_{\mathrm{rf},2}}{V_{\mathrm{rf},1}}} \quad \rightarrow \quad V_{\mathrm{rf},2} = 0.9^2 V_{\mathrm{rf},1} \approx 0.81 V_{\mathrm{rf},1}$$

The RF voltage should be reduced by 20% (useful tip: $(1-\epsilon)^n \approx 1 - n\epsilon \rightarrow (1-0.1)^2 \approx 1 - 2 \cdot 0.1$)