LONGITUDINAL BEAM DYNAMICS

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COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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SUMMARY OF FORMULAS

BASICS

- Lorentz force

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$$

$\vec{\mathcal{E}}$ to accelerate and deflect

$\vec{\mathcal{B}}$ to bend trajectories

- Definition of coordinates

x horizontal position

y vertical position

z longitudinal position

ρ local bending radius

R mean radius / orbit

θ azimuth

- Assumptions made so far: $p_z \gg p_{x,y}$ and $p \approx p_z$

FIELDS, FORCES, RELATIVITY

- Acceleration in an RF gap:

$$\delta E = \int q\mathcal{E}_z\left(\rho, z, t\right) dz = qV_{\mathrm{rf}}\left(\rho, \tau\right)$$

- Magnetic rigidity:

$$\mathcal{B}_y \rho = \frac{p}{q} \quad \rightarrow \quad p\left[\mathrm{GeV/c}\right] \approx 0.3\ Z\ \mathcal{B}_y\left[\mathrm{T}\right] \rho\left[\mathrm{m}\right]$$

- Relativistic relationships ($P = p\ c$):

$$E = E_{\mathrm{kin}} + E_0 = \sqrt{P^2 + E_0^2}, \quad \beta = \frac{v}{c} = \frac{P}{E}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{E_0}$$

FIELDS, FORCES, RELATIVITY

- Relativistic relationships:

$$\beta^2 + \frac{1}{\gamma^2} = 1$$

- Relativistic differential relationships:

$$\frac{dE}{dp} = \beta c = v$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

ENERGY GAIN

- RF energy gain

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf},0}T_t\sin\left(\omega_r\tau\right) \quad \rightarrow \delta E_{\mathrm{rf}}\left(\phi\right) = qV_{\mathrm{rf}}\sin\left(\phi\right)$$

- Transit time factor

$$T_t\left(\rho,\beta\right) = \frac{\int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho,z\right)\cos\left(\frac{\omega_r z}{\beta c}\right)dz}{\int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho,z\right)dz}$$

- Assumptions:
  - β is not changing in the computation of T_t
  - The $\left(\rho,\beta\right)$ dependence of T_t will be neglected

PILLBOX CAVITY (FUNDAMENTAL MODE)

- Pillbox cavity properties

$$\mathcal{E}_z\left(\rho, t\right) = \mathcal{E}_0 J_0\left(\chi_0 \frac{\rho}{\rho_c}\right)\cos\left(\omega_r t\right)$$

J_n Bessel function, $\chi_0 \approx 2.405, \omega_r = \chi_0 c/\rho_c$

- Transit time factor of pillbox cavity

$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

OTHER ENERGY GAIN/LOSS IN A RING

- Induction acceleration (small in large synchrotrons)

$$\delta E_\mathrm{b}\left(\rho\right) = q\int_0^{2\pi}\int_0^{\rho}\frac{\partial\mathcal{B}_y\left(\rho',\theta,t\right)}{\partial t}\rho'\,d\rho'\,d\theta$$

- Synchrotron radiation (relevant for lepton accelerators)

$$\delta E_\mathrm{sr}\left(E,\rho\right) = \frac{q^2}{3\epsilon_0}\frac{\beta^3 E^4}{\rho\,E_0^4}$$

- Self induced field

$$\delta E_\mathrm{ind}\left(\tau\right) = qV_\mathrm{ind}\left(\tau\right) = -qN_b\left(\lambda * \mathcal{W}\right)$$

SYNCHRONISM IN SYNCHROTRON

- The revolution period and frequency

$$T_0 = \frac{C}{v} = \frac{2\pi R}{\beta c} \quad , \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{\beta c}{R}$$

- Synchronism condition with RF frequency

$$\omega_r = h\,\omega_{0,s} = h\frac{\beta_s c}{R_s}$$

ACCELERATION

- Acceleration rate (subscript s for synchronous particle)

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \rightarrow \quad \phi_s = \arcsin\left(2\pi\rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\mathrm{rf}}}\right)$$

- RF frequency program

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

- Assumptions: Acceleration with constant R_s and ρ_s

RADIAL DISPLACEMENT

- Momentum compaction factor (subscript 0 for design orbit/momentum, transition gamma γ_t)

$$\alpha_p = \frac{dR/R}{dp/p} = \frac{\langle D_x \rangle_\rho}{R} = \frac{1}{\gamma_t^2} \approx \frac{\Delta R/R_0}{\Delta p/p_0} \approx \frac{\Delta R/R_s}{\Delta p/p_s}$$

- Phase slip factor

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p} = \frac{dT_0/T_0}{dp/p} = \alpha_p - \frac{1}{\gamma^2} \approx -\frac{\Delta\omega_{0,0}/\omega_{0,0}}{\Delta p/p_0} \approx -\frac{\Delta\omega_{0,s}/\omega_{0,s}}{\Delta p/p_s}$$

- Assumptions: Radial displacement with constant $\mathcal{B}_y$

SYNCHROTRON DIFFERENTIAL RELATIONSHIPS

(1)	$\mathcal{B}_y, p, R$	$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$
(2)	f_0, p, R	$\frac{dp}{p} = \gamma^2 \frac{df_0}{f_0} + \gamma^2 \frac{dR}{R}$
(3)	$\mathcal{B}_y, f_0, p$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma_t^2 \frac{df_0}{f_0} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$
(4)	$\mathcal{B}_y, f_0, R$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma^2 \frac{df_0}{f_0} + \left(\gamma^2 - \gamma_t^2\right) \frac{dR}{R}$

LONGITUDINAL EQUATIONS OF MOTION

- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

LINEAR SYNCHROTRON MOTION

- Linear synchrotron frequency

$$\omega_{s0} = 2\pi f_{s_0} = \sqrt{-\frac{qV_{\mathrm{rf}} h \omega_{0,s}^2 \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Linear synchrotron tune

$$Q_{s0} = \frac{\omega_{s0}}{\omega_{0,s}} = \sqrt{-\frac{qV_{\mathrm{rf}} h \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Phase stability condition

$$\eta \cos\phi_s < 0$$

LINEAR OSCILLATION AMPLITUDE AND EMITTANCE

- Oscillation amplitude ratio

$$\frac{(\Delta E/\omega_r)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{|\eta|\,\omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{|\eta|\,h^2\omega_{0,s}} Q_{s0}$$

- Approximate longitudinal emittance

$$\varepsilon_{l,0} = \pi\Delta E_m \frac{\tau_l}{2} = \frac{\pi\beta_s^2 E_s}{4\,|\eta|}\omega_{s0}\tau_l^2$$
$$= \frac{\pi\,|\eta|}{\beta_s^2 E_s}\frac{1}{\omega_{s0}}\Delta E_m^2$$

BUNCH PARAMETERS LINEAR SCALING LAWS

- Bunch length

$$\tau_l \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{-1/4}\, h^{-1/4}\, E_s^{-1/4}\, \eta^{1/4}$$

- Energy deviation

$$\Delta E_m \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{1/4}\, h^{1/4}\, E_s^{1/4}\, \eta^{-1/4}$$

HAMILTONIAN

$$\mathcal{H} = \frac{\eta \omega_r^2}{2\beta_s^2 E_s} \left(\frac{\Delta E}{\omega_r}\right)^2 + \frac{q V_{\mathrm{rf}}}{2\pi h} \left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

RF BUCKET PARAMETERS

- RF bucket height

$$\Delta E_{\mathrm{sep,m}} = \sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}} \left| -\cos\phi_s + \frac{(\pi - 2\phi_s)}{2}\sin\phi_s \right|^{1/2}$$

- RF bucket area (acceptance)

$$\mathcal{A}_{\mathrm{bk}} \approx \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}} \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

- For the stationary RF bucket, the RF bucket length is 2π and $\mathcal{A}_{\mathrm{bk}} = 8\Delta E_{\mathrm{sep,m}}/\omega_r$

NON-LINEAR SYNCHROTRON FREQUENCY

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2K\left(\sin\frac{\phi_b}{2}\right)} \approx 1 - \frac{\phi_b^2}{16}$$