LONGITUDINAL BEAM DYNAMICS

JUAS 2023

COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS



SUMMARY OF FORMULAS



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Formulas

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BASICS

• Lorentz force

$$rac{dec{p}}{dt} = ec{F} = q\left(ec{\mathcal{E}} + ec{v} imes ec{\mathcal{B}}
ight)$$

- Definition of coordinates
 - $x \,$ horizontal position
 - $y \,$ vertical position
 - z longitudinal position

 $\vec{\mathcal{E}}$ to accelerate and deflect $\vec{\mathcal{B}}$ to bend trajectories

- ho local bending radius
- $R\,$ mean radius / orbit
- heta azimuth
- Assumptions made so far: $p_z \gg p_{x,y}$ and $p pprox p_z$

FIELDS, FORCES, RELATIVITY

• Acceleration in an RF gap:

$$\delta E = \int q \mathcal{E}_{z}\left(
ho,z,t
ight) dz = q V_{ ext{rf}}\left(
ho, au
ight)$$

• Magnetic rigidity:

$$\mathcal{B}_y
ho = rac{p}{q} \quad o \quad p \left[{
m GeV/c}
ight] pprox 0.3 \ Z \ \mathcal{B}_y \left[{
m T}
ight]
ho \left[{
m m}
ight]$$

• Relativistic relationships (P = p c):

$$E=E_{
m kin}+E_0=\sqrt{P^2+E_0^2},\quad eta=rac{v}{c}=rac{P}{E},\quad \gamma=rac{1}{\sqrt{1-eta^2}}=rac{E}{E_0}$$

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FIELDS, FORCES, RELATIVITY

• Relativistic relationships:

$$eta^2+rac{1}{\gamma^2}=1$$

• Relativistic differential relationships:



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ENERGY GAIN

• RF energy gain

$$\delta E_{
m rf}\left(au
ight) = qV_{
m rf,0}T_t\sin\left(\omega_r au
ight) \quad o \delta E_{
m rf}\left(\phi
ight) = qV_{
m rf}\sin\left(\phi
ight)$$

• Transit time factor

$$T_t\left(
ho,eta
ight) = rac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(
ho,z
ight) \cos\left(rac{\omega_r z}{eta c}
ight) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(
ho,z
ight) dz}$$

- Assumptions:
 - eta is not changing in the computation of T_t
 - The (
 ho,eta) dependence of T_t will be neglected

PILLBOX CAVITY (FUNDAMENTAL MODE)

• Pillbox cavity properties

$$\mathcal{E}_{z}\left(
ho,t
ight)=\mathcal{E}_{0}J_{0}\left(\chi_{0}rac{
ho}{
ho_{c}}
ight)\cos\left(\omega_{r}t
ight)$$

 J_n Bessel function, $\chi_0pprox 2.405$, $\omega_r=\chi_0 c/
ho_c$

• Transit time factor of pillbox cavity

$$T_t = rac{\sin\left(rac{\chi_0 g}{2eta
ho_c}
ight)}{\left(rac{\chi_0 g}{2eta
ho_c}
ight)}$$

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OTHER ENERGY GAIN/LOSS IN A RING

• Induction acceleration (small in large synchrotrons)

$$\delta E_{\mathrm{b}}\left(
ho
ight)=q\int_{0}^{2\pi}\int_{0}^{
ho}rac{\partial\mathcal{B}_{y}\left(
ho^{\prime}, heta,t
ight)}{\partial t}
ho^{\prime}\,d
ho^{\prime}\,d heta$$

• Synchrotron radiation (relevant for lepton accelerators)

$$\delta E_{
m sr}\left(E,
ho
ight)=rac{q^2}{3\epsilon_0}rac{eta^3 E^4}{
ho \ E_0^4}$$

• Self induced field

$$\delta E_{ ext{ind}}\left(au
ight)=qV_{ ext{ind}}\left(au
ight)=-qN_{b}\left(\lambda*\mathcal{W}
ight)$$

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SYNCHRONISM IN SYNCHROTRON

• The revolution period and frequency

$$T_0 = rac{C}{v} = rac{2\pi R}{eta c} ~~,~~ \omega_0 = 2\pi f_0 = rac{2\pi}{T_0} = rac{eta c}{R}$$

• Synchronism condition with RF frequency

$$\omega_r = h \ \omega_{0,s} = h rac{eta_s c}{R_s}$$



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ACCELERATION

• Acceleration rate (subscript *s* for synchronous particle)

$$\delta E_s = 2\pi q
ho_s R_s \dot{\mathcal{B}}_y \quad o \quad \phi_s = rcsin\left(2\pi
ho_s R_s rac{\dot{\mathcal{B}}_y}{V_{
m rf}}
ight)$$

• RF frequency program

$${f_r}\left(t
ight) = rac{{hc}}{{2\pi {R_s}}}\sqrt {rac{{{\mathcal B}_y^2\left(t
ight)}}{{{\mathcal B}_y^2\left(t
ight) + {\left({rac{{{m_0}c}}{{{
ho _s}q}}}
ight)^2 }}}$$

• Assumptions: Acceleration with constant R_s and ho_s

RADIAL DISPLACEMENT

- Momentum compaction factor (subscript 0 for design orbit/momentum, transition gamma γ_t)

$$lpha_p = rac{dR/R}{dp/p} = rac{\langle D_x
angle_
ho}{R} = rac{1}{\gamma_t^2} pprox rac{\Delta R/R_0}{\Delta p/p_0} pprox rac{\Delta R/R_s}{\Delta p/p_s}$$

• Phase slip factor

$$\eta = -rac{d\omega_0/\omega_0}{dp/p} = rac{dT_0/T_0}{dp/p} = lpha_p - rac{1}{\gamma^2} pprox -rac{\Delta\omega_{0,0}/\omega_{0,0}}{\Delta p/p_0} pprox -rac{\Delta\omega_{0,s}/\omega_{0,s}}{\Delta p/p_s}$$

• Assumptions: Radial displacement with constant \mathcal{B}_y

SYNCHROTRON DIFFERENTIAL RELATIONSHIPS

(1)
$$\mathcal{B}_{y}, p, R$$
 $\frac{dp}{p} = \gamma_{t}^{2} \frac{dR}{R} + \frac{d\mathcal{B}_{y}}{\mathcal{B}_{y}}$
(2) f_{0}, p, R $\frac{dp}{p} = \gamma^{2} \frac{df_{0}}{f_{0}} + \gamma^{2} \frac{dR}{R}$
(3) $\mathcal{B}_{y}, f_{0}, p$ $\frac{d\mathcal{B}_{y}}{\mathcal{B}_{y}} = \gamma_{t}^{2} \frac{df_{0}}{f_{0}} + \frac{\gamma^{2} - \gamma_{t}^{2}}{\gamma^{2}} \frac{dp}{p}$
(4) $\mathcal{B}_{y}, f_{0}, R$ $\frac{d\mathcal{B}_{y}}{\mathcal{B}_{y}} = \gamma^{2} \frac{df_{0}}{f_{0}} + (\gamma^{2} - \gamma_{t}^{2}) \frac{dR}{R}$

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LONGITUDINAL EQUATIONS OF MOTION

• Energy

$$rac{d}{dt}\left(rac{\Delta E}{\omega_{0,s}}
ight)=rac{qV_{
m rf}}{2\pi}\left[\sin\left(\phi
ight)-\sin\left(\phi_{s}
ight)
ight]$$

• Phase

$$rac{d\phi}{dt} = rac{h\eta\omega_{0,s}^2}{eta_s^2 E_s} \left(rac{\Delta E}{\omega_{0,s}}
ight)$$



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LINEAR SYNCHROTRON MOTION

• Linear synchrotron frequency

$$\omega_{s0}=2\pi f_{s_0}=\sqrt{-rac{qV_{
m rf}h\omega_{0,s}^2\eta\cos\phi_s}{2\pieta_s^2E_s}}$$

• Linear synchrotron tune

$$Q_{s0}=rac{\omega_{s0}}{\omega_{0,s}}=\sqrt{-rac{qV_{
m rf}h\eta\cos\phi_s}{2\pieta_s^2E_s}}$$

• Phase stability condition

 $\eta\cos\phi_s < 0$

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LINEAR OSCILLATION AMPLITUDE AND EMITTANCE

• Oscillation amplitude ratio

$$rac{\left(\Delta E/\omega_r
ight)_m}{\Delta \phi_m} = rac{eta_s^2 E_s}{\left|\eta
ight| \, \omega_r^2} \omega_{s0} = rac{eta_s^2 E_s}{\left|\eta
ight| \, h^2 \omega_{0,s}} Q_{s0}$$

• Approximate longitudinal emittance

$$arepsilon_{l,0} = \pi \Delta E_m rac{ au_l}{2} = rac{\pi eta_s^2 E_s}{4 \left| \eta
ight|} \omega_{s0} au_l^2
onumber \ = rac{\pi \left| \eta
ight|}{eta_s^2 E_s} rac{1}{\omega_{s0}} \Delta E_m^2$$

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BUNCH PARAMETERS LINEAR SCALING LAWS

• Bunch length

$$au_l \propto arepsilon_{l,0}^{1/2} \ V_{
m rf}^{-1/4} \ h^{-1/4} \ E_s^{-1/4} \ \eta^{1/4}$$

• Energy deviation

$$\Delta E_m \propto arepsilon_{l,0}^{1/2} \ V_{
m rf}^{1/4} \ h^{1/4} \ E_s^{1/4} \ \eta^{-1/4}$$



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HAMILTONIAN

$$\mathcal{H} = rac{\eta \omega_r^2}{2 eta_s^2 E_s} \left(rac{\Delta E}{\omega_r}
ight)^2 + rac{q V_{
m rf}}{2 \pi h} \left[\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s
ight]$$



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RF BUCKET PARAMETERS

• RF bucket height

$$\Delta E_{
m sep,m} = \sqrt{rac{2qV_{
m rf}eta_s^2E_s}{\pi h\left|\eta
ight|}} \left|-\cos\phi_s + rac{(\pi-2\phi_s)}{2}\sin\phi_s
ight|^{1/2}$$

• RF bucket area (acceptance)

$$\mathcal{A}_{
m bk} pprox rac{8}{\omega_r} \sqrt{rac{2qV_{
m rf}eta_s^2 E_s}{\pi h \left|\eta
ight|}} rac{1-\sin\phi_s}{1+\sin\phi_s}$$

• For the stationary RF bucket, the RF bucket length is 2π and ${\cal A}_{
m bk}=8\Delta E_{
m sep,m}/\omega_r$

NON-LINEAR SYNCHROTRON FREQUENCY

$$rac{\omega_s}{\omega_{s0}} = rac{\pi}{2K\left(\sinrac{\phi_b}{2}
ight)} pprox 1 - rac{\phi_b^2}{16}$$



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