LONGITUDINAL BEAM DYNAMICS

JUAS 2023

COURSE 1: THE SCIENCE OF PARTICLE ACCELERATORS

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Joint Universities Accelerator School

INTRODUCTION

ACKNOWLEDGEMENTS

JUAS FORMER LECTURERS AND THEIR LEGACY

ELIAS, BENOIT, DANILO, DAVID AND SEBASTIEN FOR THEIR SUPPORT

THE CERN ACCELERATOR SCHOOL AND ITS NUMEROUS REFERENCES

COLLEAGUES FROM THE RF GROUP AND BR SECTION AT CERN

AND YOU!

RESOURCES

WEB

- E. Metral website, JUAS courses, exercises, exams and corrections

COURSES

- G. Dôme, Theory of RF Acceleration
- L. Rinolfi, Longitudinal Beam Dynamics Application to synchrotron
- F. Tecker, Longitudinal Beam Dynamics in Circular Accelerators
- B. Holzer, Introduction to Longitudinal Beam Dynamics
- H. Damerau, Introduction to Non-linear Longitudinal Beam Dynamics
- R. Garoby, RF Gymnastics in Synchrotrons
- B. W. Montague, Single particle dynamics : Hamiltonian formulation
- W. Pirkl, Longitudinal beam dynamics
- J. Le Duff, Longitudinal beam dynamics in circular accelerators
- E. Jensen, RF Cavity Design

RESOURCES

NOTES

- H. G. Hereward, What are the equations for the phase oscillations in a synchrotron?
- J. A. MacLachlan, Difference Equations for Longitudinal Motion in a Synchrotron
- J. A. MacLachlan, Differential Equations for Longitudinal Motion in a Synchrotron
- C. Bovet, R. Gouiran, I. Gumowski, K. H. Reich, A selection of formulae and data useful for the design of A.G. synchrotrons

BOOKS

- A. A. Kolomensky, A. N. Lebedev, Theory of Cyclic Accelerators
- H. Bruck, Accelerateurs Circulaires De Particules
- S. Y. Lee, Accelerator Physics
- S. Humphries, Principles of Charged Particle Acceleration
- T. P. Wangler, RF Linear Accelerators
- H. Wiedemann, Particle Accelerator Physics
- M. Reiser, Theory and Design of Charged Particle Beams

SCHEDULE

COURSE CONTENT

- 1 Introductory session
- 10 Teaching modules including
  - Lecture
  - Derivations
  - Computational exercises
  - Quizz
  - Interleaving exercises with lecture. The last slot of each afternoon dedicated to tutorials/questions.
- Exam preparation
- PyHEADTAIL workshop

SCHEDULE

WEEK 1

(COURSE 1)

WEEK #1

	9 Jan. Monday	10 Jan. Tuesday	11 Jan. Wednesday	12 Jan. Thursday	13 Jan. Friday
MORNING (From 9:00 to 12:00)		Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer
		Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer
	OFFICIAL OPENING: Presentation of JUAS & Introduction of students *E. Metral, B. Holland, S. Vandergooten*	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer	Transverse Beam Dynamics B. Holzer
AFTERNOON (From 13:30 onwards)	Special relativity, electromagnetism, classical and quantum mechanics: What to remember for particle accelerators *E. Metral*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*
		Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*
	Particle Accelerators in the 21st century Seminar *M. Vretenar*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*	Longitudinal Beam Dynamics *A. Lasheen*
	CHECK-IN AT THE RESIDENCE & SHOPPING FOR GROCERIES	Introduction to CERN & its Accelerator Complex Seminar *R. Alemany*			

SCHEDULE

WEEK 2

(COURSE 1)

WEEK #2

	16 Jan. Monday	17 Jan. Tuesday	18 Jan. Wednesday	19 Jan. Thursday	20 Jan. Friday
MORNING (From 9:00 to 12:00)	**Introduction to MAD-X** *N. Fuster Martinez*	**Introduction to PyHeadTail** *B. Salvant*	**PyHeadTail workshop** *B. Salvant*	**Linacs** *D. Alesini*	**Linacs** *D. Alesini*
	Transverse Beam Dynamics **(exam preparation)** B. Holzer	**Longitudinal Beam Dynamics** **(exam preparation)** *A. Lasheen*	**PyHeadTail workshop** *B. Salvant*	**Linacs** *D. Alesini*	**Linacs** *D. Alesini*
	Transverse Beam Dynamics **(exam preparation)** B. Holzer	**Longitudinal Beam Dynamics** **(exam preparation)** *A. Lasheen*	**PyHeadTail workshop** *B. Salvant*	**Linacs** *D. Alesini*	**Linacs** *D. Alesini*
AFTERNOON (From 13:30 onwards)	**MADX workshop** *N. Fuster Martinez*	**MADX workshop** *N. Fuster Martinez*	**Linacs** *D. Alesini*	**Transverse linear imperfections** *D. Gamba*	**Transverse linear imperfections** *D. Gamba*
	MADX workshop *N. Fuster Martinez*	**MADX workshop** *N. Fuster Martinez*	**Linacs** *D. Alesini*	**Transverse linear imperfections** *D. Gamba*	**Transverse linear imperfections** *D. Gamba*
	MADX workshop *N. Fuster Martinez*	**MADX workshop** *N. Fuster Martinez*	**Transverse linear imperfections** *D. Gamba*	**Transverse linear imperfections** *D. Gamba*	**Transverse linear imperfections** *D. Gamba*
			Transverse linear imperfections *D. Gamba*		

SCHEDULE

WEEK 3

(COURSE 1)

juas

WEEK #3

	23 Jan. Monday	24 Jan. Tuesday	25 Jan. Wednesday	26 Jan. Thursday	27 Jan. Friday
MORNING (From 9:00 to 12:00)	WRITTEN EXAMINATION Transverse beam dynamics	Cyclotrons & FFAs *B. Jacquot*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*
		Cyclotrons & FFAs *B. Jacquot*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*
	WRITTEN EXAMINATION Longitudinal beam dynamics	Cyclotrons & FFAs *B. Jacquot*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation (exam preparation) *R. Ischebeck*
AFTERNOON (From 13:30 onwards)	Trip to CERN	Dedicated session on COLLIDERS 1) LHC & HL-LHC *(O. Brüning)* *2) Nuclear collisions at the LHC (J. Jowett)* 3) FCC-hh *(M. Giovannozzi)* 4) Electron-positron circular colliders *(J. Keintzel)* 5) The US Electron-Ion Collider *(T. Satogata)* *6) Future high-energy linear colliders (P. Burrows)* 7) Muon collider *(D. Schulte)*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation *R. Ischebeck*	Synchrotron Radiation (exam preparation) *R. Ischebeck*
	Visit of the CERN LEIR accelerator *N. Biancacci*		Cyclotrons & FFAs *B. Jacquot*	Transverse nonlinear effects *H. Bartosik*	Transverse nonlinear effects *H. Bartosik*
	Drink at CERN		Cyclotrons & FFAs *B. Jacquot*	Transverse nonlinear effects *H. Bartosik*	Transverse nonlinear effects *H. Bartosik*
	Visit to ALICE experiment at the CERN LHC *J. Jowett*		Cyclotrons & FFAs *B. Jacquot*	Transverse nonlinear effects *H. Bartosik*	Transverse nonlinear manipulations Seminar *M. Giovannozzi*
	Intro on Colliders (for tomorrow's afternoon session on Collider) Seminar E. *Métral*				
	Dinner at CERN				

SLIDES

- The slides are provided in `html` and `pdf` formats.
- The `html` file includes animations and features such as the menu and chalkboard. Keep the `img` and `menu.css` files in the same directory as the `html` to fully profit from these features.
- The hotkeys are
  - `Esc`/`o`: Overview of the slides
  - `m`: Menu
  - `b`/`c`: Chalkboard / Notes
  - `x`: Change color
  - `Right click`: Erase
  - `Backspace`: Erase all notes
- The drawings are kept in the `html` file after closure. I advise nonetheless to keep a separate trace of the notes you estimate important!
- You can navigate `Left`/`Right` but also Up/Down for details on mathematical derivations.

COURSE LAYOUT

INTRODUCTORY SESSION

- What is longitudinal beam dynamics?
- How does this lecture relates to the others?

LESSON 1 - FUNDAMENTALS OF PARTICLE ACCELERATION

- Fields, forces
- Accelerator designs
- Relativistic relationships

COURSE LAYOUT

LESSON 2 - SYNCHROTRON DESIGN

- Equations for the synchronous particle
- One word on betatronic acceleration, synchrotron radiation, self induced fields
- Momentum compaction, differential relationships

LESSON 3 - LONGITUDINAL EQUATIONS OF MOTION

- Equations for non synchronous particles
- Introduction to tracking

COURSE LAYOUT

LESSON 4 - SYNCHROTRON MOTION

- Linearized synchrotron motion
- Phase stability and synchrotron frequency/tune
- Non-linear synchrotron motion
- RF bucket, longitudinal emittance, non-linear synchrotron frequency

LESSON 5 - REAL LIFE APPLICATIONS

- Longitudinal bunch profile measurements
- Examples of RF operation
- Introduction to RF manipulations ("gymnastics")

TEACHING AGREEMENT

WHAT YOU SHOULD KNOW AT THE END OF THE COURSE

- Understand how a beam is effectively accelerated in a particle accelerator.
- Understand fundamental concepts of longitudinal beam dynamics (i.e. synchrotron motion, the RF bucket and its parameters).
- How main equations/formulas are derived and underlying assumptions.

WHAT YOU SHOULD BE ABLE TO DO AT THE END OF THE COURSE

- Compute RF parameters and basic design parameters of a synchrotron.
- Interpret the longitudinal motion of a measured bunch of particles.

KEY ASPECTS OF LONGITUDINAL BEAM DYNAMICS

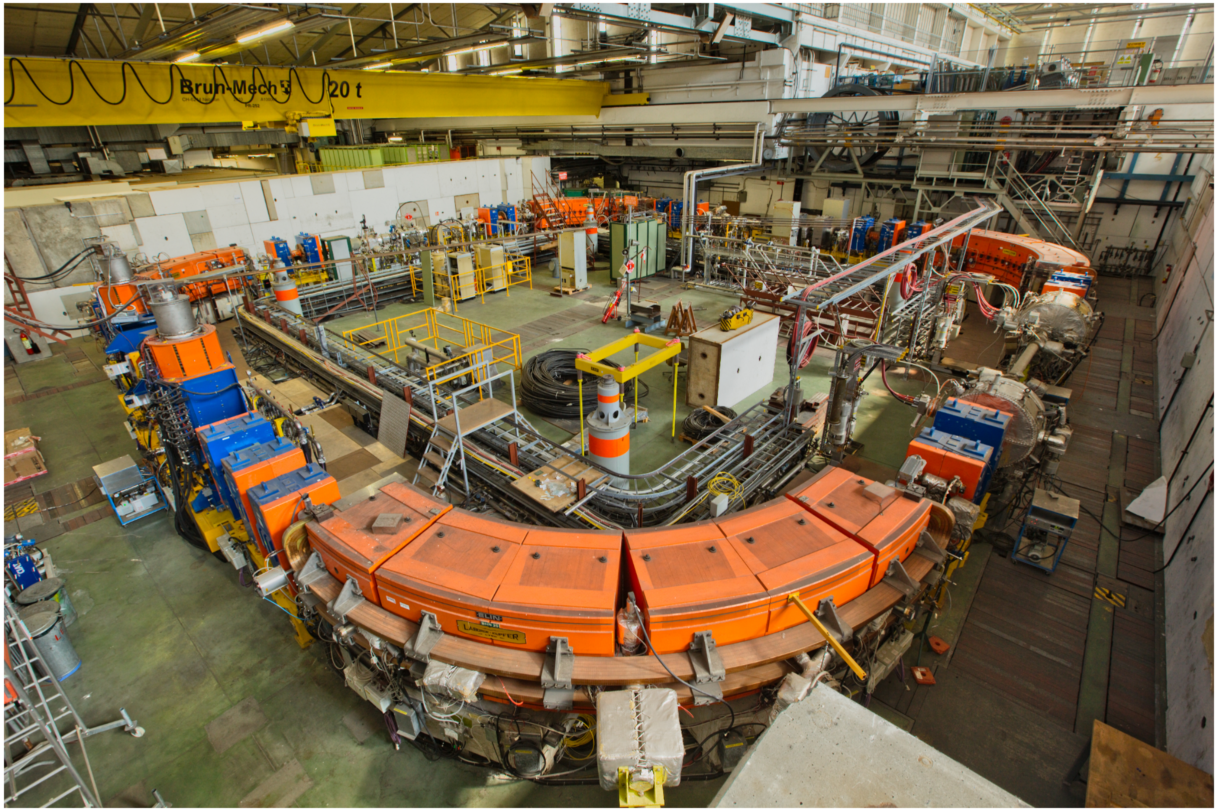
→ **Particle acceleration**

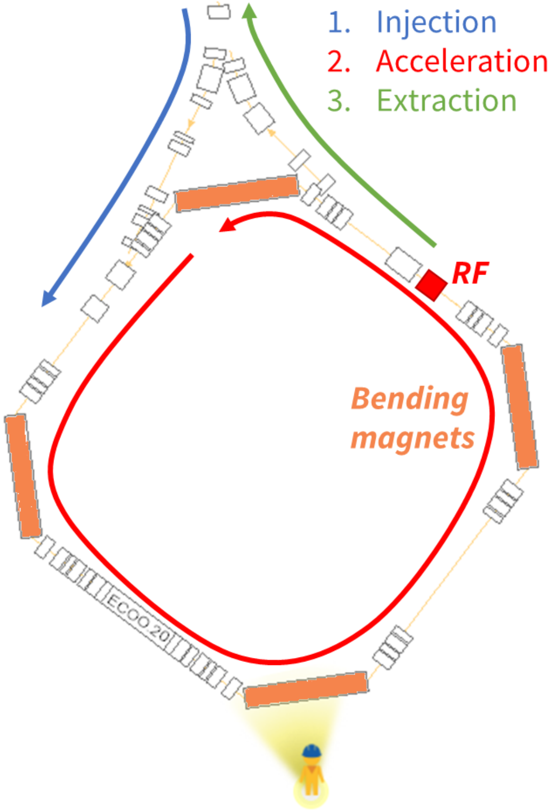
→ **Focusing of particles in the longitudinal direction (bunching)**

→ **Synchrotron motion**

LAYOUT OF A REAL ACCELERATOR

THE LOW ENERGY ION RING (LEIR) AT CERN

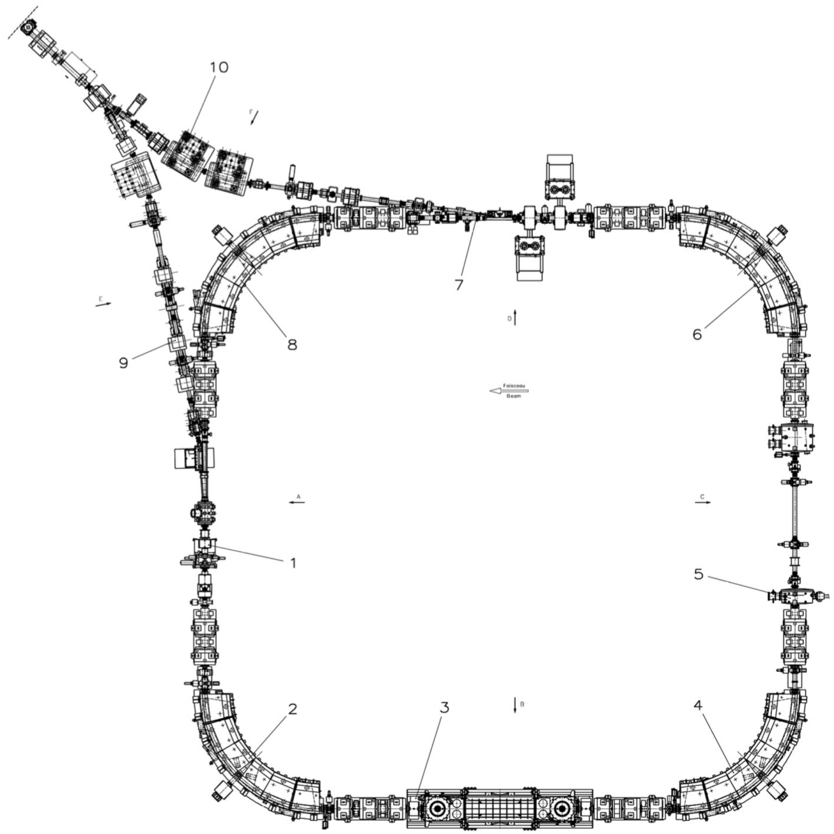


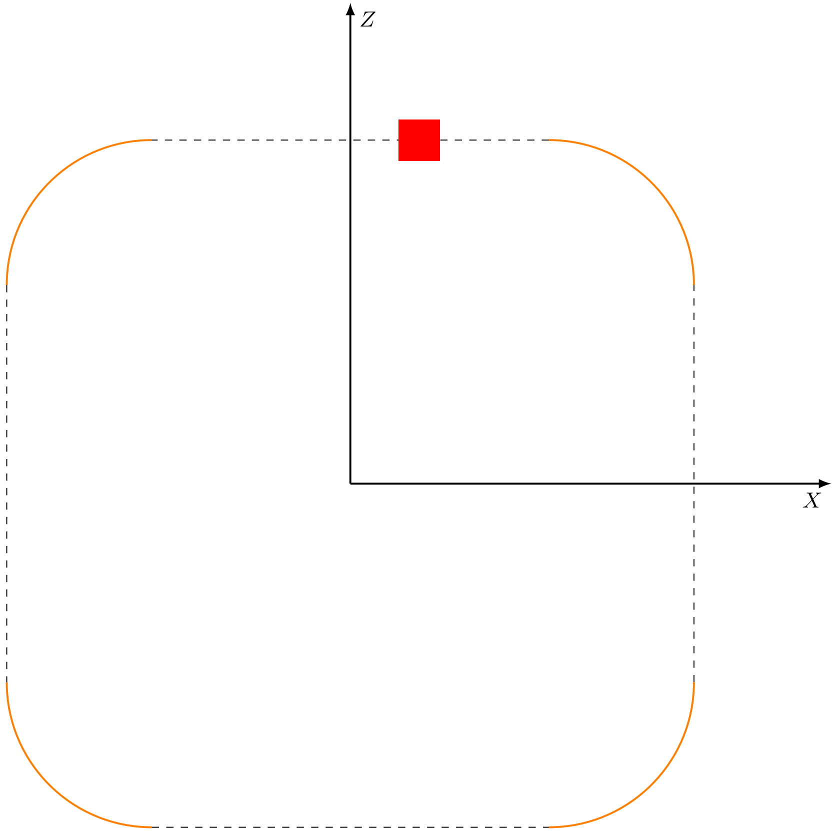


- Virtual walk around LEIR… (visit with Nicolo on 23/01!)
- To see other accelerators at CERN…

COORDINATE SYSTEM(S)

Accelerator seen from above…

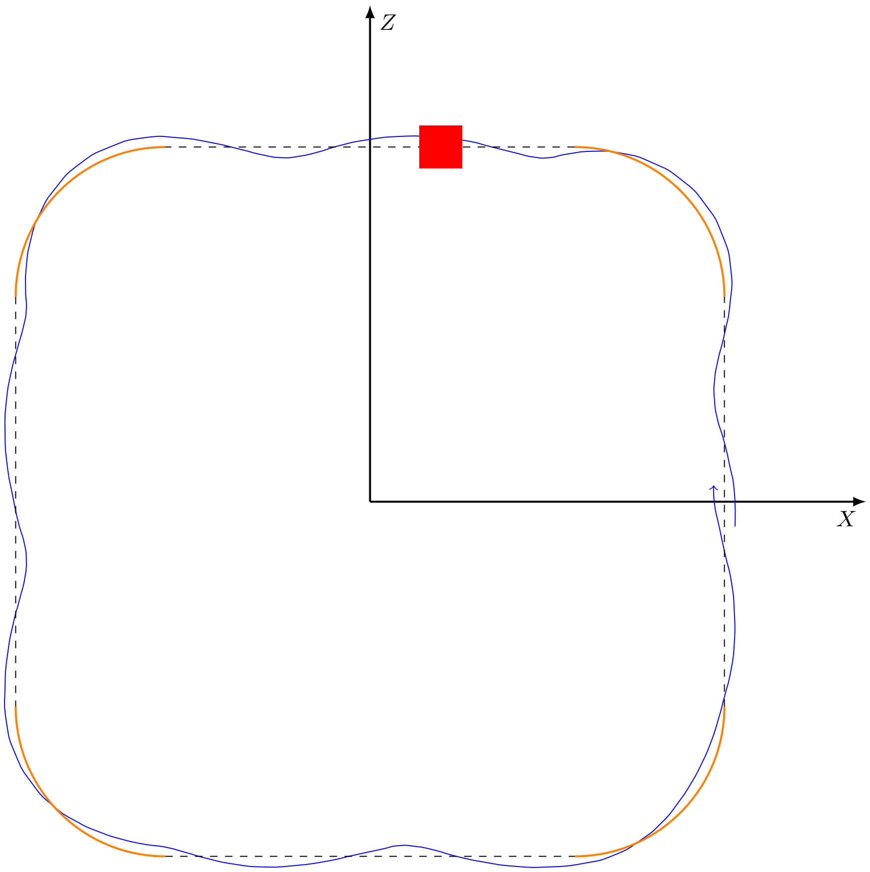




Bending magnets   Accelerating RF cavities

COORDINATE SYSTEM(S)

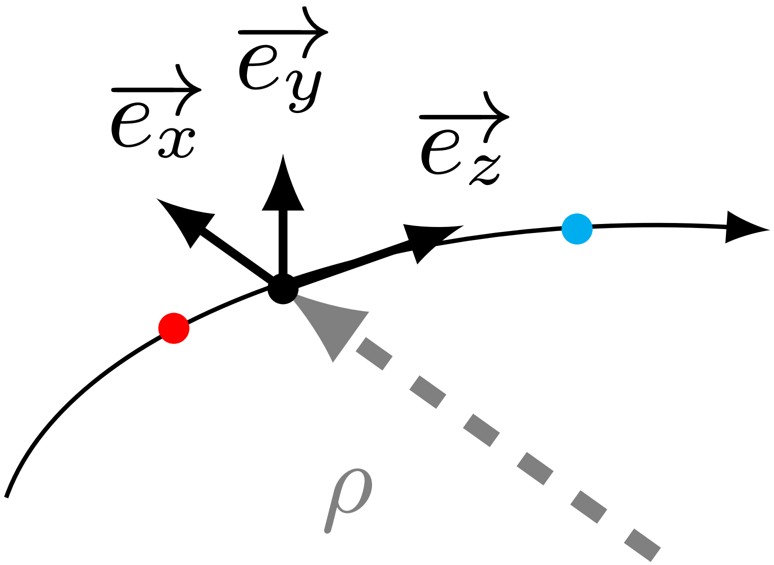
Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- The black line represents the (ideal) design trajectory of the beam around which a particle oscillate (blue).
- The accelerator layout can be described in fixed cartesian coordinates $\left(\vec{X}, \vec{Z}, \vec{Y}\right)$ where the $\vec{Y}$ direction is the vertical direction.
- However, this coordinate system is not suited to describe particle motion in circular accelerators.

COORDINATE SYSTEM(S)

FRENET-SERRET COORDINATE SYSTEM



- A particle trajectory follows a curved path, which can be described in the Frenet-Serret coordinate system.

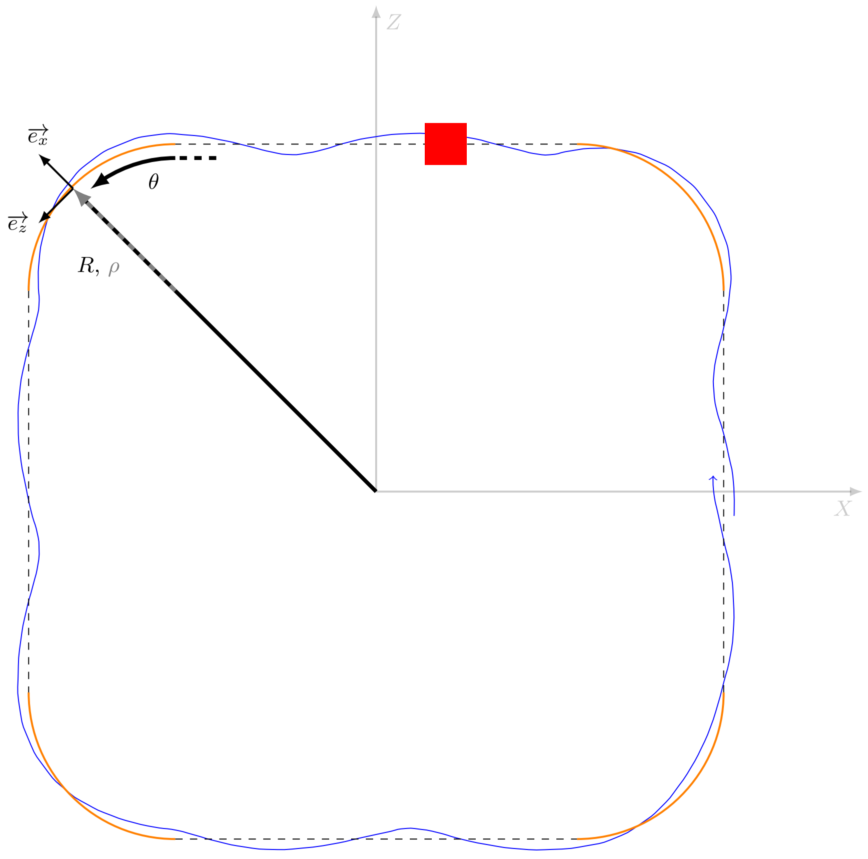
- The particle coordinates are given as offsets with respect to the design trajectory with

$$\begin{array}{ll} x & \text{Horizontal} \\ y & \text{Vertical} \\ z & \text{Longitudinal} \end{array}$$

- The curvature of the trajectory has a local bending radius ρ.

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- We use the Frenet-Serret coordinate system $(\vec{x}, \vec{z}, \vec{y})$ as reference to describe the motion of particles.

- We introduce the mean radius

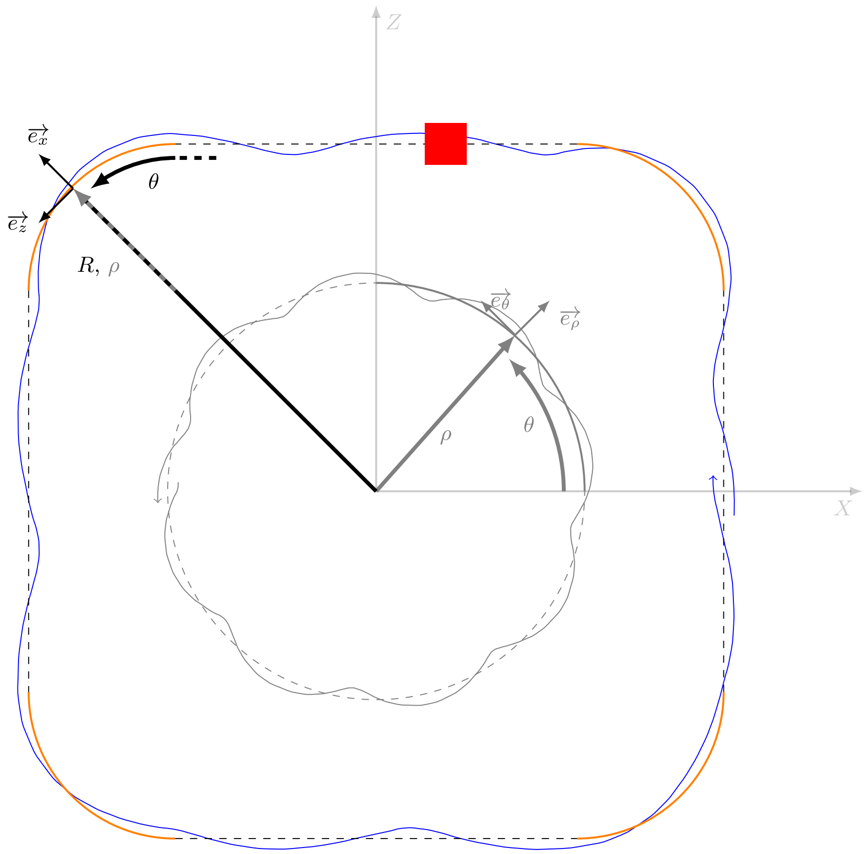
$$R = \frac{C}{2\pi}$$

where C is the path circumference and the generalized azimuth

$$\theta \in [0, 2\pi]$$

COORDINATE SYSTEM(S)

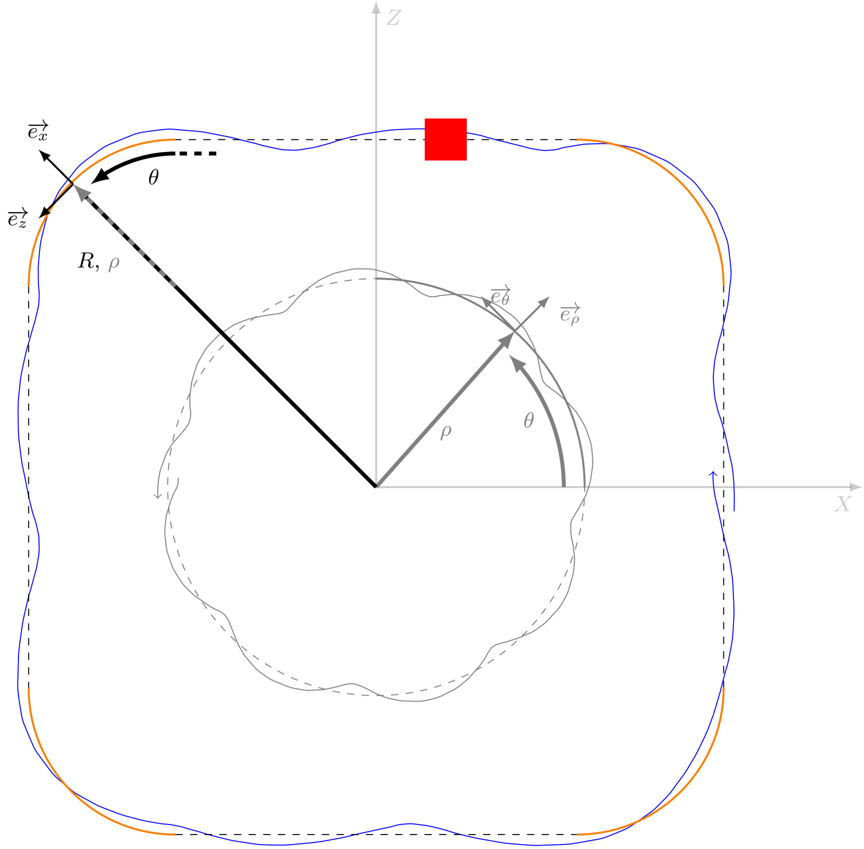
Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- For a circular accelerator, this coordinate system is comparable to the cylindrical coordinate system $\left(\vec{\rho}, \vec{\theta}, \vec{y}\right)$
- A particle orbit and horzitonal positions are equivalent, as well as the longitudinal position and azimuth.
- Beware, definitions can be interchanged!

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- It is also important to disembiguate ρ which is the bending radius and R which is the particle orbit including straight sections of total length L. We have

$$C = 2\pi R = L + 2\pi\rho$$

PARTICLE ACCELERATION

- The primary purpose of a particle accelerator is to produce a beam of particles with a precise energy E.

- The energy can be provided to the particles applying the Lorentz force to charged particles

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$$

where

- $\vec{p} = m\vec{v}$ is the particle momentum
- q is the particle charge
- m is the particle (relativistic) mass
- $\vec{v}$ is the particle velocity
- $\vec{F}$ is a force
- $\vec{\mathcal{E}}$ is an electric field
- $\vec{\mathcal{B}}$ is a magnetic field

PARTICLE ACCELERATION

ELECTRIC FIELD CONTRIBUTION

$$\vec{F}_{\mathcal{E}} = q\,\vec{\mathcal{E}}$$

- An electric field can effectively **accelerate (or decelerate) particles**.
- Electric fields can also be used to **deflect particles** if applied transversally to the particle trajectory.

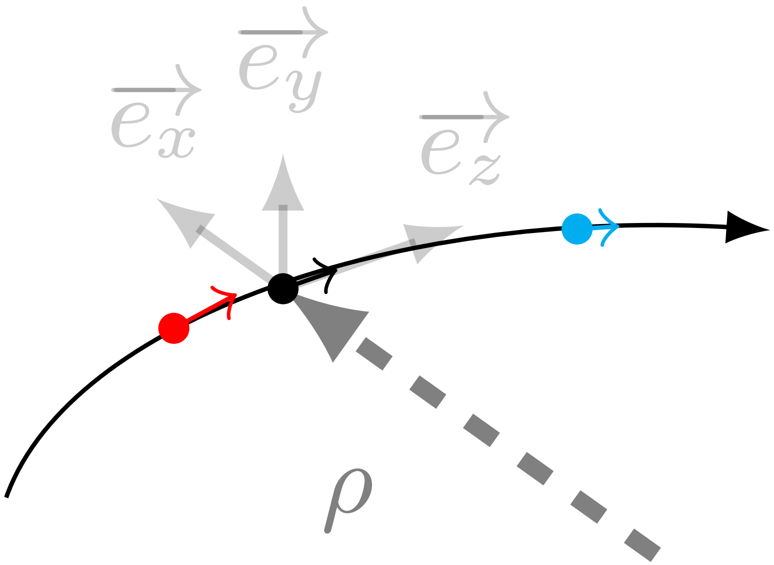
MAGNETIC FIELD CONTRIBUTION

$$\vec{F}_{\mathcal{B}} = q\left(\vec{v} \times \vec{\mathcal{B}}\right)$$

- The force applied by a magnetic field is always orthogonal to the particle trajectory and therefore **cannot accelerate the beam**.
- Magnetic fields are used to **steer the beam**.

PARTICLE TRAJECTORIES

ACCELERATION ALONG THE LONGITUDINAL DIRECTION



- The acceleration is done by applying an electric field tangential to the beam trajectory with

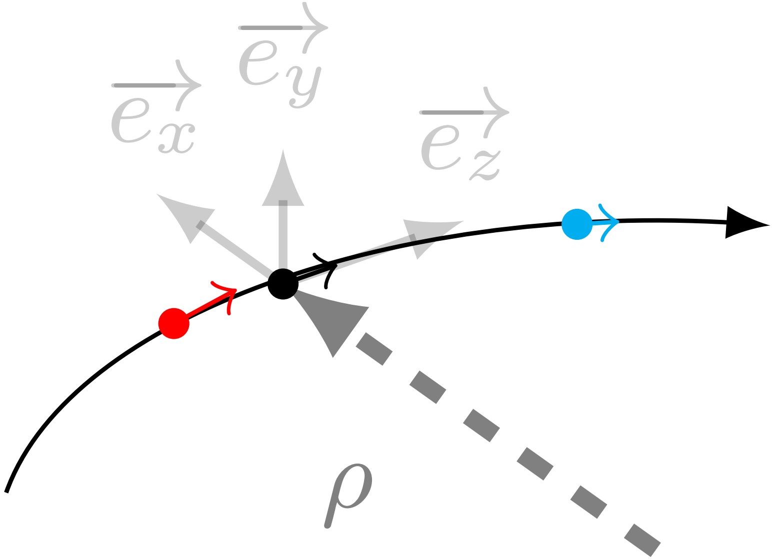
$$\vec{\mathcal{E}} = \mathcal{E}_z \vec{e_z}$$

- Except at extremely low energies (e.g. particle sources), the momentum of a particle is almost exclusively directed towards the longitudinal direction z with small angles in the transverse x and y directions.

- **Assumptions:** $p_z \gg p_{x,y}$ **and** $p \approx p_z$

PARTICLE TRAJECTORIES

STEERING THE DESIGN TRAJECTORY



- The beam trajectory is steered horizontally by applying a vertical magnetic field with

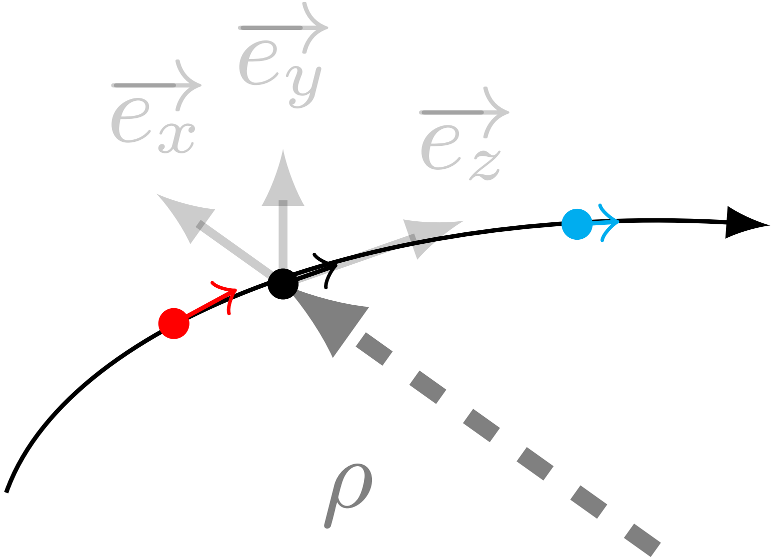
$$\vec{\mathcal{B}} = \mathcal{B}_y \vec{e_y}$$

- The applied force depends on the particle velocity v_z. For particles with different momenta, the steering and trajectories will be different than the design one.

- This effect is called dispersion and will be covered in both transverse and longitudinal beam dynamics lectures.

PARTICLE TRAJECTORIES

EVOLUTION OF RELATIVE PARTICLE POSITIONS



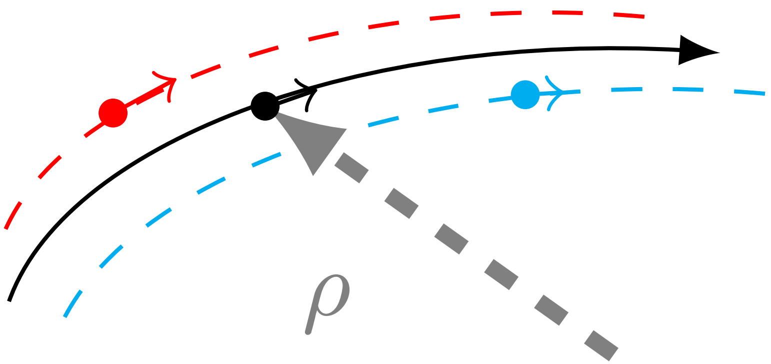
- In the longitudinal direction, a particle can be in front (in advance), or behind (late) with respect to the **ideal particle (on time)**.

- The relative distance between particles can change

  - Because a particle can also have a smaller/larger velocity v_z (and momentum p_z).

PARTICLE TRAJECTORIES

EVOLUTION OF RELATIVE PARTICLE POSITIONS

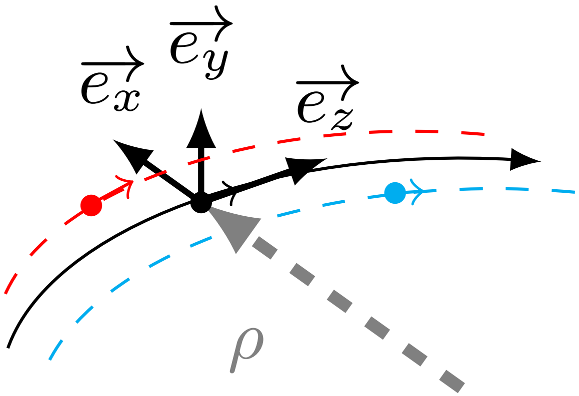


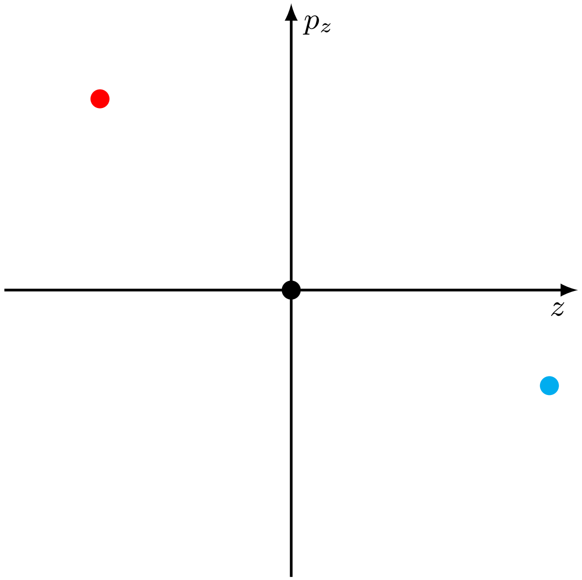
Red is faster but at larger orbit, while blue is slower but inner orbit.

How do we accelerate all three particles evenly? How do we keep these particles together?

- In the longitudinal direction, a particle can be in front (in advance), or behind (late) with respect to the **ideal particle (on time)**.
- The relative distance between particles can change
  - Because a particle can also have a smaller/larger velocity v_z (and momentum p_z).
  - Because of a shorter/longer path length in a bending (i.e. smaller/larger orbit), which depends on the particle momentum.

LONGITUDINAL PHASE SPACE

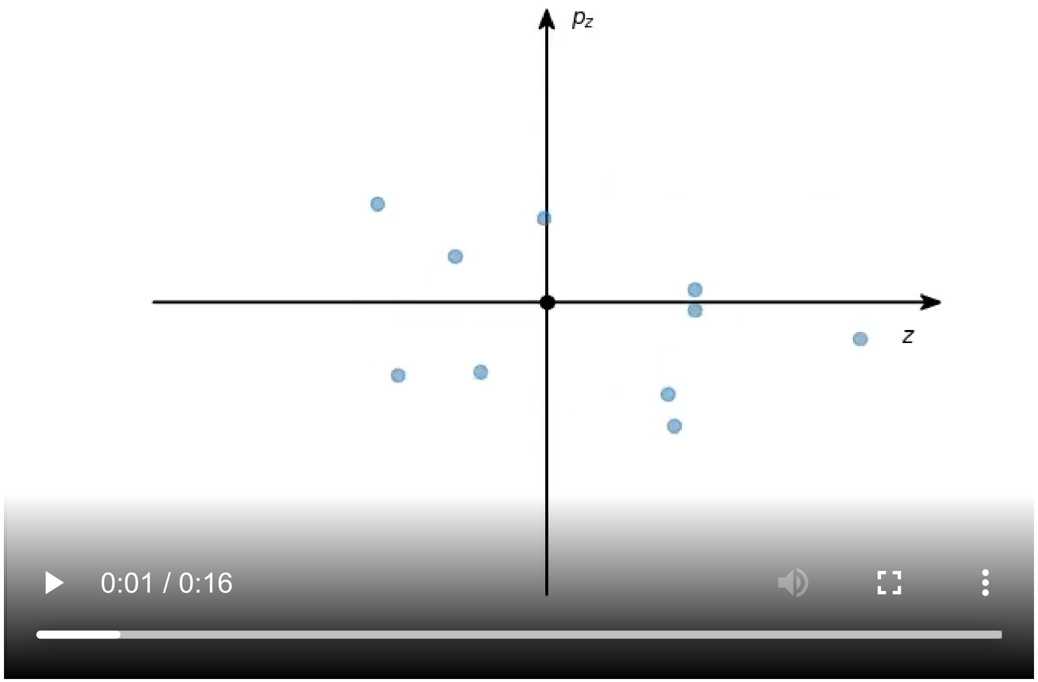




- We will introduce the notion of longitudinal phase space.
- The particle motion can be described in the (z, p_z) phase space, relative to the **ideal particle** following the design orbit and energy.
- As described before other particles can be
  - In front, or in advance in time (right)
  - In the back, or delayed in time (left)
  - Have higher momentum/velocity (top)
  - Have lower momentum/velocity (bottom)
- The motion of the particles in the longitudinal phase space is called **synchrotron motion**.

SYNCHROTRON OSCILLATIONS

WITH A FEW PARTICLES

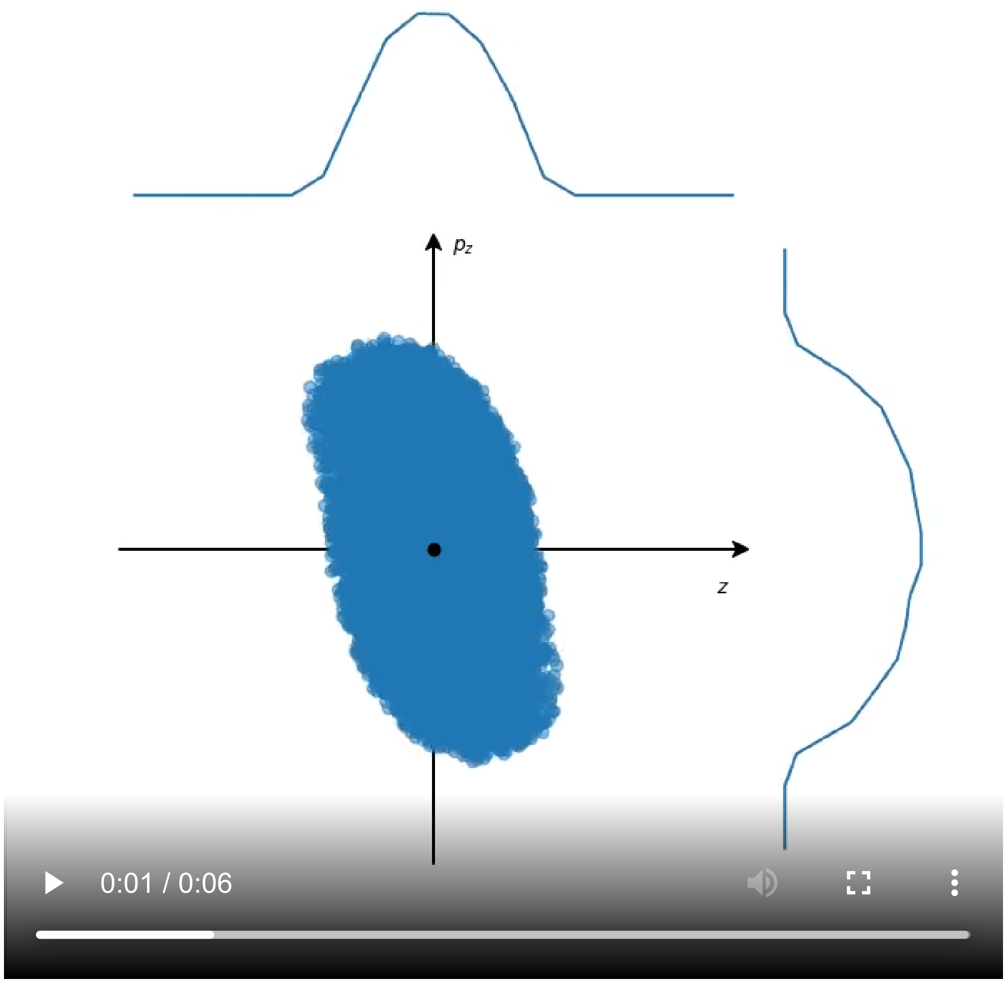


In a bunch, particles rotate around the ideal particle in black used a reference.

These are called **synchrotron oscillations**.

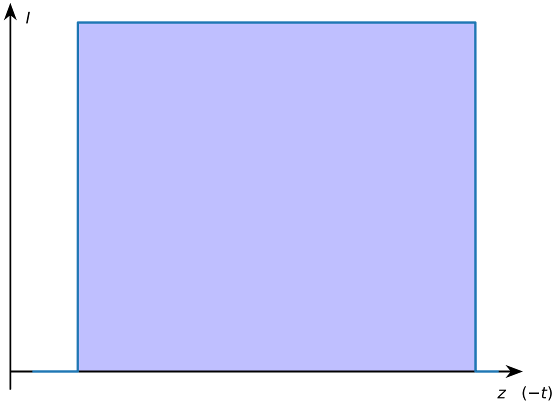
SYNCHROTRON OSCILLATIONS

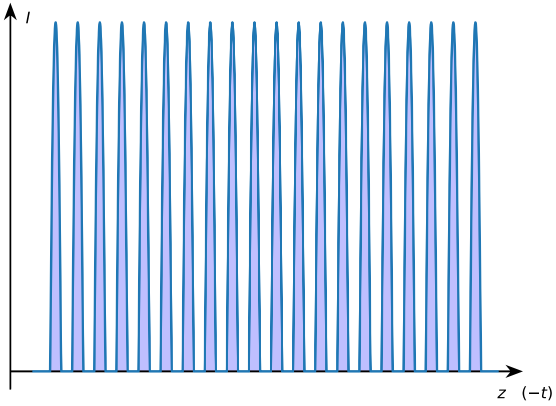
WITH MANY PARTICLES



- A bunch is usually composed of a very large number of particles, typically $\mathcal{O}\left(10^{10}-10^{12}\right)$ at CERN.
- In a real machine, the coherent motion of a bunch can be measured and analyzed from the longitudinal bunch density (top line, projection along the p_z axis, instantaneous beam current).
- You can notice the non-linear synchrotron motion in phase space at large amplitude.

TEMPORAL DEFINITION OF A BEAM





- Controlling the synchrotron motion allows to define the temporal structure of a pulse of particles.

- The beam current is

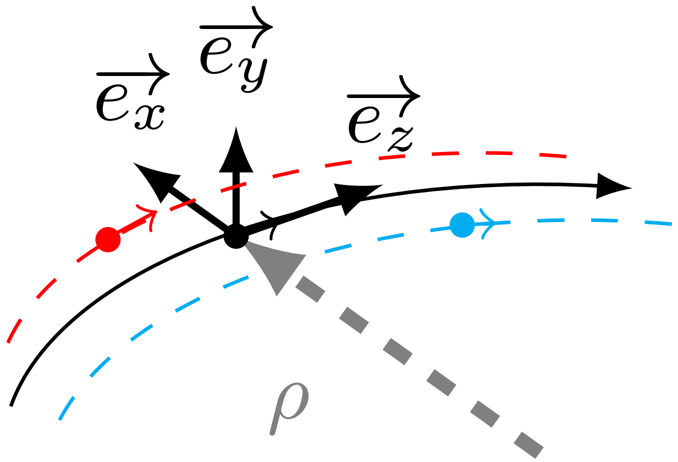
$$I = \frac{dQ}{dt}$$

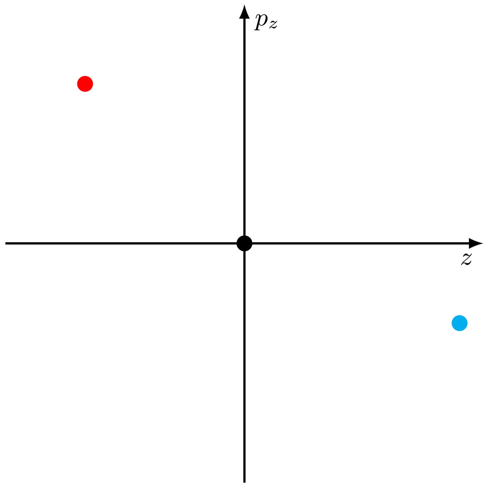
where dQ is the charge passing in a time dt.

- Depending on the destination (experiment or next machine in a chain), parameters defining the synchrotron motion can be adjusted to deliver a continuous or bunched beam.

WHAT IS LONGITUDINAL BEAM DYNAMICS?

- Longitudinal beam dynamics is the description of the acceleration and motion of particles along the forward path of the beam.
- Since the orbit of a particle also plays a role, we will see that the horizontal/radial position of a particle is an important parameter.
- We will derive the equations to describe synchrotron oscillations in longitudinal phase space.





RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 1

- How do we focus the beam in the horizontal and vertical directions, how do we transport the beam to a target?

→ **Transverse Beam Dynamics**

- Can we use the beam in another way than colliding on a target, what is the principle behind light sources ?

→ **Synchrotron radiation**

- Do charged particles interact with each other, can we accelerate an infinite amount of particles?

→ **Collective Effects - Space Charge and Instabilities**

RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 1

- This course is devoted to describe fundamentals of longitudinal beam dynamics with specifities linked to the design of **Synchrotrons**.

- Dedicated courses are devoted to the specificities of **Linacs** and **Cyclotrons**.

- You will find similar concepts between the courses. Nonetheless, beware of definitions, conventions and assumptions used to derive formulas!

RELATIONSHIP WITH OTHER COURSES

JUAS COURSE 2

- What systems do we use to provide the beam with an electric field, how are they designed ?

→ **RF Engineering and Superconducting RF Cavities**

- How do we measure a bunch, specificially in the longitudinal plane ?

→ **Beam Instrumentation**

TAKE AWAY MESSAGE

- Lorentz force

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right)$$

$\vec{\mathcal{E}}$ to accelerate and deflect

$\vec{\mathcal{B}}$ to bend trajectories

- Definition of coordinates

x horizontal position

y vertical position

z longitudinal position

ρ local bending radius

R mean radius / orbit

θ azimuth

- Assumptions made so far: $p_z \gg p_{x,y}$ and $p \approx p_z$

LESSON 1: FUNDAMENTALS OF PARTICLE ACCELERATION

MODULE 1: FIELDS AND FORCES

→ **Acceleration in electric fields**

→ **Electrostatic, induction, and RF acceleration**

→ **Circular accelerators and magnetic rigidity**

MAXWELL EQUATIONS

DIFFERENTIAL EQUATIONS IN VACUUM

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}} = \frac{\rho_q}{\epsilon_0} \qquad \text{Gauss' law}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{B}} = 0 \qquad \text{Flux/Thomson's law}$$

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} = -\frac{\partial \overrightarrow{\mathcal{B}}}{\partial t} \qquad \text{Faraday's law}$$

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}} = \mu_0 \left(\overrightarrow{j} + \epsilon_0 \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t} \right) \qquad \text{Ampère's law}$$

ϵ_0 Vacuum permittivity , μ_0 Vacuum permeability

ρ_q Charge density, $\overrightarrow{j}$ Current density

MAXWELL EQUATIONS

INTEGRAL FORM EQUATIONS IN VACUUM

$$\oiint_S \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dS} = \frac{1}{\epsilon_0} \iiint \rho_q dV$$ Gauss' law

$$\oiint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS} = 0$$ Flux/Thomson's law

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = -\frac{d}{dt} \iint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS}$$ Faraday's law

$$\oint_C \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dz} = \mu_0 \iint_S \overrightarrow{j} \cdot \overrightarrow{dS} + \mu_0 \epsilon_0 \iint_S \frac{\partial \overrightarrow{\mathcal{E}}}{\partial t} \cdot \overrightarrow{dS}$$ Ampère's law

dz Line element, dS Surface element, dV Volume element

ACCELERATION IN ELECTROSTATIC FIELDS (DC)

The simplest form of acceleration can be obtained using electrostatic fields along the longitudinal direction

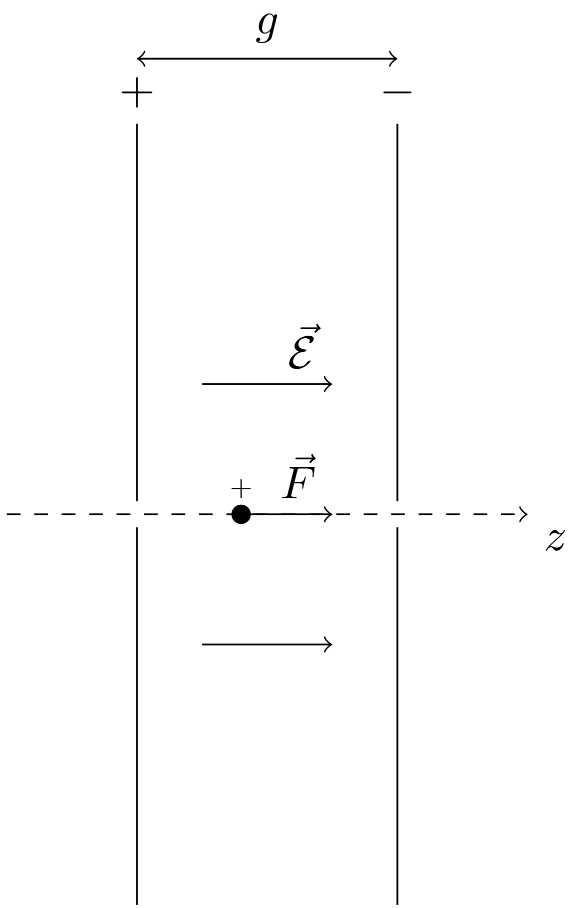
$$\frac{dp}{dt} = \frac{dE}{dz} = q\,\mathcal{E}_z$$

giving an increment in energy

$$\delta E = \int q\mathcal{E}_z dz = q\mathcal{E}_z g = qV_g$$

where the scalar potential V is defined as

$$\overrightarrow{\mathcal{E}} = -\overrightarrow{\nabla} V \implies \mathcal{E}_z = -\frac{\partial V}{\partial z}$$



DEFINITIONS OF ENERGY AND POWER

PARTICLE ENERGY

The energy of particles in accelerators is expressed in electronvolts eV corresponding to the energy gain by a particle with elementary charge e in a potential $V_g = 1V$

$$1\ \mathrm{eV} = 1.602\ 176\ 634 \times 10^{-19}\ \mathrm{J}$$

POWER TRANSFERRED TO THE BEAM

The average power transferred to the beam in W is defined as the total accelerated beam energy $N_p E_{\mathrm{acc}}$ (N_p being the number of particles and E_{acc} expressed in J) delivered in an acceleration time T_{acc}.

$$\langle P_b \rangle = \frac{N_p E_{\mathrm{acc}}}{T_{\mathrm{acc}}}$$

EXERCISES ON THE EV

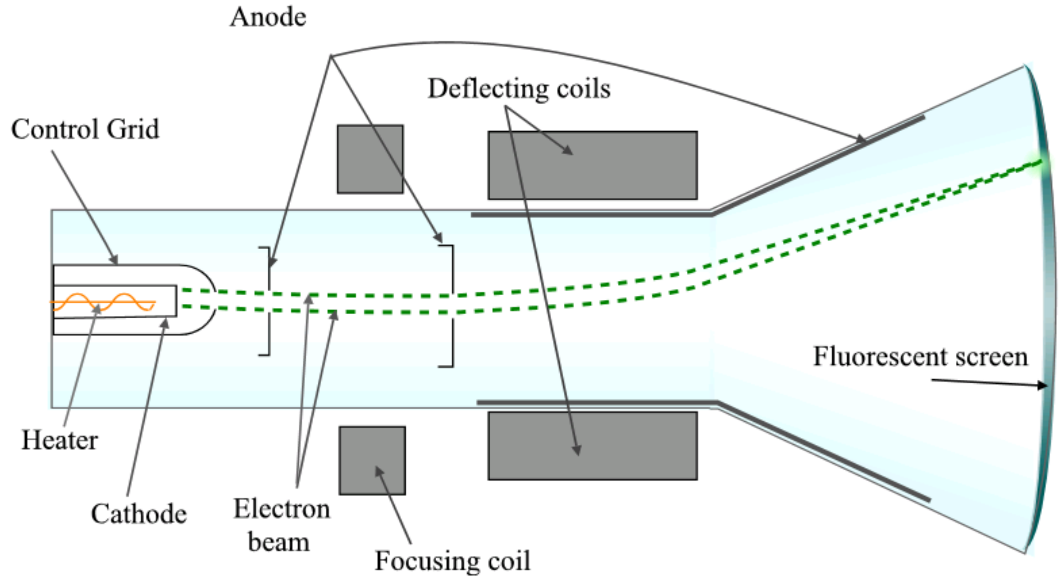
- **An accelerator has a potential of 20 MV, what is the corresponding energy gain of a particle in Joules?**
- **What is the total energy of the beam stored in the LHC?** (The beam is composed of 2808 bunches of 1.15×10^{11} protons each at 7 TeV)
- **What is the equivalent speed of a high speed train?** (Assume a 400 tons (200 m long) TGV train)
- **What is the beam power delivered to the LHC beam?** (Consider an acceleration from 450 GeV to 7 TeV in 30 minutes)

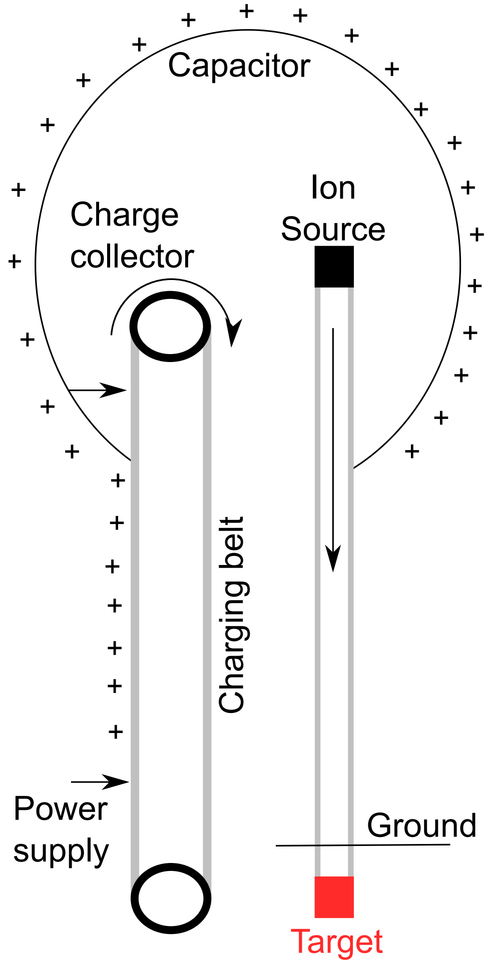
EXERCISE ON THE EV

CORRECTION

- **An accelerator has a potential of 20 MV, what is the corresponding energy gain of the beam in Joules?**
  - $20 \cdot 10^6 \cdot 1.609 \cdot 10^{-19} = 3.2 \cdot 10^{-12}\ \mathrm{J}$
- **What is the total energy of the beam stored in the LHC**
  - $2808 \cdot 1.15 \cdot 10^{11} \cdot 7 \cdot 10^{12} \cdot 1.609 \cdot 10^{-19} = 364\ \mathrm{MJ}$
- **What is the equivalent speed of a high speed train ($E_{\mathrm{LHC}} = E_{\mathrm{kin,train}}$)**
  - $v_{\mathrm{train}} = \sqrt{2E_{\mathrm{LHC}}/m_{\mathrm{train}}} = \sqrt{2 \cdot 364 \cdot 10^6 / \left(400 \cdot 10^3\right)} = 154\ \mathrm{km/h}$
- **What is the power delivered to the LHC beam (1800 s)**
  - $2808 \cdot 1.15 \cdot 10^{11} \cdot (7 - 0.450) \cdot 10^{12} \cdot 1.609 \cdot 10^{-19}/1800 = 189\ \mathrm{kW}$

EXAMPLES OF ELECTROSTATIC ACCELERATORS





- Various designs exist for extraction from a particle source, high field DC acceleration (e.g. Cockroft-Walton, Van de Graaf, Tandem).
- Various applications exist such as cathode ray tudes for (old) TVs, industrial/medical applications...
- See CAS - Electrostatic accelerators for more details.

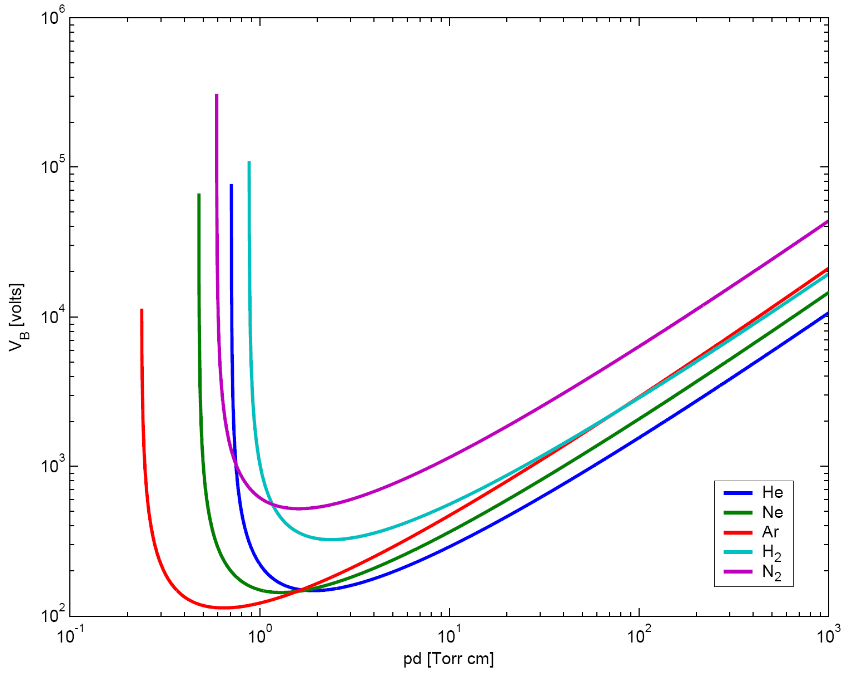
LIMITATIONS OF ELECTROSTATIC ACCELERATORS

- Maximum electric field limited to the MV range due to discharge/arcs.

- The maximum voltage reached depends on the gas nature and pressure and follows the Paschen law.

- Moreover from Faraday's law for static fields implies

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = 0$$



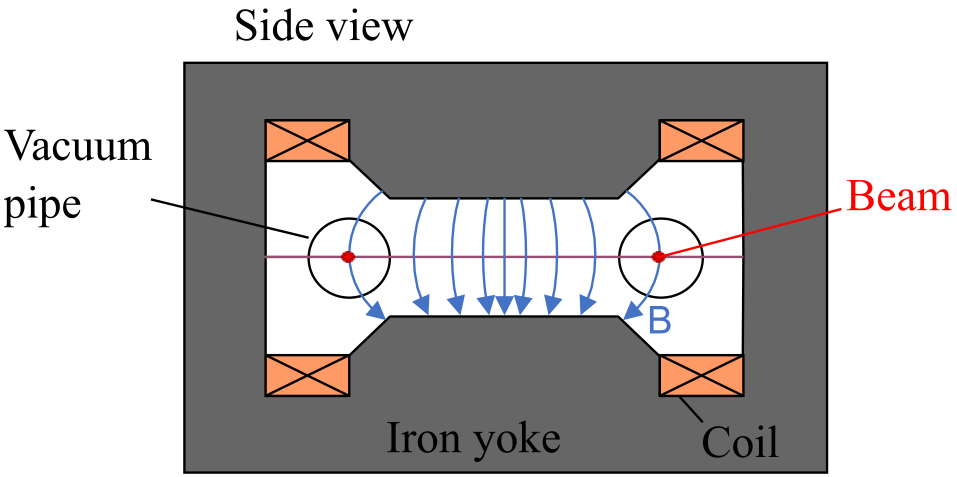
- Single pass accelerator only, cannot reach higher energies than the tens of MeV level (high energy hadron colliders ~TeV!).

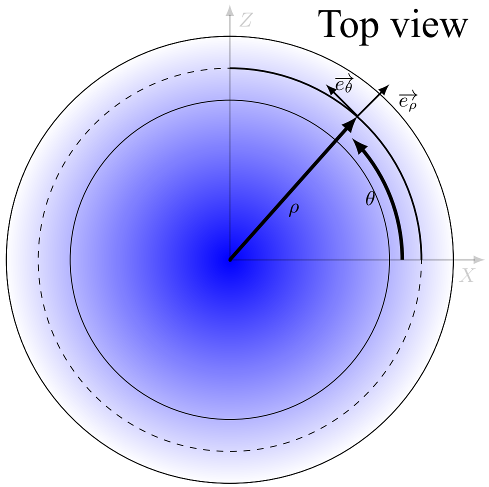
INDUCTION ACCELERATION

An electric field can be obtained with a ramping magnetic field. Again from Faraday's law for induction

$$\oint_C \overrightarrow{\mathcal{E}} \cdot \overrightarrow{dz} = -\frac{d}{dt} \iint_S \overrightarrow{\mathcal{B}} \cdot \overrightarrow{dS}$$

This is the principle behind the betatron accelerator design sketched below, with $B\left(\rho\right)$ in blue.





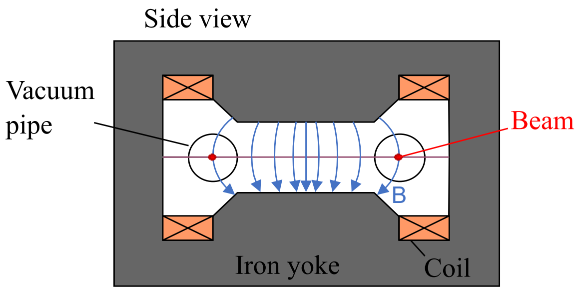
INDUCTION ACCELERATION

BETATRON CONDITION, 2:1 RULE

Assuming azimuthal symmetry, vertical magnetic field $\vec{\mathcal{B}} = -\mathcal{B}_y\left(\rho, t\right)\vec{e_y}$ at a constant orbit ρ_0, we get

$$\mathcal{B}_y\left(\rho_0\right) = \frac{1}{2}\frac{\Phi_{S,\rho_0}}{\pi\rho_0^2} = \frac{1}{2}\left\langle\mathcal{B}_y\right\rangle_{S,\rho_0}$$

→ If the particles move in a circular path of orbit ρ_0, the averaged magnetic field (flux) in the surface enclosed in the orbit ρ_0 should be twice the magnetic field on the particle trajectory. This is also stated as the 2:1 rule.



INDUCTION ACCELERATION

DERIVATION OF THE BETATRON CONDITION

Assuming azimuthal symmetry, vertical magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y\left(\rho, t\right)\vec{e_y}$ at a constant orbit ρ_0, can you derive an equation for $\mathcal{E}_\theta$ and the corresponding dp_θ/dt ?

We will introduce the magetic flux Φ_{S,ρ_0} and an averaged magnetic field in the betatron core $\left\langle\mathcal{B}_y\right\rangle_{S,\rho_0}$

$$\Phi_{S,\rho_0} = 2\pi\int_0^{\rho_0}\mathcal{B}_y\left(\rho\right)\rho\, d\rho = \pi\rho_0^2\left\langle\mathcal{B}_y\right\rangle_{S,\rho_0}$$

What is the equilibrium condition for a constant p_θ if

$$\mathcal{B}_y\rho = \frac{p_\theta}{q} = \frac{p}{q}$$

INDUCTION ACCELERATION

DERIVATION OF THE BETATRON CONDITION

Assuming azimuthal symmetry, vertical magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y\left(\rho, t\right)\vec{e_y}$ at a constant orbit ρ_0, Faraday's law for induction give

$$\int_0^{2\pi} \mathcal{E}_\theta\ \rho\ d\theta = \frac{d}{dt}\int_0^{2\pi}\int_0^{\rho_0} \mathcal{B}_y\left(\rho, t\right)\rho\ d\rho\ d\theta$$

$$\implies\ 2\pi\rho_0\mathcal{E}_\theta = \frac{d\Phi_{S,\rho_0}}{dt}$$

$$\implies\ \mathcal{E}_\theta = \frac{1}{2\pi\rho_0}\frac{d\Phi_{S,\rho_0}}{dt}$$

where Φ_{S,ρ_0} is the magnetic flux in the contour enclosed in the orbit ρ_0

INDUCTION ACCELERATION

DERIVATION OF THE BETATRON CONDITION

The obtained acceleration is

$$\frac{dp_\theta}{dt} = q\mathcal{E}_\theta = \frac{q}{2\pi\rho_0}\frac{d\Phi_{S,\rho_0}}{dt}$$

$$\implies p_\theta = \frac{q}{2\pi\rho_0}\Phi_{S,\rho_0}$$

Using the magnetic rigidity $p_\theta = q\mathcal{B}_y\left(\rho_0\right)\rho_0$ (derivation here), we obtain

$$q\mathcal{B}_y\left(\rho_0\right)\rho_0 = \frac{q}{2\pi\rho_0}\Phi_{S,\rho_0}$$

$$\implies \mathcal{B}_y\left(\rho_0\right) = \frac{1}{2}\frac{\Phi_{S,\rho_0}}{\pi\rho_0^2}$$

INDUCTION ACCELERATION

DERIVATION OF THE BETATRON CONDITION

We introduce an averaged magnetic field in the betatron core $\left<\mathcal{B}_y\right>_{S,\rho_0}$

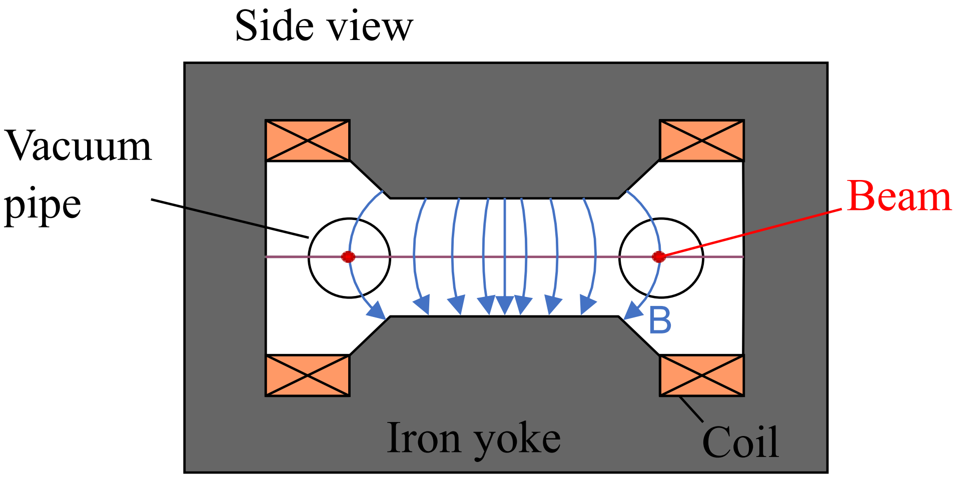
$$\Phi_{S,\rho_0} = 2\pi \int_0^{\rho_0} \mathcal{B}_y\left(\rho\right)\rho\, d\rho = \pi\rho_0^2 \left<\mathcal{B}_y\right>_{S,\rho_0}$$

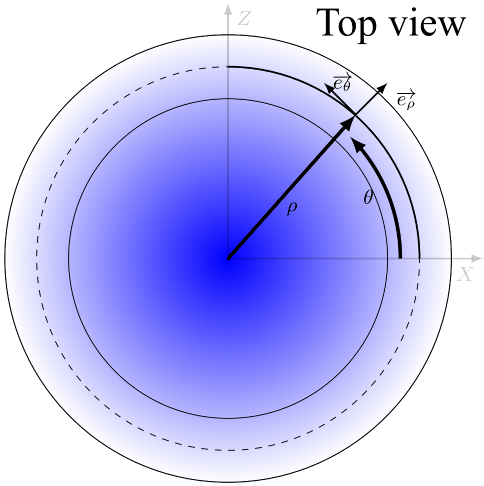
we finally get

$$\mathcal{B}_y\left(\rho_0\right) = \frac{1}{2}\frac{\Phi_{S,\rho_0}}{\pi\rho_0^2} = \frac{1}{2}\left<\mathcal{B}_y\right>_{S,\rho_0}$$

LIMITATIONS OF INDUCTION ACCELERATION

- The accelerator is covered by large magnets
  - Limited size of the accelerator
  - Saturation of the iron yoke
- The maximum energy reached is about 300 MeV with electrons (high energy lepton synchrotrons ~100s GeV!)





ELECTROMAGNETIC WAVE ACCELERATION

Combining Maxwell's equation in vacuum (no charge, no current)

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = 0 \qquad \text{Gauss' law}$$

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0 \qquad \text{Flux/Thomson's law}$$

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \qquad \text{Faraday's law}$$

$$\vec{\nabla} \times \vec{\mathcal{B}} = \mu_0 \epsilon_0 \frac{\partial \vec{\mathcal{E}}}{\partial t} \qquad \text{Ampère's law}$$

an electric field can be obtained in the form of a wave

$$\Delta \vec{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0 \quad , \left(c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \right)$$

ELECTROMAGNETIC WAVE ACCELERATION

DERIVATION OF THE ELECTRIC WAVE

$$\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}} = -\frac{\partial \overrightarrow{\mathcal{B}}}{\partial t}$$

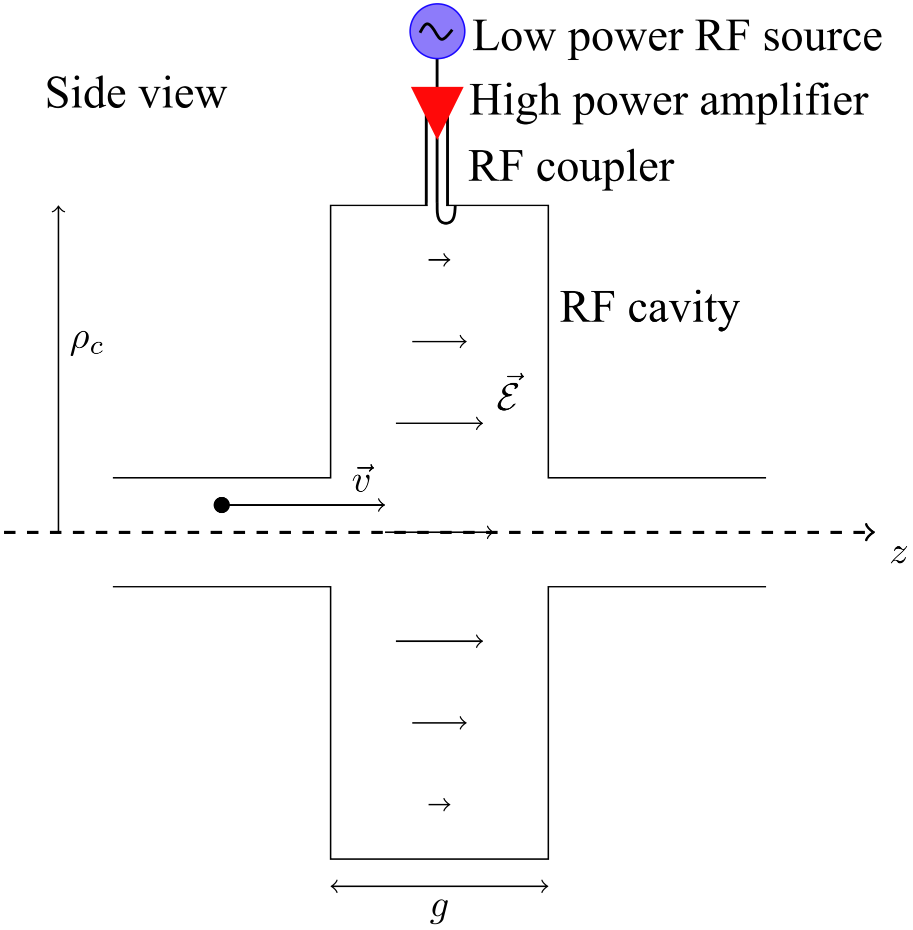
$$\implies \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{E}}\right) = -\overrightarrow{\nabla} \times \left(\frac{\partial \overrightarrow{\mathcal{B}}}{\partial t}\right)$$

$$\implies \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}}\right) - \overrightarrow{\nabla}^2 \overrightarrow{\mathcal{E}} = -\frac{\partial}{\partial t}\left(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}}\right)$$

$$\implies \Delta \overrightarrow{\mathcal{E}} - \mu_0 \epsilon_0 \frac{\partial^2 \overrightarrow{\mathcal{E}}}{\partial t^2} = 0 \quad , \left(c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\right)$$

A similar equation can be obtained for $\overrightarrow{\mathcal{B}}$, $\overrightarrow{\mathcal{E}}$ **and** $\overrightarrow{\mathcal{B}}$ **propagate together.**

RF SYSTEMS



- An electromagnetic wave can be confined in a cavity, with an opening to let the beam pass through the oscillating electric field with

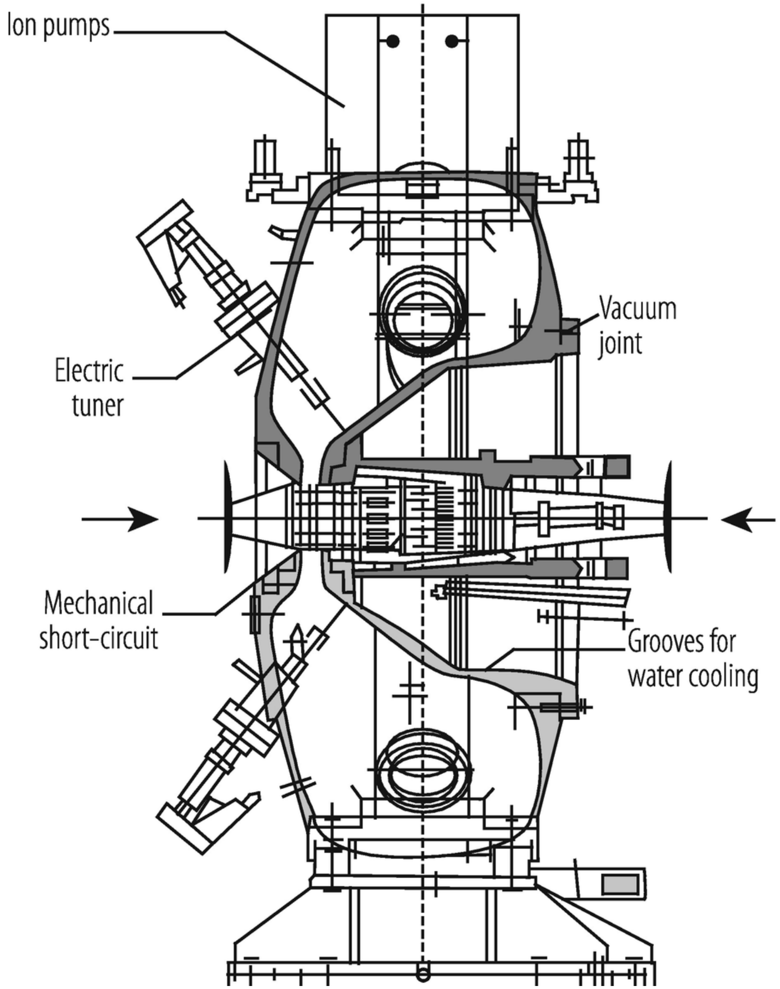
$$\vec{\mathcal{E}} = \mathcal{E}_z(\rho, z)\cos(\omega_r t)\,\vec{e_z}$$

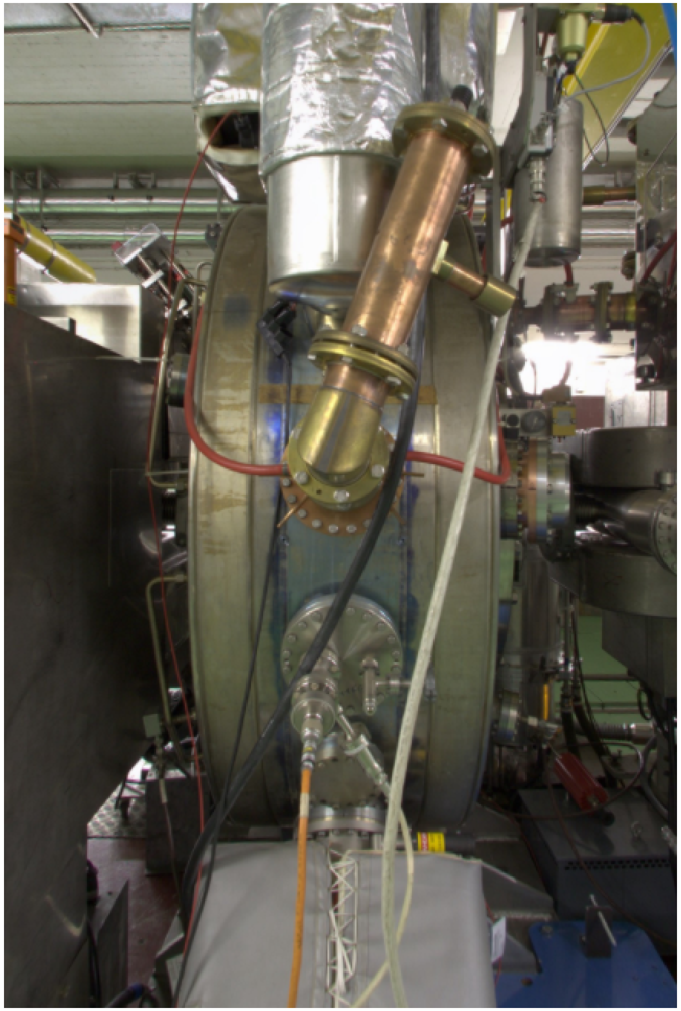
where $\omega_r = 2\pi f_r$ is the (angular) frequency of the field and depends on the geometry of the cavity.

- A low power RF signal is amplified and coupled to the cavity. The amplitude and phase of the wave with respect to the beam can be controlled.

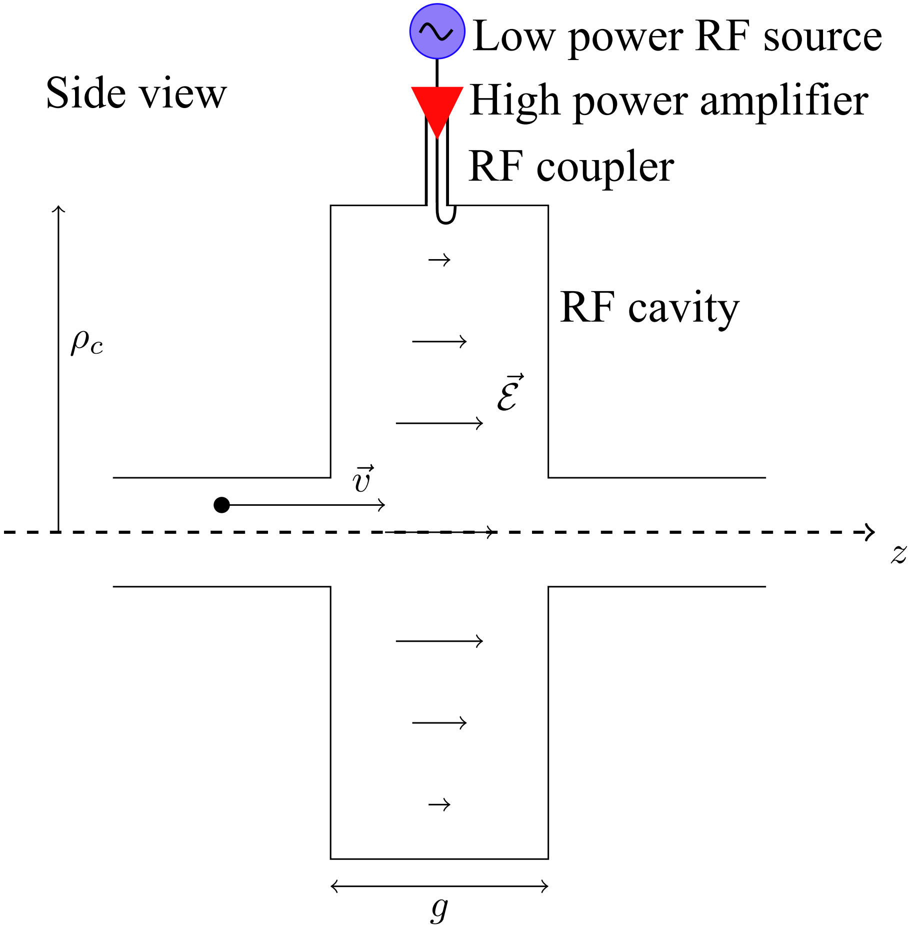
RF SYSTEMS

EXAMPLE OF REAL RF CAVITY IN THE PS (VIEW)





RF ACCELERATION



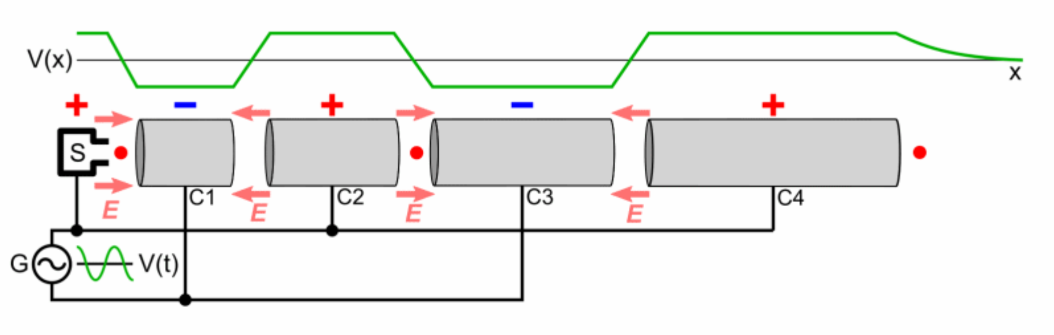
- The increment in energy of a particle passing through an RF cavity gap is

$$\delta E_{\text{rf}} = \int q\mathcal{E}_z\left(\rho, z, t\right) dz$$
$$= qV_{\text{rf}}\left(\rho, \tau\right)$$

where V_{rf} is the total accelerating potential of a particle arriving at a time τ in the cavity (we will derive a relevant expression of V_{rf} during the next lesson!).

- Unlike electrostatic fields, cavities can be installed consecutively to accelerate the particles.

LINEAR ACCELERATORS (LINACS)



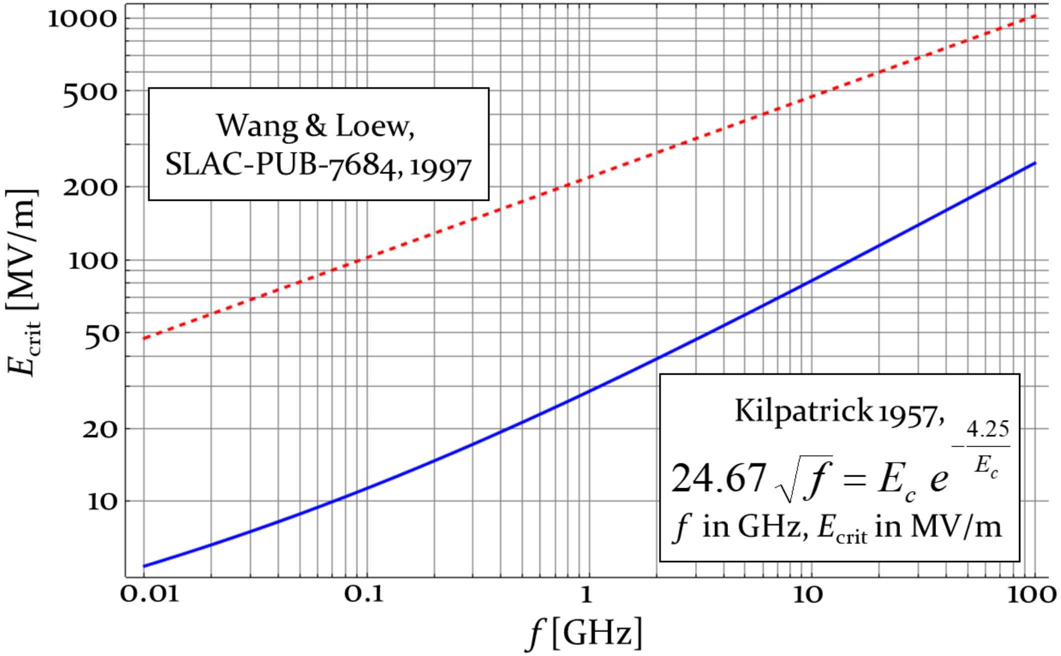
- The basic principle of linear accelerators is a single pass in many RF systems to accumulate energy.

- The distance between two accelerating gaps depends on the particle velocity (synchronism condition for Linacs).

- The maximum energy reach scales with the length of the linac and the RF accelerating gradient.

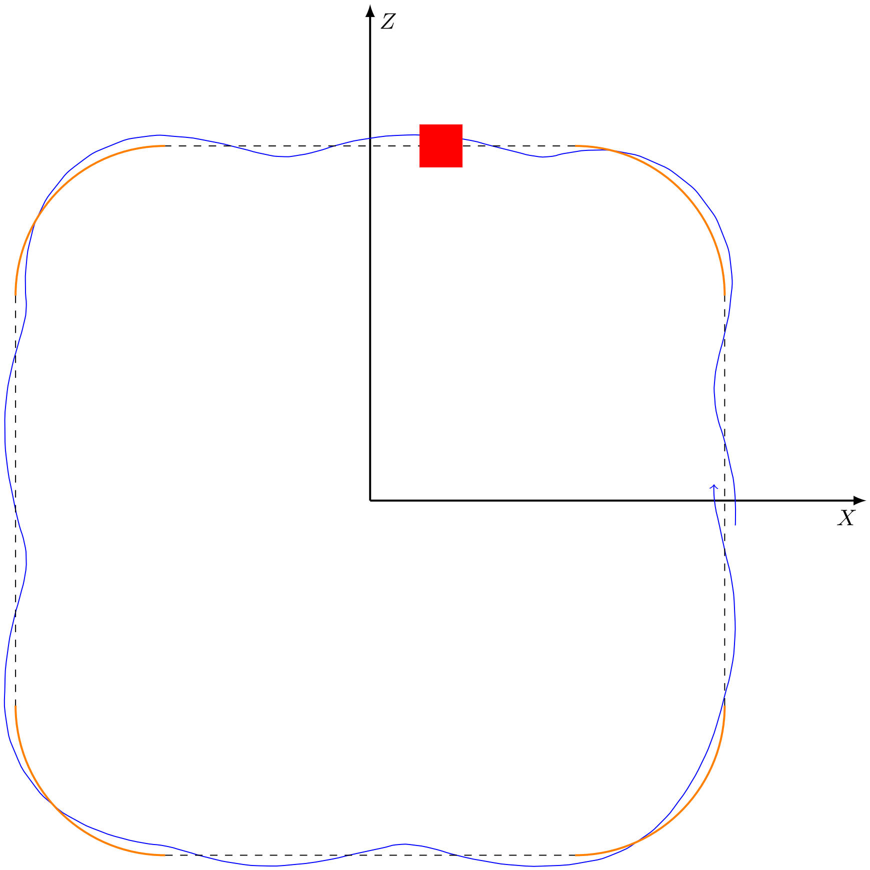
- Dedicated **JUAS Lecture on Linacs** and walk along LINAC4.

BREAKDOWN AND RF



- The maximum accelerating gradient in RF cavities is limited by breakdown.
- The observed frequency dependence was formulated empirically by Kilpatrick.
- Breakdown is dependent on the cavity surface quality and conditioning. Present cavities go beyond the Kilpatrick criterion (ratio expressed in "Kilpatrick" unit).
- Typical range for RF cavities $\sim \mathcal{O}\left(1\text{-}100\ \mathrm{MV/m}\right)$

CIRCULAR ACCELERATORS



For circular accelerators the principle is to steer the beam back to the RF cavity and passing multiple time. We need to introduce the concept of **magnetic rigidity**.

MAGNETIC RIGIDITY

The applied force in bending magnets to shape a circular accelerator is

$$\vec{F}_{\mathcal{B}} = q\left(\vec{v} \times \vec{\mathcal{B}}\right)$$

which gives the vertical magnetic field required to keep particles with a given momentum on a given orbit

$$\mathcal{B}_y \rho = \frac{p}{q}$$

This relationship is called the magnetic rigidity or more trivially the "$\mathcal{B}\rho$".

MAGNETIC RIGIDITY

DERIVATION

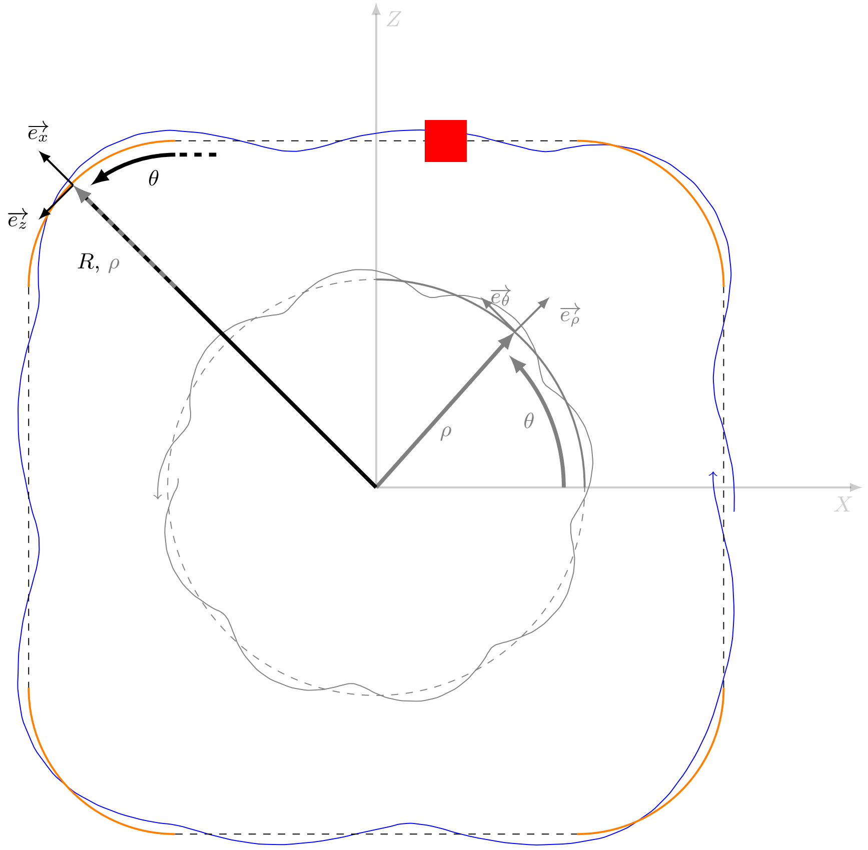
The force from a magnetic field is always orthogonal to the particle direction, in cylindrical coordinates and with a constant $v = v_\theta$ (implying $p = p_\theta, \dot{m} = 0$), and a magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y \vec{e_y}$.

Demonstrate that

$$\mathcal{B}_y \rho = \frac{p_\theta}{q} = \frac{p}{q}$$

MAGNETIC RIGIDITY

COORDINATE SYSTEM REMINDER



- Reminder, in cylindrical coordinates

$$v_\rho = \dot{\rho}$$
$$v_\theta = \rho\dot{\theta} = \rho\omega$$
$$v_y = \dot{y}$$

and if $\dot{m} = 0$

$$\dot{p}_\rho = m\left(\ddot{\rho} - \rho\dot{\theta}^2\right)$$
$$\dot{p}_\theta = m\left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)$$
$$\dot{p}_y = m\,\ddot{y}$$

(overdot is derivative with time d/dt)

MAGNETIC RIGIDITY

DERIVATION

The force from a magnetic field is always orthogonal to the particle direction, in cylindrical coordinates and with a constant $v = v_\theta$ (implying $p = p_\theta, \dot{m} = 0$), and a magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y \vec{e_y}$.

Demonstrate that

$$\mathcal{B}_y \rho = \frac{p_\theta}{q} = \frac{p}{q}$$

MAGNETIC RIGIDITY

DERIVATION

The force from a magnetic field is always orthogonal to the particle direction, in cylindrical coordinates and with a constant $v = v_\theta$ (implying $p = p_\theta, \dot{m} = 0$), and a magnetic field $\overrightarrow{\mathcal{B}} = -\mathcal{B}_y \vec{e_y}$, we get

$$\frac{d\vec{p}}{dt} = \vec{F_\mathcal{B}} = q\left(\vec{v} \times \vec{\mathcal{B}}\right)$$

$$\text{in } \vec{e_\rho} \implies m\left(\ddot{\rho} - \rho\dot{\theta}^2\right) = -qv_\theta\mathcal{B}_y \quad , (\dot{\rho} = 0)$$

$$\implies m\frac{v_\theta^2}{\rho} = qv_\theta\mathcal{B}_y \quad , \left(v_\theta = \rho\dot{\theta}\right)$$

$$\implies p_\theta = q\mathcal{B}_y\rho$$

$$\implies \mathcal{B}_y\rho = \frac{p_\theta}{q} = \frac{p}{q}$$

STRATEGY FOR CIRCULAR ACCELERATORS

Two possibilities to reach high energies with

$$\mathcal{B}_y \rho = \frac{p}{q}$$

- Increase $\mathcal{B}_y$ at fixed $\rho \rightarrow$ Synchrotron
- Increase ρ at fixed $\mathcal{B}_y \rightarrow$ **Cyclotron (dedicated JUAS Lecture)**

- **In the next lessons, we will focus on the synchrotron design.**
- The maximum energy of a circular accelerator is in principle limited by the maximum $\mathcal{B}_y$ in the bending magnets or the radial size of the accelerator (e.g. FCC 100km!). The typical range for bending magnetic field is $\sim \mathcal{O}\left(1\text{-}10\ \mathrm{T}\right)$.
- In presence of **synchrotron radiation** (for lepton machines), the maximum energy is limited by RF power.

VERY HIGH GRADIENT ACCELERATION

- How do we go beyond the limits fixed by present accelerator technologies? Can we have more compact accelerators? Can we reach GV/m accelerating gradient using fields provided by lasers and plasmas?

→ **Follow the JUAS Seminar on Novel High Gradient Particle Accelerators**

→ In the context of this lecture, we will concentrate on conventional **RF acceleration**.

MODULE 2: RELATIVISTIC KINEMATICS

→ **Recap on relativistic parameters**

→ **Useful differential relationships**

DEFINITION OF PARAMETERS

Reminder: we now assume that the momentum of the particle is $p \approx p_z$

Particle energy and momentum

$$E = E_{\text{kin}} + E_0 = \sqrt{P^2 + E_0^2}$$

where E total energy, $E_0 = m_0 c^2$ rest energy (particle rest mass m_0), $p = P/c$ is the momentum

Relativistic parameters

$$\beta = \frac{v}{c} = \frac{P}{E}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{E_0}$$

where β relativistic velocity, γ Lorentz factor and $p = mv = \beta\gamma m_0 c$

UNITS

- The energies E, E_{kin}, E_0 and P can be expressed in eV
- The momentum p can be expressed in $\mathrm{eV/c}$
- The mass m can be expressed in $\mathrm{eV/c^2}$
- β and γ are unitless

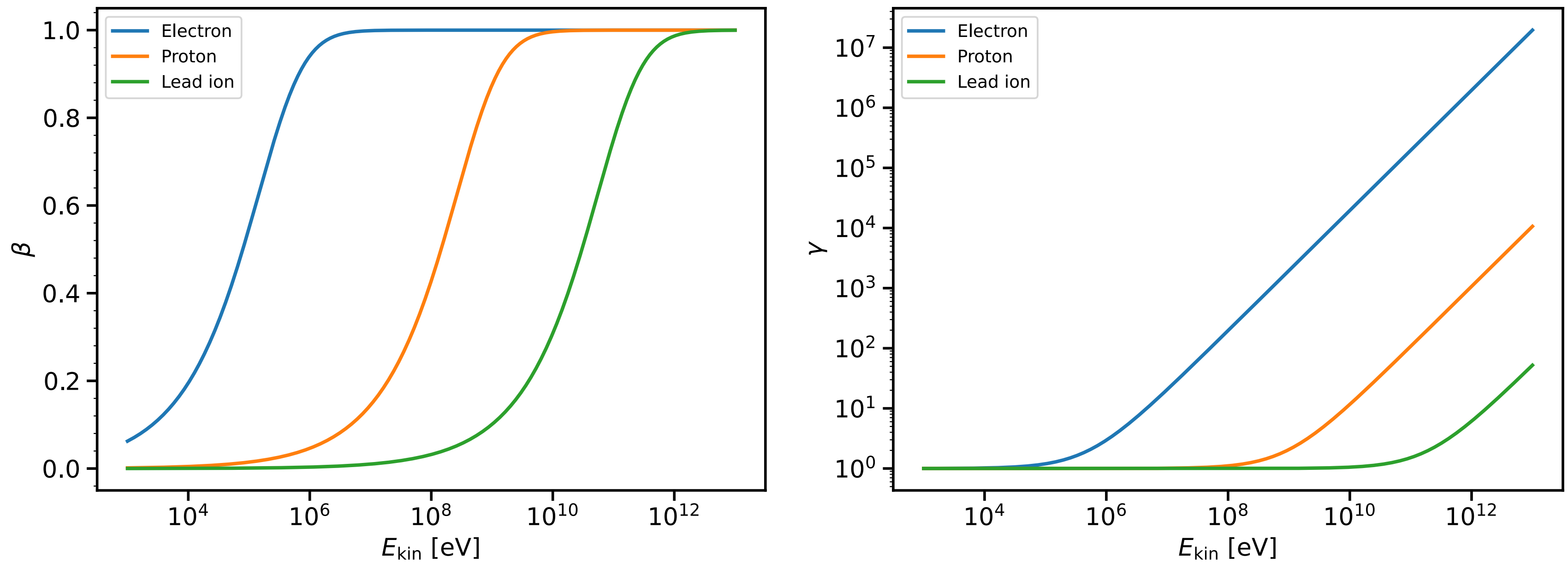
Practical magnetic rigidity formula

We will demonstrate that

$$p\,[\mathrm{GeV/c}] \approx 0.3\ Z\ \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

where Z is the number of elementary charges e ($Z = +1$ for protons and $Z = -1$ for electrons).

RELATIVISTIC PARAMETERS AND PARTICLE REST MASS



- Electrons can be considered with $v \approx c$ at moderate kinetic energy, but not heavier particles.
- The evolution of the particle velocity during acceleration has an important influence on the design of an accelerator.

USEFUL RELATIONSHIPS

Practical relationships that will be used in further derivations.

$$\beta^2 + \frac{1}{\gamma^2} = 1$$

Differential forms

$$\frac{dE}{dp} = \beta c = v$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

EXERCISES

- **Show that**

$$p\,[\mathrm{GeV/c}] \approx 0.3\; Z\; \mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

- **Compute the relativistic parameters for the following CERN machines**

Machine	E_0 [MeV]	E_{kin} [GeV]	E [GeV]	γ	β	p [GeV/c]	$\mathcal{B}_y\rho$ [Tm]
PSB inj (p+)		0.160					
PSB ext (p+)		2					
SPS ($^{208}\mathrm{Pb}^{82+}$)							86.4
LHC (p+)			7000				
LEP (e+/e-)			100				

$$m_p = 1.6726\times10^{-27}\ \mathrm{kg}, m_e = 9.1094\times10^{-31}\ \mathrm{kg}, u = 1.661\times10^{-27}\ \mathrm{kg}$$

- **Derive the differential relationships from the previous slide**

EXERCISES

BENDING RADIUS PRACTICAL EQUATION

The magnitude of a variable (unitless) is noted in $||$

$$p\,[\mathrm{Ns}] = e\,[\mathrm{C}]\;\; Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

$$p\,[\mathrm{Ns}]\,c\,[\mathrm{m/s}] = c\,[\mathrm{m/s}]\,e\,[\mathrm{C}]\;\; Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

$$P\,[\mathrm{Nm}] = c\,[\mathrm{m/s}]\,e\,[\mathrm{C}]\;\; Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

$$P\,[\mathrm{Nm}]\,/\,|e| = 1\,[\mathrm{C}]\,c\,[\mathrm{m/s}]\;\; Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

$$P\,[\mathrm{eV}]\,/\,(1\,[\mathrm{m/s}]) = |c|\,1\,[\mathrm{C}]\;\; Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

$$p\,[\mathrm{GeV/c}] \approx 0.3\;Z\,\mathcal{B}_y\,[\mathrm{T}]\,\rho\,[\mathrm{m}]$$

EXERCISES

MACHINE PARAMETERS

Machine	E_0 [MeV]	E_{kin} [GeV]	E [GeV]	γ	β	p [GeV/c]	$\mathcal{B}_y \rho$ [Tm]
PSB inj (p+)	938	0.160	1.098	1.17	0.52	0.57	1.90
PSB ext (p+)	938	2	2.938	3.13	0.95	2.78	9.30
SPS ($^{208}\text{Pb}^{82+}$)	193751	1940.50	2134.25	11.0	0.996	2125.44	86.4
LHC (p+)	938	6999	7000	7460	0.999..	6999.99..	23333
LEP (e+/e-)	0.511	99.99	100	195695	0.999..	99.99..	333.33

EXERCISES

DEMONSTRATION OF DIFFERENTIAL RELATIONSHIPS

$$E^2 = P^2 + E_0^2$$
$$\implies d\left(E^2\right) = d\left(P^2\right) + d\left(E_0^2\right)$$
$$\implies 2EdE = 2PdP = 2pdpc^2$$
$$\implies \frac{dE}{dp} = \frac{pc^2}{E}$$
$$\implies \frac{dE}{dp} = \beta c = v$$

EXERCISES

DEMONSTRATION OF DIFFERENTIAL RELATIONSHIPS

$$EdE = pdpc^2$$

$$\implies \frac{dE}{E} = \frac{pc^2}{E^2}dp$$

$$\implies \frac{dE}{E} = \left(\frac{pc}{E}\right)^2 \frac{dp}{p}$$

$$\implies \frac{dE}{E} = \beta^2 \frac{dp}{p}$$

$$\implies \frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

EXERCISES

DEMONSTRATION OF DIFFERENTIAL RELATIONSHIPS

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

$$\implies d\left(\beta^2\right) = d\left(1 - \frac{1}{\gamma^2}\right)$$

$$\implies 2\beta d\beta = 2\gamma^{-3} d\gamma$$

$$\implies \frac{d\beta}{\beta} = \left(\frac{1}{\beta\gamma}\right)^2 \frac{d\gamma}{\gamma}$$

$$\implies \frac{d\beta}{\beta} = \frac{1}{\gamma^2}\frac{dp}{p}$$

TAKE AWAY MESSAGE

- Acceleration in an RF gap:

$$\delta E = \int q\mathcal{E}_z\left(\rho, z, t\right) dz = qV_{\mathrm{rf}}\left(\rho, \tau\right)$$

- Magnetic rigidity:

$$\mathcal{B}_y \rho = \frac{p}{q} \quad \rightarrow \quad p\left[\mathrm{GeV/c}\right] \approx 0.3\ Z\ \mathcal{B}_y\left[\mathrm{T}\right]\rho\left[\mathrm{m}\right]$$

- Relativistic relationships ($P = p\ c$):

$$E = E_{\mathrm{kin}} + E_0 = \sqrt{P^2 + E_0^2}, \quad \beta = \frac{v}{c} = \frac{P}{E}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{E_0}$$

TAKE AWAY MESSAGE

- Relativistic relationships:

$$\beta^2 + \frac{1}{\gamma^2} = 1$$

- Relativistic differential relationships:

$$\frac{dE}{dp} = \beta c = v$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

LESSON 2: SYNCHROTRON DESIGN

MODULE 3: ACCELERATION IN A SYNCHROTRON

→ **Fundamental mode in a pillbox cavity**

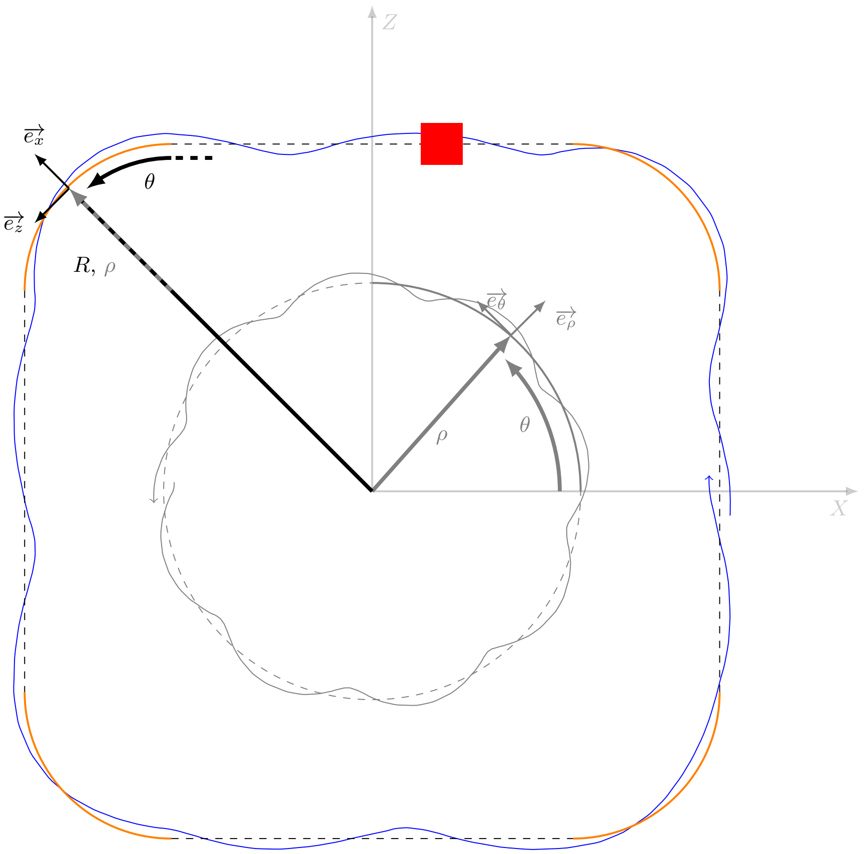
→ **Energy gain**

→ **Transit time factor**

→ **Other sources of energy gain/loss**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ axis...

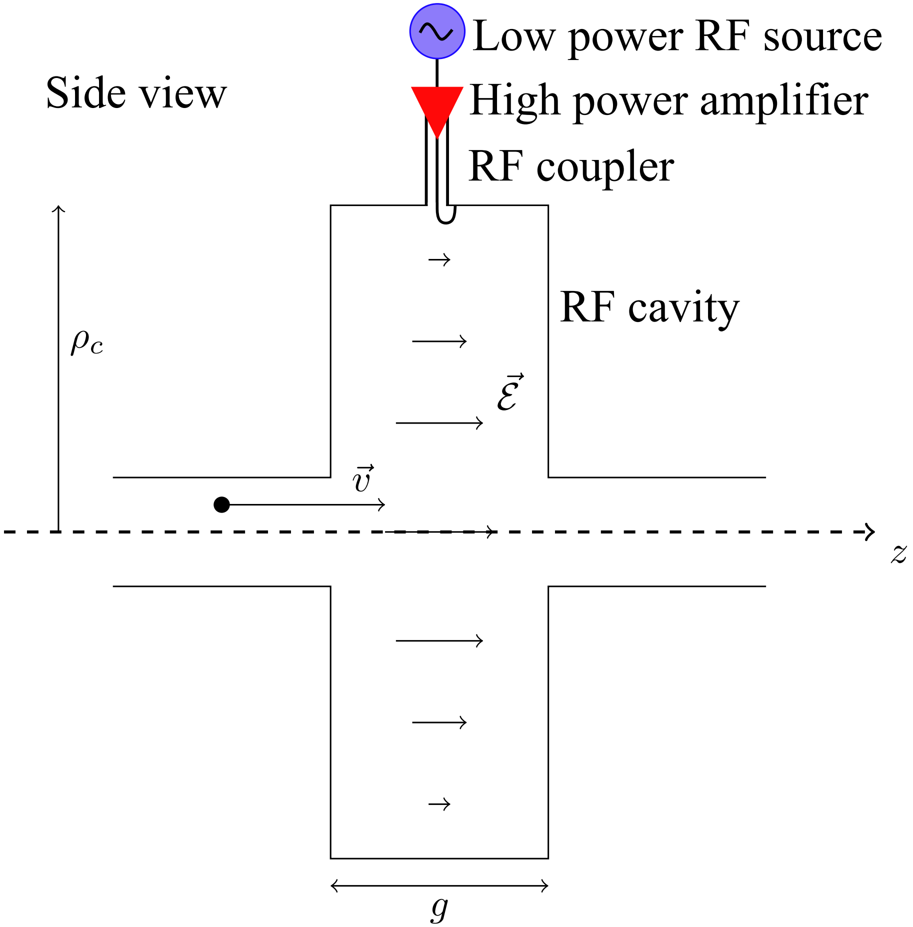


- We start by describing the acceleration of a particle in the RF cavity.
- In the previous lesson we found the expression of the energy gain in the cavity for a single pass

$$\delta E_{\mathrm{rf}} = \int q\mathcal{E}_z\left(\rho, z, t\right) dz$$
$$= qV_{\mathrm{rf}}\left(\rho, \tau\right)$$

- We will find a common expression of $\mathcal{E}_z$ for a simple form of RF cavity.

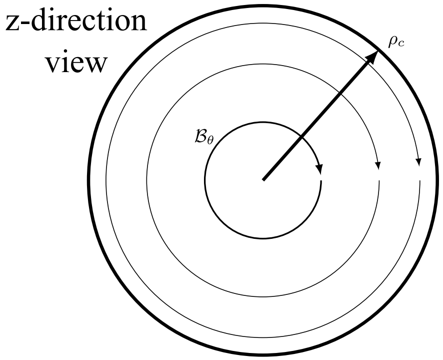
FUNDAMENTAL MODE OF A PILLBOX CAVITY



- A convenient model is the so-called "pillbox cavity", where the fields of the fundamental mode are

$$\mathcal{E}_z(\rho, t) = \mathcal{E}_0 J_0\left(\chi_0 \frac{\rho}{\rho_c}\right) \cos(\omega_r t)$$

$$\mathcal{B}_\theta(\rho, t) = -\frac{\mathcal{E}_0}{c} J_1\left(\chi_0 \frac{\rho}{\rho_c}\right) \sin(\omega_r t)$$

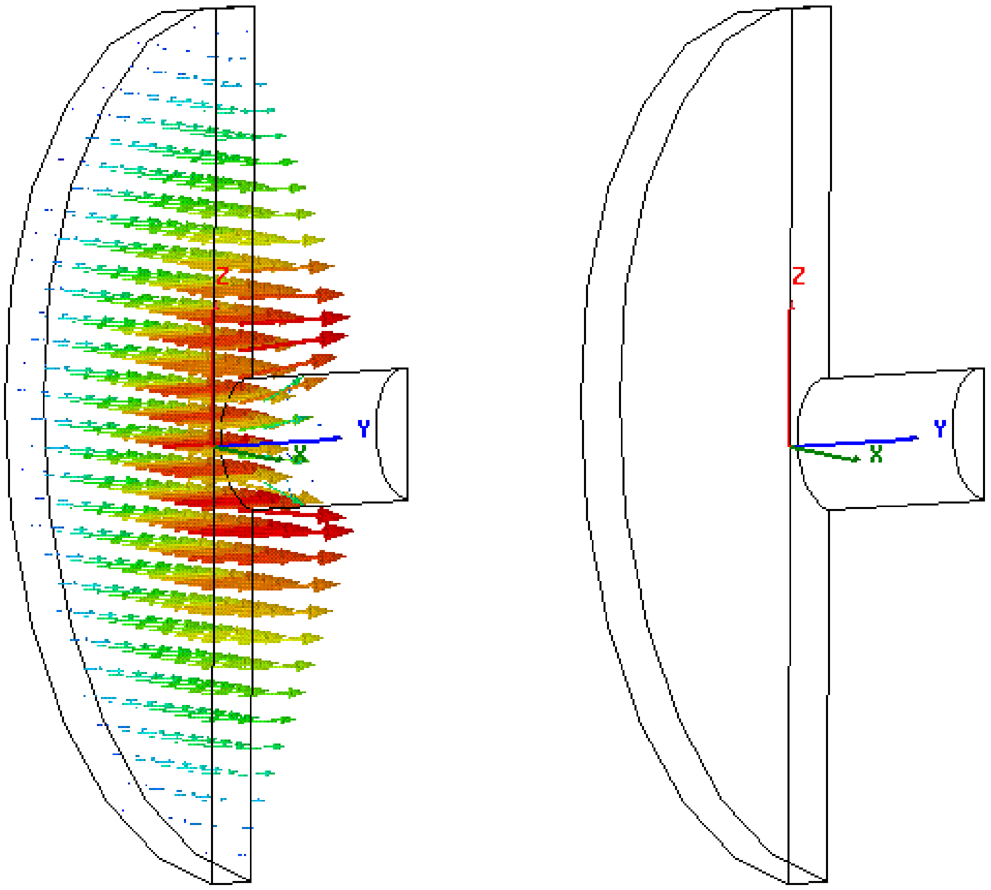


J_n Bessel function

$\chi_0 \approx 2.405$

$\omega_r = \chi_0 c / \rho_c$

FUNDAMENTAL MODE OF A PILLBOX CAVITY



Animation: E. Jensen

- The maximum electric field $\mathcal{E}_0$ achievable in a cavity depends on many parameters including the
  - Cavity material
  - Power amplification
  - Coupling in transmission lines and reflections

- The frequency of a pillbox cavity depends on the radial size, not on the length!

- Dedicated courses in the JUAS Course 2: Introduction and RF Engineering lectures.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

From the wave equation

$$\Delta \overrightarrow{\mathcal{E}} - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{\mathcal{E}}}{\partial t^2} = 0$$

Two conditions on the fields on the boundaries of the cavity (conductor material)

- The electric field is orthogonal to the surface.
- The magnetic field is parallel to the surface.

We neglect the aperture due to the beam pipe and the power coupler.

A large number of modes of oscillation can exist in the cavity, we are interested only in the fundamental mode for which $\overrightarrow{\mathcal{E}} = \mathcal{E}_z \vec{e}_z$ and $\overrightarrow{\mathcal{B}} = \mathcal{B}_\theta \vec{e}_\theta$.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

We will assume a solution of the form $\mathcal{E}_z = \mathcal{E}_0\left(\rho\right)\cos\left(\omega_r t\right)$

Reminder: In cylindrical coordinates

$$\Delta\overrightarrow{\mathcal{E}} = \frac{\partial^2\mathcal{E}_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial\mathcal{E}_z}{\partial\rho} + \frac{\partial^2\mathcal{E}_z}{\partial\rho^2}$$

Reminder: The Bessel differential equation

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + \left(\frac{x}{x_0} - n\right)^2 y = 0$$

$$\rightarrow \quad y = y_0 J_n\left(\frac{x}{x_0}\right)$$

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The wave equation in cylindrical coordinates becomes

$$\frac{\partial^2 \mathcal{E}_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial \mathcal{E}_z}{\partial \rho} + \frac{\partial^2 \mathcal{E}_z}{\partial \rho^2} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}_z}{\partial t^2} = 0$$

Assuming a solution of the form $\mathcal{E}_z = \mathcal{E}_{z,\rho}\left(\rho\right)\cos\left(\omega_r t\right)$ lead to

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho}\cos\left(\omega_r t\right) + \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2}\cos\left(\omega_r t\right) - \frac{1}{c^2}\frac{\partial^2 \cos\left(\omega_r t\right)}{\partial t^2}\mathcal{E}_{z,\rho} &= 0 \\
\implies \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2}\cos\left(\omega_r t\right) + \frac{1}{\rho}\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho}\cos\left(\omega_r t\right) + \left(\frac{\omega_r}{c}\right)^2 \mathcal{E}_{z,\rho}\cos\left(\omega_r t\right) &= 0 \\
\implies \rho^2\frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2} + \rho\frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho} + \left(\frac{\rho\omega_r}{c}\right)^2 \mathcal{E}_{z,\rho} = 0
\end{aligned}$$

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The differential equation

$$\rho^2 \frac{\partial^2 \mathcal{E}_{z,\rho}}{\partial \rho^2} + \rho \frac{\partial \mathcal{E}_{z,\rho}}{\partial \rho} + \left(\frac{\rho \omega_r}{c}\right)^2 \mathcal{E}_{z,\rho} = 0$$

is the Bessel differential equation which has a solution for $\mathcal{E}_{z,\rho}$

$$\mathcal{E}_{z,\rho} = \mathcal{E}_0 J_0 \left(\frac{\rho \omega_r}{c}\right)$$

where $\mathcal{E}_0$ is the amplitude of the field at $\rho = 0$.

FUNDAMENTAL MODE OF A PILLBOX CAVITY

DERIVATION

The boundary condition for electrical fields implies that $\mathcal{E}_z\left(\rho=\rho_c\right)=0$. We reformulate the electric field

$$\mathcal{E}_{z,\rho}=\mathcal{E}_0 J_0\left(\chi_0\frac{\rho}{\rho_c}\right)$$

where $\chi_0=\rho_c\omega_r/c\approx 2.405$ is the first zero of the Bessel function J_0.

Finally

$$\mathcal{E}_z\left(\rho,t\right)=\mathcal{E}_0 J_0\left(\chi_0\frac{\rho}{\rho_c}\right)\cos\left(\omega_r t\right)$$

with $\omega_r=\chi_0 c/\rho_c\approx 2.405\ c/\rho_c$.

ENERGY GAIN IN AN RF CAVITY

We express the energy gain of a particle passing through a cavity

$$\delta E_{\mathrm{rf}} = \int q\mathcal{E}_z\left(\rho, z, t\right) dz = q\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right)\cos\left(\omega_r t\right) dz$$

For a particle passing through the center of the cavity ($\rho = 0$), the energy gain becomes

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf},0}T_t\cos\left(\omega_r\tau\right) \quad \text{Linac convention}$$
$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf},0}T_t\sin\left(\omega_r\tau\right) \quad \text{Synchrotron convention}$$

The maximum potential $V_{\mathrm{rf},0}$ (denominator below) and the transit time factor are

$$T_t = \frac{\int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho, z\right)\cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho, z\right) dz}$$

ENERGY GAIN IN AN RF CAVITY

DERIVATION

Starting from

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left(\omega_r t\right) dz$$

The longitudinal position of the particle with respect to the cavity is

$$z\left(t\right) = \int_{\tau}^{t} \beta\left(t\right) c \, dt \approx \beta c \left(t - \tau\right)$$

Assumption: the change in velocity of the particle is neglected here. This is not valid for high gradient cavities with non-relativistic particles!

Derive the energy gain and the expression of the transit time factor.

ENERGY GAIN IN AN RF CAVITY

DERIVATION

Starting from

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left[\omega_r \left(\frac{z}{\beta c}\right) - \omega_r \tau\right] dz$$

Using the trigonometric relationship

$$\delta E_{\mathrm{rf}} = q \int_{-g/2}^{g/2} \mathcal{E}_0 \left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) \cos\left(\omega_r \tau\right) dz + \sin\left(\frac{\omega_r z}{\beta c}\right) \sin\left(\omega_r \tau\right) dz$$

The $\sin$ function is odd and cancels in the integral.

ENERGY GAIN IN AN RF CAVITY

DERIVATION

We get

$$\delta E_{\mathrm{rf}} = q\cos\left(\omega_r \tau\right)\int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho, z\right)\cos\left(\frac{\omega_r z}{\beta c}\right)dz$$

We define the maximum possible accelerating potential (no variation with time during particle passage) as

$$V_{\mathrm{rf},0} = \int_{-g/2}^{g/2}\mathcal{E}_0\left(\rho, z\right)dz$$

ENERGY GAIN IN AN RF CAVITY

DERIVATION

We define the transit time factor as the ratio between the accelerating potential including the time variation of the field and the maximum potential

$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{V_{\mathrm{rf},0}}$$

The energy gain in the gap finally becomes

$$\delta E_{\mathrm{rf}}\left(\tau\right) = q V_{\mathrm{rf},0} T_t \cos\left(\omega_r \tau\right) = q V_{\mathrm{rf}} \cos\left(\omega_r \tau\right)$$

The $\cos$ which can be interchanged with $\sin$ depending on the convention used (linac vs. synchrotrons).

TRANSIT TIME FACTOR

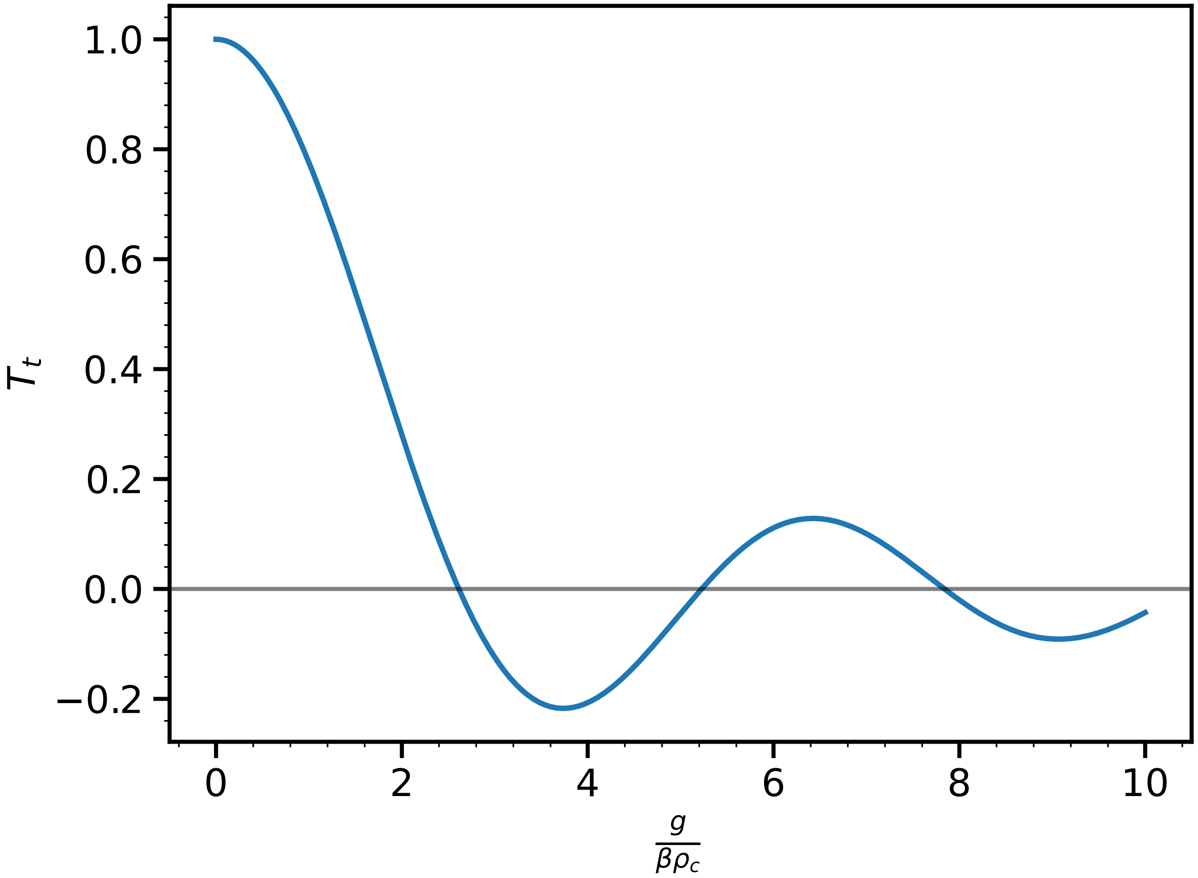
$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

The transit time factor is the ratio between the effective accelerating potential including the time variation of the field (top term) and the maximum potential if a particle would pass instantaneously in the cavity (bottom term, $V_{\mathrm{rf},0}$).

The transit time factor is $T_t \leqslant 1$ and depends on principle on the particle transverse position. For a pillbox cavity, the transit time factor becomes

$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

TRANSIT TIME FACTOR



$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

- The electric field oscillates while the particle goes through the RF cavity
- If the gap is too long, the acceleration potential is effectively reduced.

- A compromise in the design of a cavity is needed to maximize the accelerating potential.

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Derive the transit time factor for a pillbox cavity using the expression of the electric field

$$\mathcal{E}_0 J_0\left(\rho\right)\cos\left(\frac{\omega_r z}{\beta c}\right)$$

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Including the expression of the pillbox cavity field in the transit time factor

$$T_t = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho, z\right) dz}$$

$$\implies T_t = \frac{\mathcal{E}_0 J_0\left(\rho\right) \int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\mathcal{E}_0 J_0\left(\rho\right) \int_{-g/2}^{g/2} dz}$$

$$\implies T_t = \frac{1}{g} \int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz$$

TRANSIT TIME FACTOR FOR A PILLBOX CAVITY

DERIVATION

Including the expression of the pillbox cavity field in the transit time factor

$$\implies T_t = \frac{1}{g}\int_{-g/2}^{g/2} \cos\left(\frac{\omega_r z}{\beta c}\right) dz$$

$$\implies T_t = \frac{\beta c}{\omega_r g}\left[\sin\left(\frac{\omega_r g}{2\beta c}\right) - \sin\left(\frac{-\omega_r g}{2\beta c}\right)\right]$$

$$\implies T_t = \frac{\sin\left(\frac{\omega_r g}{2\beta c}\right)}{\left(\frac{\omega_r g}{2\beta c}\right)} = \frac{\sin\left(\frac{\chi_0 g}{2\beta \rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta \rho_c}\right)}$$

ENERGY GAIN IN AN RF CAVITY

SYNCHROTRON CONVENTION

For the rest of the course we will use

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf}}\sin\left(\omega_r\tau\right) \quad \rightarrow \quad \delta E_{\mathrm{rf}}\left(\phi\right) = qV_{\mathrm{rf}}\sin\left(\phi\right)$$

where the transit time factor is included in the definition of V_{rf} (this parameter is often noted $\hat{V}_{\mathrm{rf}}$ in the litterature) and ϕ is the phase of arrival in the cavity.

Assumption: The transit time factor depends on the particle radial position and β. These dependencies will be neglected in the coming derivations.

INDUCTION FORCE IN SYNCHROTRONS

"BETATRONIC" ACCELERATION

During acceleration in a synchrotron the magnetic field is ramped to keep the beam on a constant orbit with

$$\dot{p} = q\dot{\mathcal{B}}_y\rho$$

With the same principle as in the betatron, an azimuthal electric field is induced. This leads to the energy gain (assuming ρ constant)

$$\delta E_\mathrm{b}\left(\rho\right) = q\oint_C \overrightarrow{\mathcal{E}}\cdot\overrightarrow{dz} = q\int_0^{2\pi}\int_0^{\rho}\frac{\partial\mathcal{B}_y\left(\rho',\theta,t\right)}{\partial t}\rho'\,d\rho'\,d\theta$$

Assumption: this force is usually negligible in large synchrotrons, although it may not be overlooked to derive precisely longitudinal equations of motion.

SYNCHROTRON RADIATION

ENERGY LOSS IN BENDING MAGNETS

The power of the light emitted by a particle in a curved trajectory is

$$P_{\mathrm{sr}} = \frac{q^2 c}{6\pi\epsilon_0} \frac{(\beta\gamma)^4}{\rho^2}$$

The energy loss over a turn, by multiplying by the time spent in the bending magnet T_b

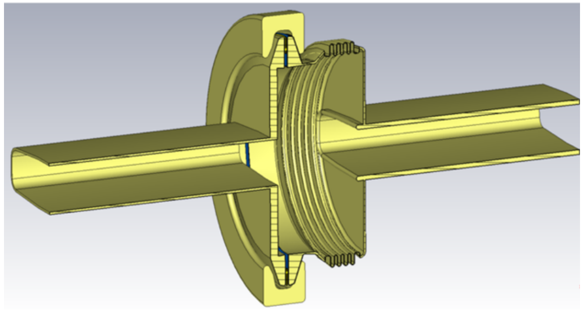
$$\delta E_{\mathrm{sr}}(E,\rho) = \frac{q^2}{3\epsilon_0} \frac{\beta^3 E^4}{\rho\, E_0^4} \quad , \left(T_b = \frac{2\pi\rho}{\beta c} \right)$$

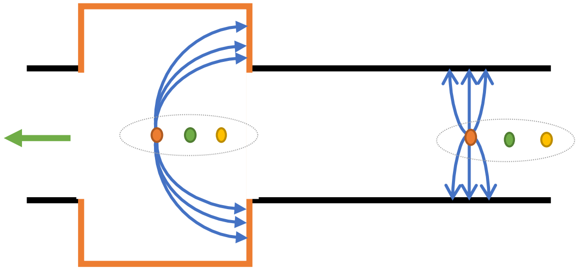
Note the important dependence on E_0^4. Synchrotron radiation is usually neglected for hadron synchrotrons and is predominant for lepton machines.

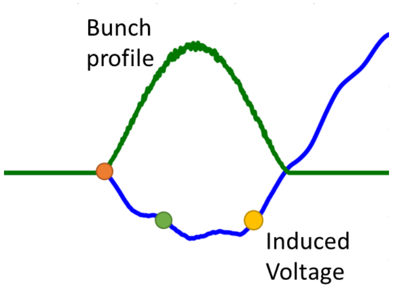
→ Dedicated JUAS course on Synchrotron radiation

SELF INDUCED FIELDS ALONG THE RING

A real accelerator is composed of many equipment, which can lead for example to discontinuities in the beam pipe aperture.







A single particle passing through a cavity-like gap will induce a wakefield $\mathcal{W}(\tau)$. A bunch with a longitudinal charge density $\lambda(\tau)$ (number of particles N_b) will induce a voltage $V_{\text{ind}}(\tau)$, as a convolution product of all the particles single wakes

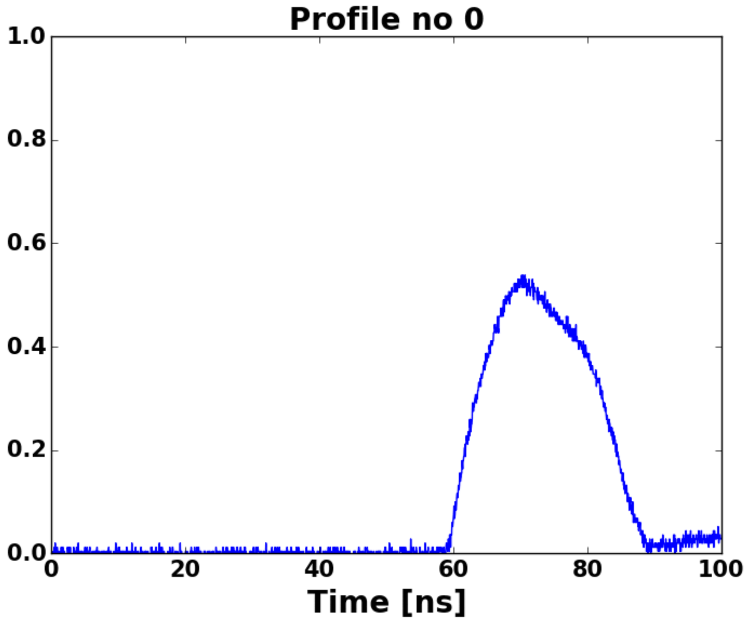
$$\delta E_{\text{ind}}(\tau) = qV_{\text{ind}}(\tau) = -qN_b\left(\lambda * \mathcal{W}\right)$$

$\longrightarrow$ Dedicated JUAS course on Collective effects

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity.



- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?

FUNDAMENTAL MODE OF A PILLBOX CAVITY

EXERCISES

- Compute the expected radius of the 80 MHz cavity shown in lesson 1, assuming it's a pillbox cavity (Slide 11)
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\, 80 \cdot 10^6\right) = 1.4\ \mathrm{m}$
- The animation is a measured bunch profile modulation by a 1.4 GHz wakefield, what is the size of the device responsible for wakefields?
  - $\rho_c = 2.405 \cdot 3 \cdot 10^8 / \left(2\pi\, 1.4 \cdot 10^9\right) = 8.2\ \mathrm{cm}$

MODULE 4: THE SYNCHRONOUS PARTICLE

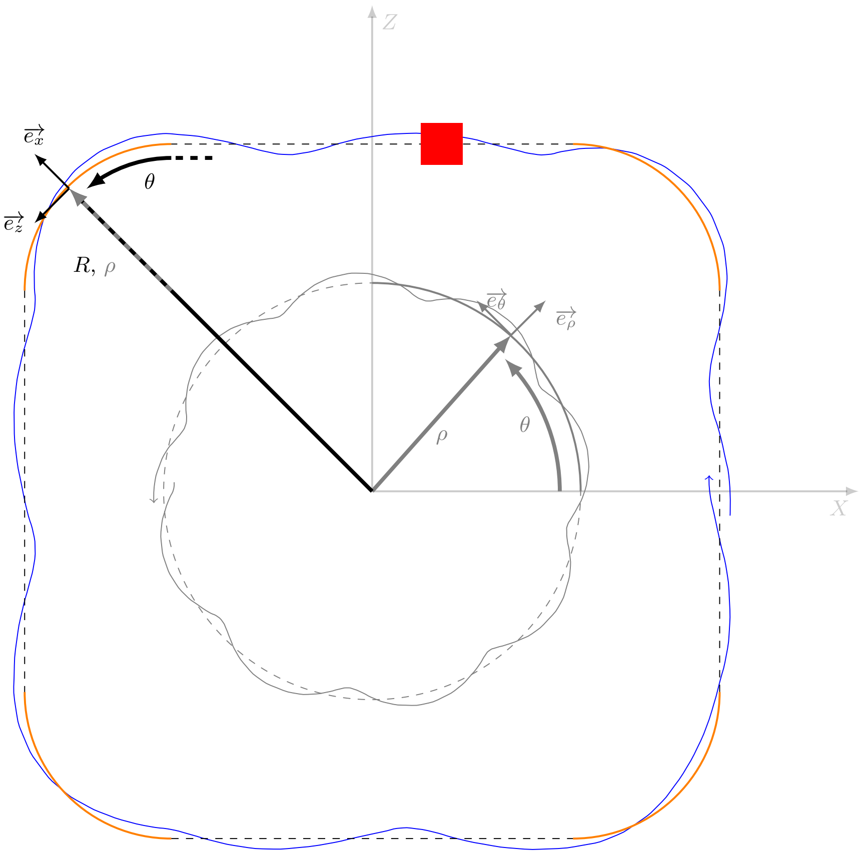
→ **Synchronism condition in synchrotrons**

→ **Acceleration rate**

→ **Magnetic and RF frequency programs**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- The revolution period of an arbitrary particle in a circular machine is

$$T_0 = \frac{C}{v} = \frac{2\pi R}{\beta c}$$

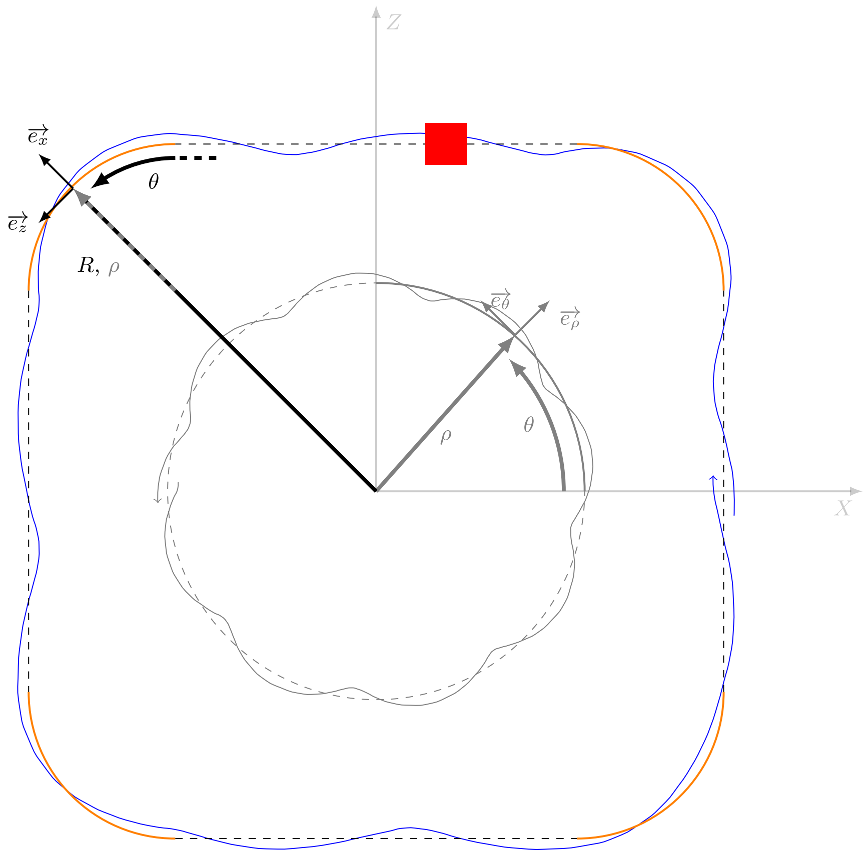
- The corresponding revolution (angular) frequency is

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{\beta c}{R}$$

- We will derive the relationships for the **synchronous** particle (subscript s).

SYNCHRONISM CONDITION

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- A particle is synchronous to the RF if

$$\omega_r = h\,\omega_{0,s} = h\frac{\beta_s c}{R_s}$$

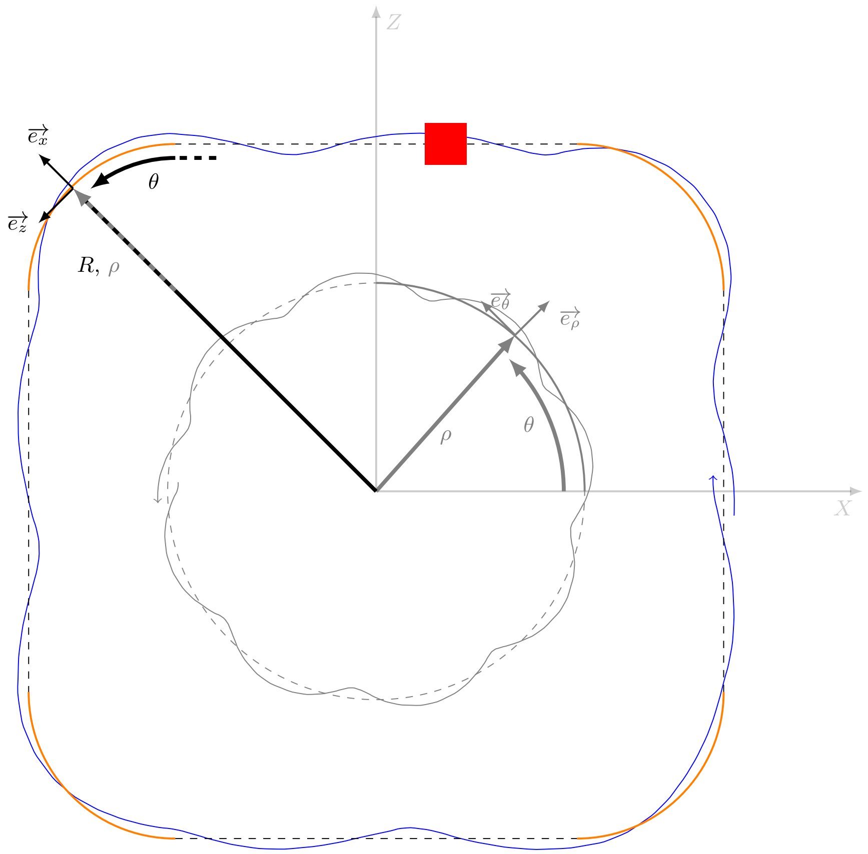
where h is the **RF harmonic number** (integer number).

- There are h different synchronous particles in a synchrotron (and effectively up to h bunches).

- The synchronous particle is fictitious, it is in reality an ideal reference point.

SYNCHRONISM CONDITION

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- The total energy variation over one turn is

$$\delta E_s = \delta E_{\text{rf},s} + \delta E_{\text{b,s}} + \delta E_{\text{sr,s}} + \delta E_{\text{ind,s}}$$

- For the following derivations we will only consider the RF contribution, the energy gain per turn of the synchronous particle is

$$\delta E_s = qV_{\text{rf}} \sin\left(h\omega_{0,s}\tau_s\right) = qV_{\text{rf}} \sin\left(\phi_s\right)$$

where ϕ_s is the synchronous phase.

ACCELERATION RATE

Assumption: The acceleration rate with time of the synchronous particle is assumed to be a smooth function with time. The energy gain per turn is usually small (not in Rapid Cycling Synchrotrons!).

The acceleration rate is

$$\dot{E}_s \approx \frac{\delta E_s}{T_{0,s}} \quad \rightarrow \quad \dot{E}_s = \frac{q V_{\mathrm{rf}}}{T_{0,s}} \sin\left(\phi_s\right)$$

The bending field must be increased synchronously, keeping a constant orbit ρ_s, R_s

$$\dot{\mathcal{B}}_y \rho_s = \frac{\dot{p}_s}{q}$$

ACCELERATION RATE

Using the differential relationship

$$\frac{dE}{dp} = \beta c \quad \rightarrow \quad \dot{E} = \beta c \dot{p}$$

and assuming that $\dot{\rho}_s = 0$, we get

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \text{and} \quad \phi_s = \arcsin\left(2\pi \rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\text{rf}}}\right)$$

Reminder: This equation is not related to the induction acceleration. In this case, the acceleration is obtained from the electric field in the RF cavity. The magnetic field in bending magnets is increased so that the synchronous particle keeps fulfilling the synchronism condition.

ACCELERATION RATE

DERIVATION

Derive the acceleration per turn and the synchronous phase assuming $\dot{\rho}_s = 0$

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \text{and} \quad \phi_s = \arcsin\left(2\pi \rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_{\mathrm{rf}}}\right)$$

ACCELERATION RATE

DERIVATION

$$\frac{d}{dt}\left(\mathcal{B}_y \rho_s\right) = \frac{\dot{p}_s}{q}$$

$$\implies \dot{\mathcal{B}}_y \rho_s + \mathcal{B}_y \dot{\rho}_s = \frac{\dot{E}_s}{q\beta_s c}$$

Assuming $\dot{\rho}_s = 0$

$$\implies \dot{\mathcal{B}}_y \rho_s = \frac{V_{\mathrm{rf}}}{\beta_s c T_{0,s}} \sin\left(\phi_s\right) = \frac{V_{\mathrm{rf}}}{2\pi R_s} \sin\left(\phi_s\right)$$

$$\implies 2\pi q R_s \rho_s \dot{\mathcal{B}}_y = q V_{\mathrm{rf}} \sin\left(\phi_s\right) = \delta E_s$$

RF FREQUENCY SWEEP

To preserve the synchronism condition, the RF frequency must also be adjusted to follow the evolution of β_s during acceleration

$$\omega_r(t) = \frac{hc}{R_s}\beta_s(t)$$

The RF frequency program is linked to the magnetic field

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

The principle of the synchrotron acceleration is to ramp the bending field and the RF frequency synchronously, providing constant R_s and the synchronous phase ϕ_s.

RF FREQUENCY SWEEP

DERIVATION

From

$$\omega_r(t) = \frac{hc}{R_s}\beta_s(t)$$

Express $\beta_s(t)$ as a function of $\mathcal{B}_y(t)$ using the definition of the magnetic rigidity with constant ρ_s (and R_s).

Obtain

$$f_r(t) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2(t)}{\mathcal{B}_y^2(t) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

RF FREQUENCY SWEEP

DERIVATION

Expression of β_s

$$\beta_s\left(t\right) = \frac{pc}{E} = \frac{pc}{\sqrt{\left(pc\right)^2 + E_0^2}}$$

$$= \frac{\mathcal{B}_y \rho_s q c}{\sqrt{\left(\mathcal{B}_y \rho_s q c\right)^2 + \left(m_0 c^2\right)^2}}$$

$$= \frac{\mathcal{B}_y}{\sqrt{\left(\mathcal{B}_y\right)^2 + \left(\frac{m_0 c^2}{\rho_s q c}\right)^2}}$$

RF FREQUENCY SWEEP

DERIVATION

Expression of β_s

$$\beta_s\left(t\right) = \sqrt{\frac{\mathcal{B}_y^2}{\mathcal{B}_y^2 + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

Leading to

$$\begin{aligned} f_r\left(t\right) &= \frac{\omega_r\left(t\right)}{2\pi} = \frac{hc}{2\pi R_s}\beta_s\left(t\right) \\ &= \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2\left(t\right)}{\mathcal{B}_y^2\left(t\right) + \left(\frac{m_0 c}{\rho_s q}\right)^2}} \end{aligned}$$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

- Revolution period/frequency of the SPS ($\rho_0 = 741.25$ m, $C_0 = 6911.50$ m)
- RF frequency of the SPS ($h = 4620$)
- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
- Smallest RF voltage to accelerate the synchronous particle
- Compute the same parameters with Lead ions $^{208}\mathrm{Pb}^{82+}$

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

Compute the following parameters for protons in the SPS at injection and extraction energies (momentum 26 GeV/c $\longrightarrow$ 450 GeV/c)

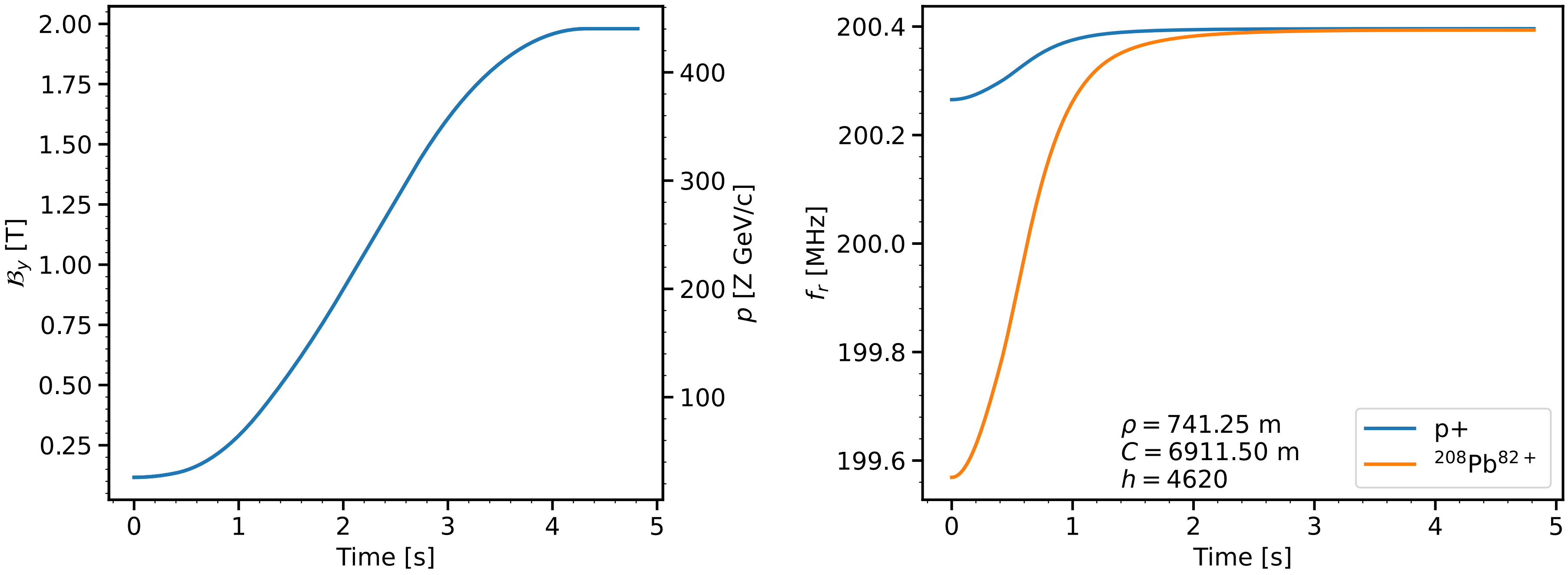
Machine	SPS inj. p+	SPS ext. p+	SPS inj. Pb	SPS ext. Pb
p [GeV/c]	26	450	2132	36900
E [GeV]	26.0169	450.001	2140.786	36900.509
β	0.99935	0.999...	0.9959	0.999..
T_0 [μs]	23.0693	23.0543	23.1493	23.0546
f_r [MHz]	200.266	200.396	199.574	200.394

EXAMPLE PROGRAMS IN THE SPS

EXERCISES

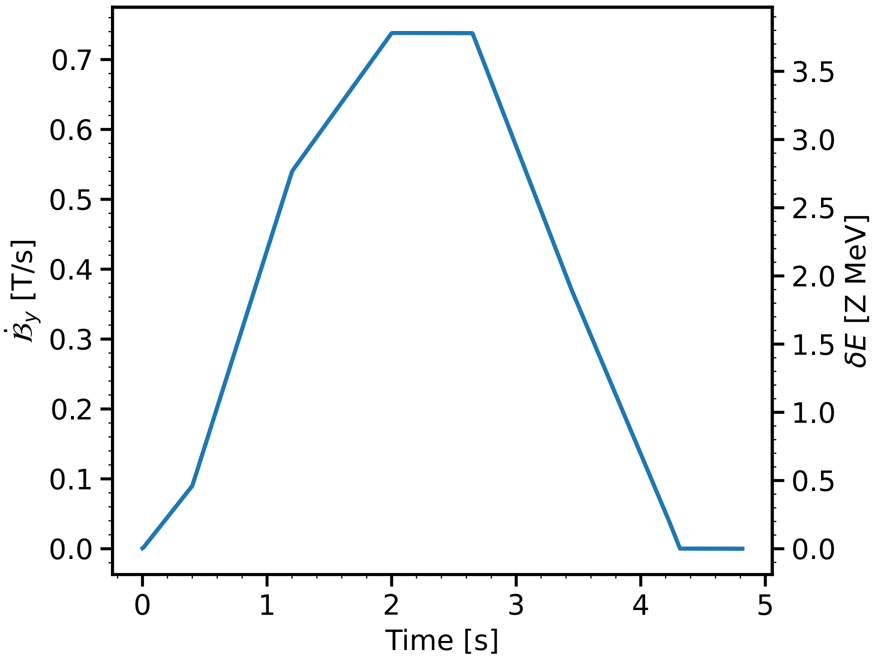
- Energy gain per turn in the SPS ($\dot{\mathcal{B}}_y = 0.7$ T/s)
  - $6911.50 \cdot 741.35 \cdot 1 \cdot 0.7 = 3.59\ \mathrm{MeV}$ for p+
  - $6911.50 \cdot 741.35 \cdot 82 \cdot 0.7 = 294\ \mathrm{MeV}$ for Pb
- Smallest RF voltage to accelerate the synchronous particle
  - $3.59\ \mathrm{MV}$ for p+ and Pb ($\delta E_{\mathrm{rf}}/q$)

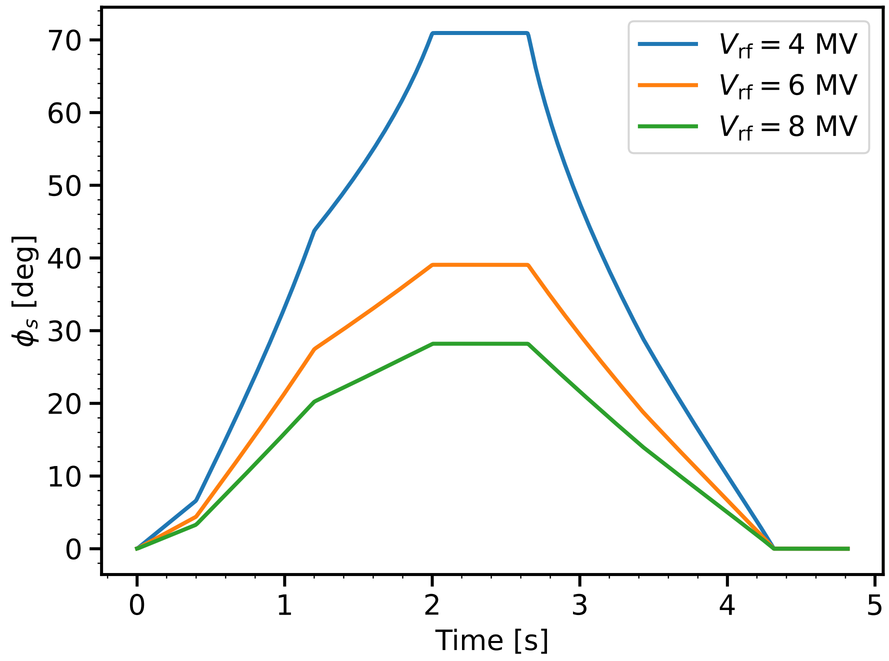
EXAMPLE PROGRAMS IN THE SPS



- The magnetic (and expected momentum) program together with the RF frequency program.
- Due to their larger mass the lead ions have a lower β_s and hence f_r for the same $\mathcal{B}_y \rho_s$

EXAMPLE PROGRAMS IN THE SPS





- The energy gain per turn defines the minimal RF voltage required to accelerate the synchronous particle ($\phi_s < \pi/4$, NB: independent of the charge Z!). Increasing V_{rf} allows to design a shorter cycle in time.

- We will see that there is in reality more considerations to design a V_{rf} program!

MODULE 5: DIFFERENTIAL RELATIONSHIPS IN A SYNCHROTRON

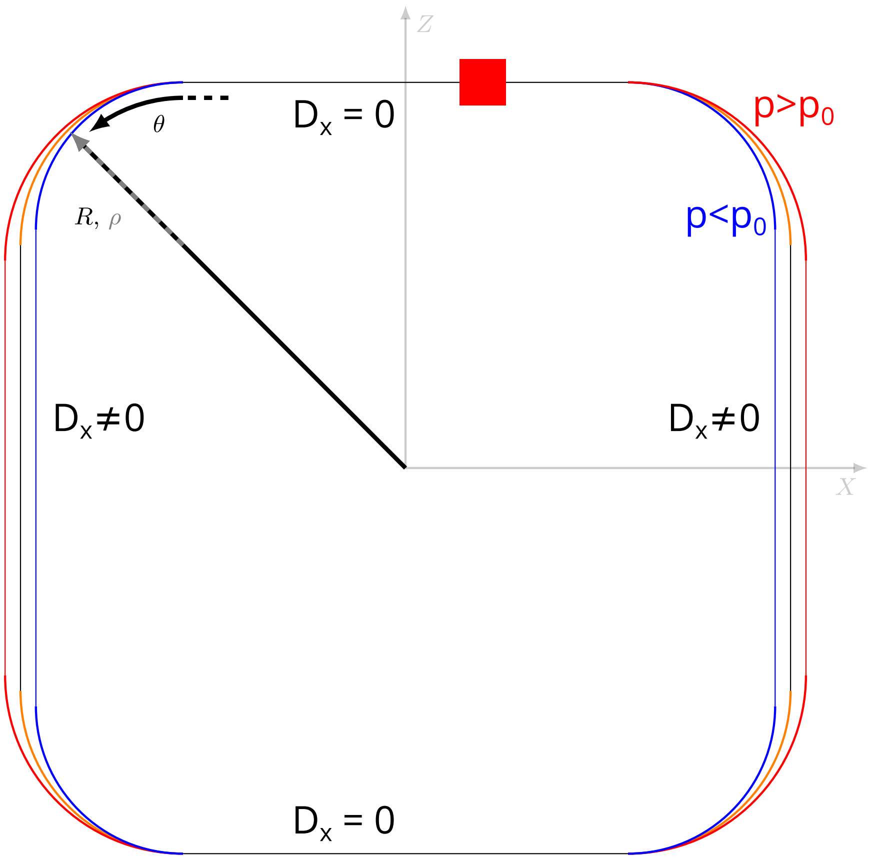
→ **Momentum compaction factor**

→ **Phase slip factor, transition gamma**

→ **Derivation of differential relationships**

COORDINATE SYSTEM(S)

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



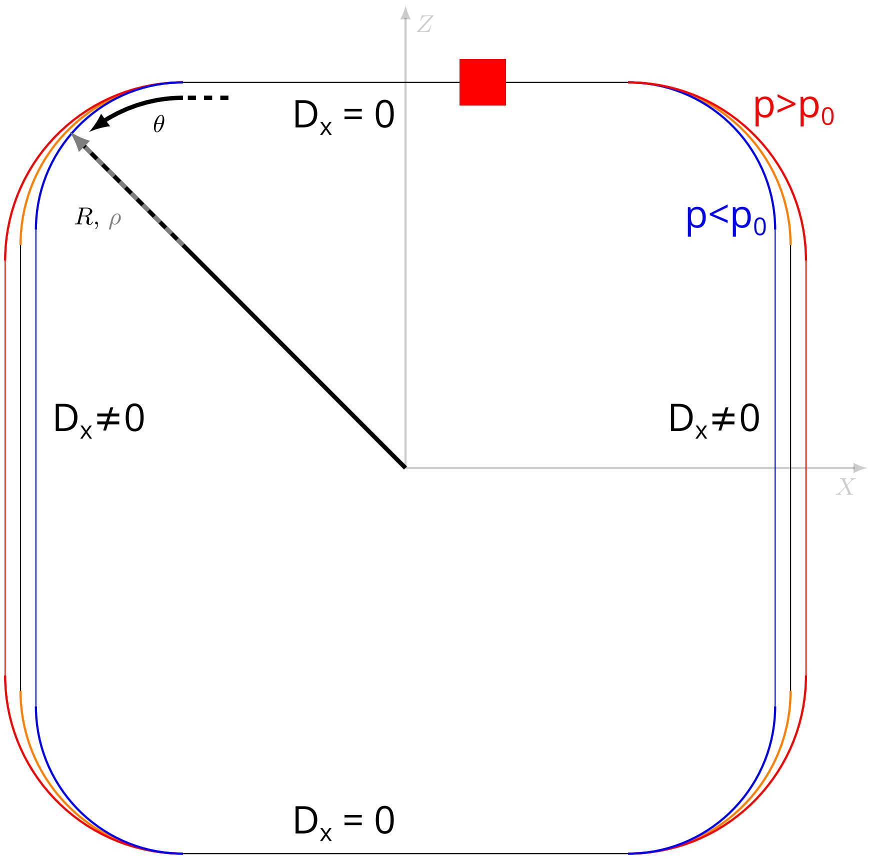
- In the previous module we assumed acceleration with a constant ρ_s.
- Nonetheless, like in cyclotrons, a particle can also be accelerated at fixed magnetic field with

$$\begin{aligned} dp_s &= q\left(d\mathcal{B}_y \rho_s + \mathcal{B}_y d\rho_s\right) \\ &= q\mathcal{B}_y d\rho_s \end{aligned}$$

- The synchronism condition $\omega_r = h\omega_{0,s}$ remains valid, and the RF frequency can be adjusted to accelerate the beam.

ORBIT AND DISPERSION

Accelerator seen from above, along the vertical $\vec{Y}$ axis...

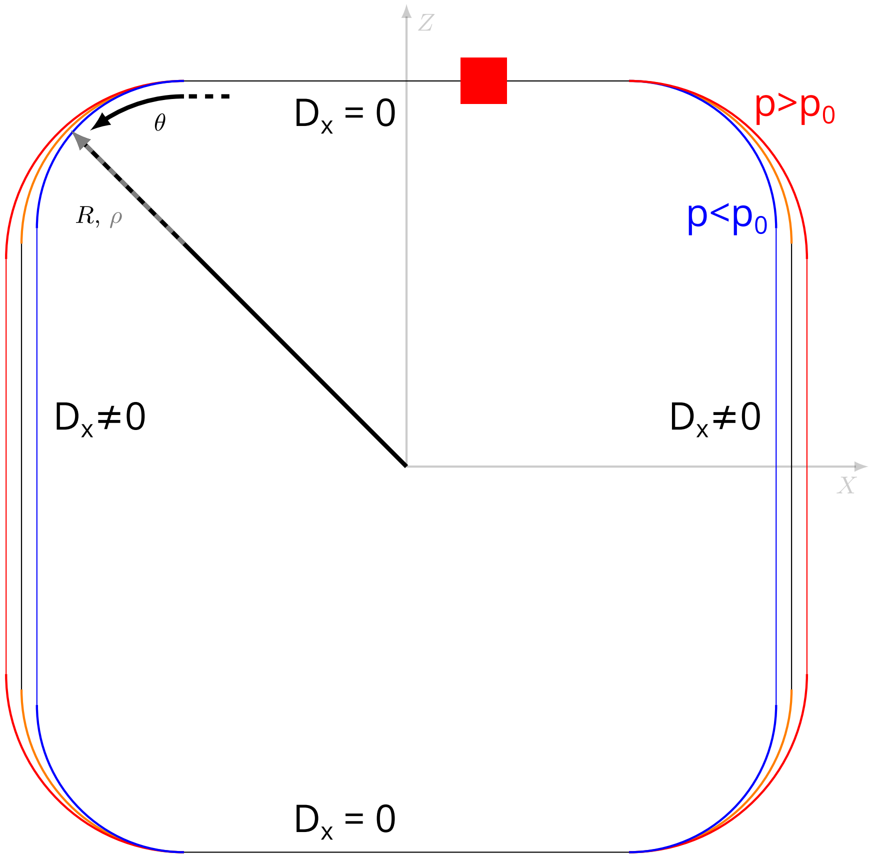


- In synchrotrons, the small beam pipe aperture only allows for small orbit changes.
- Nonetheless, this is used to steer the beam radially or to do very fine adjustments of the beam energy.
- The radial/horizontal offset is given by the transverse dispersion function

$$x_D(z) = D_x(z)\,\frac{dp}{p}$$

MOMENTUM COMPACTION AND PHASE SLIP FACTOR

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



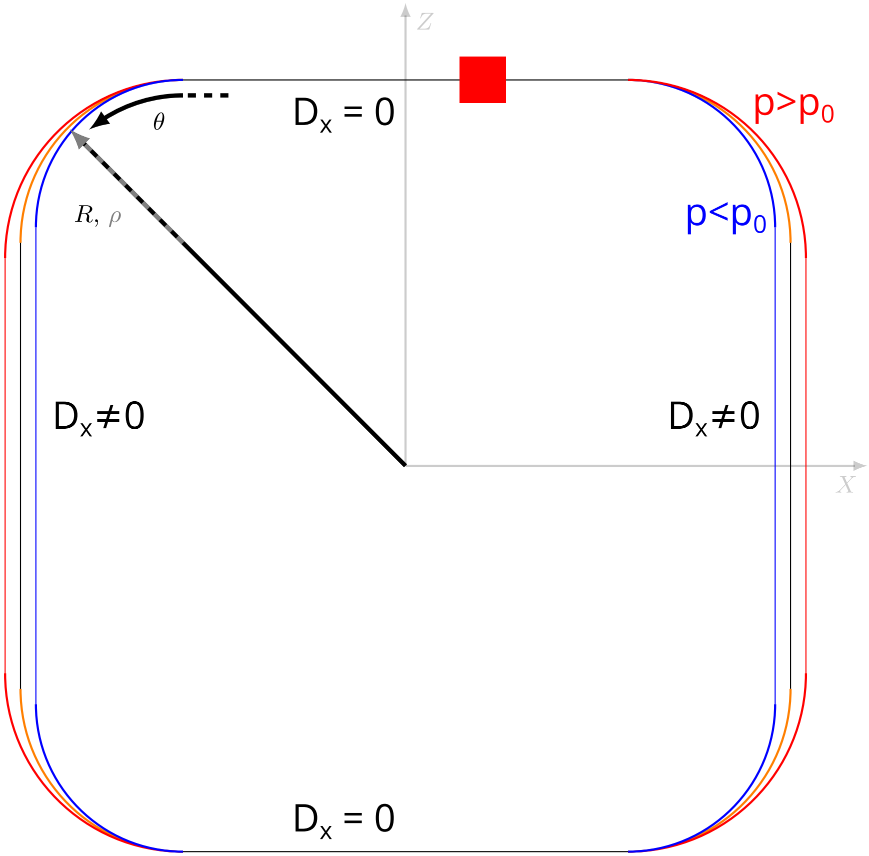
- The relative elongation of the mean radius due to a momentum relative offset is called the "momentum compaction factor"

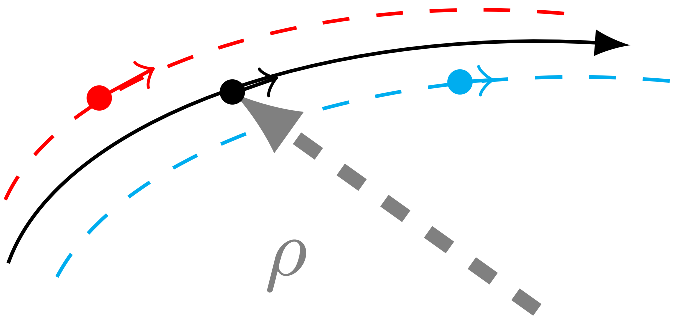
$$\alpha_p = \frac{dR/R}{dp/p}$$

- This parameter is linked to the relative change in revolution frequency, the phase slip factor

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p} = \frac{dT_0/T_0}{dp/p}$$

MOMENTUM COMPACTION AND PHASE SLIP FACTOR





- Is the blue particle arriving before or after the red one after a turn?

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p}$$

- The phase slip factor is fundamental to longitudinal beam dynamics!

MOMENTUM COMPACTION FACTOR

DEFINITION

The momentum is a function of $(\rho, \mathcal{B}_y)$, and consequently of $(R, \mathcal{B}_y)$. It can be differentiated as

$$\frac{dp}{p} = \left(\frac{\partial p}{\partial R}\right)_{\mathcal{B}_y} \frac{R}{p} \frac{dR}{R} + \left(\frac{\partial p}{\partial \mathcal{B}_y}\right)_R \frac{\mathcal{B}_y}{p} \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

$$\implies \frac{dp}{p} = \frac{1}{\alpha_p} \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

The momentum compaction factor **assumes a constant** $\mathcal{B}_y$ and is defined as

$$\alpha_p = \left(\frac{\partial p}{\partial R}\right)^{-1}_{\mathcal{B}_y} \frac{p}{R} = \left(\frac{\partial R}{\partial p}\right)_{\mathcal{B}_y} \frac{p}{R}$$

REMINDER DIFFERENTIATION

Differentiating a function $f\left(x, y\right)$

$$df\left(x, y\right) = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

partial derivatives are taken with all other parameters constant.

Under certain considerations (continuous injective functions), we have

$$\frac{\partial f}{\partial x} = \left(\frac{\partial x}{\partial f}\right)^{-1}$$

MOMENTUM COMPACTION FACTOR

DERIVATION OF GENERAL DEFINITION

Let's remind that $p = \mathcal{B}_y \rho q$, therefore

$$\left(\frac{\partial p}{\partial \mathcal{B}_y}\right)_R \frac{\mathcal{B}_y}{p} = \rho q \frac{\mathcal{B}_y}{p} = 1$$

MOMENTUM COMPACTION FACTOR

COMPUTATION FROM DISPERSION FUNCTION

The horizontal (i.e. radial) offset of a particle closed orbit is obtained from the machine optics (see JUAS Transverse Beam Dynamics Course 3)

$$x_D\left(z\right) = D_x\left(z\right)\frac{dp}{p}$$

The momentum compaction factor can be computed from

$$\alpha_p = \frac{1}{2\pi R}\int_0^{2\pi R}\frac{D_x\left(z\right)}{\rho\left(z\right)}dz = \frac{\left\langle D_x\right\rangle_\rho}{R}$$

where $\left\langle D_x\right\rangle_\rho$ is an averaged dispersion value in the bending magnets (*NB:* $\rho \rightarrow \infty$ *in straight sections, and* $\alpha_p = 0$ *in linacs*). For azimuthally symmetric fields, $\alpha_p = 1/Q_x^2$ (which is a reasonable scaling law in general).

MOMENTUM COMPACTION FACTOR

DERIVATION FROM DISPERSION

The total path length increase due to dispersion is

$$dC = 2\pi dR = \int_0^{2\pi} x_D\left(z\right) d\theta$$

Reminder: x_D *is a horizontal and i.e. radial offset.*

Give an equation to α_p as a function of D_x.

MOMENTUM COMPACTION FACTOR

DERIVATION FROM DISPERSION

The total path length increase due to dispersion is

$$dC = 2\pi dR = \int_0^{2\pi} x_D\left(z\right) d\theta$$

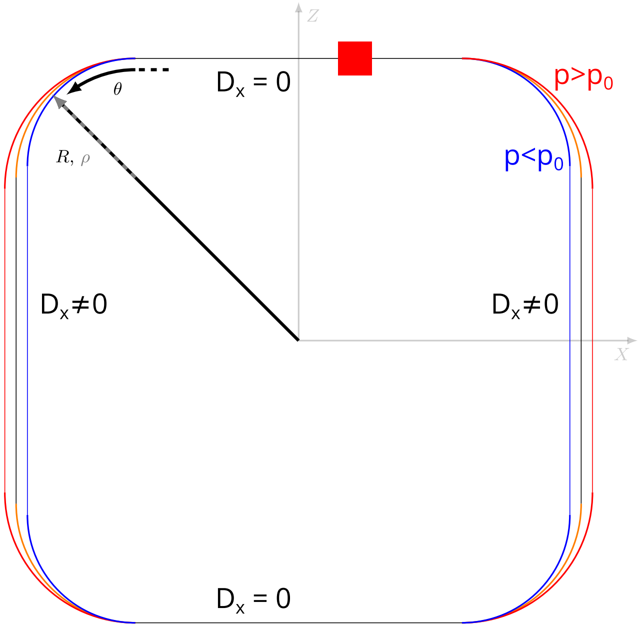
$$\implies dR = \frac{1}{2\pi}\int_0^{2\pi R} D_x\left(z\right)\frac{dp}{p}\frac{dz}{\rho\left(z\right)} \quad ,\left(dz = \rho\left(z\right) d\theta\right)$$

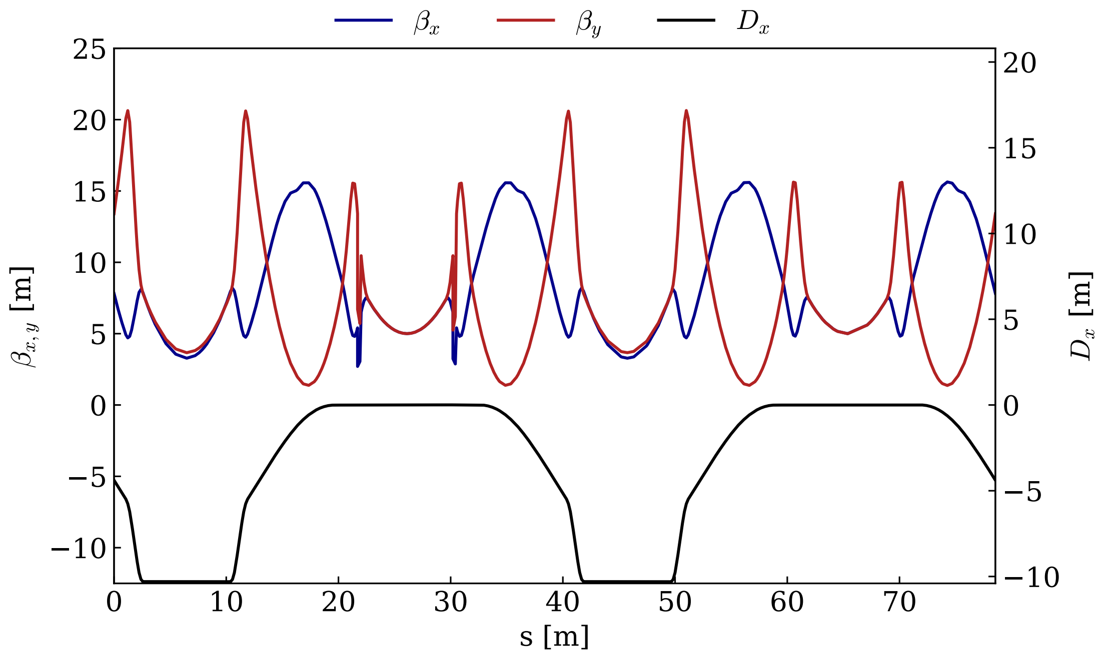
$$\implies \frac{dR}{R}\frac{p}{dp} = \frac{1}{2\pi R}\int_0^{2\pi R}\frac{D_x\left(z\right)}{\rho\left(z\right)} dz$$

$$\implies \alpha_p = \frac{1}{2\pi R}\int_0^{2\pi R}\frac{D_x\left(z\right)}{\rho\left(z\right)} dz$$

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)

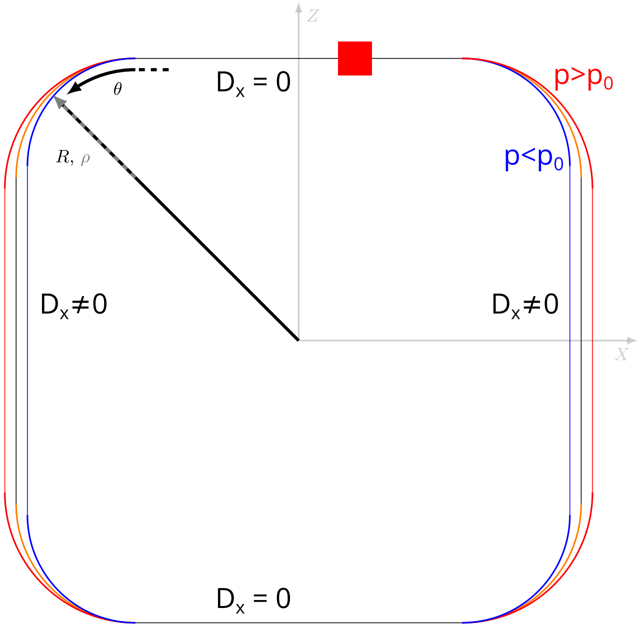


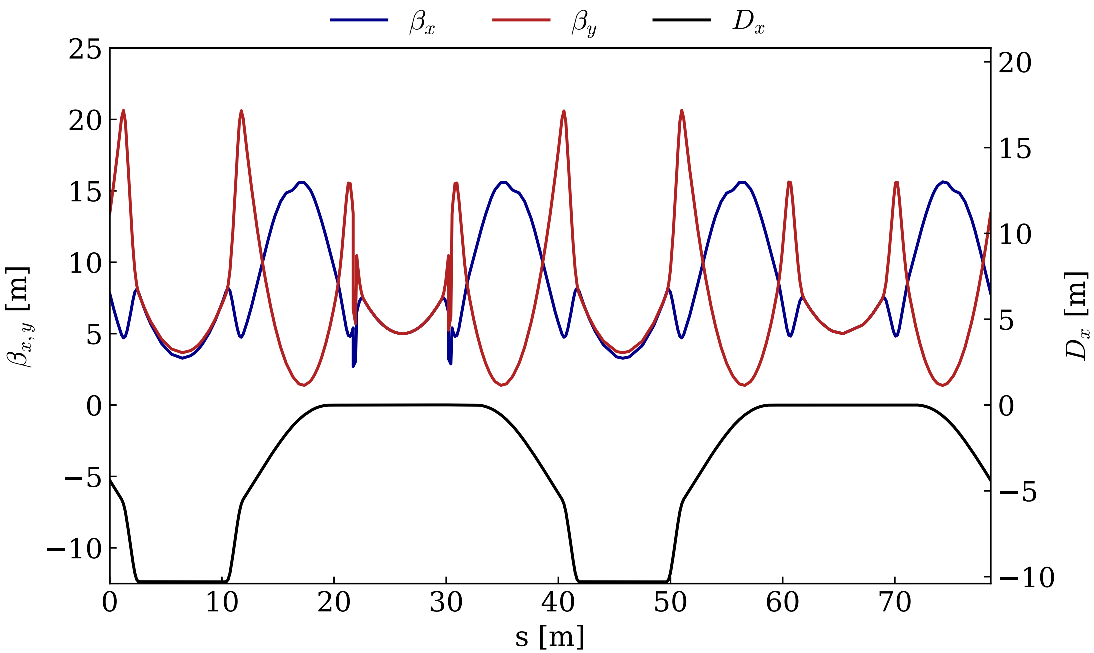


- Dispersion is 0 in two straight sections and large in the other two straight sections.
- NB: Dispersion is displayed as negative but should be taken as positive. Due to convention, integrating from $2\pi R \to 0$.

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)

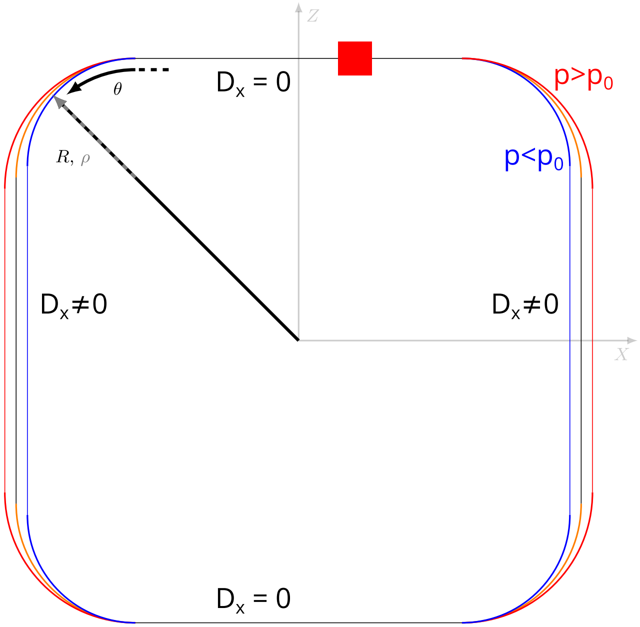


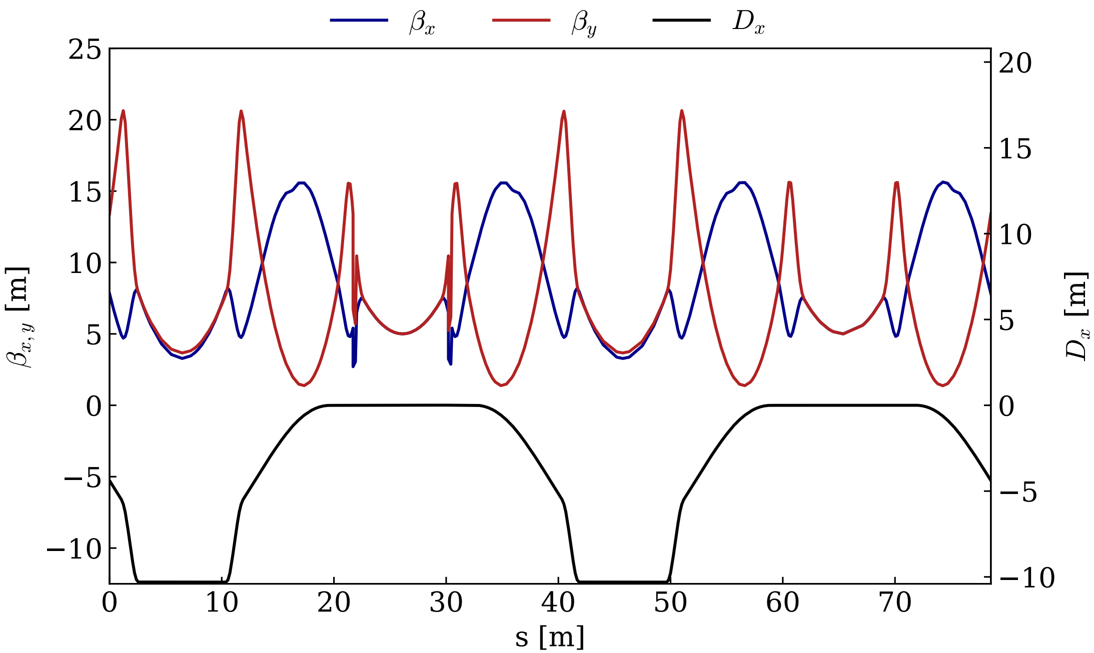


- What is the local radial offset for a particle with $\frac{\Delta p}{p_0} = 10^{-3}$ at large dispersion?

MOMENTUM COMPACTION FACTOR

REALISTIC SCENARIO, LEIR (CERN OPTICS REPOSITORY)





- What is the local radial offset for a particle with $\frac{\Delta p}{p_0} = 10^{-3}$ at large dispersion?

- $x_D \approx 1$ cm.

PHASE SLIP FACTOR

The momentum compaction factor expresses the orbit variation due to a momentum offset. The revolution period/frequency of the particle (and RF frequency via the synchronism condition) is also changed.

Differentiating the revolution frequency

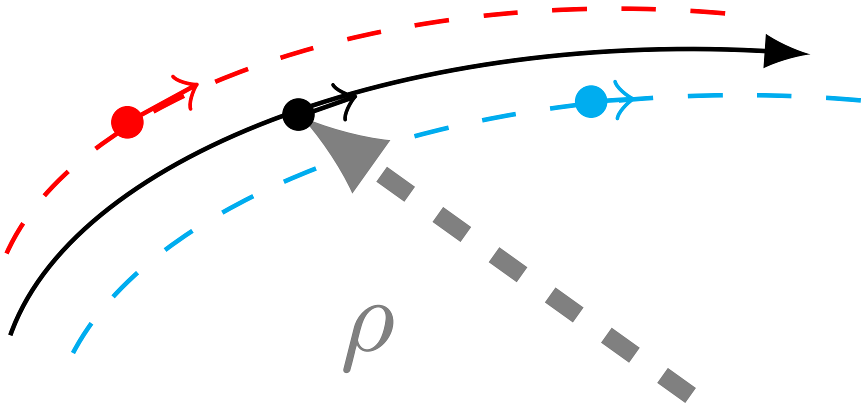
$$\omega_0 = \frac{\beta c}{R} \quad \rightarrow \quad \frac{d\omega_0}{\omega_0} = \frac{d\beta}{\beta} - \frac{dR}{R} \left(= -\frac{dT_0}{T_0} \right)$$

we obtain the phase slip factor

$$\eta = \frac{dT_0/T_0}{dp/p} = -\frac{d\omega_0/\omega_0}{dp/p} = \alpha_p - \frac{1}{\gamma^2}$$

Note: The phase slip is sometimes defined with an opposite sign in the litterature, beware of the used conventions!

TRANSITION ENERGY



- Two regimes

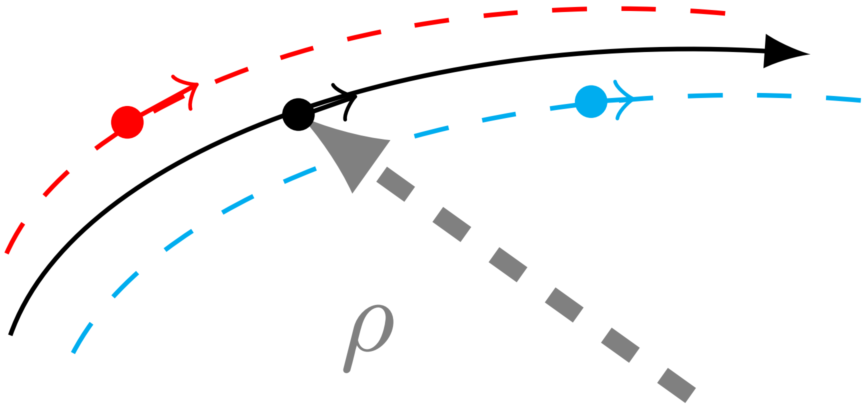
$$\eta > 0 \quad \text{if} \quad \alpha_p > \frac{1}{\gamma^2}, \qquad \gamma > \gamma_t$$

$$\eta < 0 \quad \text{if} \quad \alpha_p < \frac{1}{\gamma^2}, \qquad \gamma < \gamma_t$$

We introduced the transition gamma $\gamma_t = 1/\sqrt{\alpha_p}$.

In the two different regimes, which particle is circulating faster/slower in the machine? What happens for very large γ?

TRANSITION ENERGY



- Two regimes

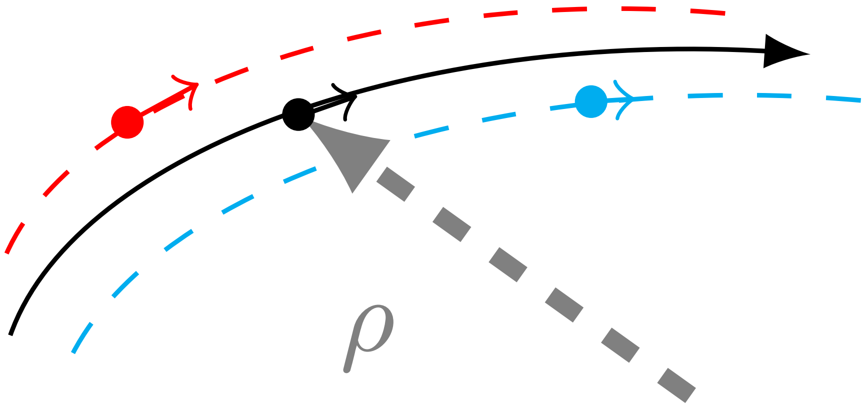
$$\eta > 0 \quad \text{if} \quad \alpha_p > \frac{1}{\gamma^2}, \qquad \gamma > \gamma_t$$

$$\eta < 0 \quad \text{if} \quad \alpha_p < \frac{1}{\gamma^2}, \qquad \gamma < \gamma_t$$

We introduced the transition gamma γ_t.

- For $\gamma < \gamma_t$ (below transition), the particles with **increasing orbit/momentum arrives earlier** as the velocity gain is more important than the increased path length. This happens in particular in **low energy synchrotrons**.
- For $\gamma > \gamma_t$ (above transition), the particles with **increasing orbit/momentum arrives later** as the velocity gain is less important than the increased path length. This happens in particular in **high energy synchrotrons**.

TRANSITION ENERGY



- Special regime

$$\eta = 0 \quad \text{if} \quad \alpha_p = \frac{1}{\gamma^2}, \qquad \gamma = \gamma_t$$

- At transition energy, all particles circulate with the same revolution period.
- This change or regime during acceleration requires a special treatment which requires further derivations.
- To avoid transition crossing, special optics with $\langle D_x \rangle < 0 \rightarrow \alpha_p < 0$ can be made, leading mathematically to an imaginary γ_t.

USUAL APPROXIMATIONS

- The momentum compaction factor is computed in synchrotrons with respect to the design orbit R_0 (subscript for design synchrotron parameters 0). Note that the synchronous particle can be offset in orbit with respect to the design trajectory.
- The momentum compaction factor can be expanded in series around p_0 and coefficients computed from the non-linear dispersion function (JUAS Lecture on Transverse Non Linearities)

$$\alpha_{p_0} = \alpha_0 + \alpha_1 \frac{\Delta p}{p_0} + \alpha_2 \left(\frac{\Delta p}{p_0}\right)^2 + \ldots$$

Assumption: we will assume linear dispersion, momentum compaction factor $\alpha_0 = \left(\Delta R / R_0\right) / \left(\Delta p / p_0\right)$**, phase slip factor** $\eta_0 = -\left(\Delta \omega_0 / \omega_{0,0}\right) / \left(\Delta p / p_0\right)$ **for the rest of the course.**

AVERAGE MAGNETIC FIELD INDEX

The magnetic rigidity formula can also be written

$$p = q\mathcal{B}_y\rho = q\left\langle\mathcal{B}_y\right\rangle R$$

where we define the average magnetic field along a particle path

$$\left\langle\mathcal{B}_y\right\rangle = \frac{1}{2\pi R}\int_0^{2\pi R}\mathcal{B}_y dz$$

We define the average magnetic field index

$$\left\langle n\right\rangle = -\frac{d\left\langle\mathcal{B}_y\right\rangle / \left\langle\mathcal{B}_y\right\rangle}{dR/R} = 1 - \frac{1}{\alpha_p}$$

These definitions are found in litterature for derivations.

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Show that the definition of the average magnetic field leads to

$$p = q\mathcal{B}_y\rho = q\left\langle \mathcal{B}_y \right\rangle R$$

By differentiating the formula above, demonstrate the relationship between $\langle n \rangle$ and α_p

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Starting from

$$\begin{aligned}
\langle \mathcal{B}_y \rangle &= \frac{1}{2\pi R}\int_0^{2\pi R} \mathcal{B}_y dz \\
&= \frac{1}{2\pi R}\int_0^{2\pi R} \frac{p}{q\rho} dz \\
&= \frac{1}{2\pi R}\frac{p}{q}\int_0^{2\pi R} \frac{dz}{\rho\left(z\right)} \\
&= \frac{1}{2\pi R}\frac{p}{q}\frac{2\pi\rho}{\rho} && \rho \rightarrow \infty \text{ in straigt sections} \\
\langle \mathcal{B}_y \rangle R &= \frac{p}{q} && \text{otherwise constant}
\end{aligned}$$

AVERAGE MAGNETIC FIELD INDEX

DERIVATION

Differentiating

$$p = q \langle \mathcal{B}_y \rangle R$$

$$\implies \frac{dp}{p} = \frac{d\langle \mathcal{B}_y \rangle}{\langle \mathcal{B}_y \rangle} + \frac{dR}{R}$$

$$\implies \frac{dp/p}{dR/R} = \frac{d\langle \mathcal{B}_y \rangle / \langle \mathcal{B}_y \rangle}{dR/R} + 1$$

$$\implies \frac{1}{\alpha_p} = -\langle n \rangle + 1$$

$$\implies \langle n \rangle = 1 - \frac{1}{\alpha_p}$$

SYNCHROTRON DIFFERENTIAL EQUATIONS

(1)	$\mathcal{B}_y, p, R$	$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$
(2)	f_0, p, R	$\frac{dp}{p} = \gamma^2 \frac{df_0}{f_0} + \gamma^2 \frac{dR}{R}$
(3)	$\mathcal{B}_y, f_0, p$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma_t^2 \frac{df_0}{f_0} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$
(4)	$\mathcal{B}_y, f_0, R$	$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma^2 \frac{df_0}{f_0} + \left(\gamma^2 - \gamma_t^2\right) \frac{dR}{R}$

SYNCHROTRON DIFFERENTIAL EQUATIONS

OTHER USEFUL RELATIONSHIPS

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

$$\frac{dp}{p} = \frac{dR}{R} + \frac{d\left\langle \mathcal{B}_y \right\rangle}{\left\langle \mathcal{B}_y \right\rangle}$$

$$\frac{dR}{R} = \frac{1}{\alpha_p}\frac{d\rho}{\rho}$$

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]		
γ		
T_0 [μs]		
α_p $[10^{-3}]$		
E_t [GeV]		
η $[10^{-3}]$		

EXERCISES

PARAMETER COMPUTATION

- Fill the table for the SPS ($C_0 = 6911.50$ m) with $\gamma_t = 18$ for a proton beam

Machine	SPS injection	SPS extraction
Momentum [GeV/c]	14	450
E [GeV]	14.03	450
γ	14.95	479.6
T_0 [μs]	23.11	23.05
α_p [10^{-3}]	3.086	3.086
E_t [GeV]	16.89	16.89
η [10^{-3}]	-1.385	3.082

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?

EXERCISES

PARAMETER COMPUTATION

- Is transition crossed in the SPS during acceleration?
  - Transition is crossed since η changes sign
- What is the mean radial offset ΔR of a particle with $\Delta p/p_0 = -10^{-4}$ with a constant $\mathcal{B}_y$?
  - $\Delta R = 3.086 \cdot 10^{-3} \cdot 6911.50/(2 \cdot 3.14) \cdot \left(-10^{-4}\right) =$-0.34 mm
- What is the corresponding change in the revolution period ΔT_0? Is the particle delayed or in advance after a turn, with respect to the reference?
  - Low E: $\Delta T_0 = -1.385 \cdot 10^{-3} \cdot 23.11 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$ 3.2 ps (late)
  - High E: $\Delta T_0 = 3.082 \cdot 10^{-3} \cdot 23.05 \cdot 10^{-6} \cdot \left(-10^{-4}\right) =$-7.1 ps (early)

EXERCISES

- Demonstrate the differential equations (1), (2), (3), (4)

EXERCISES

DERIVATION

(1) Definition in the lecture

(2) Combining

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta} \quad \text{and} \quad \frac{d\beta}{\beta} = \frac{df_0}{f_0} + \frac{dR}{R}$$

(3) Substituting $\frac{dR}{R}$ from (1) in (2)

(4) Substituting $\frac{dp}{p}$ from (1) in (2)

TAKE AWAY MESSAGE

ENERGY GAIN

- RF energy gain

$$\delta E_{\mathrm{rf}}\left(\tau\right) = qV_{\mathrm{rf},0}T_t \sin\left(\omega_r \tau\right) \quad \rightarrow \delta E_{\mathrm{rf}}\left(\phi\right) = qV_{\mathrm{rf}} \sin\left(\phi\right)$$

- Transit time factor

$$T_t\left(\rho,\beta\right) = \frac{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho,z\right)\cos\left(\frac{\omega_r z}{\beta c}\right) dz}{\int_{-g/2}^{g/2} \mathcal{E}_0\left(\rho,z\right) dz}$$

- Assumptions:
  - β is not changing in the computation of T_t
  - The (ρ,β) dependence of T_t will be neglected

TAKE AWAY MESSAGE

PILLBOX CAVITY (FUNDAMENTAL MODE)

- Pillbox cavity properties

$$\mathcal{E}_z\left(\rho,t\right) = \mathcal{E}_0 J_0\left(\chi_0 \frac{\rho}{\rho_c}\right)\cos\left(\omega_r t\right)$$

J_n Bessel function, $\chi_0 \approx 2.405, \omega_r = \chi_0 c/\rho_c$

- Transit time factor of pillbox cavity

$$T_t = \frac{\sin\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}{\left(\frac{\chi_0 g}{2\beta\rho_c}\right)}$$

TAKE AWAY MESSAGE

OTHER ENERGY GAIN/LOSS IN A RING

- Induction acceleration (small in large synchrotrons)

$$\delta E_{\mathrm{b}}\left(\rho\right) = q \int_0^{2\pi} \int_0^{\rho} \frac{\partial \mathcal{B}_y\left(\rho', \theta, t\right)}{\partial t} \rho' \, d\rho' \, d\theta$$

- Synchrotron radiation (relevant for lepton accelerators)

$$\delta E_{\mathrm{sr}}\left(E, \rho\right) = \frac{q^2}{3\epsilon_0} \frac{\beta^3 E^4}{\rho \, E_0^4}$$

- Self induced field

$$\delta E_{\mathrm{ind}}\left(\tau\right) = q V_{\mathrm{ind}}\left(\tau\right) = -q N_b \left(\lambda * \mathcal{W}\right)$$

TAKE AWAY MESSAGE

SYNCHRONISM IN SYNCHROTRON

- The revolution period and frequency

$$T_0 = \frac{C}{v} = \frac{2\pi R}{\beta c} \quad , \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{\beta c}{R}$$

- Synchronism condition with RF frequency

$$\omega_r = h\,\omega_{0,s} = h\frac{\beta_s c}{R_s}$$

TAKE AWAY MESSAGE

ACCELERATION

- Acceleration rate (subscript s for synchronous particle)

$$\delta E_s = 2\pi q \rho_s R_s \dot{\mathcal{B}}_y \quad \rightarrow \quad \phi_s = \arcsin\left(2\pi\rho_s R_s \frac{\dot{\mathcal{B}}_y}{V_\mathrm{rf}}\right)$$

- RF frequency program

$$f_r\left(t\right) = \frac{hc}{2\pi R_s}\sqrt{\frac{\mathcal{B}_y^2\left(t\right)}{\mathcal{B}_y^2\left(t\right) + \left(\frac{m_0 c}{\rho_s q}\right)^2}}$$

- Assumptions: Acceleration with constant R_s and ρ_s

TAKE AWAY MESSAGE

RADIAL DISPLACEMENT

- Momentum compaction factor (subscript 0 for design orbit/momentum, transition gamma γ_t)

$$\alpha_p = \frac{dR/R}{dp/p} = \frac{\left\langle D_x \right\rangle_\rho}{R} = \frac{1}{\gamma_t^2} \approx \frac{\Delta R/R_0}{\Delta p/p_0} \approx \frac{\Delta R/R_s}{\Delta p/p_s}$$

- Phase slip factor

$$\eta = -\frac{d\omega_0/\omega_0}{dp/p} = \frac{dT_0/T_0}{dp/p} = \alpha_p - \frac{1}{\gamma^2} \approx -\frac{\Delta\omega_{0,0}/\omega_{0,0}}{\Delta p/p_0} \approx -\frac{\Delta\omega_{0,s}/\omega_{0,s}}{\Delta p/p_s}$$

- Assumptions: Radial displacement with constant $\mathcal{B}_y$

TAKE AWAY MESSAGE, DIFFERENTIAL EQS.

(1) $\mathcal{B}_y, p, R$

$$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{d\mathcal{B}_y}{\mathcal{B}_y}$$

(2) f_0, p, R

$$\frac{dp}{p} = \gamma^2 \frac{df_0}{f_0} + \gamma^2 \frac{dR}{R}$$

(3) $\mathcal{B}_y, f_0, p$

$$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma_t^2 \frac{df_0}{f_0} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$$

(4) $\mathcal{B}_y, f_0, R$

$$\frac{d\mathcal{B}_y}{\mathcal{B}_y} = \gamma^2 \frac{df_0}{f_0} + \left(\gamma^2 - \gamma_t^2\right) \frac{dR}{R}$$

LESSON 3: LONGITUDINAL EQUATIONS OF MOTION

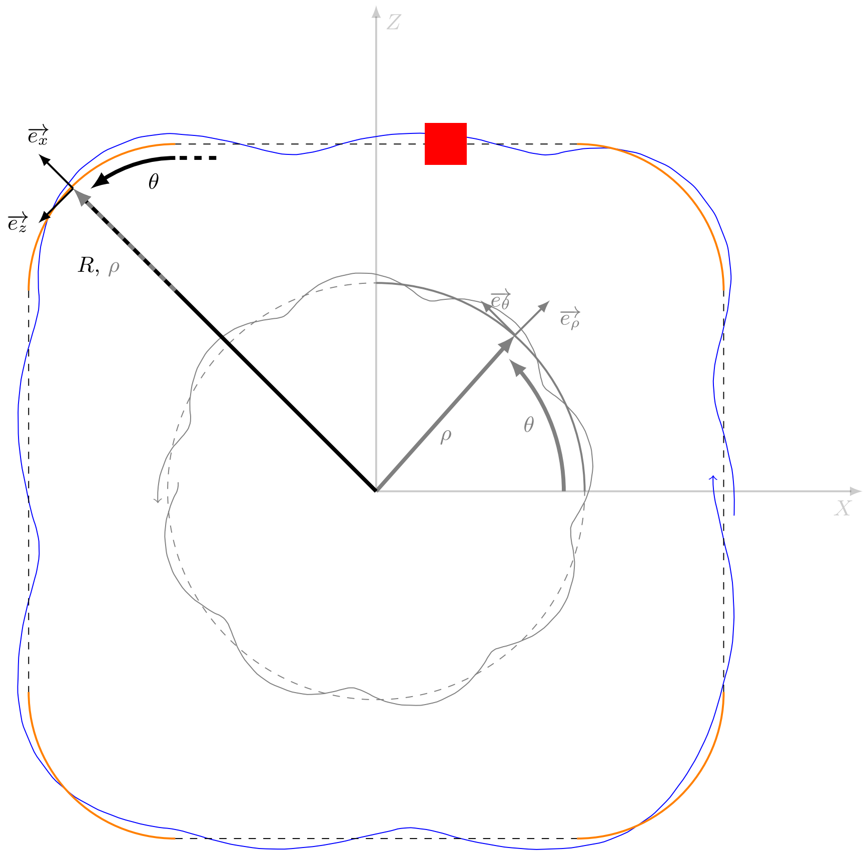
MODULE 6: THE NON-SYNCHRONOUS PARTICLES

$\longrightarrow$ **Energy gain equation of motion**

$\longrightarrow$ **Phase slippage equation of motion**

NON-SYNCHRONOUS PARTICLE

Accelerator seen from above, along the vertical $\vec{Y}$ *axis...*



- In the last module we considered only the idealized synchronous particle with subscript s.

- We will consider from now on the equations for a non-synchronous particle with

$$E = E_s + \Delta E$$

$$\omega_0 = \omega_{0,s} + \Delta\omega_0$$

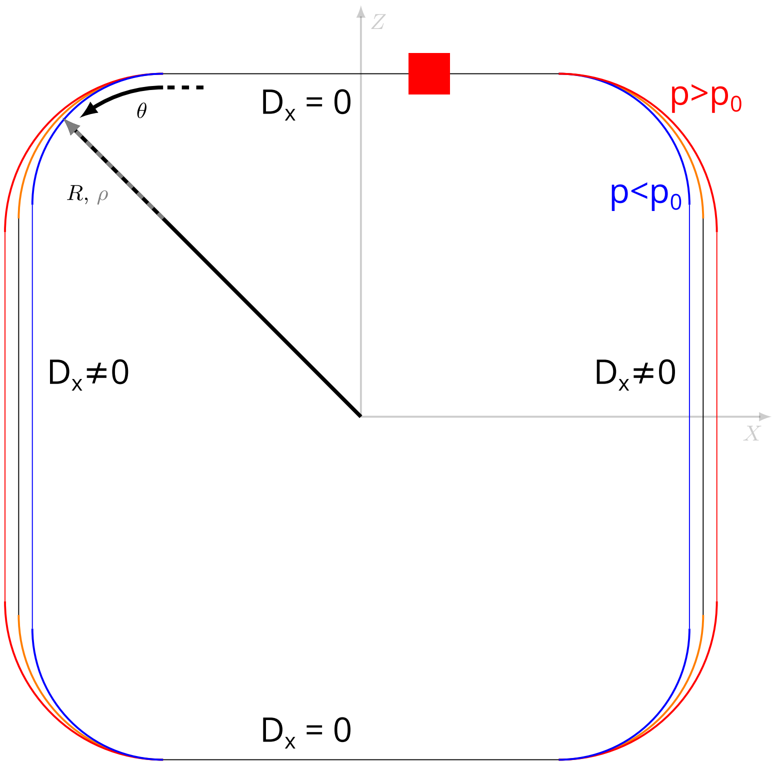
$$\theta = \theta_s + \Delta\theta$$

$$\rho\left(z\right) = \rho_s\left(z\right) + x_D\left(z\right)\ (\text{Dispersion})$$

$$\dots$$

EQUATIONS OF MOTION

Accelerator seen from above, along the vertical $\vec{Y}$ axis...



f and g arbitrary mathematical functions

- We will describe the evolution of the energy gain of an arbitrary particle arriving at a phase ϕ in the cavity, compared to the synchronous particle.

$$\frac{d\left(\Delta E\right)}{dt} = f\left(\phi\right)$$

- We will then derive the evolution of the phase of an arbitrary particle with different energy ΔE with respect to the synchronous particle.

$$\frac{d\left(\phi\right)}{dt} = g\left(\Delta E\right)$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

In the previous lesson we derived the acceleration rate of the synchronous particle. The acceleration rate is **first approximated to consider only the RF contribution** (no induction force, synchrotron radiation, wakefields...). For the synchronous particle

$$\dot{E}_s \approx \frac{\delta E_s}{T_{0,s}} \quad \rightarrow \quad \dot{E}_s = \frac{qV_{\mathrm{rf}}}{T_{0,s}} \sin\left(\phi_s\right) = \omega_{0,s} \frac{qV_{\mathrm{rf}}}{2\pi} \sin\left(\phi_s\right)$$

The acceleration rate for an arbitrary particle is

$$\dot{E} = \omega_0 \frac{qV_{\mathrm{rf}}}{2\pi} \sin\left(\phi\right)$$

The difference in acceleration rate is

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_{\mathrm{rf}}}{2\pi} \left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

ALL FORCES EXCEPT RF NEGLECTED

From

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

Re-organizing the term on the left hand side provides us with the following equation of motion

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

The second term on the left hand side can be obtained after thourough derivations.

Including the induction forces, the equation of motion can actually be made simpler!

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Demonstrate we can write

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\Delta E}{\omega_{0,s}} \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}$$

NB: Remember that ω_0 and $\omega_{0,s}$ are functions of time.

Hint: It may easier to handle T_0 than ω_0 for the initial derivations.

Hint 2: Keep linear orders in Δ.

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Starting from

$$\begin{aligned}\dot{E}\,T_0 - \dot{E}_s\,T_{0,s} &= \dot{E}_s\,T_{0,s} + \Delta\dot{E}\,T_{0,s} + \dot{E}_s\,\Delta T + \Delta\dot{E}\,\Delta T - \dot{E}_s\,T_{0,s} \\ &= \Delta\dot{E}\,T_{0,s} + \dot{E}_s\,\Delta T + \text{2nd order}\end{aligned}$$

replacing T_0 by ω_0 and removing second order terms we get

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} \approx \frac{\Delta\dot{E}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\,\Delta T$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Including $\omega_{0,s}$ inside the derivative with time

$$
\begin{aligned}
\frac{\Delta \dot{E}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\, \Delta T &= \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) - \Delta E \frac{d}{dt}\left(\frac{1}{\omega_{0,s}}\right) + \frac{1}{2\pi}\dot{E}_s \Delta T \\
&= \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \frac{\Delta E}{\omega_{0,s}}\frac{\dot{\omega}_{0,s}}{\omega_{0,s}} + \frac{1}{2\pi}\dot{E}_s\, \Delta T \\
&= (1) + (2) + (3)
\end{aligned}
$$

Expressing (2), using $dE = v\ dp = \omega R\ dp$ and differentiating $\omega = \beta c/R$

$$
\frac{\Delta E}{\omega_{0,s}}\frac{\dot{\omega}_{0,s}}{\omega_{0,s}} = R_s \Delta p \left(\frac{\dot{\beta}_s}{\beta_s} - \frac{\dot{R}_s}{R_s}\right)
$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Expressing (3), using $dE = v\,dp = \omega R\,dp$ and $\eta = \alpha_p - \gamma^{-2} = (dT/T)\,/\,(dp/p)$

$$\frac{1}{2\pi}\dot{E}_s\,\Delta T = \frac{1}{2\pi}\omega_{0,s}R_s\,\dot{p}_s\eta\frac{\Delta p}{p_s}T_s$$

$$= R_s\Delta p\left(\alpha_p - \frac{1}{\gamma_s^2}\right)\frac{\dot{p}_s}{p_s}$$

Summing (2) + (3) and with $d\beta/\beta = \gamma^{-2}dp/p$

$$(2) + (3) = R_s\Delta p\left(\alpha_p\frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right)$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Using the synchrotron differential equation (1) from Module 5

$$R_s \Delta p \left(\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s} \right) = R_s \Delta p \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}$$

From the relativistic differential relationship $dE = \beta c dp, \beta = P/E$ and the definition of the angular revolution frequency $\omega = \beta c / R$

$$\begin{aligned} \ldots &= R_s \frac{\Delta E}{\beta_s c} \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}} \\ &= \left(\frac{\Delta E}{\omega_{0,s}} \right) \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}} \end{aligned}$$

ENERGY GAIN FOR AN ARBITRARY PARTICLE

DERIVATION

Finally

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

The first equation of motion with only the RF contribution is

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) + \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

CONTRIBUTION OF INDUCTION FORCES

WHEN THE BETATRON MEETS THE SYNCHROTRON

- Induction forces were presumed to be negligible in the previous lesson.
- The acceleration of the synchronous particle thanks to induction is indeed very small compared to the acceleration obtained from the RF cavity.
- The difference in acceleration for an **arbitrary particle with respect to synchronous particle** to express ΔE is relevant!

CONTRIBUTION OF INDUCTION FORCES

We add the induction contribution on the previous equation of motion to the right hand side

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right] \qquad (1)$$

$$- \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) \qquad (2)$$

$$+ \frac{q}{2\pi}\int_0^{2\pi}\frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta \qquad (3)$$

It can be demonstrated that (3) is equal to (2) to the first order!

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

Taking into account the induction force, the acceleration rate for the synchronous particle is

$$\dot{E}_s \approx \frac{\delta E_{\mathrm{rf},s} + \delta E_{\mathrm{b},s}}{T_{0,s}}$$

$$\dot{E}_s = \frac{\omega_{0,s}}{2\pi}\left[qV_{\mathrm{rf}}\sin\left(\phi_s\right) + q\int_0^{2\pi}\int_0^{\rho_s}\frac{\partial \mathcal{B}_y}{\partial t}\rho'\,d\rho'\,d\theta\right]$$

The acceleration rate for an arbitary particle is

$$\dot{E} = \frac{\omega_0}{2\pi}\left[qV_{\mathrm{rf}}\sin\left(\phi\right) + q\int_0^{2\pi}\int_0^{\rho}\frac{\partial \mathcal{B}_y}{\partial t}\rho'\,d\rho'\,d\theta\right]$$

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

The difference in the magnetic flux in the surface between the paths of an arbitrary particle and the synchronous one is

$$\begin{aligned}
\int_0^{2\pi}\int_{\rho_s}^{\rho}\frac{\partial\mathcal{B}_y}{\partial t}\rho'\,d\rho'\,d\theta &\approx \int_0^{2\pi}\frac{\partial\mathcal{B}_{y,s}}{\partial t}\int_{\rho_s}^{\rho}\rho'\,d\rho'\,d\theta \\
&= \int_0^{2\pi}\frac{\partial\mathcal{B}_{y,s}}{\partial t}\frac{\rho^2-\rho_s^2}{2}\,d\theta \\
&= \int_0^{2\pi}\frac{\partial\mathcal{B}_{y,s}}{\partial t}\frac{\left(\rho_s+x_D\right)^2-\rho_s^2}{2}\,d\theta \\
\text{(2nd order in } \rho \text{ neglected)} &\approx \int_0^{2\pi}\frac{\partial\mathcal{B}_{y,s}}{\partial t}\rho_s\,x_D\,d\theta
\end{aligned}$$

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

The difference in acceleration rate between an arbitrary and the synchronous particle becomes to the first order

$$\frac{\dot{E}}{\omega_0} - \frac{\dot{E}_s}{\omega_{0,s}} = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right] + \frac{q}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta$$

We now demonstrate that the induction term on the right hand side is equal to

$$\frac{q}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta = \alpha_p \frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

and hence compensate with the term (2) here.

CONTRIBUTION OF INDUCTION FORCES

DERIVATION

Using the definition of the momentum compaction factor (from dispersion)

$$\frac{q}{2\pi}\int_0^{2\pi}\frac{\partial\mathcal{B}_{y,s}}{\partial t}\rho_s x_D d\theta = \frac{q}{2\pi}\dot{\mathcal{B}}_{y,s}\rho_s\int_0^{2\pi}D_x\frac{\Delta p}{p}d\theta$$

$$= \frac{q}{2\pi}\dot{\mathcal{B}}_{y,s}\rho_s\; 2\pi R_s\;\alpha_p\frac{\Delta p}{p_s}$$

$$= \alpha_p\frac{q\rho_s}{p_s}\dot{\mathcal{B}}_{y,s}R_s\;\Delta p$$

$$= \alpha_p\frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}R_s\frac{\Delta E}{\beta_s c}$$

$$= \alpha_p\frac{\dot{\mathcal{B}}_{y,s}}{\mathcal{B}_{y,s}}\left(\frac{\Delta E}{\omega_{0,s}}\right) \quad = \text{-}\ (2)$$

FIRST LONGITUDINAL EQUATION OF MOTION

EVOLUTION OF THE RELATIVE ENERGY OF AN ARBITRARY PARTICLE

We obtain finally

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

This is the first fundamental longitudinal equation of motion

Note that $\omega_{0,s}$ is inside of the time derivative!

PHASE SLIPPAGE

In the previous lesson, we considered the synchronous particle which is by definition always synchronous to the RF and arrives at the phase ϕ_s.

The evolution of the azimuth of an arbitrary particle with time is

$$\theta\left(t\right) = \int \omega_0 dt$$

The phase of an arbitrary particle with respect to the RF at all time is

$$\phi\left(t\right) = \int \omega_r dt - h\theta\left(t\right) = -h \int \Delta\omega_0 dt$$

Remember that $\omega_r = h\omega_{s,0}$. Notice the $-$ sign, a particle in front in azimuth (higher θ) will arrive earlier in the cavity (lower ϕ)

FIRST LONGITUDINAL EQUATION OF MOTION

EVOLUTION OF THE PHASE WITH RESPECT TO THE RF OF AN ARBITRARY PARTICLE

By differentiating with time and including the phase slip factor η

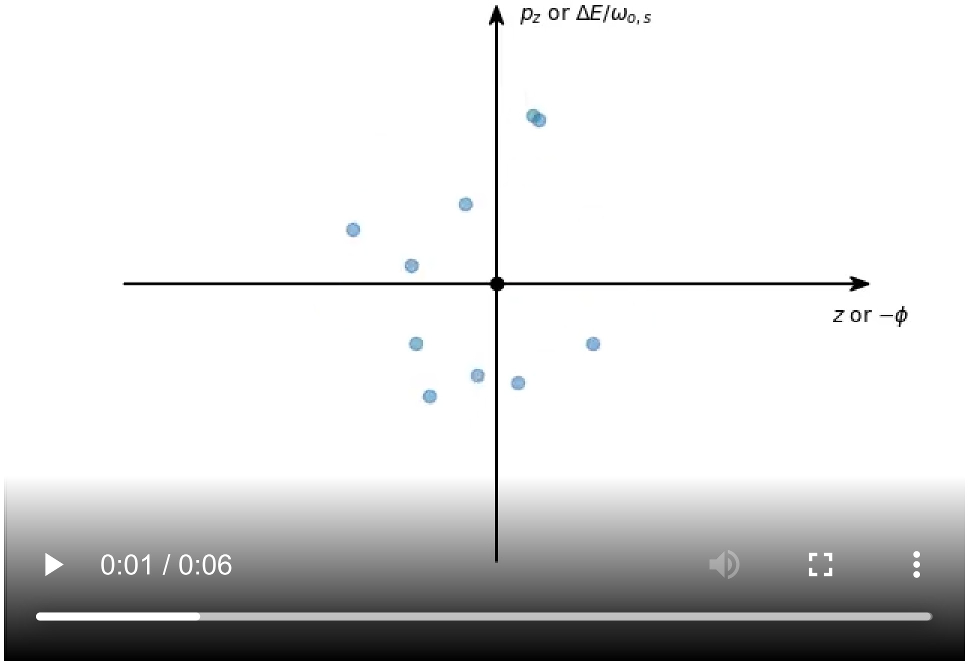
$$\frac{d\phi}{dt} = -h\Delta\omega_0$$
$$= h\eta\omega_{0,s}\frac{\Delta p}{p_s}$$

Using the differential relationship $dE/E = \beta^2 dp/p$

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

This is the second fundamental longitudinal equation of motion

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

- From now on we will describe the motion in the longitudinal phase space $\left(\phi, \frac{\Delta E}{\omega_{0,s}}\right)$, instead of (z, p_z)

MODULE 7: INTRODUCTION TO PARTICLE TRACKING

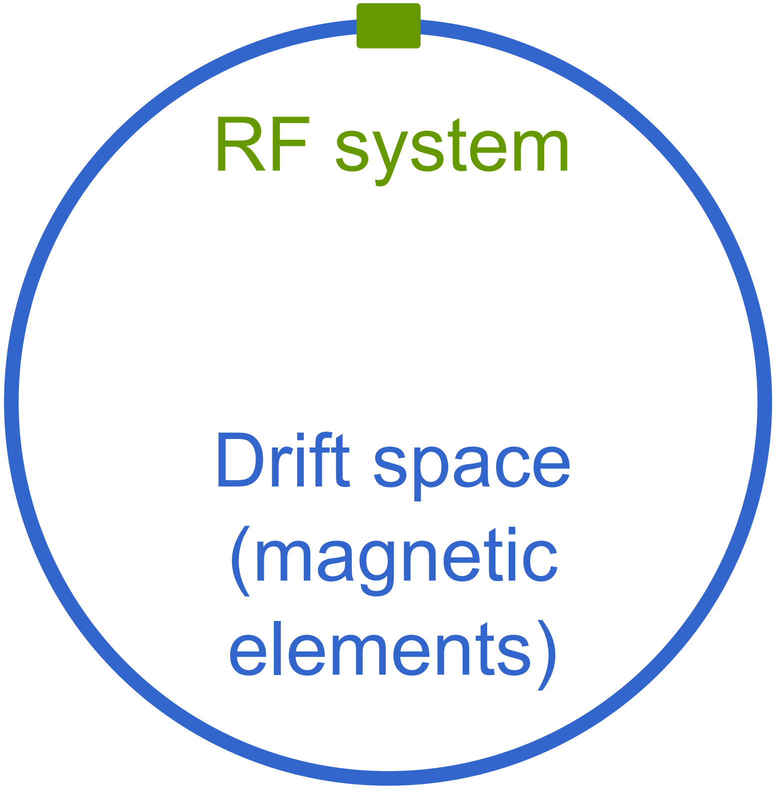
→ **Defining the accelerator**

→ **Implementing and iterating the equations of motion**

→ **Examples**

SYNCHROTRON DEFINITION

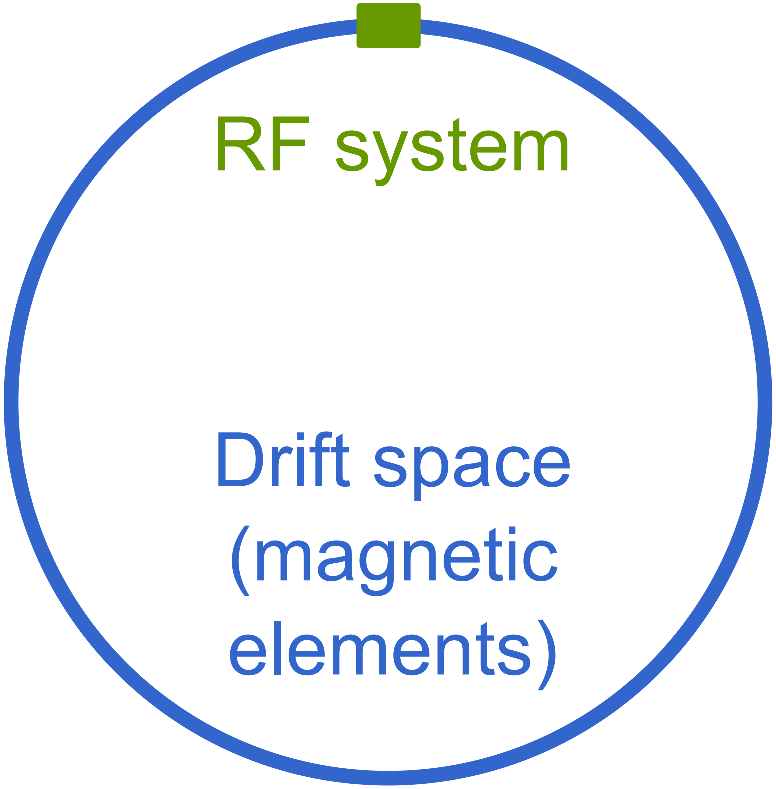
Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- In this part we implement numerically the equations of motion.
- The synchrotron motion can be complex to grasp, a tracking code can help to visualize easily the longitudinal motion in phase space.
- A rather simple tracking code can allow to do very accurate simulations, in a rather small number of code lines!

SYNCHROTRON DEFINITION

Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- Definition of "design" energy (momentum)

$$p_d = q\mathcal{B}_y \rho_d$$

- Definition of "design" revolution period

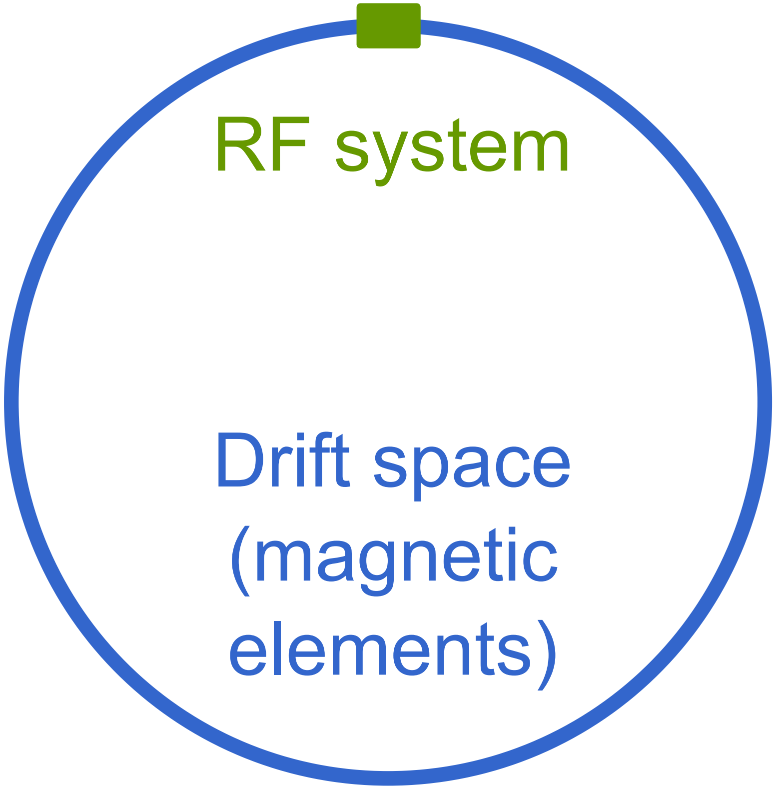
$$\mathcal{B}_y, p_d \rightarrow T_{0,d} = \frac{2\pi}{\omega_{0,d}} = \frac{C_d}{\beta_d c}$$

- Definition of RF parameters

$$\omega_r = h\omega_{0,d}$$

SYNCHROTRON DEFINITION

Simplest configuration, starting with a single RF system and assuming no longitudinal energy gain/loss in the rest of the ring



- In the context of the tracking code, we will use the $(\phi, \Delta E)$ coordinate system
- Relative energy of an arbitrary particle

$$\Delta E = E - E_d$$

- Phase of the particle relative to the RF

$$\phi$$

- NB: $(\phi, \Delta E)$ are not canonical variables, $(\tau, \Delta E)$ or $(\phi, \Delta E/\omega_{0,s})$ are canonical.

ENERGY GAIN IN RF CAVITY

- The longitudinal equation of motion (continuous in t) is commonly called the "kick" equation (NB: we neglect $\dot{\omega}_{0,d}$ for simplicity)

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,d}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_d\right)\right] \approx \frac{\Delta E^{n+1} - \Delta E^{n}}{\omega_{0,d} T_{0,d}}$$

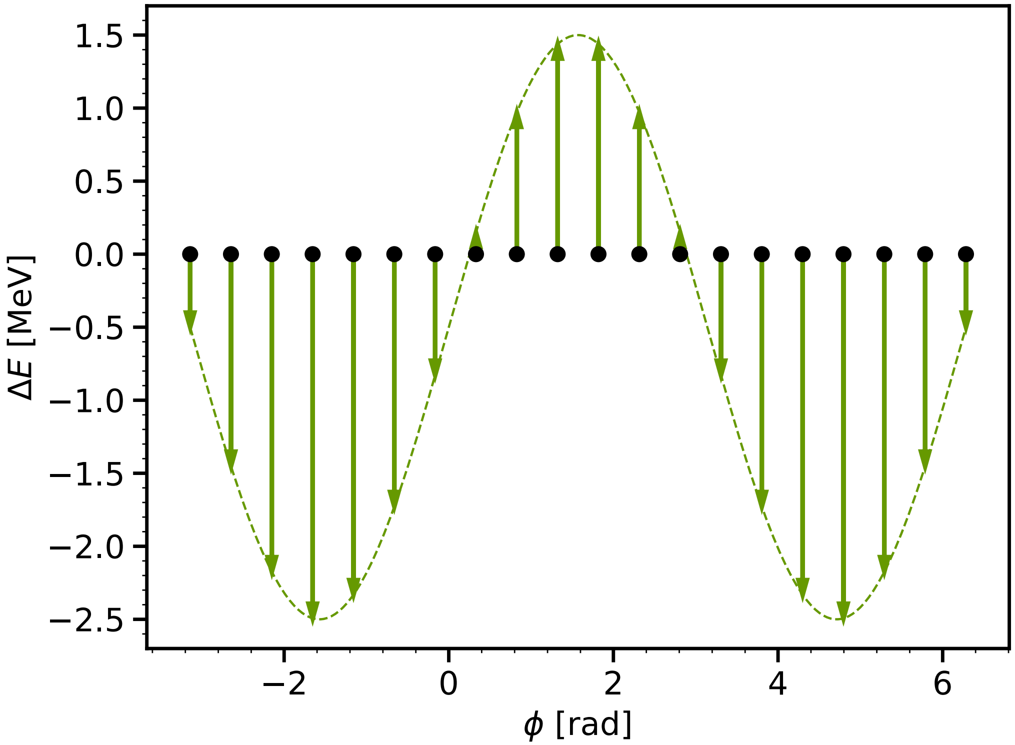
- The equation of motion is actually discrete by nature, the relative energy gain every turn $T_{0,d}$ is

$$\Delta E^{n+1} = \Delta E^{n} + qV \sin\left(\phi\right) - \delta E_d^{n\rightarrow n+1}$$

- The acceleration per turn is

$$\delta E_d^{n\rightarrow n+1} = 2\pi q \rho_d R_d \frac{\mathcal{B}_y^{n+1} - \mathcal{B}_y^{n}}{T_{0,d}}$$

ENERGY GAIN IN RF CAVITY



- Example for protons passing in RF system with $V = 2\mathrm{MV}$, $f_{\mathrm{rf}} = 200\mathrm{MHz}$, $\delta E_d = 0.5\mathrm{MeV}$

DRIFT

- The phase slip equation is commonly called the "drift" equation, neglectic any source of change in ΔE along the magnetic elements

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{\phi^{n+1} - \phi^n}{T_{0,d}}$$

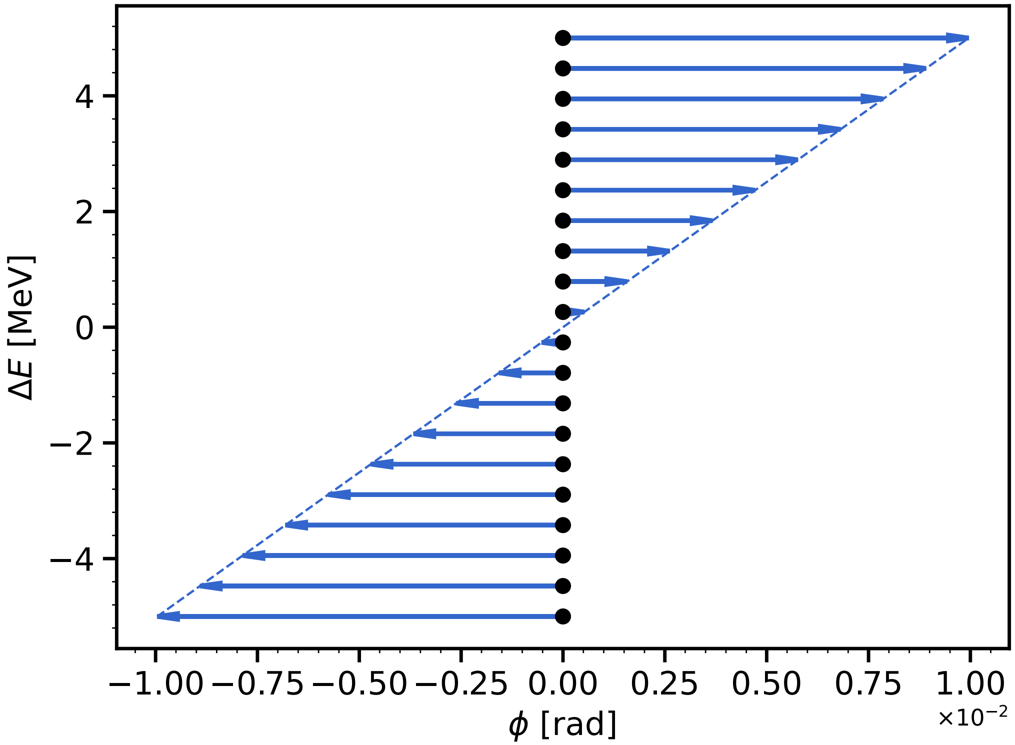
- Drift in time of an arbitrary particle with respect to the design particle after $T_{0,d}$

$$\phi^{n+1} = \phi^n + \left(\frac{2\pi h\eta_0}{\beta^2 E}\right)_d \Delta E$$

with

$$\eta_{0,d} = \frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma_d^2} = \alpha_0 - \frac{1}{\gamma_d^2}$$

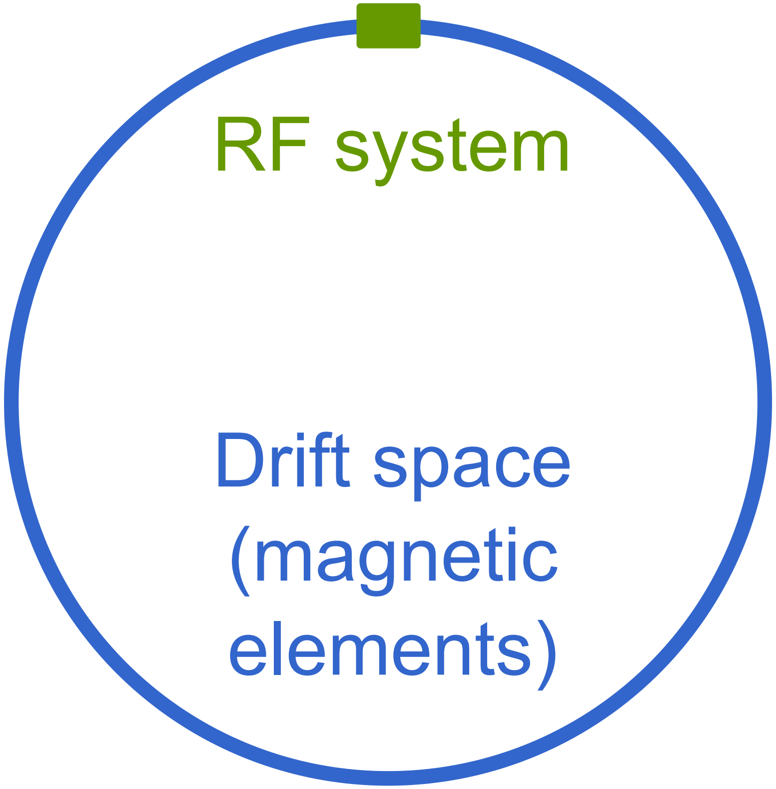
DRIFT



- Example for protons passing along a ring with $C_d = 6911.50\mathrm{m}$, $E_d = 26\mathrm{GeV}$, $\gamma_{\mathrm{tr}} = 18$

TRACKING

The two equations of motion are sufficient to build a simple **(yet very useful)** tracking code



- The tracking can be coded as

```
for n_turns:
    for n_particles:
        dE += rf_kick(phi)
        phi += drift(dE)
```

- Where

```
def rf_kick(phi):
    return q*Vrf*sin(phi) - q*Vrf*sin(phi_s)
```

```
def drift(dE):
    return 2*pi*h*eta_0 / (beta**2 * E) * dE
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON

The present example is using Python, but any language could be used following the pseudo-code layout from the previous slide.

- We import useful libraries

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.constants import m_p, c, e
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON

- Define our machine parameters

```
Ekin = 26e9  # eV

charge = 1
E0 = m_p * c**2. / e
circumference = 6911.5  # m
energy = Ekin + E0
momentum = np.sqrt(energy**2. - E0**2.)
beta = momentum / energy
gamma = energy / E0

t_rev = circumference / (beta * c)
f_rev = 1 / t_rev

harmonic = 4620
voltage = 4.5e6  # V
f_rf = harmonic * f_rev
t_rf = 1 / f_rf

gamma_t = 18
alpha_c = 1 / gamma_t**2.
eta = alpha_c - 1 / gamma**2.
```

- Print the parameters of the machine

```
print("Beta: " +
      str(beta))
print("Gamma: " +
      str(gamma))
print("Revolution period: " +
      str(t_rev * 1e6) + " mus")
print("RF frequency: " +
      str(f_rf / 1e6) + " MHz")
print("RF period: " +
      str(t_rf * 1e9) + " ns")
print("Momentum compaction factor: " +
      str(alpha_c))
print("Phase slippage factor: " +
      str(eta))
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- Define our machine parameters

```
Ekin = 26e9  # eV

charge = 1
E0 = m_p * c**2. / e
circumference = 6911.5  # m
energy = Ekin + E0
momentum = np.sqrt(energy**2. - E0**2.)
beta = momentum / energy
gamma = energy / E0

t_rev = circumference / (beta * c)
f_rev = 1 / t_rev

harmonic = 4620
voltage = 4.5e6  # V
f_rf = harmonic * f_rev
t_rf = 1 / f_rf

gamma_t = 18
alpha_c = 1 / gamma_t**2.
eta = alpha_c - 1 / gamma**2.
```

- Define your tracking functions

```
def drift(dE, harmonic, eta, beta, energy):

    return 2 * np.pi * harmonic * \
        eta * dE / (beta**2 * energy)

def rf_kick(phi, charge, voltage, phi_s=0):

    return charge * voltage * (
        np.sin(phi) - np.sin(phi_s))
```

- Define your initial particle positions (test example)

```
n_particles = 10  # or millions ? :)

phase_coordinates = np.linspace(
    0, 2 * np.pi, n_particles)

dE_coordinates = np.zeros(n_particles)
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- Track!!!

```
n_turns = 10  # or millions? :)

for idx_turn in range(n_turns):
    dE_coordinates += rf_kick(
        phase_coordinates, charge, voltage)

    phase_coordinates += drift(
        dE_coordinates, harmonic, eta, beta, energy)
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- The particle coordinates can then be monitored at each turn during the tracking.

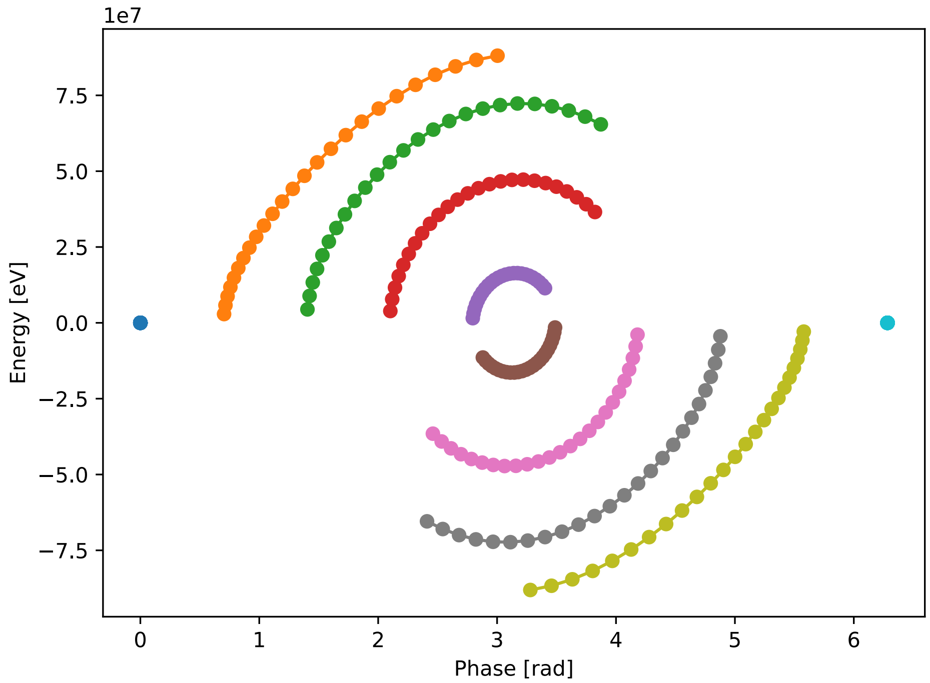
```python
n_turns = 25  # or millions? :)

saved_positions_phi = np.zeros((n_particles, n_turns))
saved_positions_dE = np.zeros((n_particles, n_turns))

for idx_turn in range(n_turns):
    dE_coordinates += rf_kick(
        phase_coordinates, charge, voltage)

    phase_coordinates += drift(
        dE_coordinates, harmonic, eta, beta, energy)

    saved_positions_dE[:, idx_turn] = dE_coordinates
    saved_positions_phi[:, idx_turn] = phase_coordinates
```

TRACKING

A REALISTIC WORKING CODE IN PYTHON (AND NUMPY)

- The trajectory of the particles can be visualized

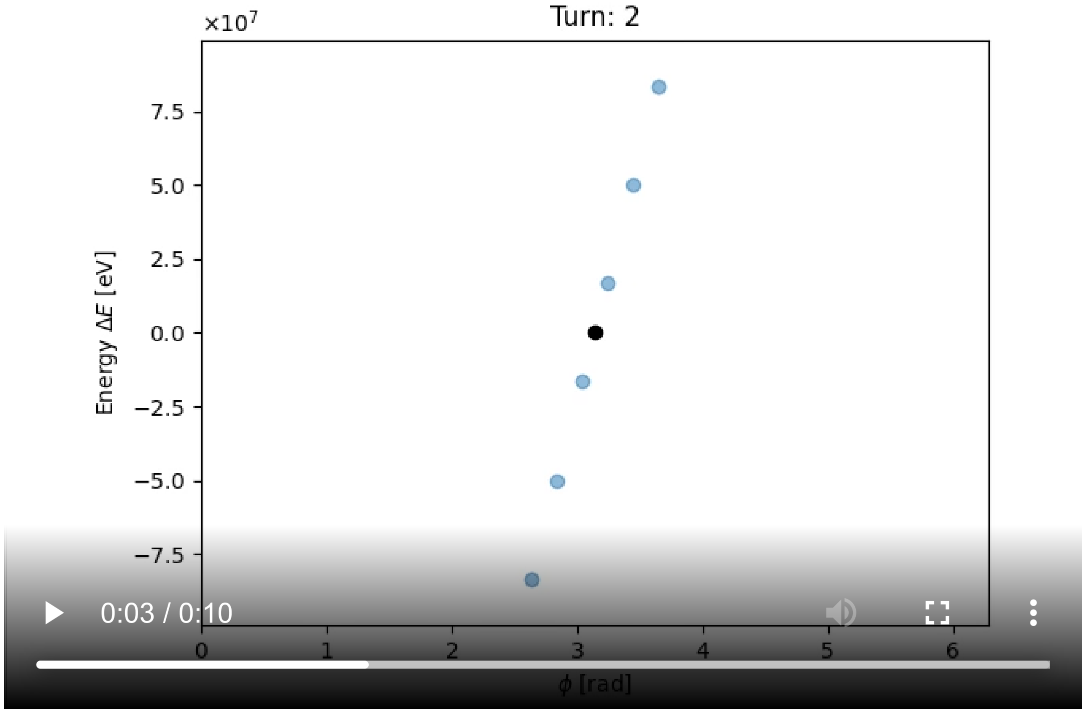
```
plt.figure('phase space')
plt.clf()
for idx_particle in range(n_particles):
    plt.plot(
        saved_positions_phi[idx_particle, :],
        saved_positions_dE[idx_particle, :],
        '-o')
plt.xlabel('Phase [rad]')
plt.ylabel('Energy [eV]')
```



- The final script is less than 100 lines long!

EXAMPLES

DRIFT ONLY

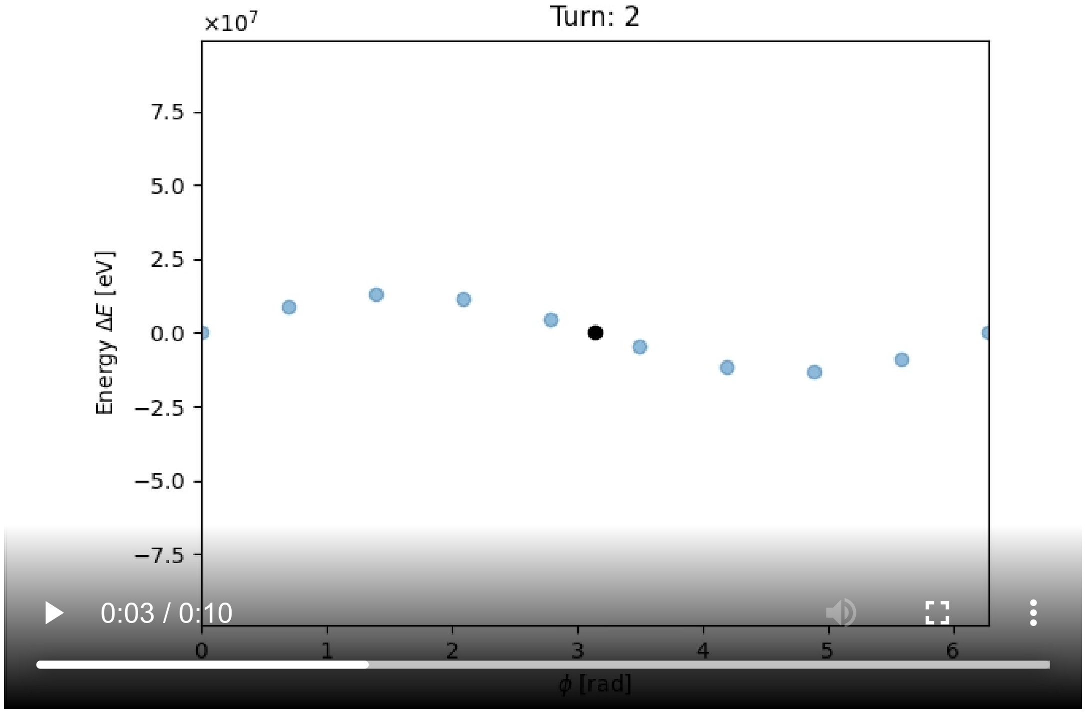


Only with the drift equation of motion, no RF.

Particles get distant from each other.

EXAMPLES

RF KICK ONLY

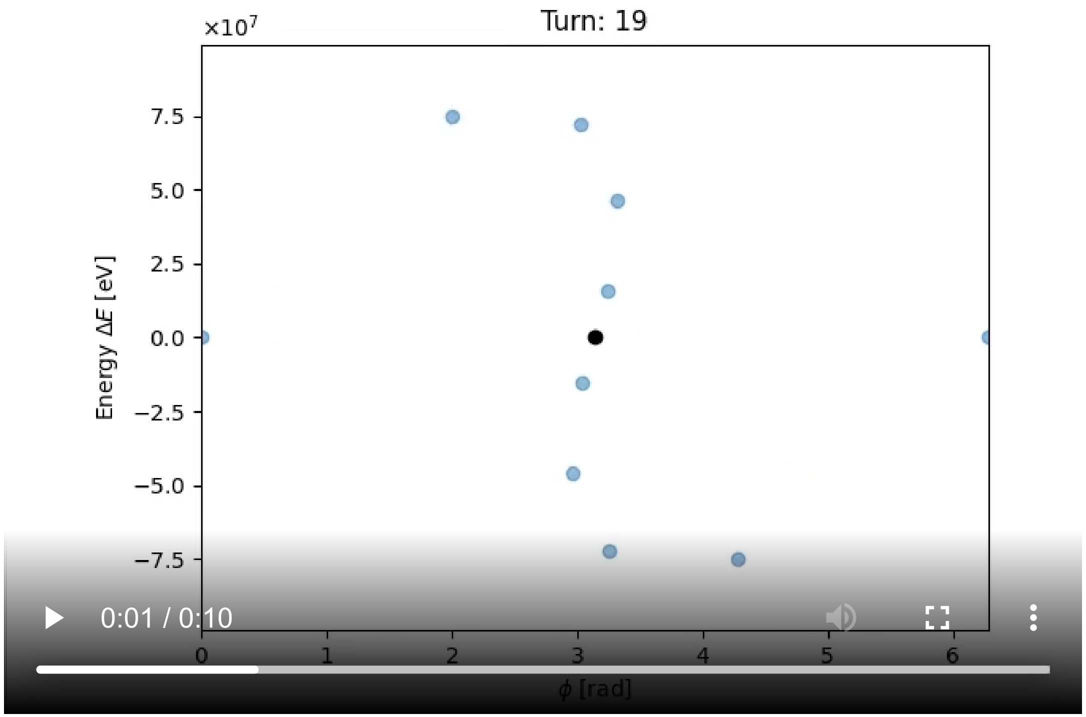


Only RF, no drift.

Particles all get accelerated/decelerated with respect to the synchronous particle.

EXAMPLES

RF KICK AND DRIFT

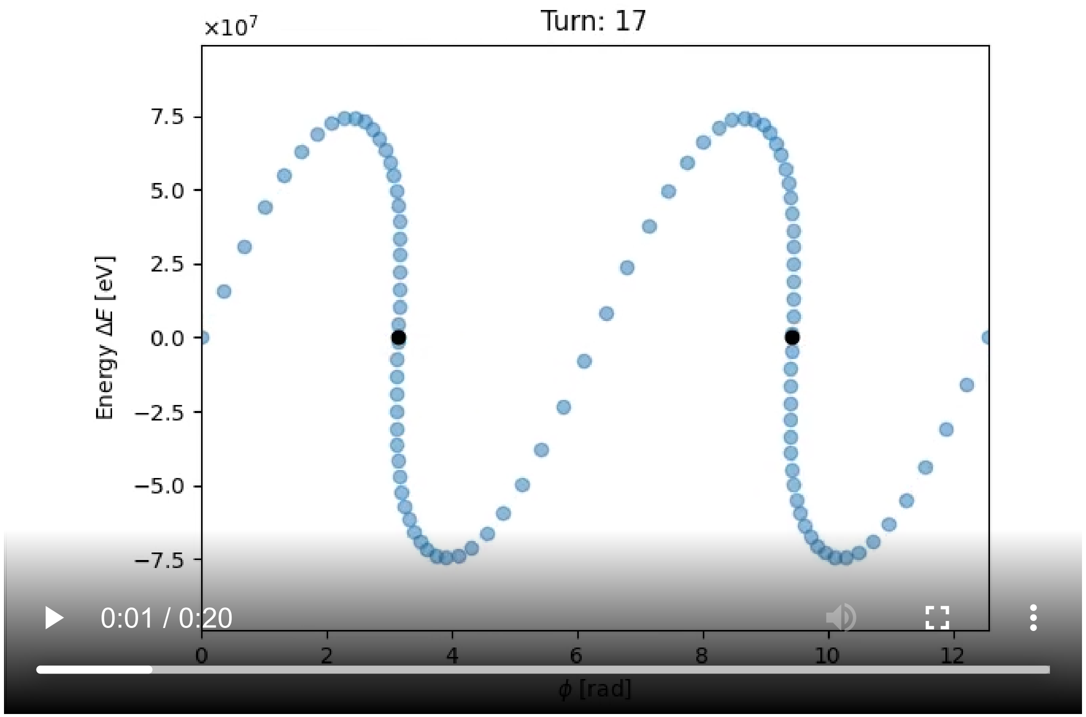


Combining RF kick and drift, particles start to oscillate around the synchronous particle.

Linear close to the synchronous particle, non-linear at large amplitude.

EXAMPLES

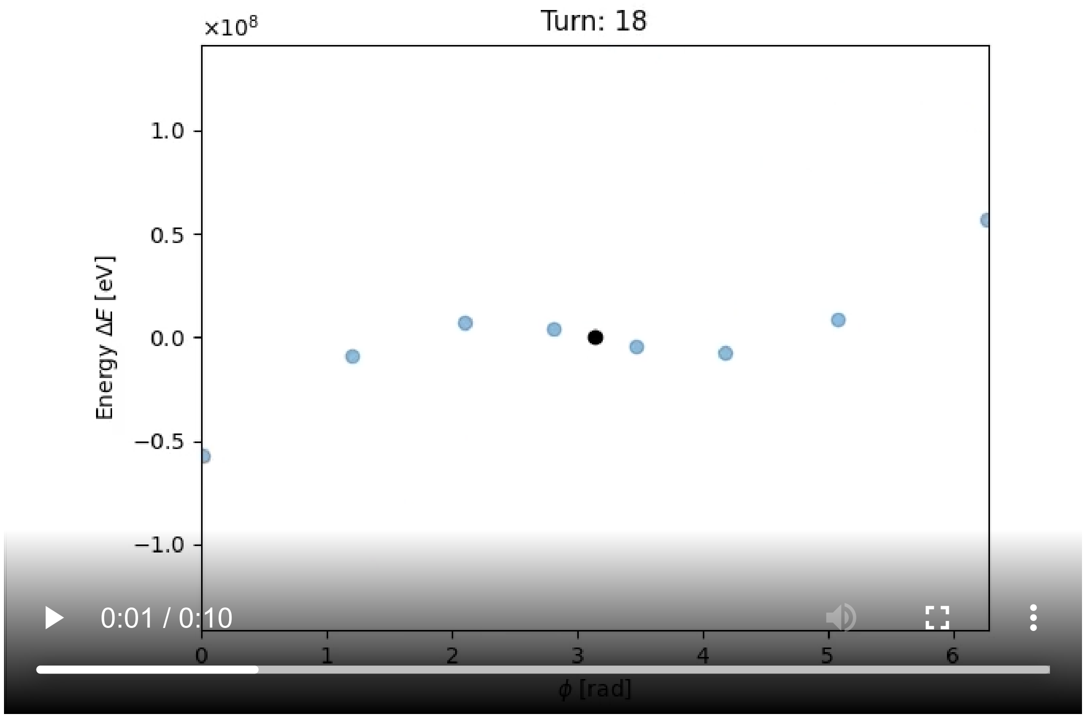
AROUND OTHER SYNCHRONOUS PARTICLES



Particles oscillate around one of the h possible synchronous particles.

EXAMPLES

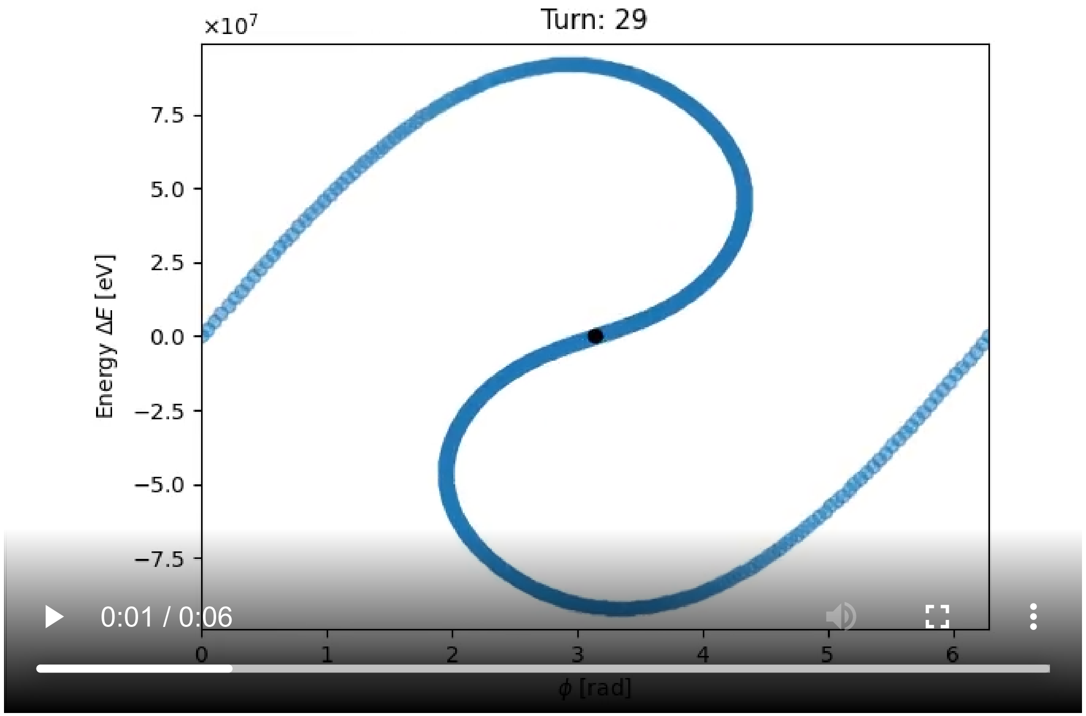
VERY LARGE AMPLITUDES



If particles have very different energies than the synchronous particle, they don't oscillate around a stable point anymore.

EXAMPLES

LIMIT OF PHASE STABILITY

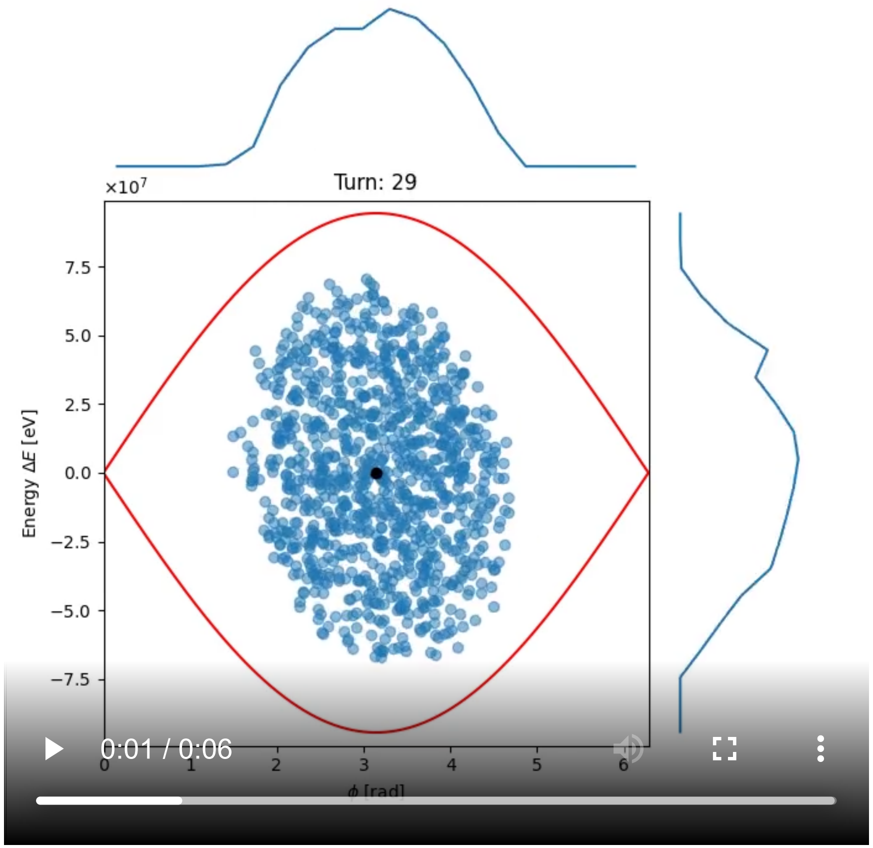


A contour of the limit of phase stability is easily obtained with tracking.

It is called the RF bucket.

EXAMPLES

REALISTIC BUNCH IN THE RF BUCKET



- The only measurable in reality is the line density (top line), corresponding to the histogram in ϕ.
- Tracking is an essential tool to compare computations with measurements!

TAKE AWAY MESSAGE

LONGITUDINAL EQUATIONS OF MOTION

- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\mathrm{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

LESSON 4: SYNCHROTRON MOTION

MODULE 8: LINEAR SYNCHROTRON MOTION

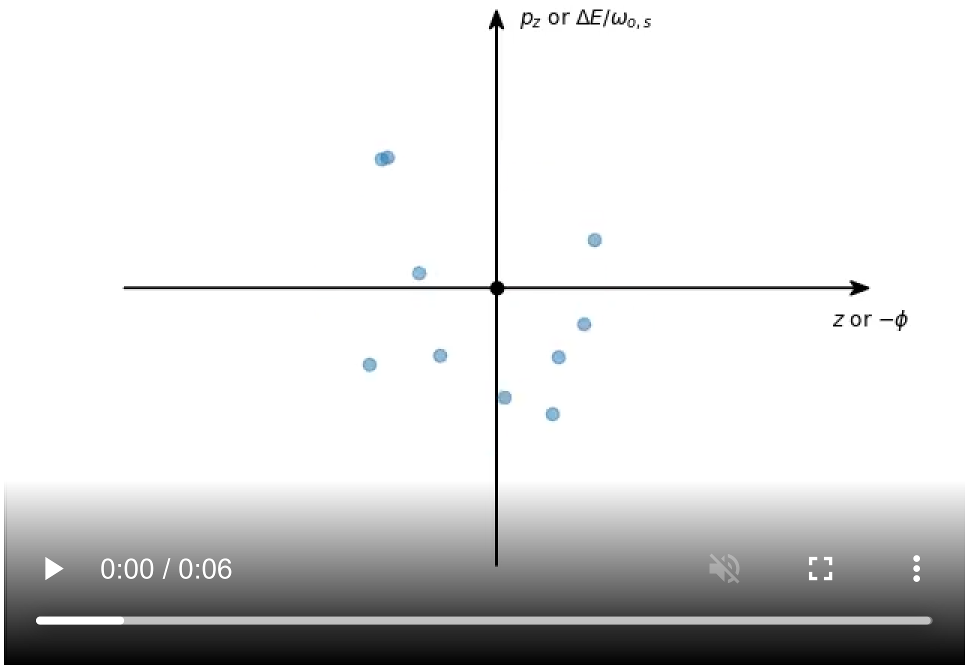
→ **Combined linear equations of motion**

→ **Linear synchrotron frequency, tune**

→ **Phase stability, transition crossing**

→ **Emittance, adiabaticity**

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_{0,s}}\right) = \frac{qV_{\text{rf}}}{2\pi}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{h\eta\omega_{0,s}^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_{0,s}}\right)$$

- The longitudinal equations of motion describe the evolution of the phase ϕ and energy ΔE of an arbitrary particle compared to the synchronous particle.

COMBINING THE EQUATIONS OF MOTION

The two equations of motion are inter-dependant and can be combined (*note that we replaced* $\omega_{0,s}$ *by* ω_r/h*, which is equivalent and will become relevant later*)

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_r}\right) = \frac{qV_{\text{rf}}}{2\pi h}\left[\sin(\phi) - \sin(\phi_s)\right] \qquad (1)$$

$$\frac{d\phi}{dt} = \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right) \qquad (2)$$

By incorporating (2) in (1), we get

$$\frac{d}{dt}\left(\frac{d\phi}{dt}\frac{\beta_s^2 E_s}{\eta\omega_r^2}\right) = \frac{qV_{\text{rf}}}{2\pi h}\left[\sin(\phi) - \sin(\phi_s)\right]$$

COMBINING THE EQUATIONS OF MOTION

We will first make two important approximations:

- The machine and beam parameters s are changing slowly with time (only ϕ and ΔE are functions of time)
- We consider small phase oscillations $\Delta\phi = \phi - \phi_s$ (reminder: $\dot{\phi}_s = 0$ by definition)

The $\sin$ functions on the right hand side are linearized

$$\begin{aligned}\sin(\phi) - \sin(\phi_s) &= \sin(\phi_s + \Delta\phi) - \sin(\phi_s) \\ &= \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi - \sin\phi_s \\ &\approx \cos\phi_s \Delta\phi\end{aligned}$$

COMBINING THE EQUATIONS OF MOTION

The approximations lead to

$$\frac{d^2\Delta\phi}{dt^2} = \frac{qV_{\text{rf}}\eta\omega_r^2}{2\pi h\beta_s^2 E_s}\cos\phi_s\Delta\phi$$

$$\implies \frac{d^2\Delta\phi}{dt^2} + \omega_{s0}^2\Delta\phi = 0$$

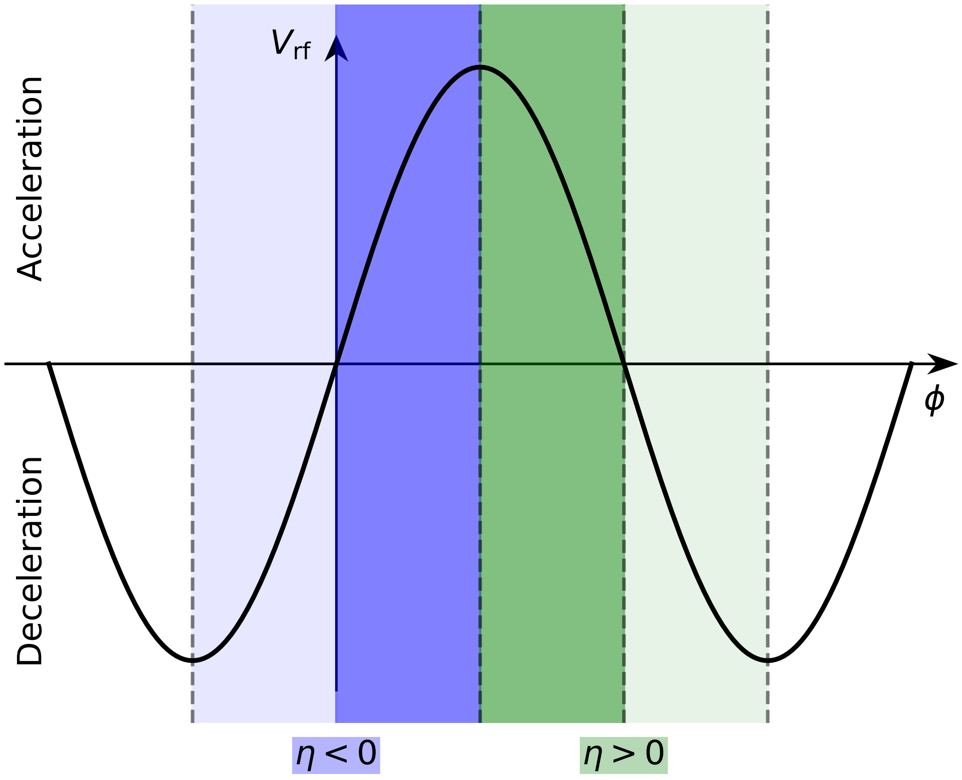
where the linear synchrotron (angular) frequency is defined as (**beware $\boldsymbol{\omega_{s0} \neq \omega_{0,s}}$**)

$$\omega_{s0} = 2\pi f_{s_0} = \sqrt{-\frac{qV_{\text{rf}}\omega_r^2\eta\cos\phi_s}{2\pi h\beta_s^2 E_s}}$$

The motion of the particles in the longitudinal phase space (synchrotron motion) is a harmonic oscillator for small $\Delta\phi$, under the condition that $\boldsymbol{\eta\cos\phi_s < 0}$.

PHASE STABILITY

EXAMPLE FOR POSITIVELY CHARGED PARTICLES



- The phase stability condition

$$\eta \cos \phi_s < 0$$

imposes that the synchronous phase is

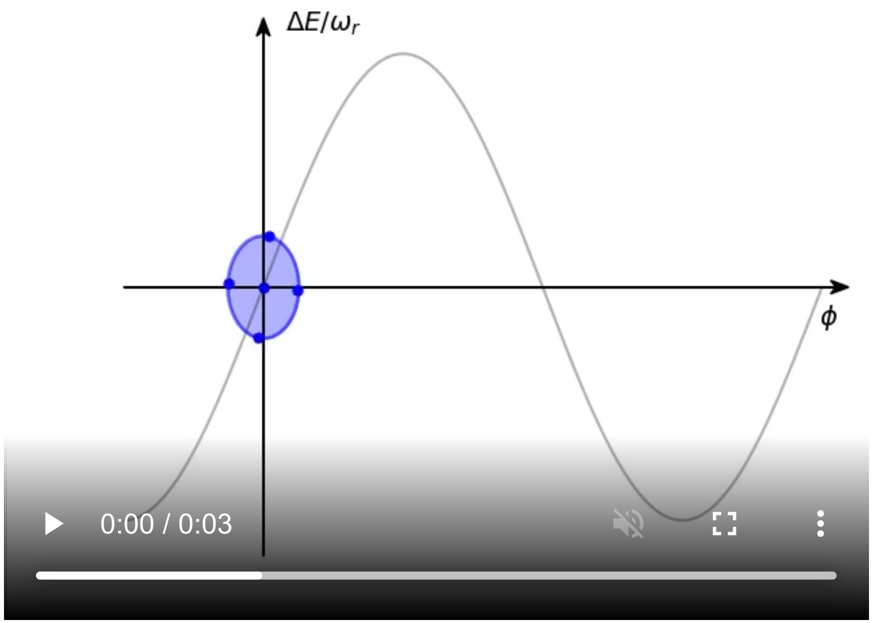
$$\eta < 0 \rightarrow \phi_s \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

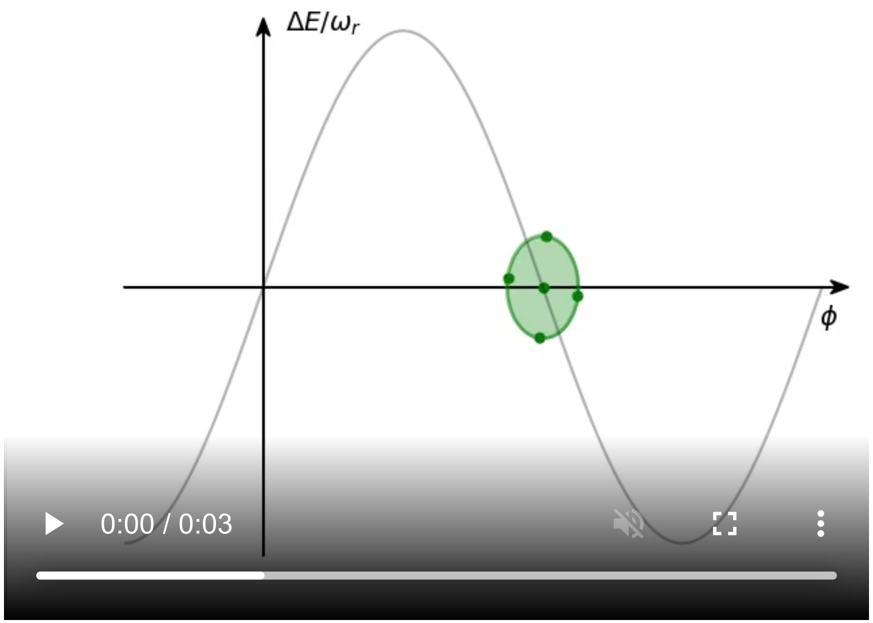
$$\eta > 0 \rightarrow \phi_s \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

- A non-synchronous particle rotates around the synchronous particle only if the phase stability condition is fulfilled.

PHASE STABILITY

EXAMPLE FOR POSITIVELY CHARGED PARTICLES, INTUITIVELY

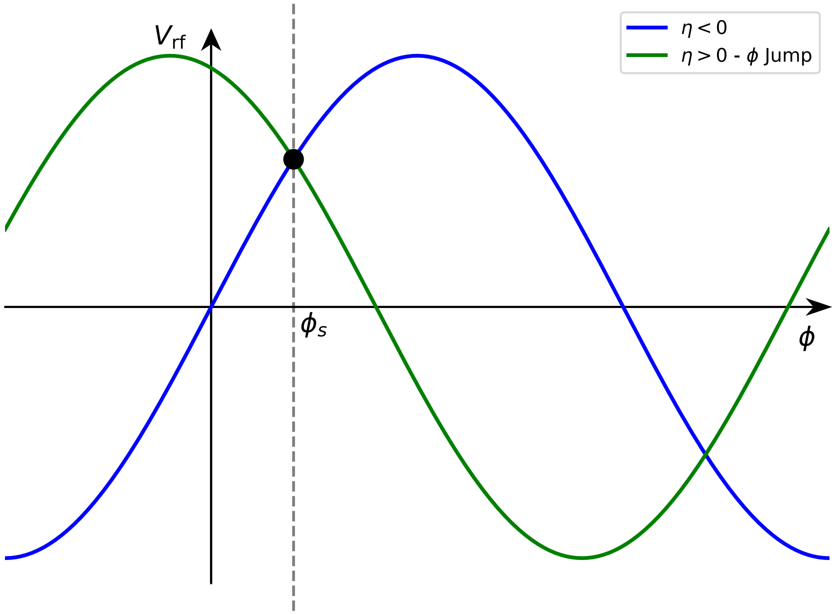




- Below transition $\eta < 0$, early particles should lose energy (velocity) and be delayed.
- Above transition $\eta > 0$, early particles should gain energy to travel a longer orbit and be delayed.

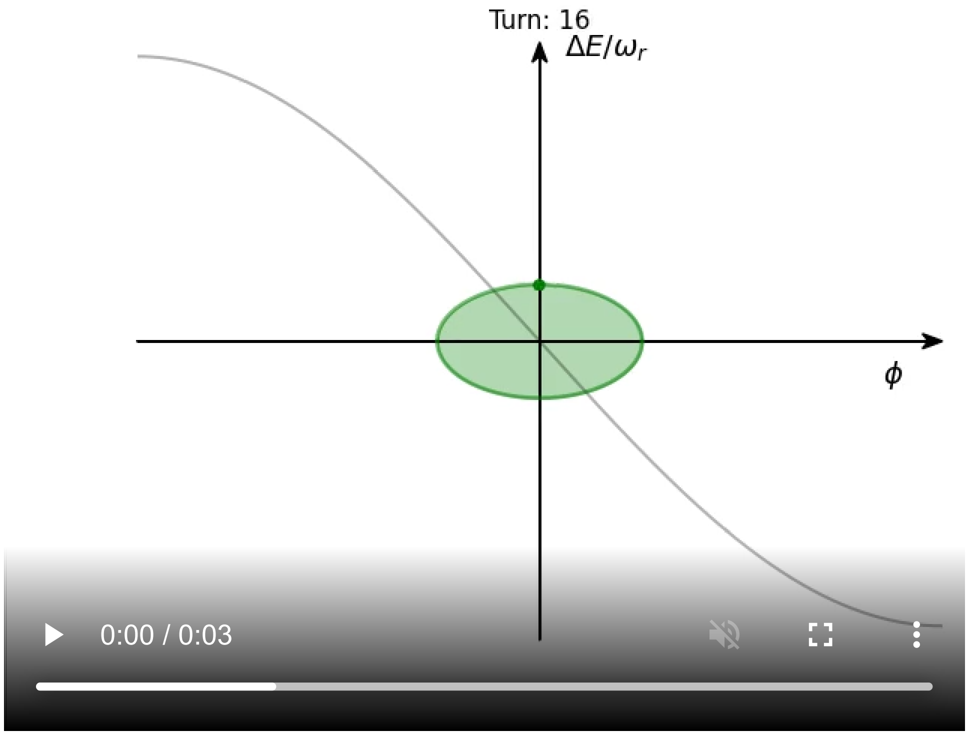
TRANSITION CROSSING

During acceleration, transition energy is crossed and η changes sign. As a reminder, $\eta(t) = \alpha_p - 1/\gamma_s^2(t)$.



In that occurence, the phase of the RF phase after transition must be rapidly shifted by $\pi - 2\phi_s$ to preserve the phase stability.

LINEAR SYNCHROTRON TUNE



- The linear synchrotron tune is defined as the ratio of the synchrotron frequency to the revolution frequency

$$Q_{s0} = \frac{\omega_{s0}}{\omega_{0,s}} = \sqrt{-\frac{qV_{\mathrm{rf}} h\eta\cos\phi_s}{2\pi\beta_s^2 E_s}}$$

- The inverse of the synchrotron tune gives the number of machine turns needed to perform one full period in longitudinal phase space.

- The synchrotron tune (longitudinal, $\mathcal{O}\left(10^{-3} - 10^{-2}\right)$) is usually much smaller than the betatron tune (transverse, $\mathcal{O}\left(1 - 10^{2}\right)$).

AMPLITUDE OF OSCILLATIONS

The solutions for the evolution of the parameters of the non-synchronous particle are

$$\Delta\phi\left(t\right) = \Delta\phi_m \sin\left(\omega_{s0} t\right)$$
$$\left(\frac{\Delta E}{\omega_r}\right)\left(t\right) = \left(\frac{\Delta E}{\omega_r}\right)_m \cos\left(\omega_{s0} t\right)$$

where the maximum amplitudes of oscillations in phase and energy are noted with the subscript m. The synchrotron angle is noted $\psi = \omega_{s0} t$.

The ratio in the amplitudes of oscillation is

$$\frac{\left(\Delta E/\omega_r\right)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{\left|\eta\right| \omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{\left|\eta\right| h^2 \omega_{0,s}} Q_{s0}$$

AMPLITUDE OF OSCILLATIONS

DERIVATION

Demonstrate the ratio of maximum amplitudes in phase/energy

$$\frac{(\Delta E/\omega_r)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{|\eta|\,\omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{|\eta|\,h^2\omega_{0,s}}Q_{s0}$$

Hint: Include the solution for $\Delta\phi$ in the equation of motion as a start.

AMPLITUDE OF OSCILLATIONS

DERIVATION

Starting from

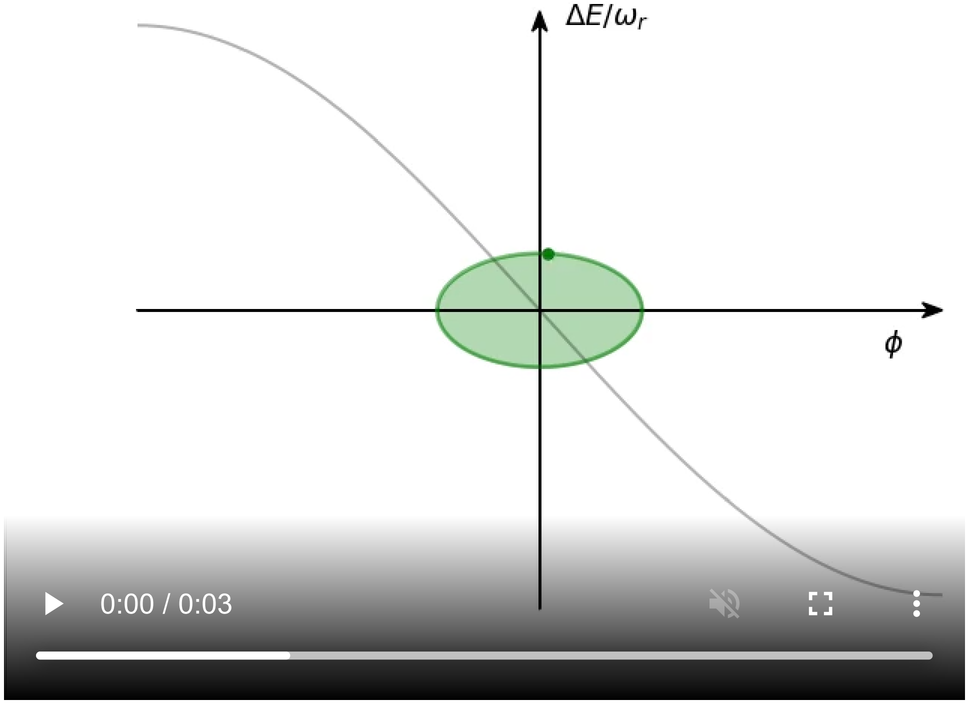
$$\begin{aligned} \Delta\phi\left(t\right) &= \Delta\phi_m \sin\left(\omega_{s0} t\right) \\ \Longrightarrow \Delta\dot{\phi} &= \Delta\phi_m \omega_{s0} \cos\left(\omega_{s0} t\right) \\ \Longrightarrow \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)_m \cos\left(\omega_{s0} t\right) &= \Delta\phi_m \omega_{s0} \cos\left(\omega_{s0} t\right) \end{aligned}$$

We obtain

$$\frac{\left(\Delta E/\omega_r\right)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{\left|\eta\right| \omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{\left|\eta\right| h^2 \omega_{0,s}} Q_{s0}$$

We took $\left|\eta\right|$ to obtain positive phase/energy maximum amplitudes.

LINEAR LONGITUDINAL EMITTANCE



- The trajectory of the particles in phase space is an ellipse of the form

$$\left(\frac{\Delta\phi}{\Delta\phi_m}\right)^2 + \left(\frac{\Delta E/\omega_r}{\left[\Delta E/\omega_r\right]_m}\right)^2 = 1$$

- The surface of the ellipse corresponds to the longitudinal emittance of a particle. A linear approximation is

$$\varepsilon_{l,0} = \frac{\pi}{\omega_r}\Delta E_m \Delta\phi_m = \pi\Delta E_m \Delta\tau_m$$

- The longitudinal emittance is expressed in the $[\mathrm{eV}\cdot\mathrm{s}]$ unit and is constant for a particle as long as machine parameters are changed slowly (adiabatically).

LINEAR LONGITUDINAL EMITTANCE

EXPRESSION

We note the bunch length $\tau_l = 2\Delta\tau_m$ corresponding to the diameter of the particle with the largest amplitude.

The linear longitudinal emittance becomes

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \frac{\pi \beta_s^2 E_s}{4\left|\eta\right|} \omega_{s0} \tau_l^2$$
$$= \frac{\pi \left|\eta\right|}{\beta_s^2 E_s} \frac{1}{\omega_{s0}} \Delta E_m^2$$

In practice, the longitudinal emittance of a bunch is estimated from the measured bunch length together with the machine parameters.

The bunch momentum spread is $\delta_p = 2\Delta p_m / p_s = 2\Delta E_m / \left(\beta_s^2 E_s\right)$.

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Demonstrate that

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \frac{\pi \beta_s^2 E_s}{4\left|\eta\right|} \omega_{s0} \tau_l^2$$
$$= \frac{\pi \left|\eta\right|}{\beta_s^2 E_s} \frac{1}{\omega_{s0}} \Delta E_m^2$$

Hint: replace the phase or energy deviation with the one obtained from the energy/phase amplitude ratios.

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Replacing ΔE_m and using $\Delta\phi_m = \omega_r \tau_l / 2$

$$\varepsilon_{l,0} = \pi \Delta E_m \frac{\tau_l}{2} = \pi \omega_r \Delta\phi_m \frac{\beta_s^2 E_s}{|\eta|\, \omega_r^2} \omega_{s0} \frac{\tau_l}{2} = \pi \frac{\omega_r \tau_l}{2} \frac{\beta_s^2 E_s}{|\eta|\, \omega_r} \omega_{s0} \frac{\tau_l}{2}$$

$$\varepsilon_{l,0} = \frac{\pi \beta_s^2 E_s}{4\,|\eta|} \omega_{s0} \tau_l^2$$

$$= \frac{\pi \beta_s^2 E_s}{4\,|\eta|} \sqrt{-\frac{q V_{\mathrm{rf}} \omega_r^2 \eta \cos\phi_s}{2\pi h \beta_s^2 E_s}} \tau_l^2$$

$$= \tau_l^2 \sqrt{-\frac{\pi}{32} \frac{\omega_{0,s}^2 \beta_s^2 E_s}{\eta} q V_{\mathrm{rf}} h \cos\phi_s}$$

LINEAR LONGITUDINAL EMITTANCE

DERIVATION

Replacing $\Delta\phi_m$

$$\varepsilon_{l,0} = \frac{\pi}{\omega_r}\Delta E_m \Delta\phi_m = \frac{\pi}{\omega_r}\Delta E_m \frac{1}{\omega_{s0}}\frac{|\eta|\,\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)_m$$

$$\varepsilon_{l,0} = \frac{\pi\,|\eta|}{\beta_s^2 E_s}\frac{1}{\omega_{s0}}\Delta E_m^2$$

$$= \frac{\pi\,|\eta|}{\beta_s^2 E_s}\sqrt{-\frac{2\pi h \beta_s^2 E_s}{q V_{\mathrm{rf}}\omega_r^2 \eta \cos\phi_s}}\Delta E_m^2$$

$$= \Delta E_m^2 \sqrt{-2\pi^3 \frac{\eta}{\omega_{0,s}^2 \beta_s^2 E_s}\frac{1}{q V_{\mathrm{rf}} h \cos\phi_s}}$$

ADIABATICITY

- The longitudinal emittance of a bunch is preserved as long as the machine parameters are changed "adiabatically". The relative variation of the synchrotron frequency with time should be small compared to the synchrotron frequency

$$\left|\frac{\dot{\omega}_{s0}}{\omega_{s0}}\right| \ll \omega_{s0}$$

- The adiabaticity parameter is then

$$\alpha_{\mathrm{ad}} = \left|\frac{1}{\omega_{s0}^2}\frac{d\omega_{s0}}{dt}\right| \ll 1$$

- Intuitively, the parameters of the machine (e.g. energy, RF voltage, RF phase...) must be done changed slower than the synchrotron motion for the bunch to adapt to its new trajectory in phase space.

SCALING LAWS

The following scaling laws allow to evaluate the change in bunch length and energy spread from relative variations in emittance and machine parameters (NB: E_s and η are interdependent).

BUNCH LENGTH

$$\tau_l \propto \varepsilon_{l,0}^{1/2}\ V_{\mathrm{rf}}^{-1/4}\ h^{-1/4}\ E_s^{-1/4}\ \eta^{1/4}$$

ENERGY DEVIATION

$$\Delta E_m \propto \varepsilon_{l,0}^{1/2}\ V_{\mathrm{rf}}^{1/4}\ h^{1/4}\ E_s^{1/4}\ \eta^{-1/4}$$

- During acceleration, with all parameters constants except E_s, the bunch length reduces with $\tau_l \propto E_s^{-1/4}$. This is adiabatic "damping" of phase oscillations.
- The energy spread scales inversely with $\Delta E_m \propto E_s^{1/4}$, $\varepsilon_{l,0}$ is constant.

EXERCISES

- Compute the linear synchrotron frequency and tune in the SPS at $p = 14$ GeV/c and $p = 450$ GeV/c, with an RF harmonic $h = 4620$ and voltage $V_{\mathrm{rf}} = 4.5$ MV (find the other SPS parameters obtained in the exercises from Module 5). The beam is not accelerated.
- Compute the approximate emittance and momentum spread at $p = 14$ GeV/c for a bunch length $\tau_l = 3$ ns.
- What would be the bunch length at $p = 450$ GeV/c if the emittance is preserved?
- What would be the bunch length and energy spread at transition energy?
- Evaluate required increase in rf voltage to shorten the bunch length by a factor 2.

EXERCISES

- Linear synchrotron frequency and tune
  - Low energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14} \sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14 / 23.11 \cdot 10^6)^2 \cdot 1.385 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}}$$

$$\approx 784\ \mathrm{Hz}$$

$$Q_{s0} = 784 \cdot 23.11 \cdot 10^{-6} \approx 1.81 \cdot 10^{-2}$$

EXERCISES

- Linear synchrotron frequency and tune
  - High energy:

$$f_{s0} = \frac{1}{2 \cdot 3.14} \sqrt{\frac{1 \cdot 4.5 \cdot 10^6 \cdot (4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6)^2 \cdot 3.082 \cdot 10^{-3}}{2 \cdot 3.14 \cdot 4620 \cdot 1 \cdot 450 \cdot 10^9}}$$

$$\approx 206 \text{ Hz}$$

$$Q_{s0} = 206 \cdot 23.05 \cdot 10^{-6} \approx 4.76 \cdot 10^{-3}$$

EXERCISES

- Linear emittance

$$\varepsilon_{l,0} = \frac{3.14 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{4 \cdot 1.385 \cdot 10^{-3}} 2 \cdot 3.14 \cdot 784 \cdot (3 \cdot 10^{-9})^2 \approx 0.35 \text{ eVs}$$

- Energy spread

$$\delta_p = 2 \frac{\Delta E_m}{\beta_s^2 E_s} = 4 \frac{\varepsilon_{l,0}}{\pi \tau_l \beta_s^2 E_s} = \frac{4 \cdot 0.35}{3.14 \cdot 3 \cdot 10^{-9} (14/14.03)^2 \cdot 14.03 \cdot 10^9}$$
$$\approx 1.06 \times 10^{-2}$$

- Adiabatic damping

$$\tau_{l,\text{high}} = \tau_{l,\text{low}} \left(\frac{E_\text{high}}{E_\text{low}} \right)^{-1/4} = 3 \cdot \left(\frac{450}{14.03} \right)^{-1/4} \approx 1.26 \text{ ns}$$

EXERCISES

- Transition

The bunch length would tend to zero while the energy spread diverge to infinity! Non-adiabatic theory needed to better evaluate bunch parameters at transition crossing.

- Adiabatic bunch shortening

$$\tau_{l,\text{high}} = \tau_{l,\text{low}} \left(\frac{V_\text{high}}{V_\text{low}}\right)^{-1/4}$$

$$\Longrightarrow V_\text{high} = V_\text{low} \left(\frac{\tau_{l,\text{high}}}{\tau_{l,\text{low}}}\right)^{-4} = V_\text{low} \times 16$$

The required voltage increase is a factor 16! Not very efficient shortening.

MODULE 9: NON-LINEAR SYNCHROTRON MOTION

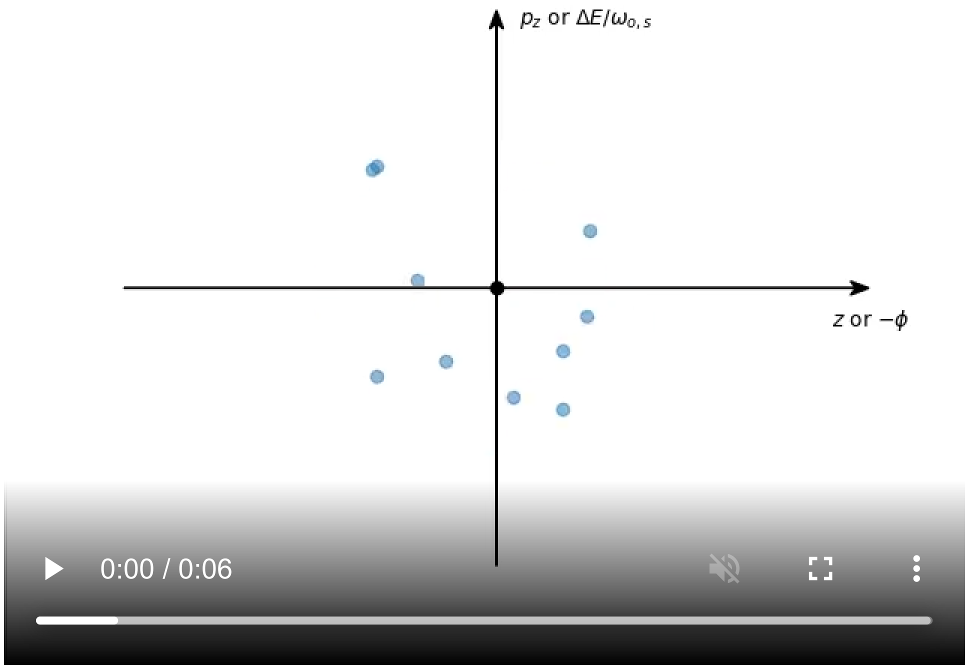
$\longrightarrow$ **Combined non-linear equations of motion (Hamiltonian)**

$\longrightarrow$ **RF bucket parameters and bunch emittance**

$\longrightarrow$ **Non-linear synchrotron frequency**

$\longrightarrow$ **Matching**

LONGITUDINAL EQUATIONS OF MOTION



- Energy

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_r}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\sin\left(\phi\right) - \sin\left(\phi_s\right)\right]$$

- Phase

$$\frac{d\phi}{dt} = \frac{\eta\omega_r^2}{\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)$$

- Starting from the same equations of motion as in Module 8.

COMBINING THE EQUATIONS OF MOTION

Starting over from the same combined equation of motion

$$\frac{d}{dt}\left(\frac{d\phi}{dt}\frac{\beta_s^2 E_s}{\eta\omega_r^2}\right) = \frac{qV_{\mathrm{rf}}}{2\pi h}\left(\sin\phi - \sin\phi_s\right)$$

We assume again that the change of machine parameters with time is negligible (adiabaticity), hence

$$\frac{d^2\phi}{dt^2} + \frac{\omega_{s0}^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

We can solve for $\dot{\phi}$. By multiplying by $\dot{\phi}$ and integrating with time, we get

$$\frac{\dot{\phi}^2}{2\omega_{s0}^2} - \frac{\cos\phi + \phi\sin\phi_s}{\cos\phi_s} = \mathcal{H}$$

COMBINING THE EQUATIONS OF MOTION

DERIVATION

For the first term, we use the differential identity

$$d\left(x^2\right) = 2\,x\,dx \quad \rightarrow \quad \dot{\phi}\ddot{\phi} = \frac{1}{2}\frac{1}{dt}\frac{d\left(\phi^2\right)}{dt^2}$$

For the second term, we integrate

$$\begin{aligned}
&\frac{\omega_{s0}^2}{\cos\phi_s}\int\left(\sin\phi - \sin\phi_s\right)\frac{d\phi}{dt}dt \\
=&\frac{\omega_{s0}^2}{\cos\phi_s}\left[\int\sin\phi\,d\phi - \int\sin\phi_s d\phi\right] \\
=&-\frac{\omega_{s0}^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right)
\end{aligned}$$

COMBINING THE EQUATIONS OF MOTION

The integration constant $\mathcal{H}$ can be offset so that its value is zero for the synchronous particle. Since $\dot{\phi}_s = 0$, we get

$$\mathcal{H} = \frac{\dot{\phi}^2}{2\omega_{s0}^2} - \frac{\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s}{\cos\phi_s}$$

Replacing $\dot{\phi}$ using the phase differential equation definition and ω_{s0}, we finally obtain

$$\mathcal{H} = \frac{\eta\omega_r^2}{2\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)^2 + \frac{qV_{\mathrm{rf}}}{2\pi h}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

Linearized for small $\Delta\phi$, the equation describes the same ellipse as obtained in the previous module.

HAMILTONIAN OF SYNCHROTRON MOTION

The same result can be obtained from the expression of the Hamiltonian using

$$\frac{d\phi}{dt} = \frac{\partial \mathcal{H}}{\partial \left(\Delta E / \omega_r\right)} \quad \text{and} \quad \frac{d\left(\Delta E / \omega_r\right)}{dt} = -\frac{\partial \mathcal{H}}{\partial \phi}$$

In the previous equations, $\mathcal{H}$ effectively represents the Hamiltonian of a particle in our system, corresponding to the **energy of synchrotron oscillations** (beware: this is not the actual particle energy!)

The Hamiltonian is composed of two parts

$$\mathcal{H} = \mathcal{T}\left(\frac{\Delta E}{\omega_r}\right) + \mathcal{U}\left(\phi\right)$$

where $\mathcal{T}$ is the "kinetic" energy of synchrotron oscillations and $\mathcal{U}$ the "potential" energy.

HAMILTONIAN MECHANICS

APARTÉ

The Hamiltonian of a particle can be obtained from the canonical Hamilton equations

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}$$

where p and q are the conjugate momentum and coordinate.

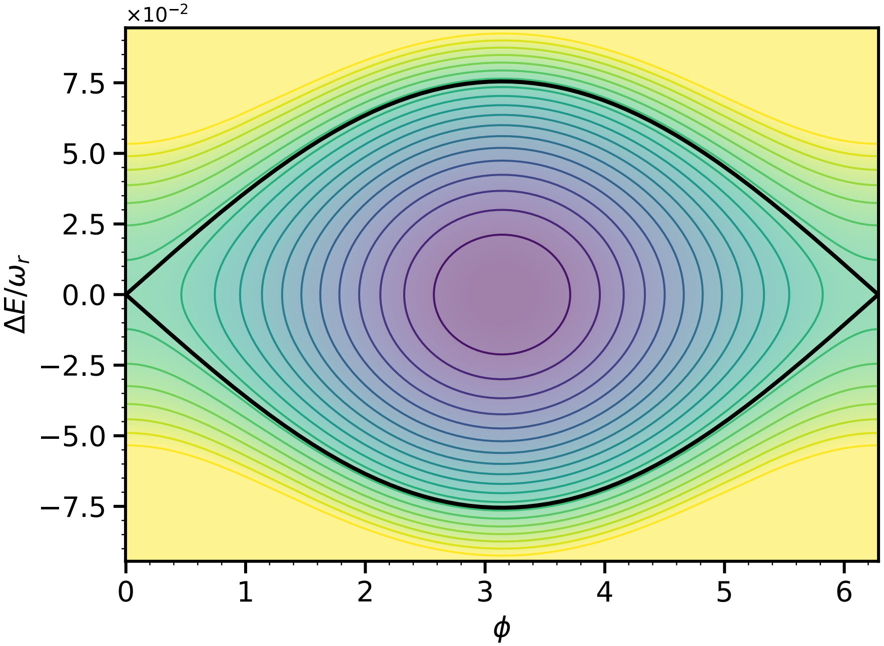
A time invariant Hamiltonian is then expressed

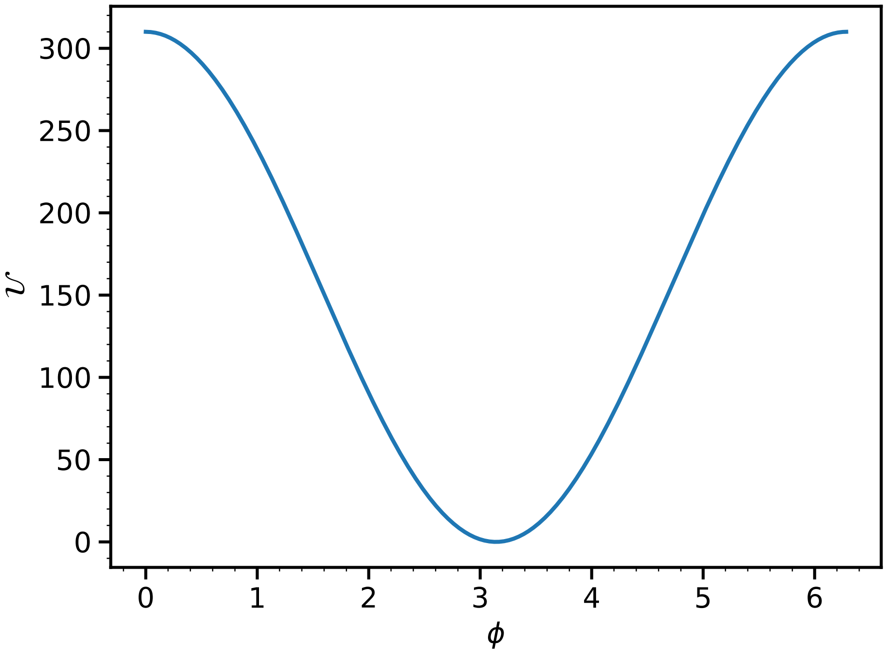
$$\mathcal{H} = \int \frac{\partial \mathcal{H}}{\partial p} dp + \int \frac{\partial \mathcal{H}}{\partial q} dq$$

A time invariant $\mathcal{H}$ is a constant of motion.

HAMILTONIAN OF SYNCHROTRON MOTION

STATIONNARY BUNCH, $\Phi_S = 0$

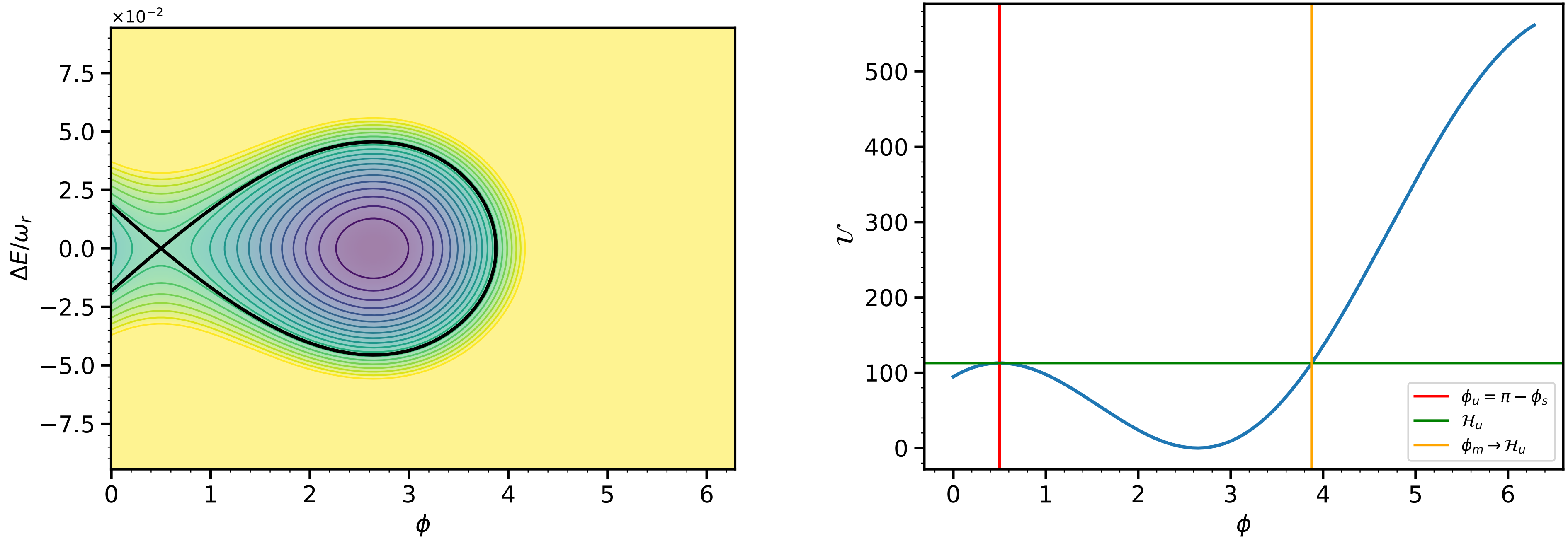




- The Hamiltonian gives the trajectory of the particle in phase space (ellipse at low $\Delta\phi$).
- A particle oscillates in phase space and performs a bounded motion if its energy $\mathcal{H}$ is lower than the maximum of the potential well.

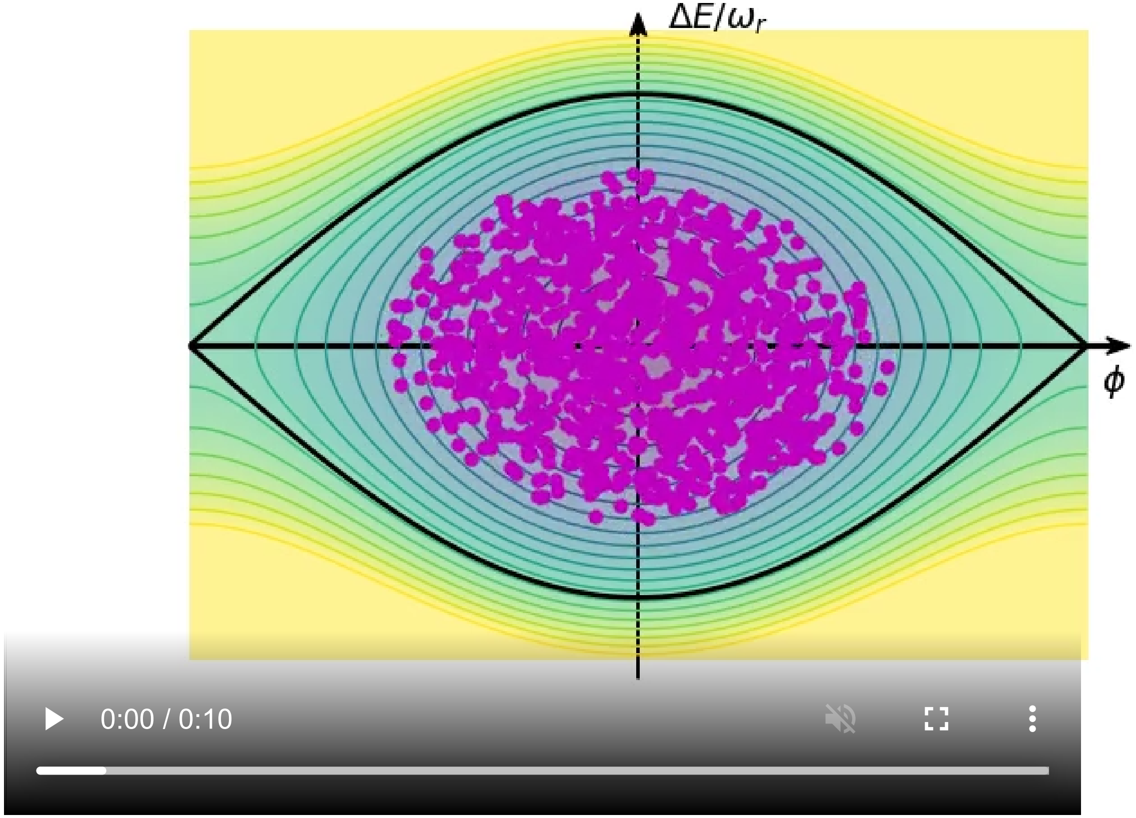
HAMILTONIAN OF SYNCHROTRON MOTION

ACCELERATION Φ_S



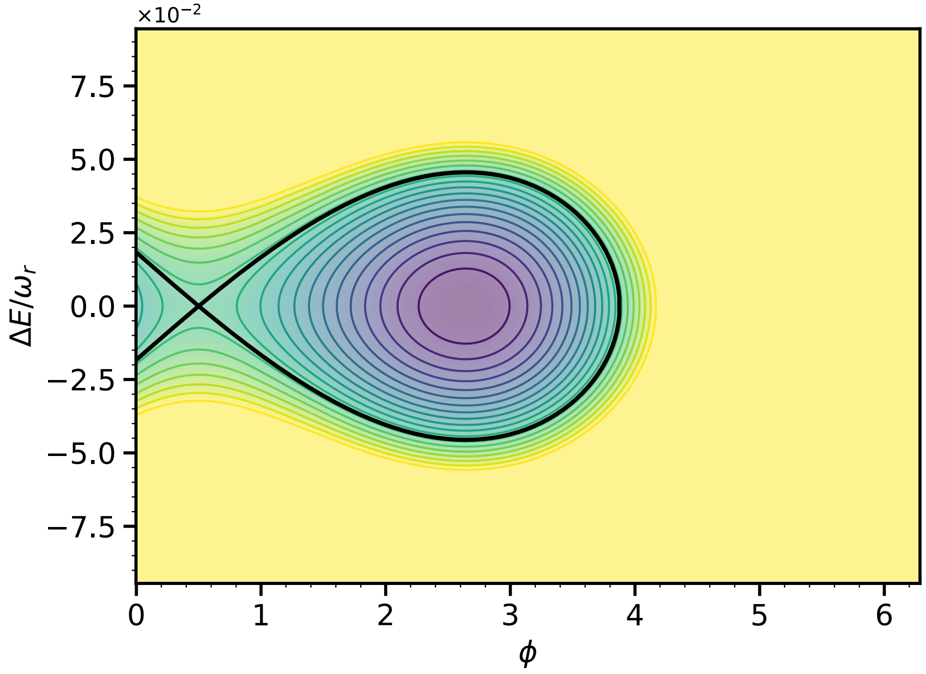
- During acceleration, the potential well is modified with ϕ_s.
- The stable fixed point (center of the RF bucket) is shifted to ϕ_s, the trajectories of non-synchronous particles are asymetric.

MOTION OF PARTICLES IN THE RF BUCKET



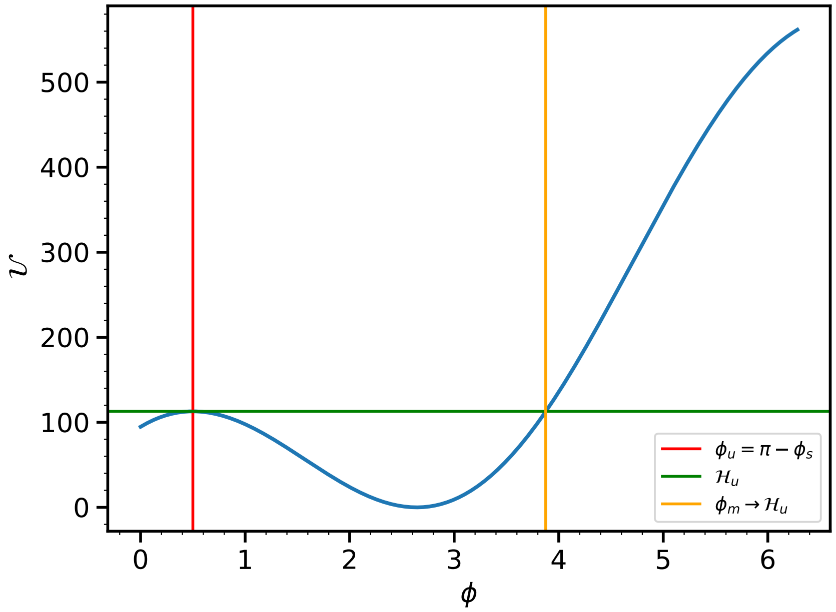
- Example of particles rotating in longitudinal phase space, with non-linear synchrotron motion.

SEPARATRIX



- The maximum contour in which the particles have a bounded motion around the synchronous phase is the **separatrix**.
- The separatrix is the limit of the **RF bucket**, where particles can be captured in a bunch.

SEPARATRIX



- The limit of the separatrix is given by the unstable fixed point $\phi_u = \pi - \phi_s$ (obtained from $d\mathcal{U}/d\phi = 0$) on one side.
- On the other side, the phase ϕ_m corresponds to the turning point where $\mathcal{U}_m = \mathcal{U}\left(\pi - \phi_s\right) = \mathcal{H}_u\left(\pi - \phi_s, \Delta E = 0\right)$

SEPARATRIX

EXPRESSION

The expression for the separatrix is obtained from the Hamiltonian

$$\mathcal{H} = \frac{\eta\omega_r^2}{2\beta_s^2 E_s}\left(\frac{\Delta E}{\omega_r}\right)^2 + \mathcal{U}\left(\phi\right)$$

$$\mathcal{H}_u = \mathcal{U}\left(\pi - \phi_s\right)$$

The maximum trajectory in energy is

$$\Delta E_{\mathrm{sep}} = \pm\sqrt{\frac{2\beta_s^2 E_s}{\left|\eta\right|}}\sqrt{\mathcal{U}\left(\pi - \phi_s\right) - \mathcal{U}\left(\phi\right)}$$

RF BUCKET HEIGHT

The RF bucket height is obtained from the maximum height of the separatrix at $\Delta E_{\mathrm{sep}}\left(\phi_s\right)$

The RF bucket height in energy is (*NB: this is the half size from 0 to* $\Delta E_{\mathrm{sep,m}}$*, and should be* $\times 2$ *for the full bucket height*)

$$\Delta E_{\mathrm{sep,m}} = \sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h \left|\eta\right|}} Y\left(\phi_s\right)$$

where

$$Y\left(\phi_s\right) = \left|-\cos\phi_s + \frac{\left(\pi - 2\phi_s\right)}{2}\sin\phi_s\right|^{1/2}$$

is the reduction of the bucket height during acceleration $Y \leqslant 1$.

RF BUCKET HEIGHT

DERIVATION

The potential well at $\pi - \phi_s$, using the trigonometric identity, is

$$\begin{aligned}\mathcal{U}\left(\pi-\phi_s\right) &= \frac{qV_{\text{rf}}}{2\pi h}\left[\cos\left(\pi-\phi_s\right)-\cos\phi_s+\left(\pi-2\phi_s\right)\sin\phi_s\right] \\ &= \frac{qV_{\text{rf}}}{2\pi h}\left[\cos\pi\cos\phi_s+\sin\pi\sin\phi_s-\cos\phi_s+\left(\pi-2\phi_s\right)\sin\phi_s\right] \\ &= \frac{qV_{\text{rf}}}{2\pi h}\left[-2\cos\phi_s+\left(\pi-2\phi_s\right)\sin\phi_s\right] \\ &= \frac{qV_{\text{rf}}}{\pi h}\left[-\cos\phi_s+\frac{\left(\pi-2\phi_s\right)}{2}\sin\phi_s\right]\end{aligned}$$

RF BUCKET AREA (ACCEPTANCE)

The bucket area (acceptance) is obtained by integrating within the separatrix contour

$$\mathcal{A}_{\mathrm{bk}} = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\mathcal{U}\left(\pi-\phi_s\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

The bucket area can be reformulated as

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|}}\Gamma\left(\phi_s\right)$$

where

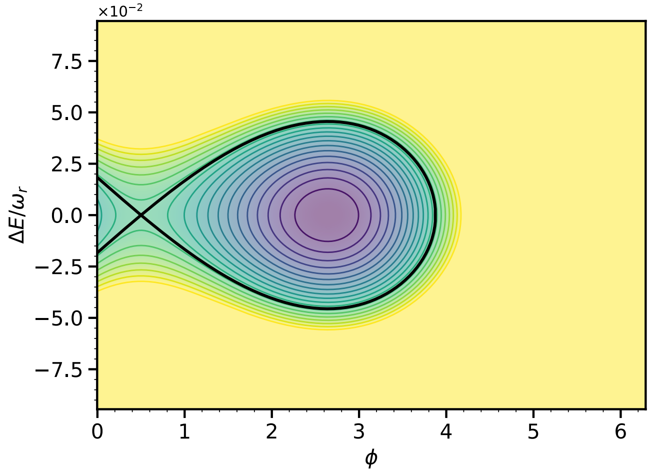
$$\Gamma\left(\phi_s\right) = \frac{1}{4\sqrt{2}} \int_{\phi_u}^{\phi_m} \sqrt{-\cos\phi_s - \cos\phi + \left(\pi - \phi - \phi_s\right)\sin\phi_s}\, d\phi$$

RF BUCKET AREA (ACCEPTANCE)

The function $\Gamma\left(\phi_s\right)$ is the reduction of the bucket area during acceleration $\Gamma \leqslant 1$ and can be approximated to give the formula

$$\mathcal{A}_{\mathrm{bk}} \approx \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}}\frac{1-\sin\phi_s}{1+\sin\phi_s}$$

For the stationary RF bucket $\mathcal{A}_{\mathrm{bk}} = 8\Delta E_{\mathrm{sep,m}}/\omega_r$



RF BUCKET AREA (ACCEPTANCE)

DERIVATION

$$\mathcal{A}_{\mathrm{bk}} = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\mathcal{U}\left(\pi-\phi_s\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

$$= 2\sqrt{\frac{qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|\,\omega_r^2}} \int_{\phi_u}^{\phi_m} \sqrt{\ldots}\, d\phi$$

$$\begin{aligned}
\ldots = &\left[\cos\left(\pi-\phi_s\right) - \cos\phi_s + \left(\pi - 2\phi_s\right)\sin\phi_s\right] \\
&- \left[\cos\phi - \cos\phi_s + \left(\phi-\phi_s\right)\sin\phi_s\right] \\
&= \left[-2\cos\phi_s + \left(\pi-2\phi_s\right)\sin\phi_s\right] \\
&- \left[\cos\phi - \cos\phi_s + \left(\phi-\phi_s\right)\sin\phi_s\right] \\
&= -\cos\phi_s - \cos\phi + \left(\pi-\phi-\phi_s\right)\sin\phi_s
\end{aligned}$$

RF BUCKET AREA (ACCEPTANCE)

DERIVATION

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}}\frac{1}{4\sqrt{2}}\int_{\phi_u}^{\phi_m}\sqrt{\ldots}\,d\phi$$

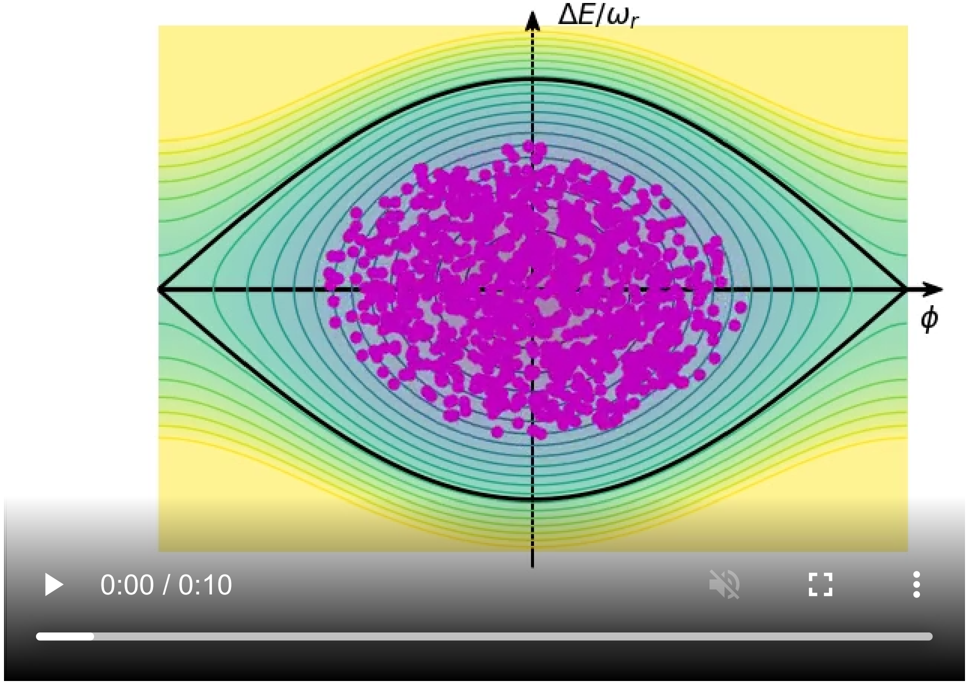
$$= \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\left|\eta\right|}}\Gamma\left(\phi_s\right)$$

where

$$\Gamma\left(\phi_s\right) = \frac{1}{4\sqrt{2}}\int_{\phi_u}^{\phi_m}\sqrt{-\cos\phi_s - \cos\phi + \left(\pi - \phi - \phi_s\right)\sin\phi_s}\,d\phi$$

$$\approx \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

LONGITUDINAL EMITTANCE, FILLING FACTOR



- The longitudinal emittance can be calculated accounting for the non-linearities of the RF bucket

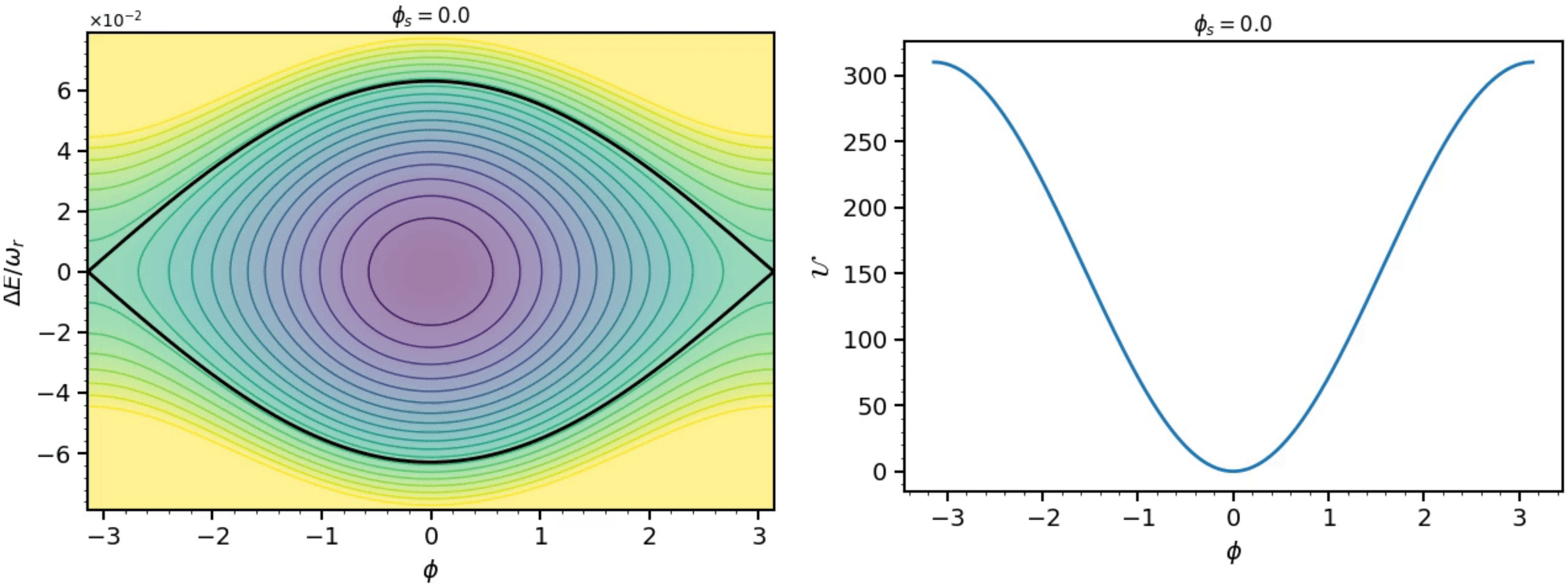
$$\varepsilon_l = 2\sqrt{\frac{2\beta_s^2 E_s}{|\eta|\,\omega_r^2}} \cdot \int_{\phi_{b,l}}^{\phi_{b,r}} \sqrt{\mathcal{U}\left(\phi_{b,lr}\right) - \mathcal{U}\left(\phi\right)}\, d\phi$$

where b, l and b, r stands for the left/right edge of the bunch in phase (amplitude ϕ_b and full length $2\phi_b$).

The filling factor is commonly defined in emittance: $\varepsilon_l / \mathcal{A}_{\mathrm{bk}}$ or in energy: $\Delta E_{b,m} / \Delta E_{\mathrm{sep},m}$

ACCELERATION

BELOW TRANSITION

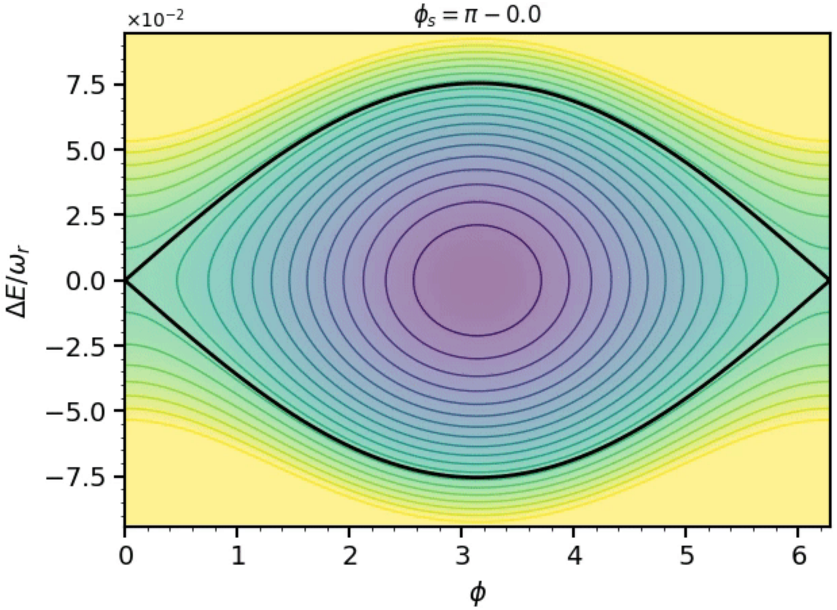


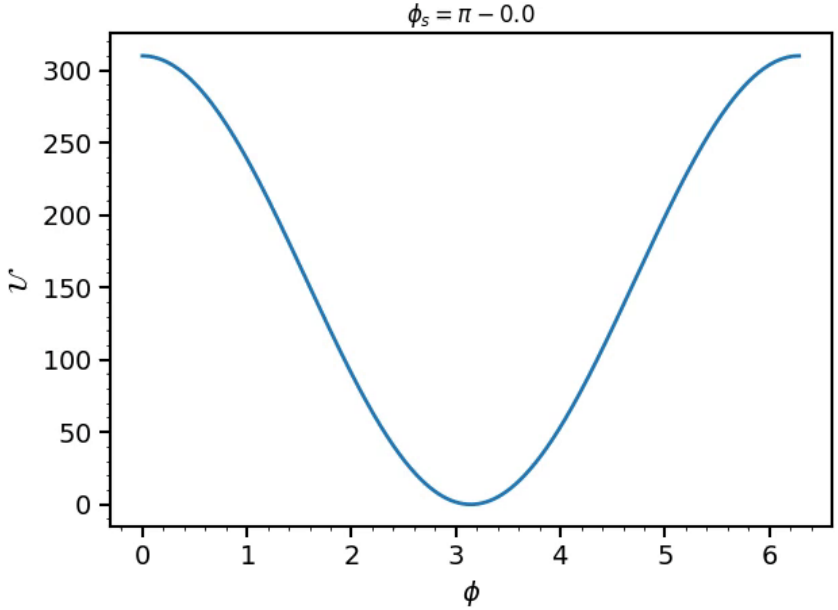
- Notice the shape of the bucket: below transition, pointing towards positive ϕ.

- The bucket shrinks if ϕ_s is too large! The acceleration should be limited or RF voltage increased to keep a reasonable filling factor (losses otherwise).

ACCELERATION

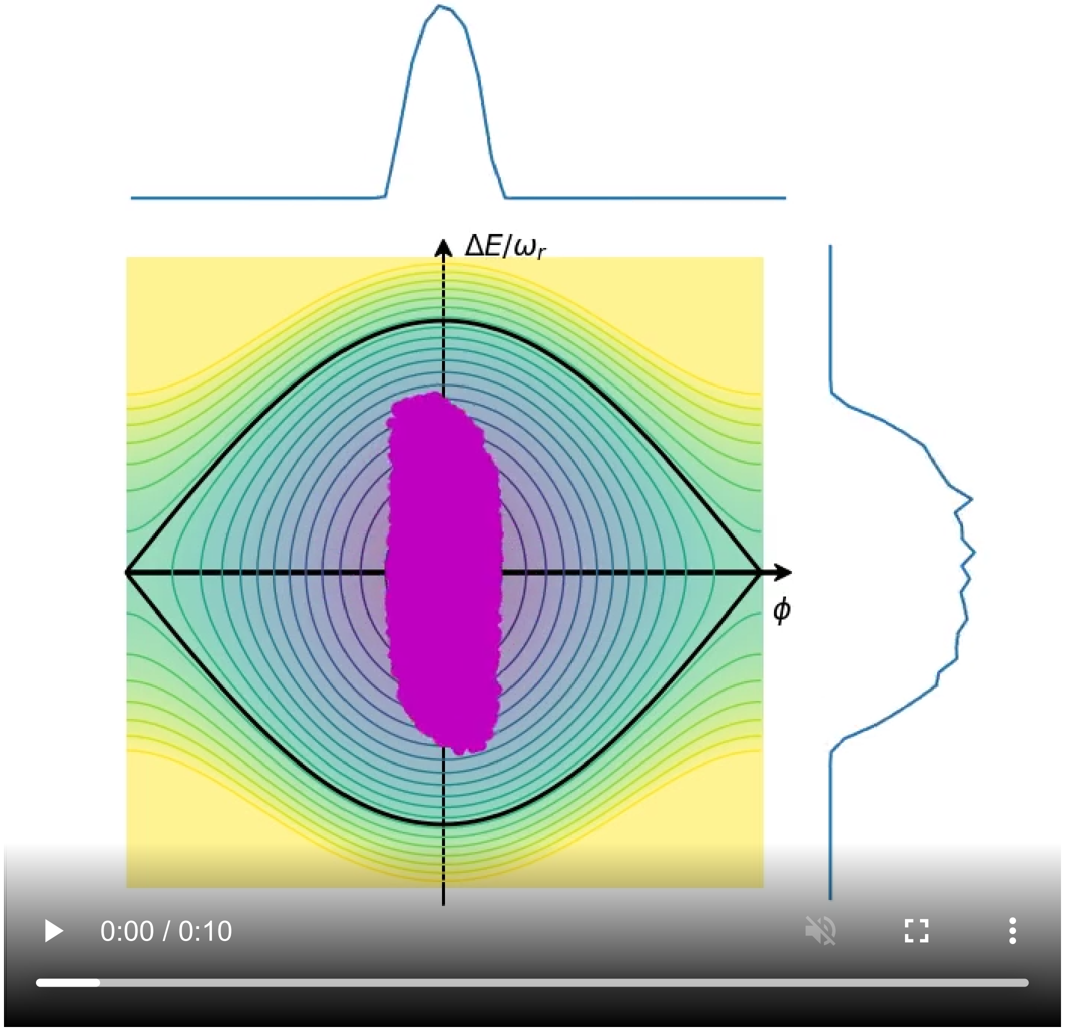
ABOVE TRANSITION





- Notice the shape of the bucket: above transition, pointing towards negative ϕ.
- The bucket shrinks if ϕ_s is too large! The acceleration should be limited or RF voltage increased to keep a reasonable filling factor (losses otherwise).

NON-LINEAR SYNCHROTRON FREQUENCY



- The non-linear frequency is obtained by integrating

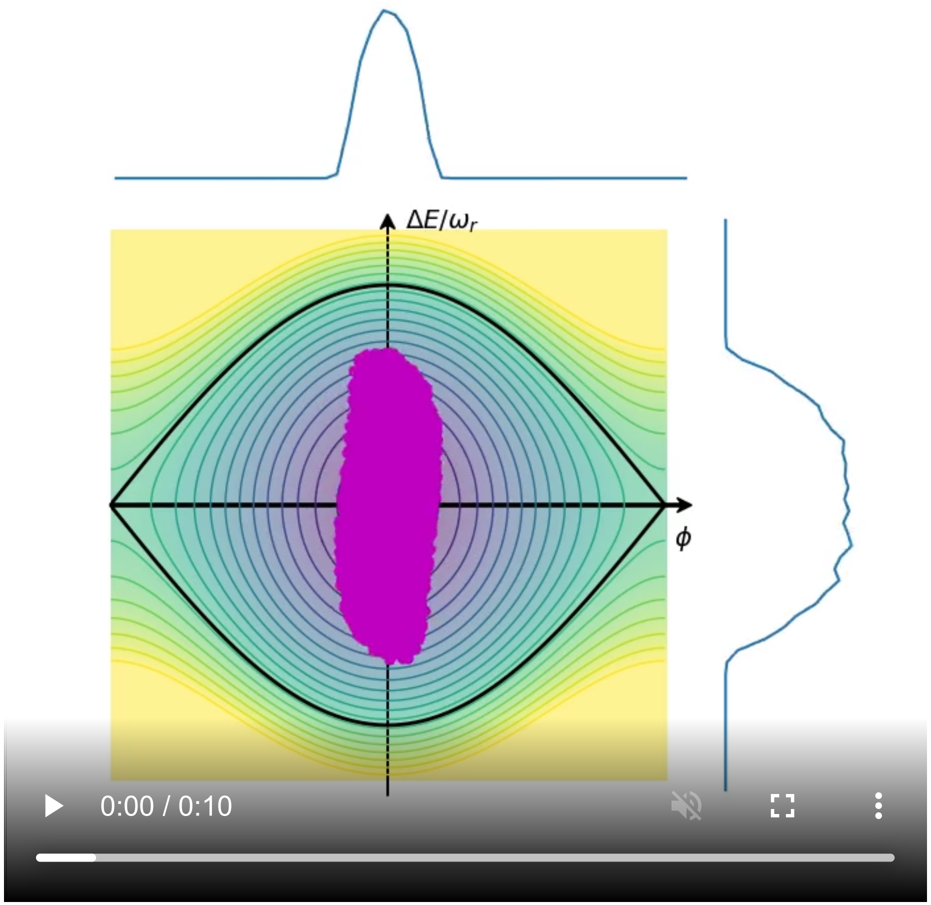
$$T_s = \int_{\phi_{b,l}}^{\phi_{b,r}} \frac{d\phi}{\dot{\phi}}$$

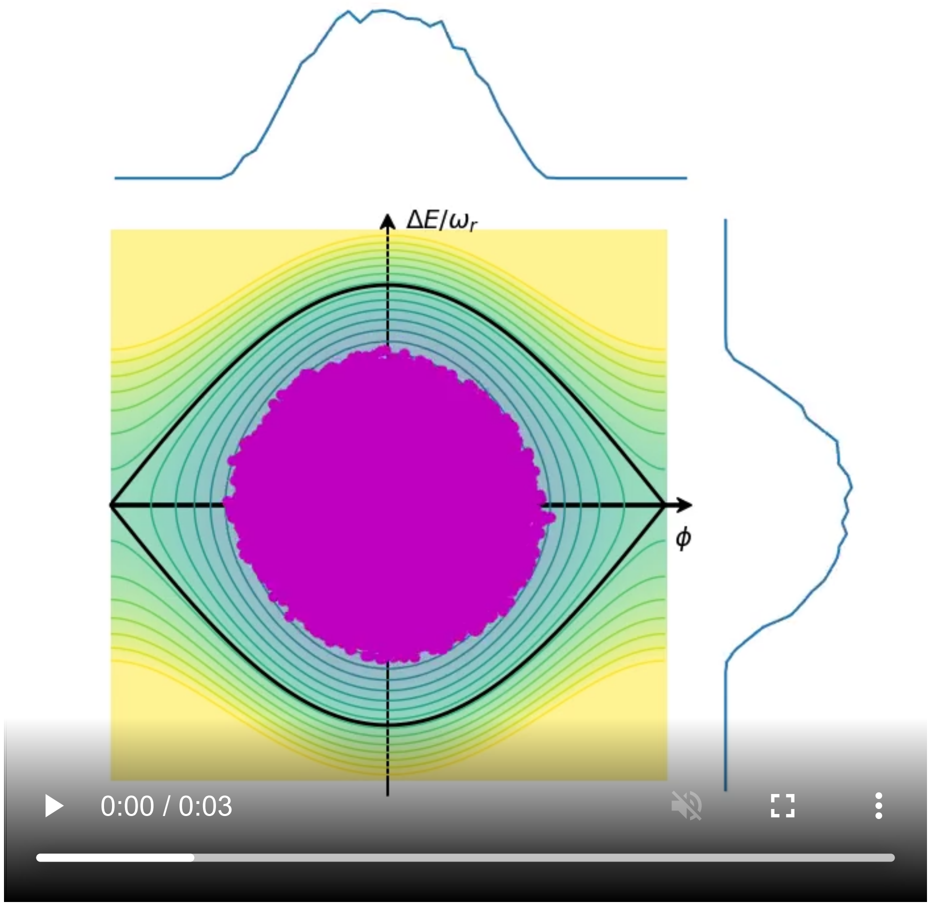
- The integration leads to the relationship

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2K\left(\sin\frac{\phi_b}{2}\right)}$$

$$\approx 1 - \frac{\phi_b^2}{16}$$

MATCHING AND FILAMENTATION





- The bunch is matched if the density along a iso-Hamiltonian line is constant. If mismatched, the bunch filaments and the statistical emittance increases.

EXERCISES

- Compute the RF bucket area (or acceptance) using the SPS parameters from Module 5 and 8.
- Compute the bucket height.
- Compute the filling factor for 3 ns bunch at 14 GeV/c (use the linear approximation for the emittance calculation)
- The bunch length oscillations at injection indicate that the energy spread is too small by 10%. How much should the RF voltage be reduced to improve the matching?

EXERCISES

- Low energy 14 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.11 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 0.50\ \mathrm{eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot (14/14.03)^2 \cdot 14.03 \cdot 10^9}{3.14 \cdot 4620 \cdot 1.385 \cdot 10^{-3}}} \approx 79.1\ \mathrm{MeV}$$

EXERCISES

- High energy 450 GeV/c, RF Bucket area

$$\mathcal{A}_{\mathrm{bk}} = \frac{8}{4620 \cdot 2 \cdot 3.14/23.05 \cdot 10^6} \cdot \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 1.91 \text{ eVs}$$

- RF Bucket height (half height)

$$\Delta E_{\mathrm{sep},m} = \sqrt{\frac{2 \cdot 1 \cdot 4.5 \cdot 10^6 \cdot 1 \cdot 450 \cdot 10^9}{3.14 \cdot 4620 \cdot 3.082 \cdot 10^{-3}}} \approx 301 \text{ MeV}$$

EXERCISES

- Filling factor

From the previous module exercise, the longitudinal emitatnce is 0.35 eVs. The filling factor in area is $0.35/0.50 \approx 70\,\%$.

- Matching

The bunch length and energy spread are fixed at injection. In orderto match the bunch, the bucket height should be reduced by 10%. The RF voltage can be reduced to reduce the bucket height, with a scaling $\sqrt{V_{\text{rf}}}$

$$\frac{\Delta E_{\text{sep},m,2}}{\Delta E_{\text{sep},m,1}} = 0.9 = \sqrt{\frac{V_{\text{rf},2}}{V_{\text{rf},1}}} \quad \rightarrow \quad V_{\text{rf},2} = 0.9^2 V_{\text{rf},1} \approx 0.81 V_{\text{rf},1}$$

The RF voltage should be reduced by 20% (useful tip: $(1-\epsilon)^n \approx 1 - n\epsilon \rightarrow (1-0.1)^2 \approx 1 - 2 \cdot 0.1$)

TAKE AWAY MESSAGE

LINEAR SYNCHROTRON MOTION

- Linear synchrotron frequency

$$\omega_{s0} = 2\pi f_{s_0} = \sqrt{-\frac{qV_{\mathrm{rf}} h \omega_{0,s}^2 \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Linear synchrotron tune

$$Q_{s0} = \frac{\omega_{s0}}{\omega_{0,s}} = \sqrt{-\frac{qV_{\mathrm{rf}} h \eta \cos\phi_s}{2\pi \beta_s^2 E_s}}$$

- Phase stability condition

$$\eta \cos\phi_s < 0$$

TAKE AWAY MESSAGE

LINEAR OSCILLATION AMPLITUDE AND EMITTANCE

- Oscillation amplitude ratio

$$\frac{\left(\Delta E/\omega_r\right)_m}{\Delta\phi_m} = \frac{\beta_s^2 E_s}{|\eta|\,\omega_r^2}\omega_{s0} = \frac{\beta_s^2 E_s}{|\eta|\,h^2\omega_{0,s}}Q_{s0}$$

- Approximate longitudinal emittance

$$\begin{aligned}\varepsilon_{l,0} = \pi\Delta E_m\frac{\tau_l}{2} &= \frac{\pi\beta_s^2 E_s}{4\,|\eta|}\omega_{s0}\tau_l^2 \\ &= \frac{\pi\,|\eta|}{\beta_s^2 E_s}\frac{1}{\omega_{s0}}\Delta E_m^2\end{aligned}$$

TAKE AWAY MESSAGE

BUNCH PARAMETERS LINEAR SCALING LAWS

- Bunch length

$$\tau_l \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{-1/4}\, h^{-1/4}\, E_s^{-1/4}\, \eta^{1/4}$$

- Energy deviation

$$\Delta E_m \propto \varepsilon_{l,0}^{1/2}\, V_{\mathrm{rf}}^{1/4}\, h^{1/4}\, E_s^{1/4}\, \eta^{-1/4}$$

TAKE AWAY MESSAGE

HAMILTONIAN

$$\mathcal{H} = \frac{\eta \omega_r^2}{2\beta_s^2 E_s} \left(\frac{\Delta E}{\omega_r} \right)^2 + \frac{q V_{\mathrm{rf}}}{2\pi h} \left[\cos\phi - \cos\phi_s + (\phi - \phi_s) \sin\phi_s \right]$$

TAKE AWAY MESSAGE

RF BUCKET PARAMETERS

- RF bucket height

$$\Delta E_{\mathrm{sep,m}} = \sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|}}\left|-\cos\phi_s + \frac{(\pi - 2\phi_s)}{2}\sin\phi_s\right|^{1/2}$$

- RF bucket area (acceptance)

$$\mathcal{A}_{\mathrm{bk}} \approx \frac{8}{\omega_r}\sqrt{\frac{2qV_{\mathrm{rf}}\beta_s^2 E_s}{\pi h\,|\eta|}\frac{1-\sin\phi_s}{1+\sin\phi_s}}$$

- For the stationary RF bucket, the RF bucket length is 2π and $\mathcal{A}_{\mathrm{bk}} = 8\Delta E_{\mathrm{sep,m}}/\omega_r$

TAKE AWAY MESSAGE

NON-LINEAR SYNCHROTRON FREQUENCY

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2K\left(\sin\frac{\phi_b}{2}\right)} \approx 1 - \frac{\phi_b^2}{16}$$

LESSON 5: APPLICATION

MODULE 10: LONGITUDINAL BEAM DYNAMICS IN ACTION

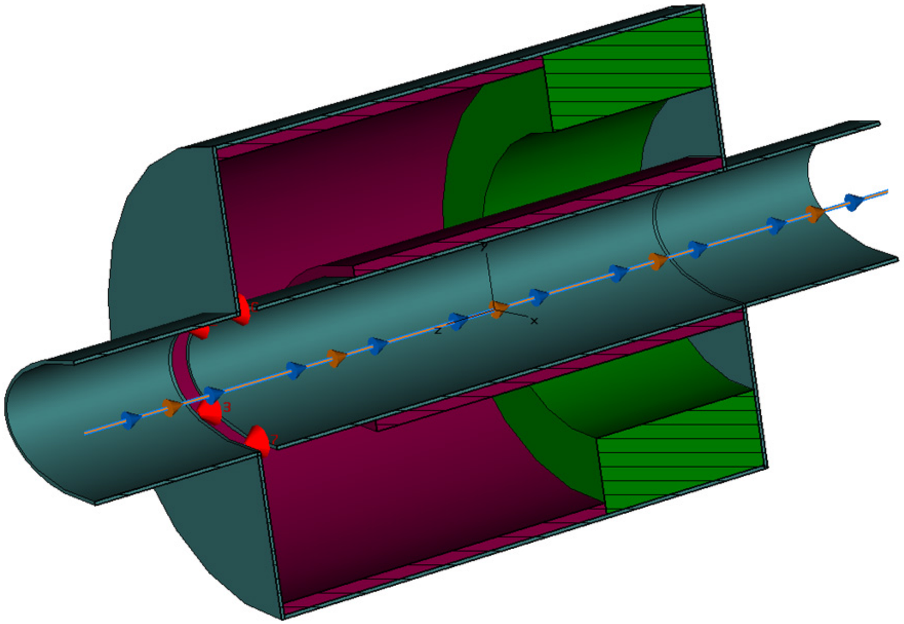
→ **Beam observation**

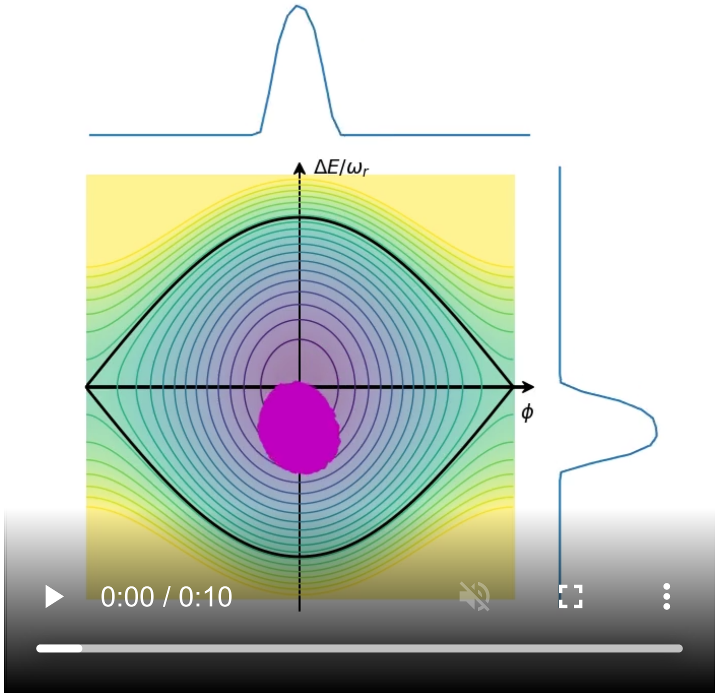
→ **Example RF operation (injection oscillations)**

→ **Introduction to RF manipulations**

→ **Beam instabilities**

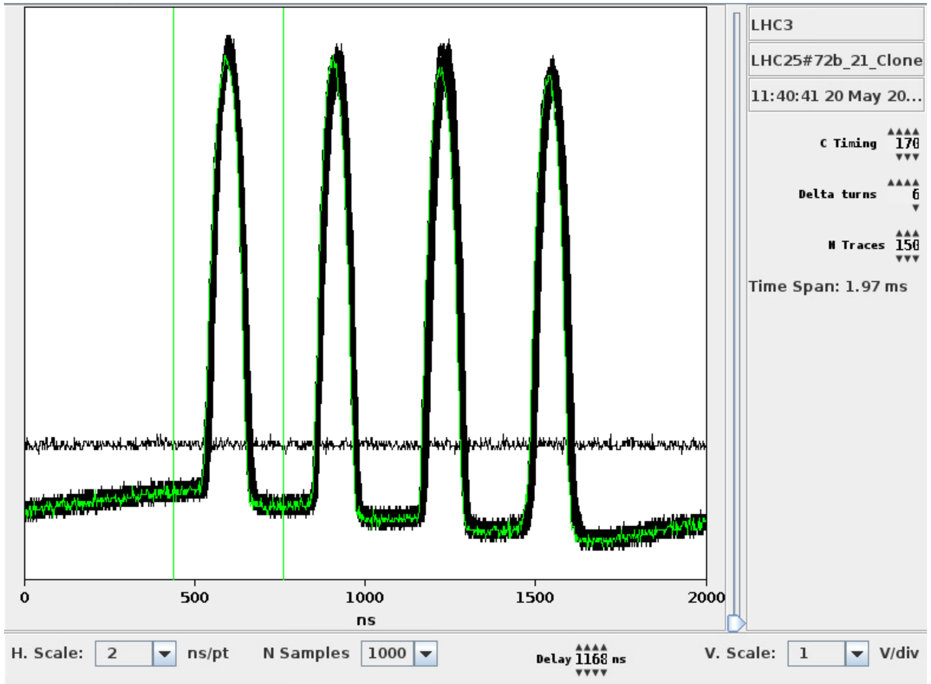
BEAM OBSERVATION

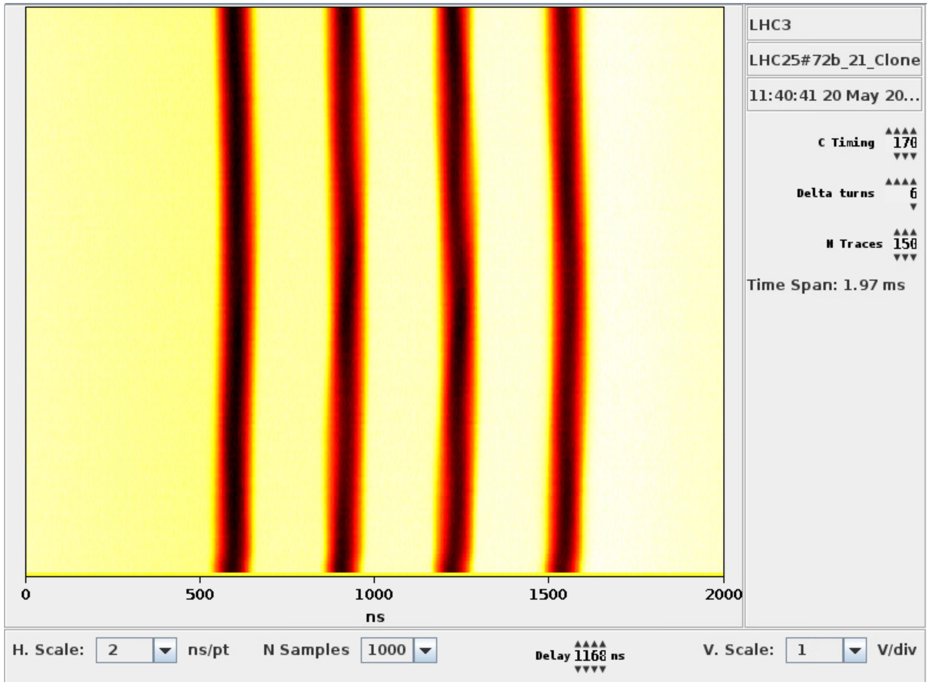




- The longitudinal bunch profile is measured using Wall Current Monitor (WCM, beam current converted in voltage).
- The WCM is connected to a digitizer or an oscilloscope, which is triggered before the bunch passage to acquire the bunch profile.

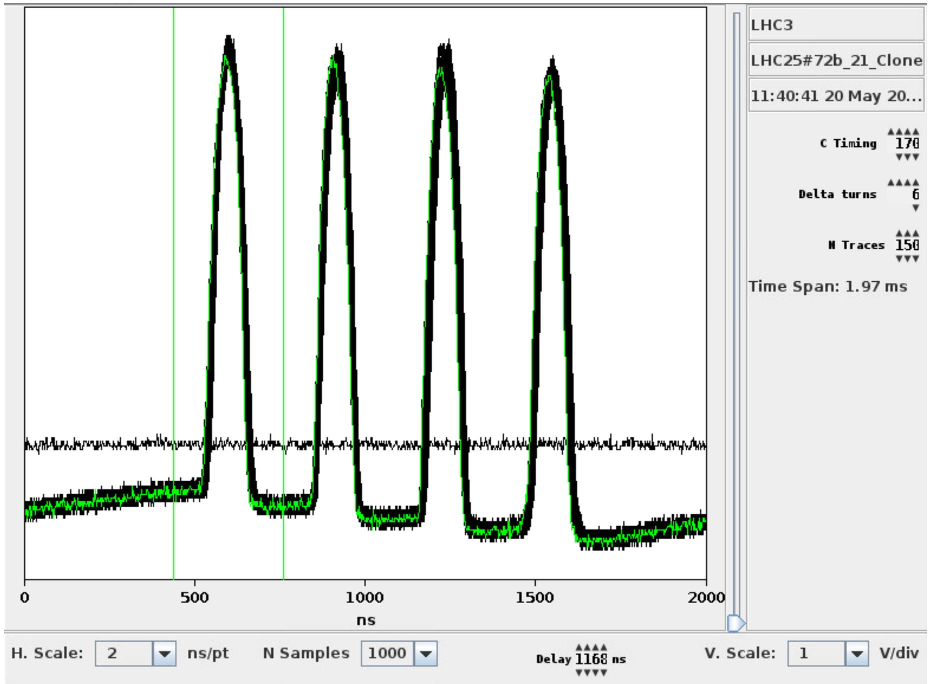
BEAM OBSERVATION

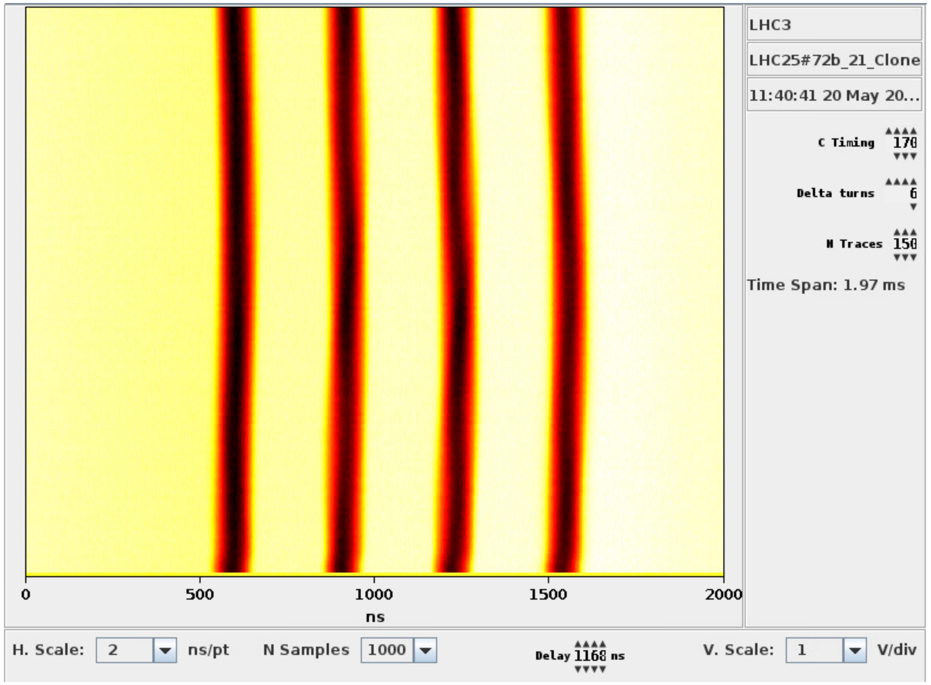




- Here is a real example of an acquisition software. The acquisition starts 170 ms after the beginning of the cycle (corresponding to injection).
- The acquisition lasts for 2000 ns, enough to measure the profiles of the 4 bunches in the machine.
- The acquisition is repeated 150 times, every 6 machine turns.

BEAM OBSERVATION

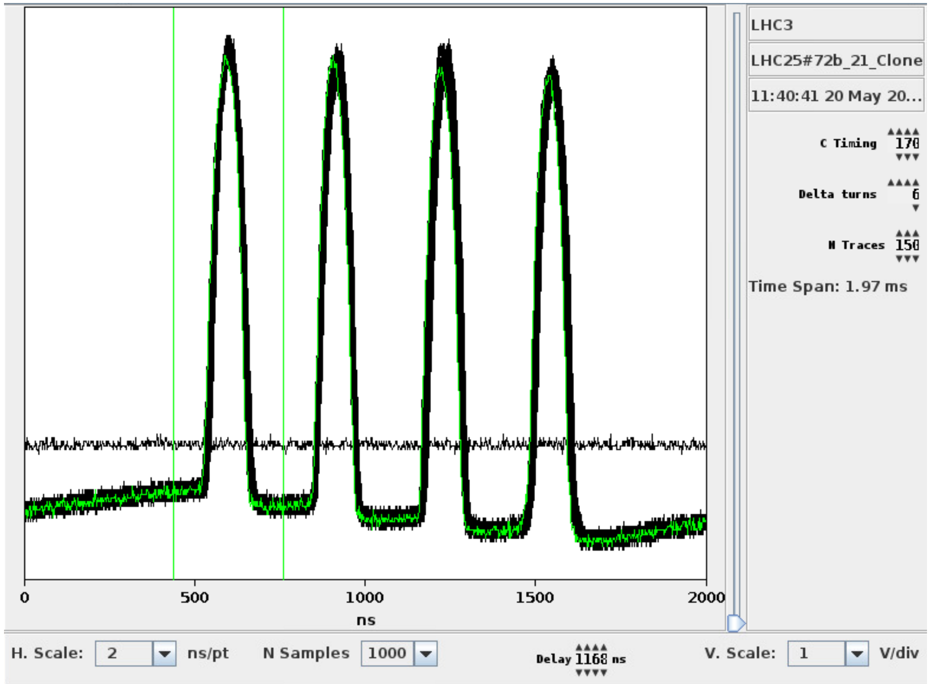


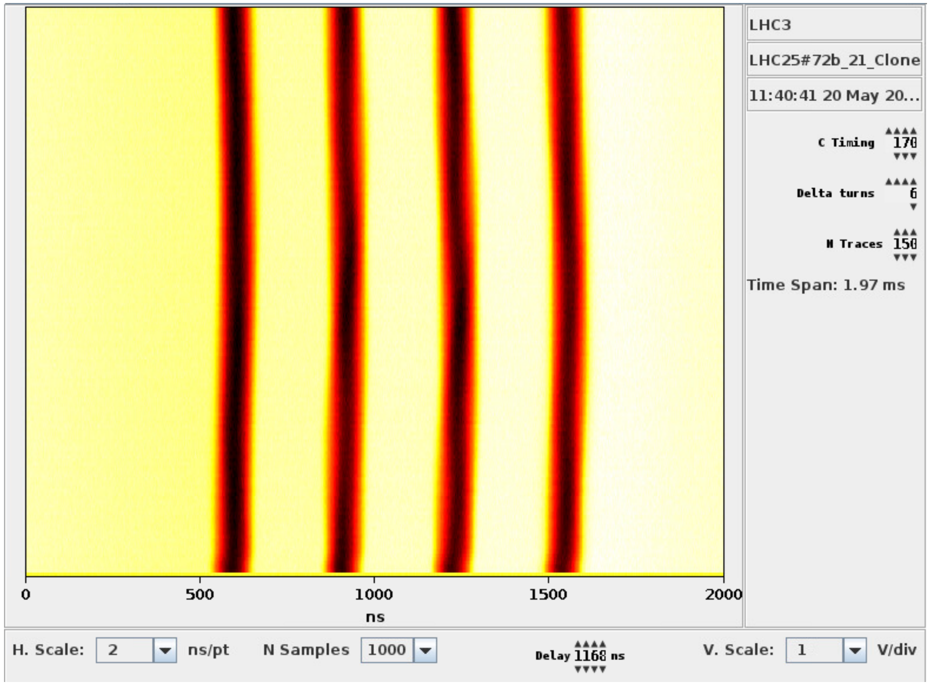


- In the left figure are all profiles are shown overlapped (the trigger is synchronous with the RF).

- The right figure shows the evolution of the profiles (horizontal, 1 trace is 2000 ns long) vs time in the cycle (vertical, 1 line = 1 trace every 6 turns).

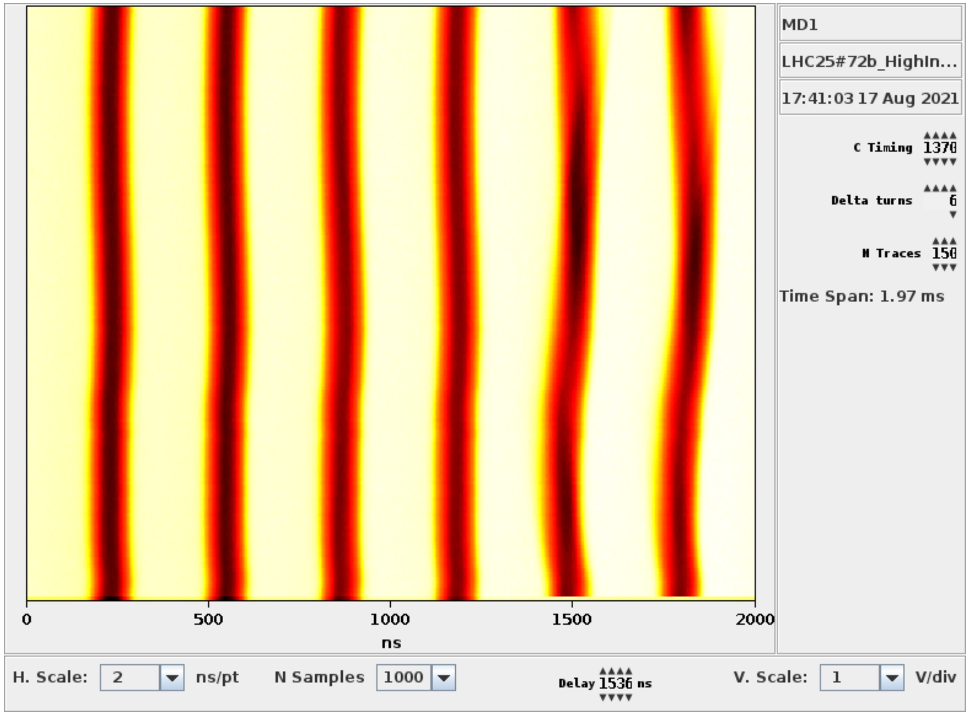
BEAM OBSERVATION





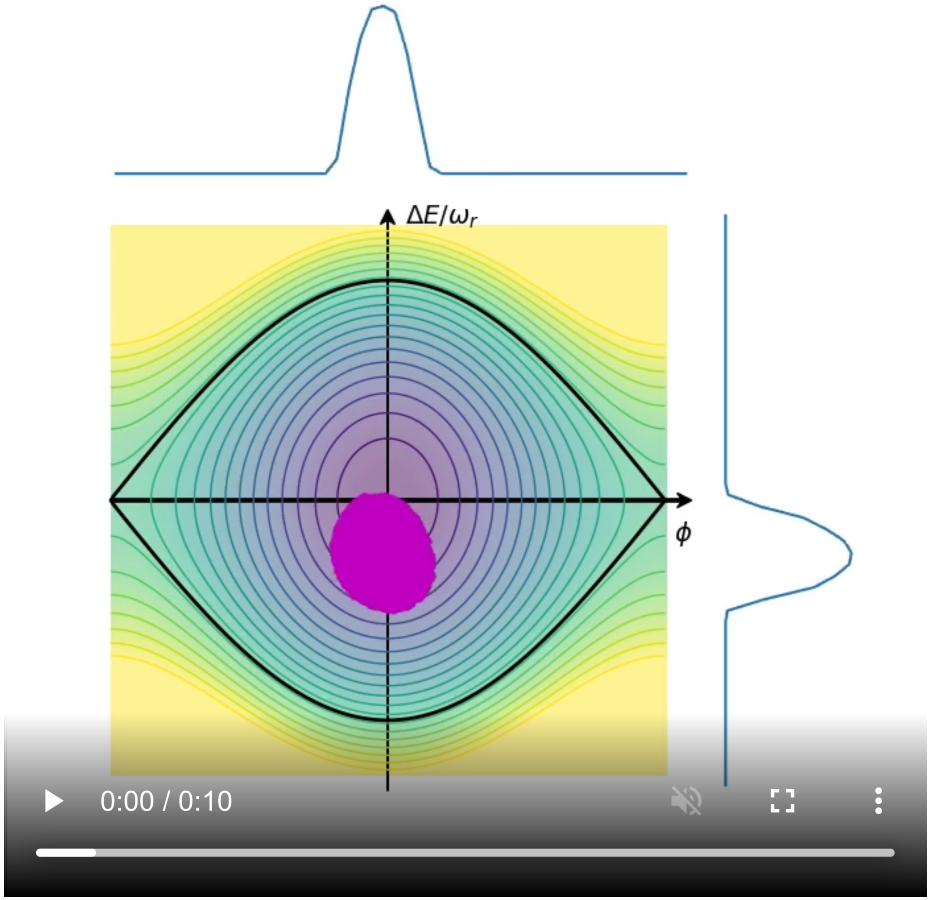
- The first trace is empty, before beam injection.
- The bunches are well matched, no signs of oscillations.

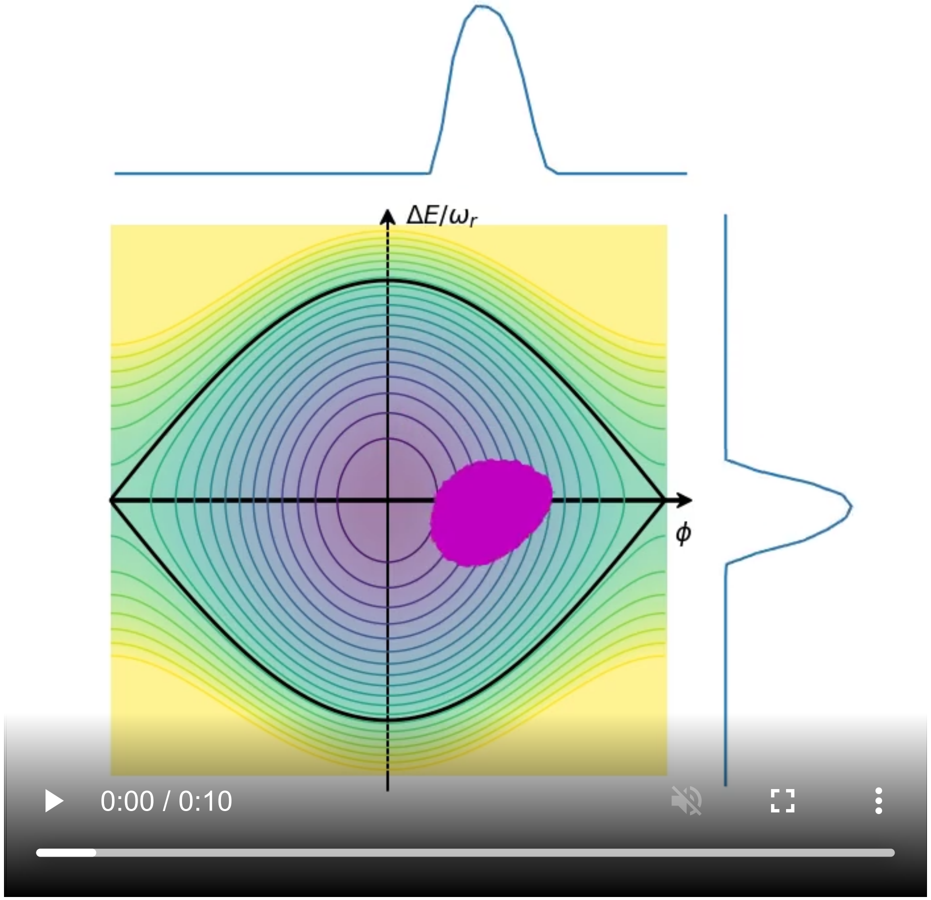
INJECTION OSCILLATIONS (PHASE)



- Two more bunches are injected, 1370 ms after the beginning of the cycle (low energy).
- The two extra bunches are not matched, they oscillate more that the 4 first bunches.
- What is wrong?

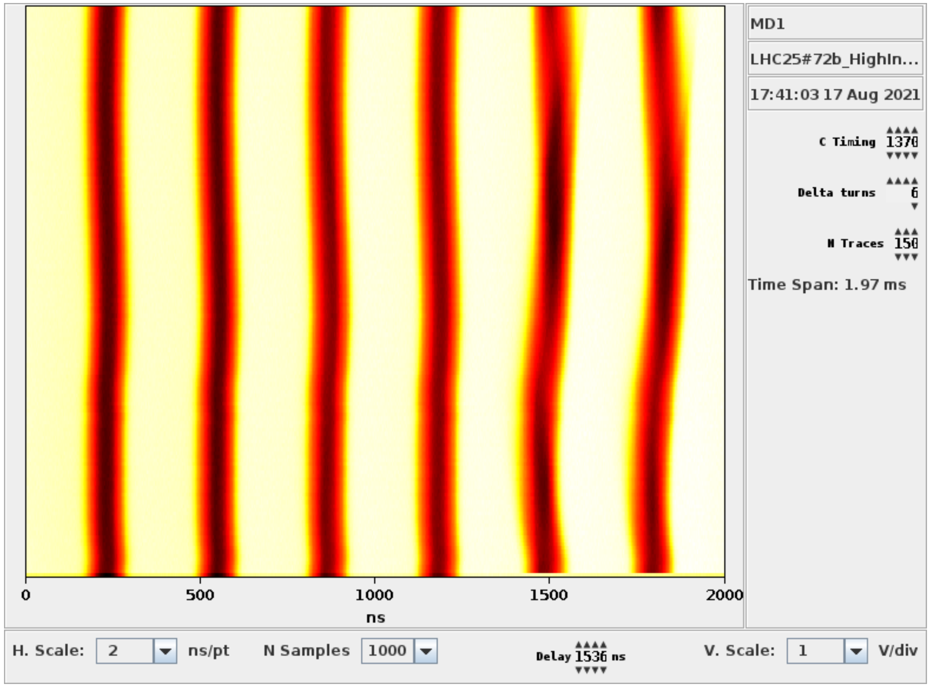
INJECTION OSCILLATIONS (PHASE)

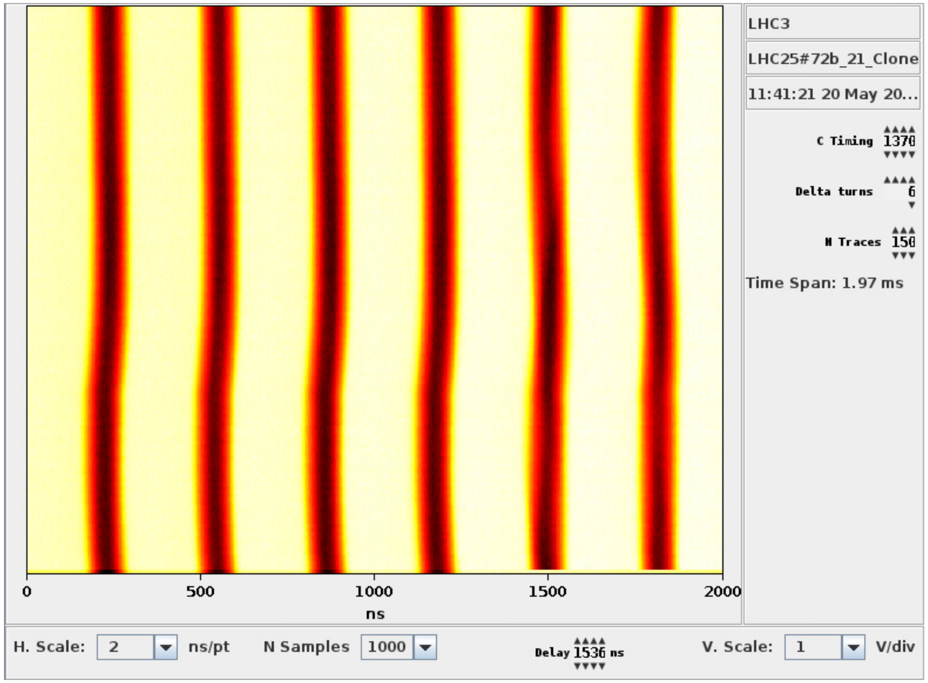




- A bunch can be mismatch because injected at wrong RF phase (left), or wrong energy (right). The bunch phase (and energy) oscillates after injection.

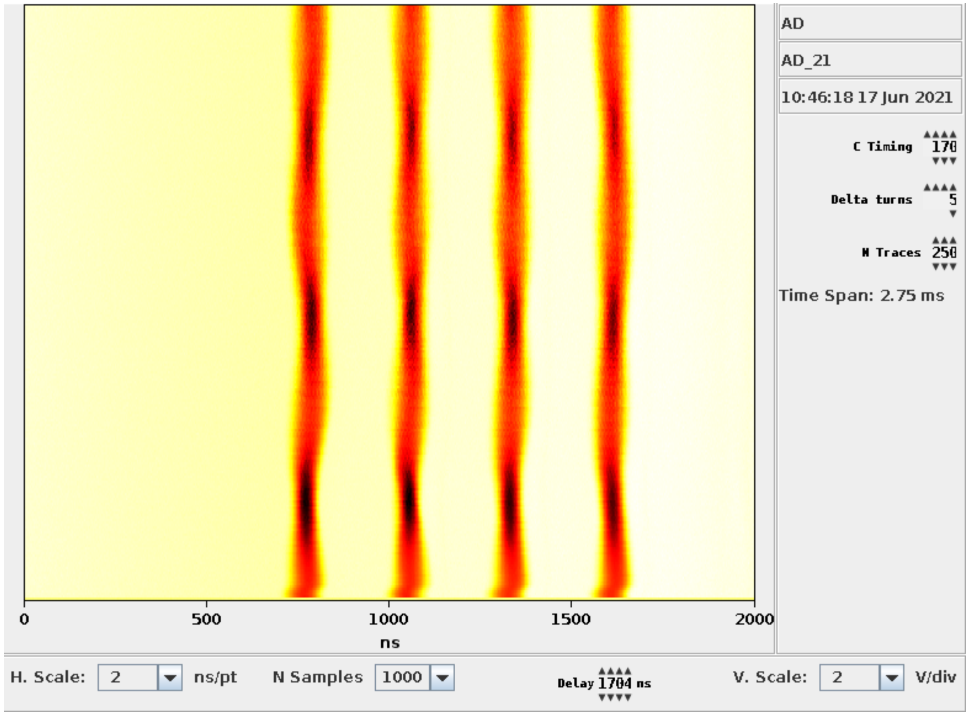
INJECTION OSCILLATIONS (PHASE)





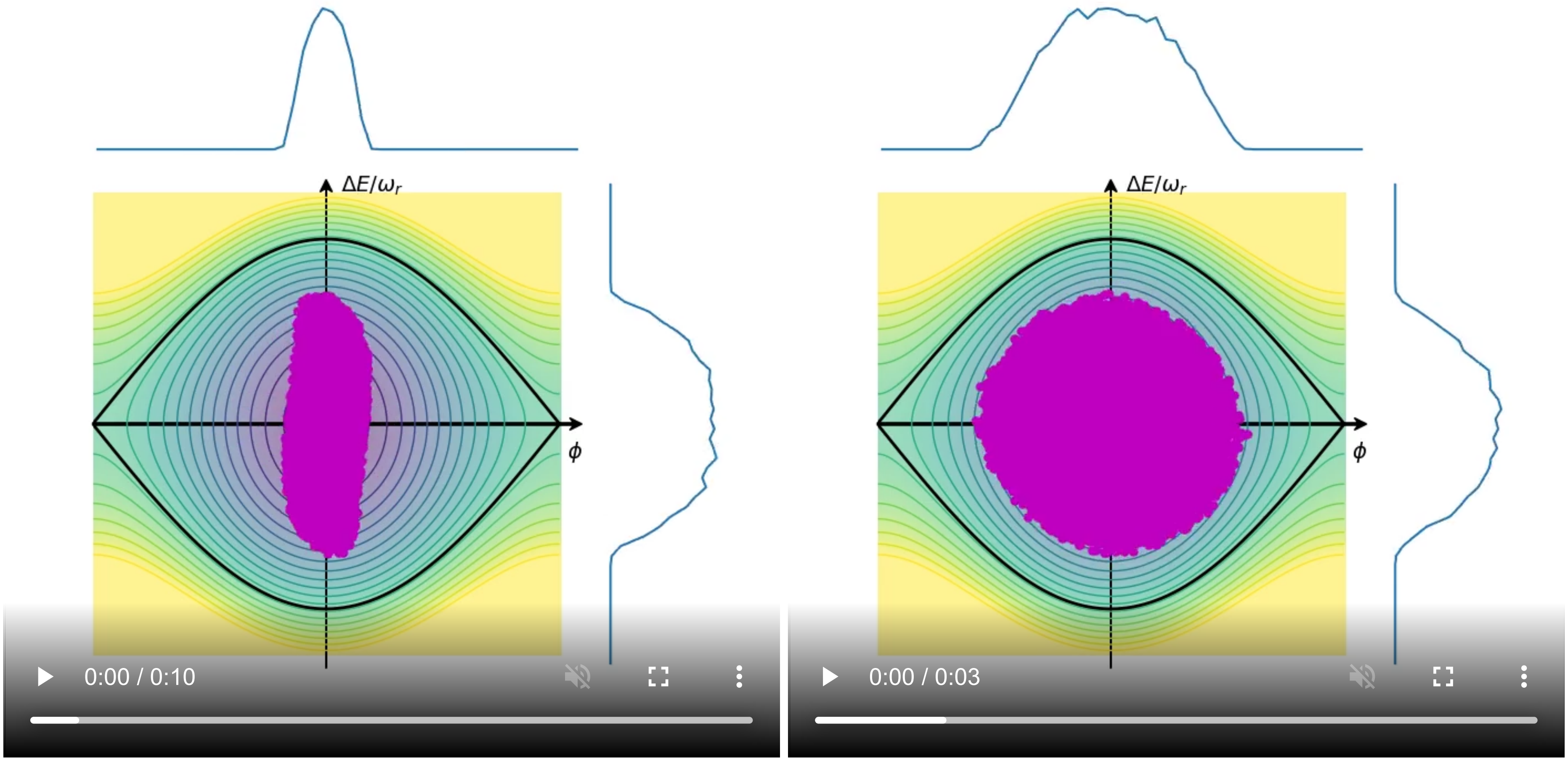
- The injection phase of the 2 extra bunches, or the energy of the circulating beam can be adjusted.
- In that case, the energy of the circulating beam was adjusted by changing the RF frequency at fixed magnetic field

INJECTION OSCILLATIONS (AMPLITUDE)



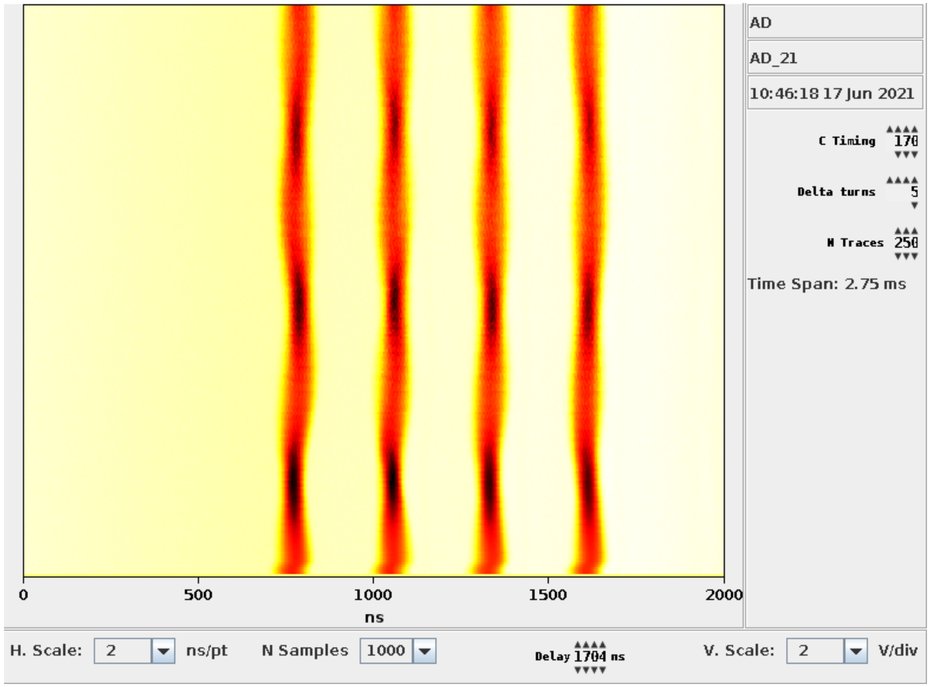
- In another cycle, 4 new bunches are injected 170 ms after the beginning of the cycle.
- The peak amplitude of the bunches (and the bunch lengths) oscillate.
- What is wrong?

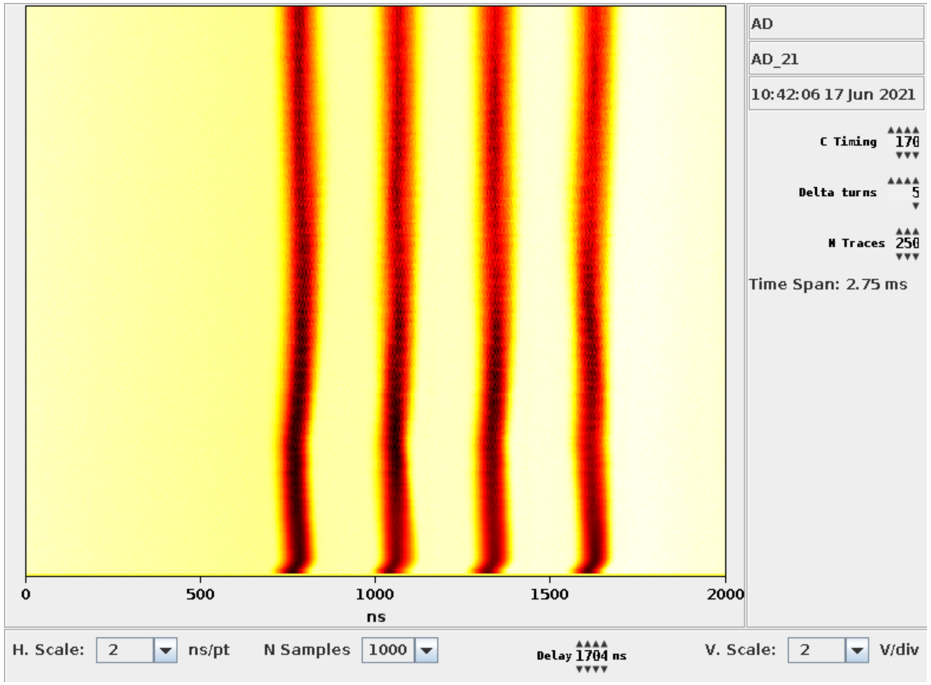
INJECTION OSCILLATIONS (AMPLITUDE)



- The bucket is too high in amplitude, the bunch is mismatched (left). After reduction of the voltage, the bunch is matched (NB: different scale in energy!)

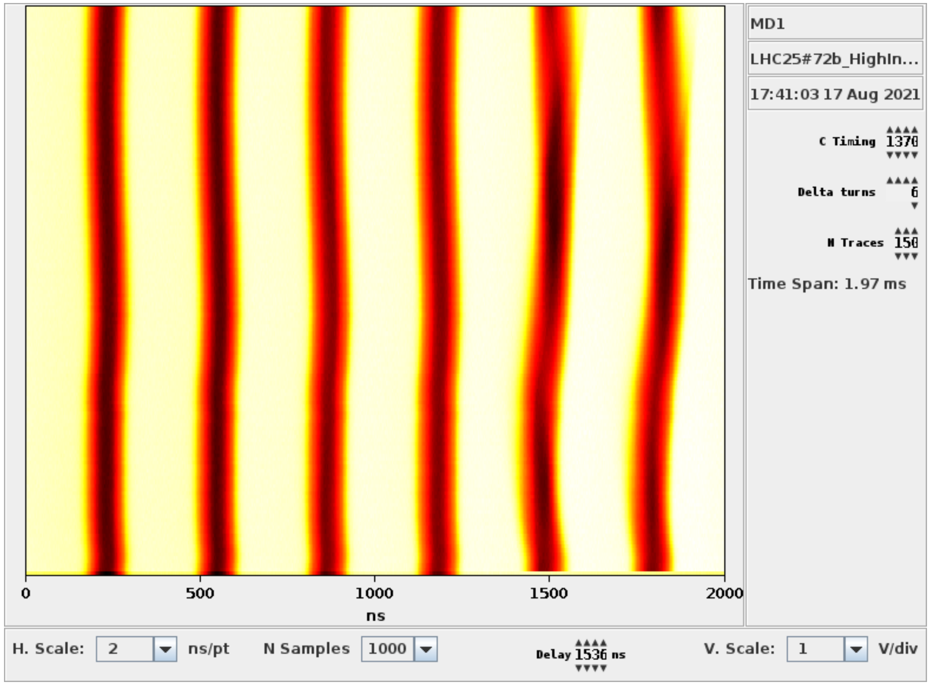
INJECTION OSCILLATIONS (AMPLITUDE)

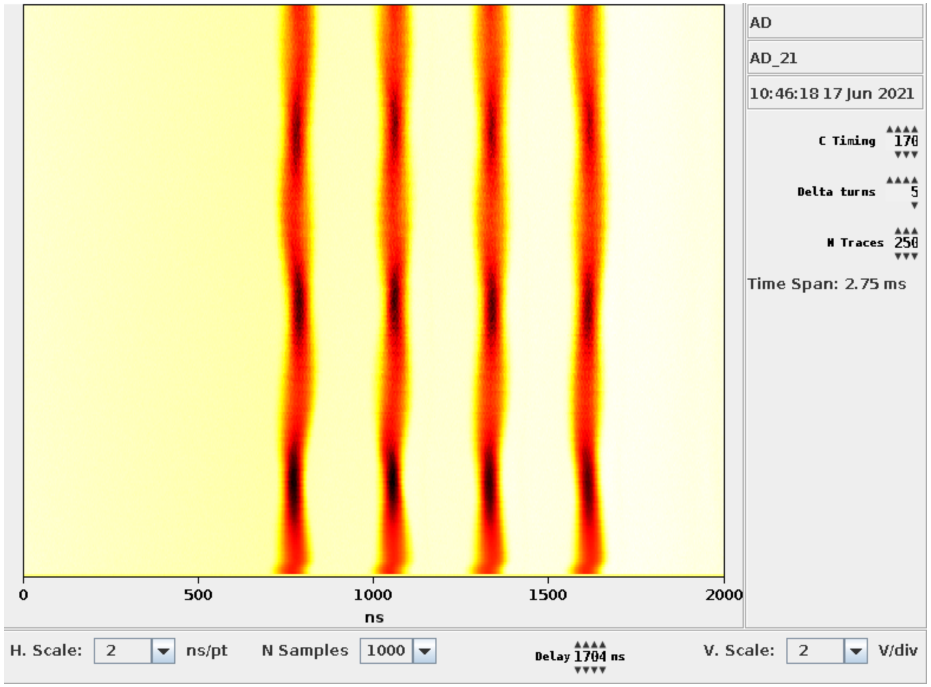




- The RF voltage can be adjusted to increase/reduce the amplitude of the bucket for matching.
- In that case, the RF voltage was reduced by a factor of 2 to improve the matching.

INJECTION OSCILLATIONS



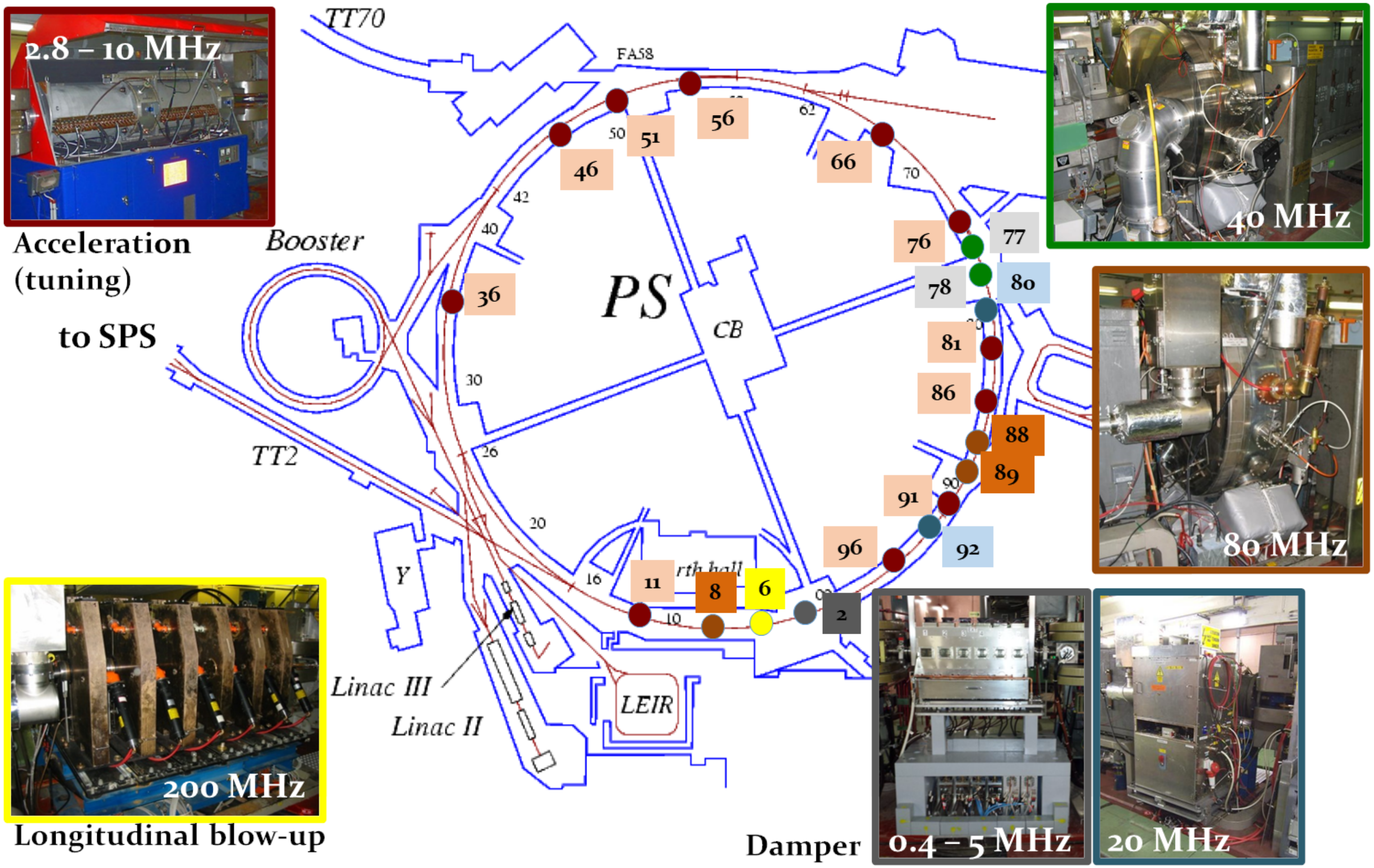


- Adjusting injection oscillations is a concrete example of routine operation to adjust machine parameters.

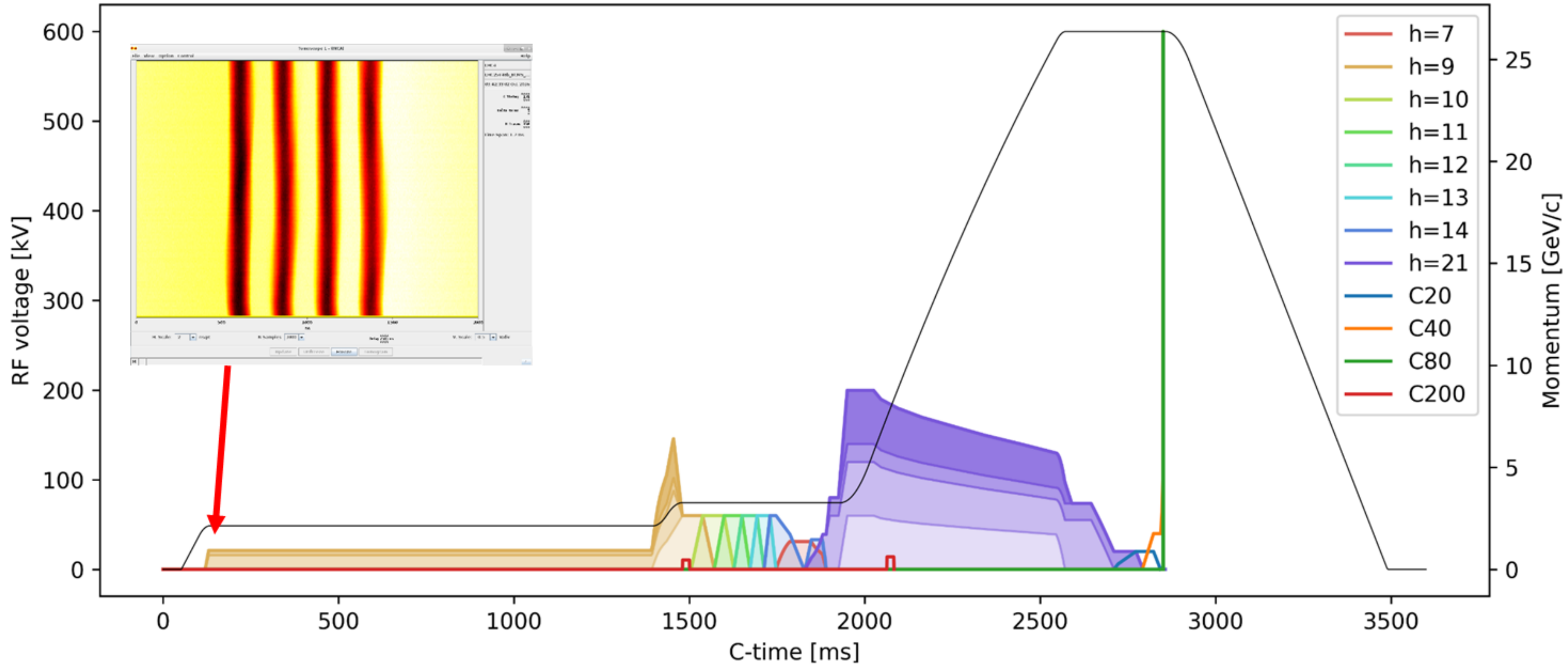
- The goal is to avoid filamentation and emittance blow-up, and fine tune the beam quality right from the start.

RF MANIPULATIONS

THE PS, ONE RING TO RULE THEM ALL

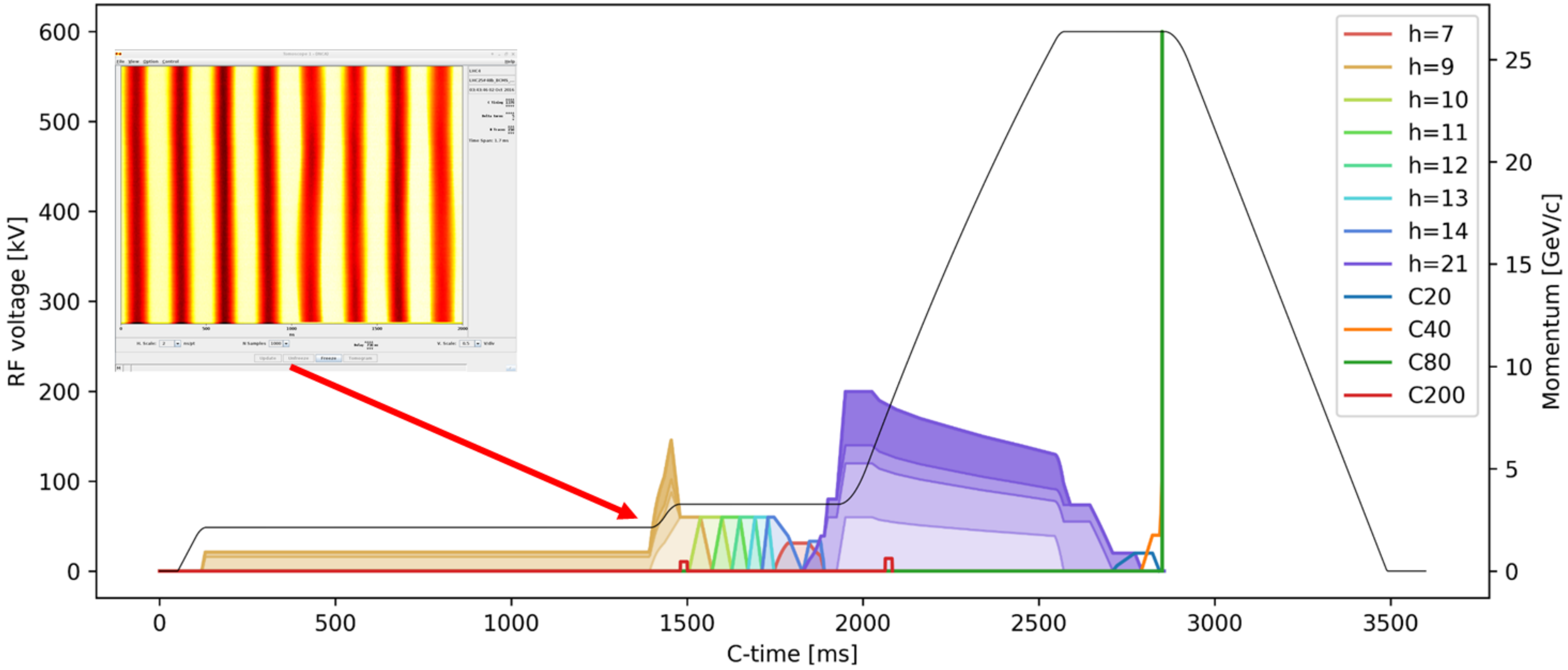


RF MANIPULATIONS OVERVIEW



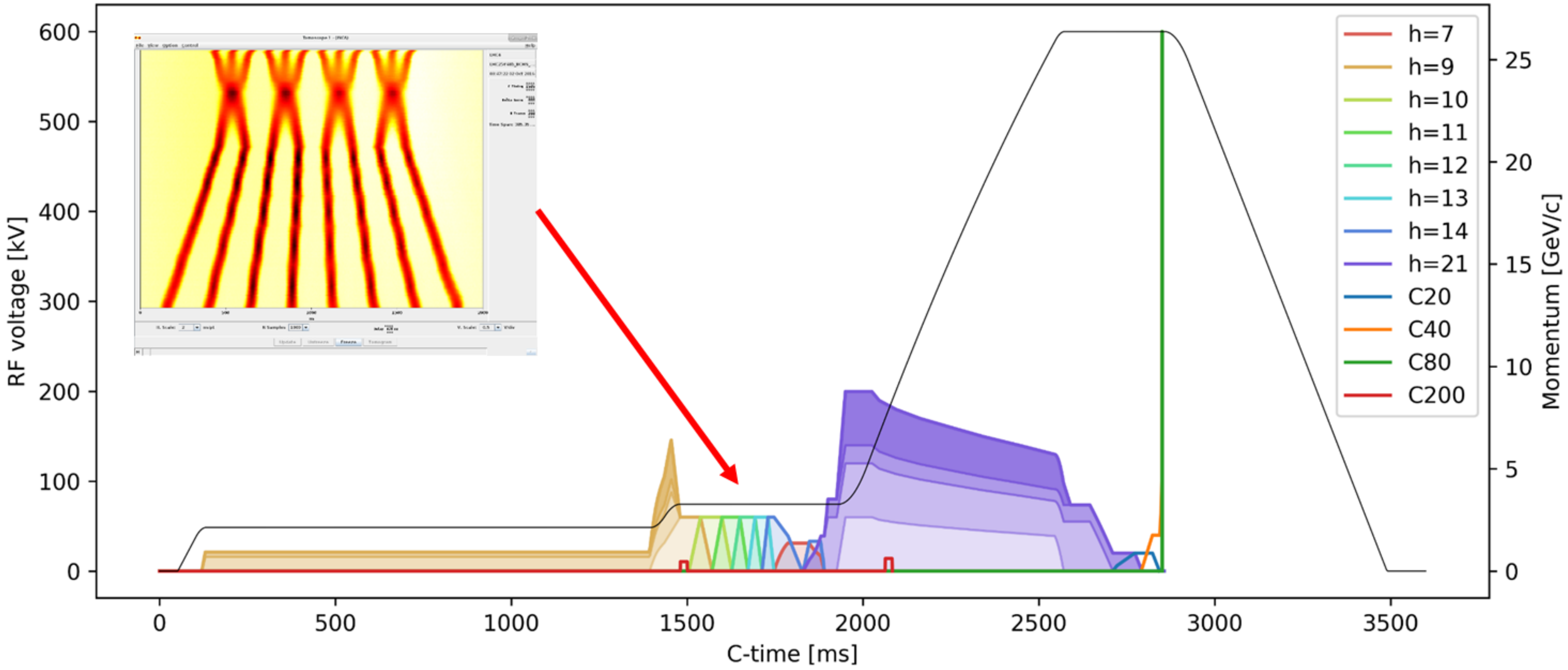
- Four bunches are injected from the pre-injector (PSB)

RF MANIPULATIONS OVERVIEW



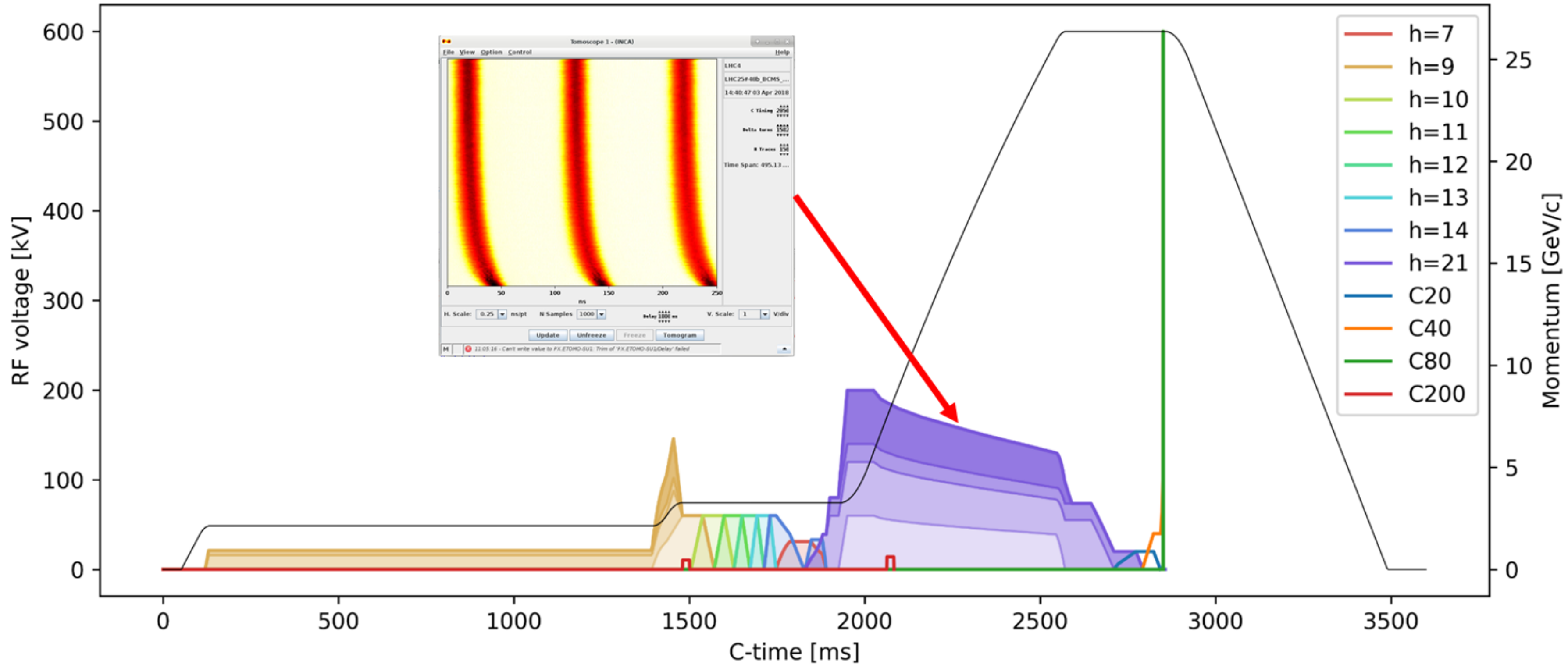
- Four more bunches are injected from the pre-injector (PSB)

RF MANIPULATIONS OVERVIEW



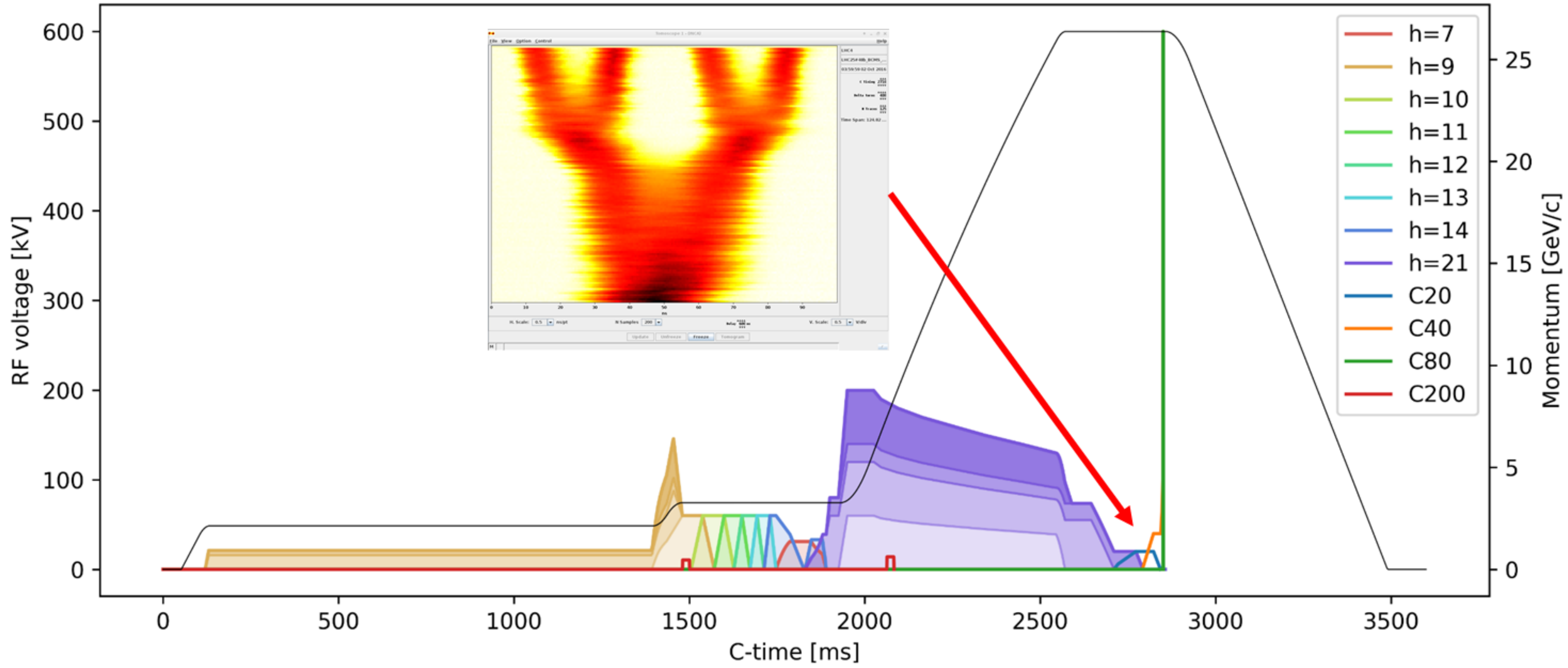
- The beam is accelerated to a plateau and undergoes many RF manipulations.
- The batch is compressed, bunches are merged, and split again

RF MANIPULATIONS OVERVIEW



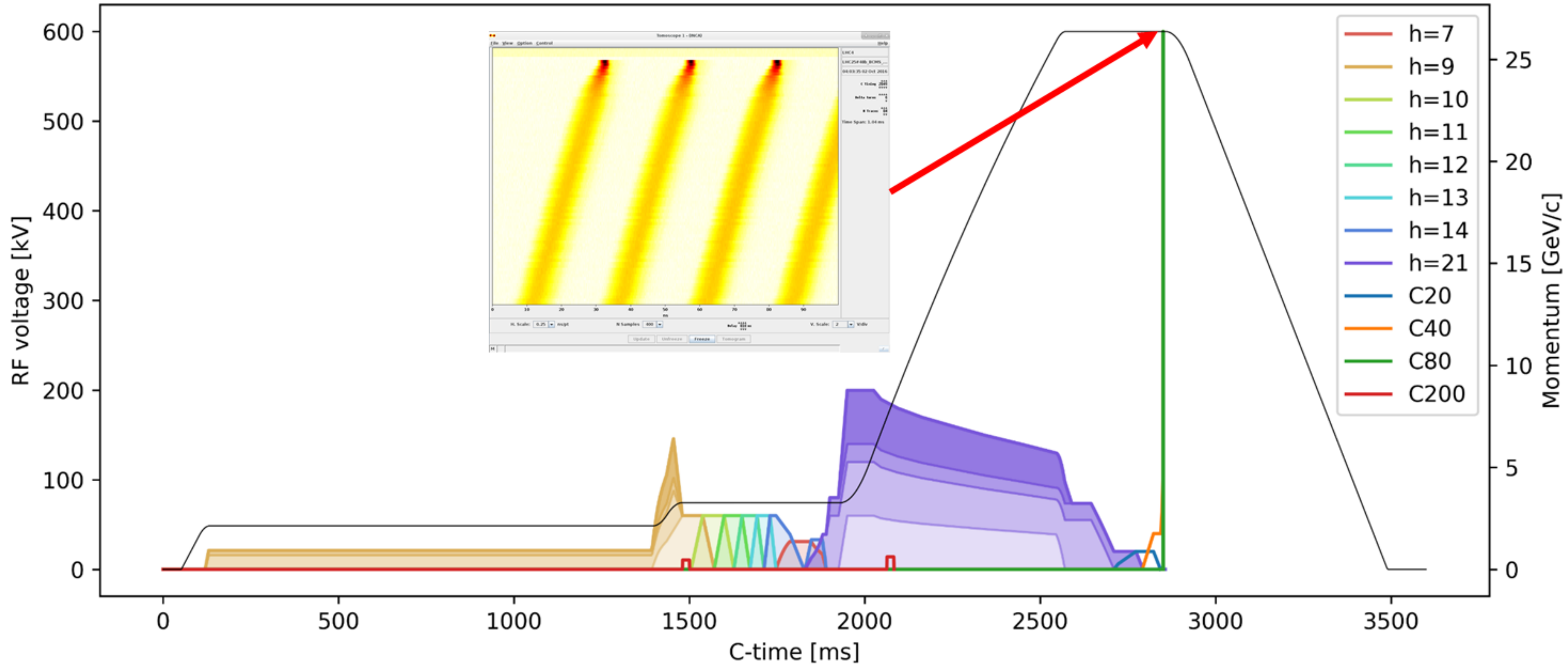
- The beam is accelerated, no (heavy) RF manipulation during the ramp.

RF MANIPULATIONS OVERVIEW



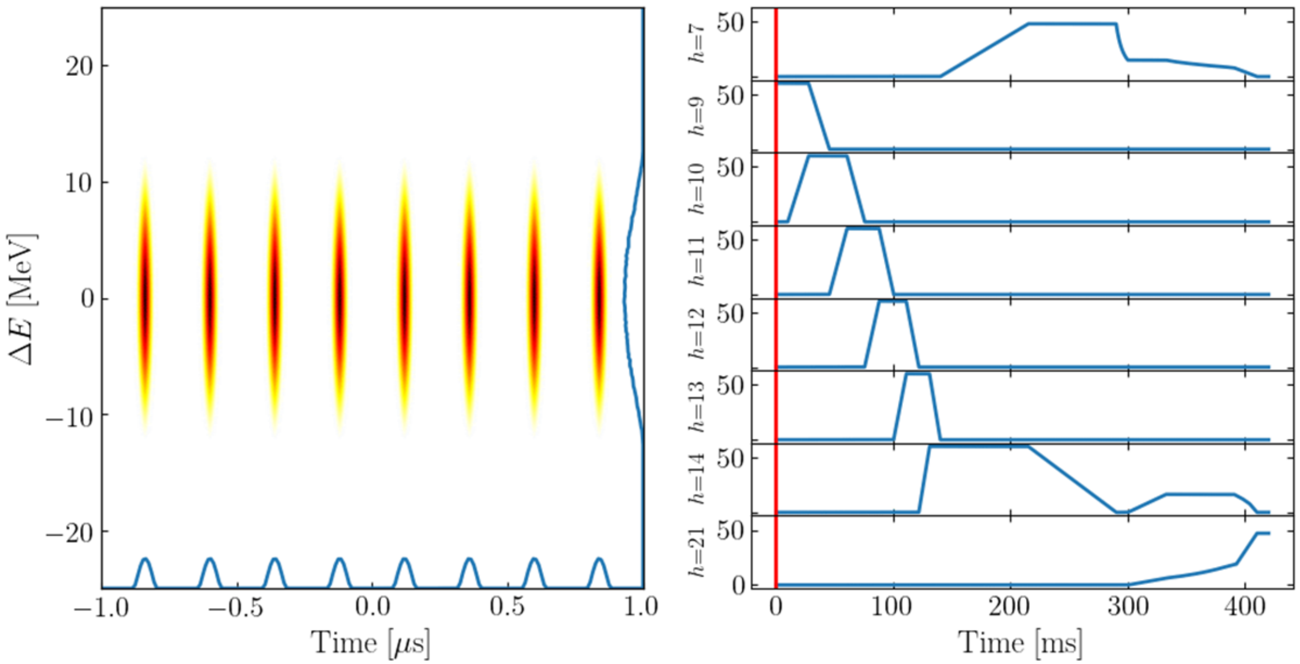
- The bunches are split again twice.

RF MANIPULATIONS OVERVIEW



- The bunches are compressed and extracted to the next machine, the SPS.
- The RF manipulations serve one purpose, define the 25 ns bunch spacing required by the final destination, the LHC!

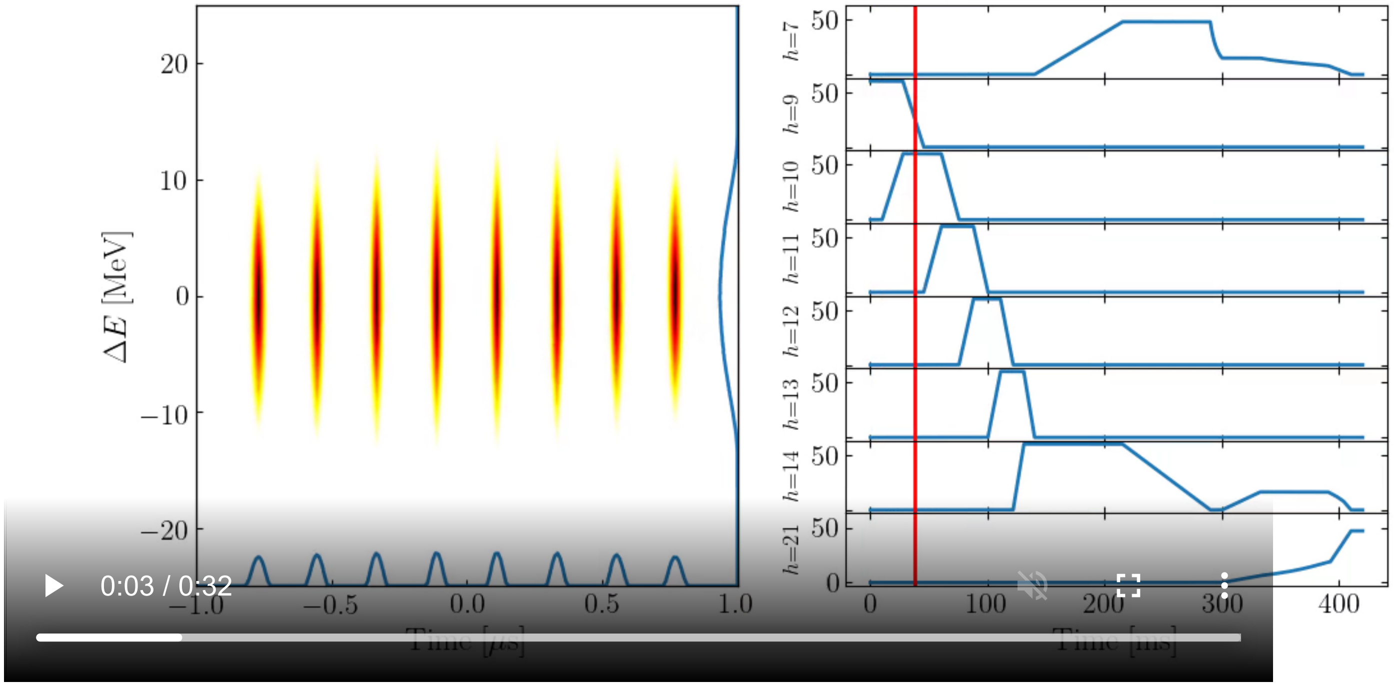
BATCH COMPRESSION, MERGING, SPLITTING



Batch compression h=9 to 14, Merging h=14to 7,

Triple Splitting h=7 to 21, with intermediate 14

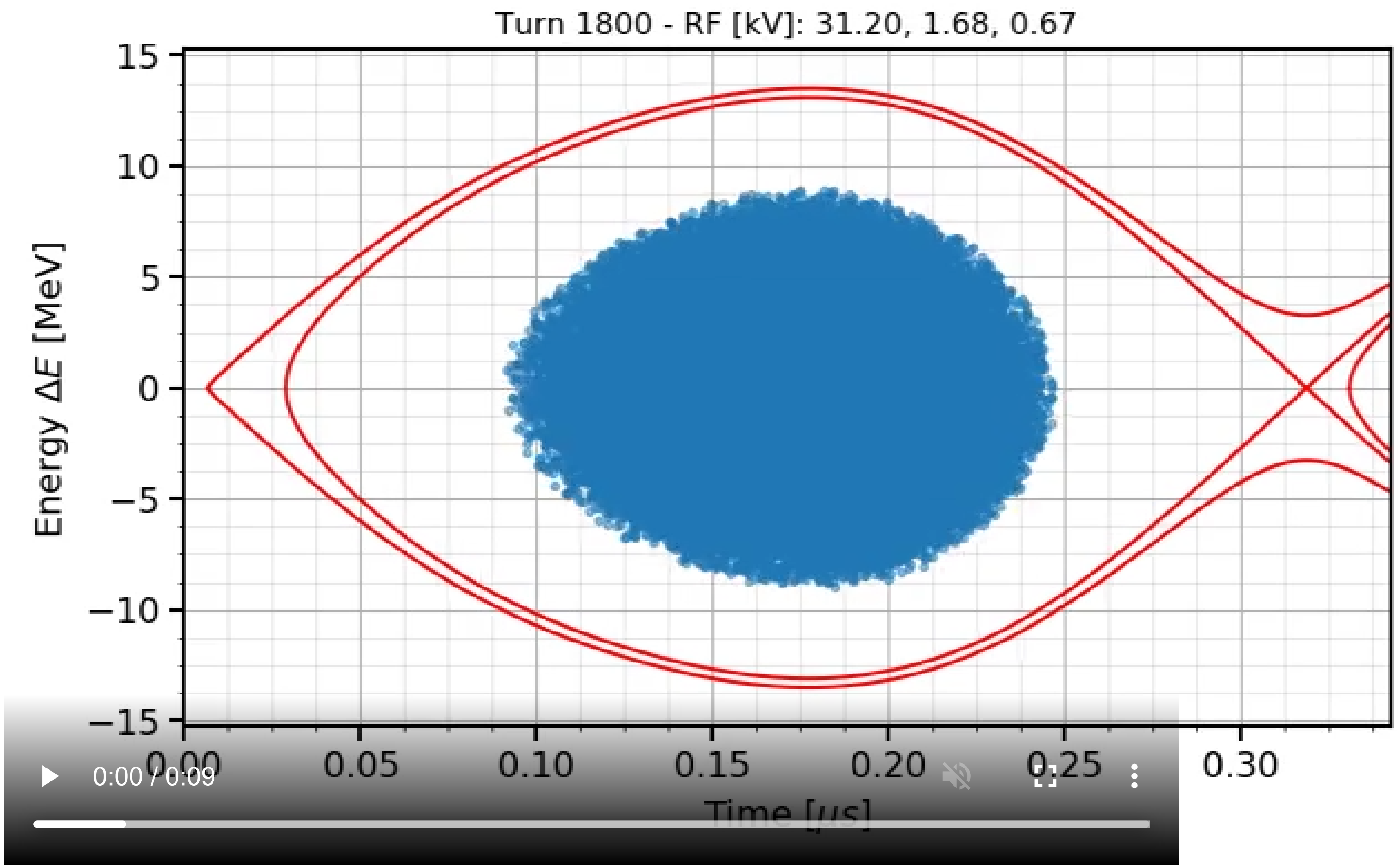
BATCH COMPRESSION, MERGING, SPLITTING



Eventually, 2 bunches merged and then split in 3. Emittance is preserved ideally (divided when split, multiplied when merged).

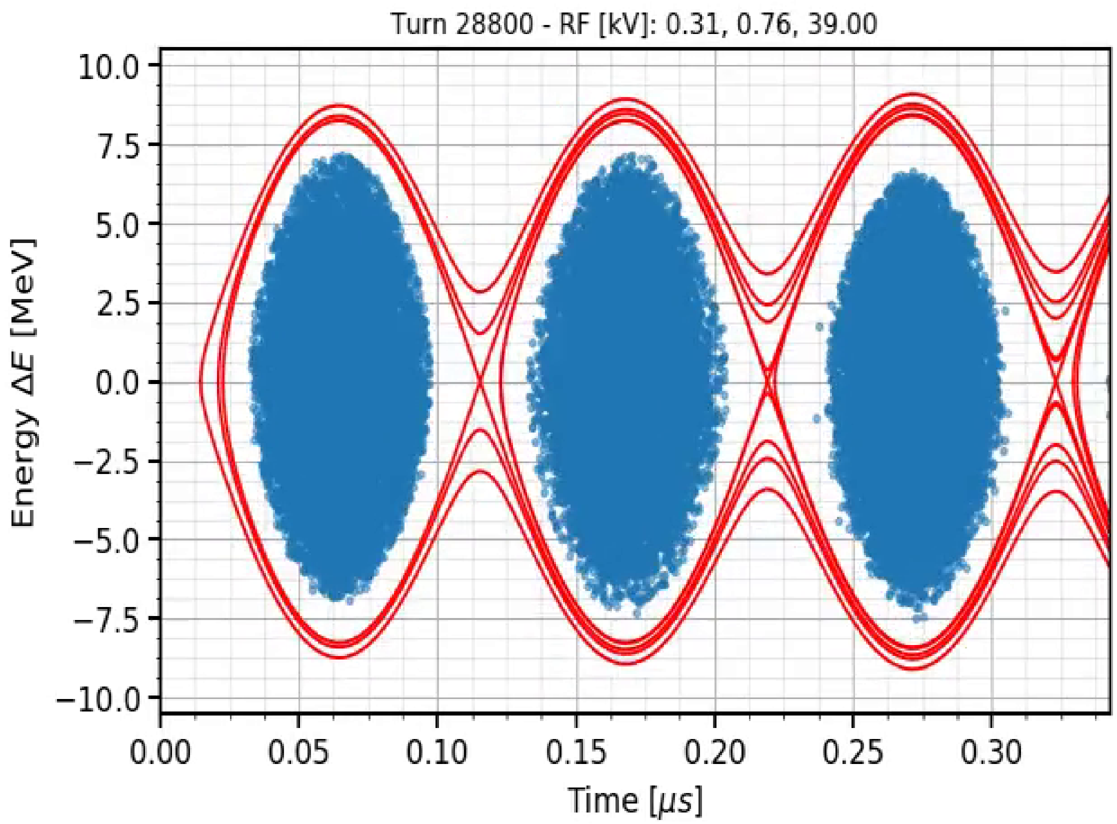
Animation: H. Damerau

ZOOM ON THE TRIPLE SPLITTING



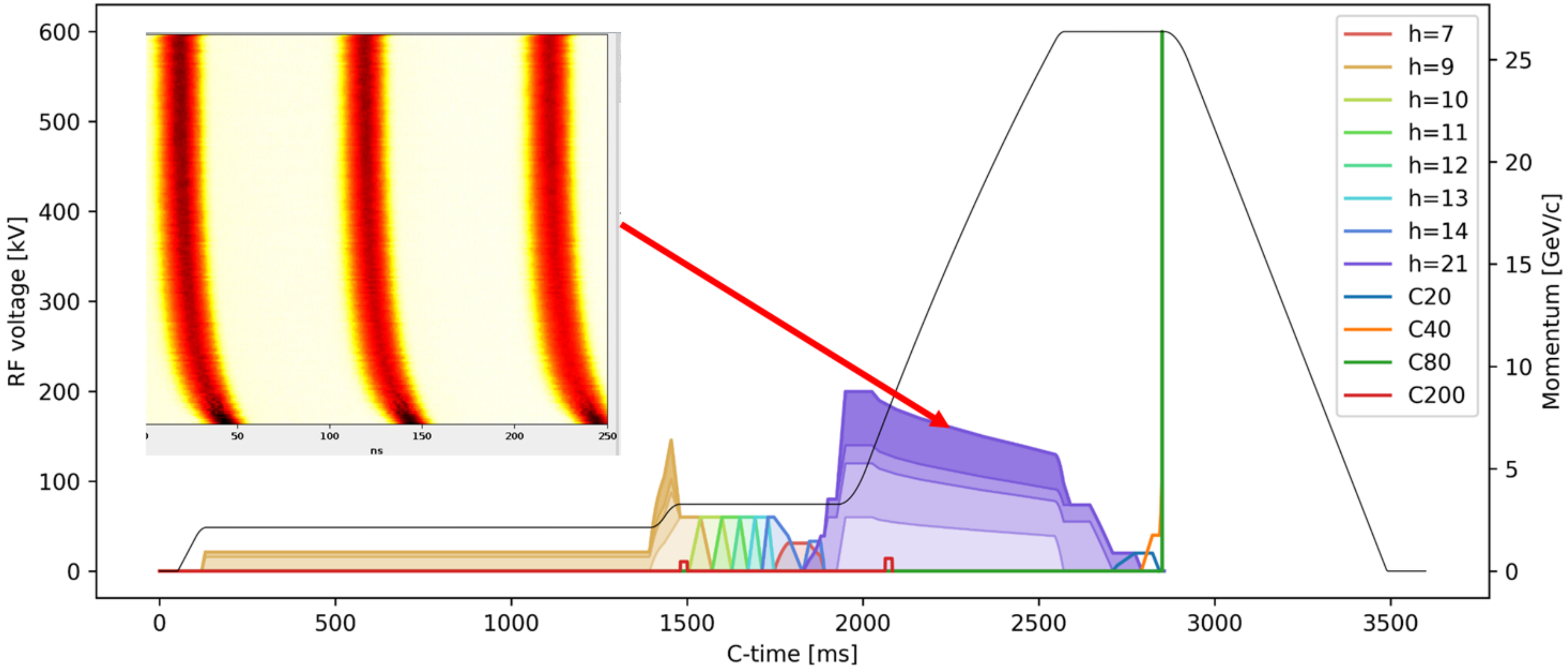
The separatrices are reprensented in red (several inner/outer separatrices, including intensity effects).

ZOOM ON THE TRIPLE SPLITTING



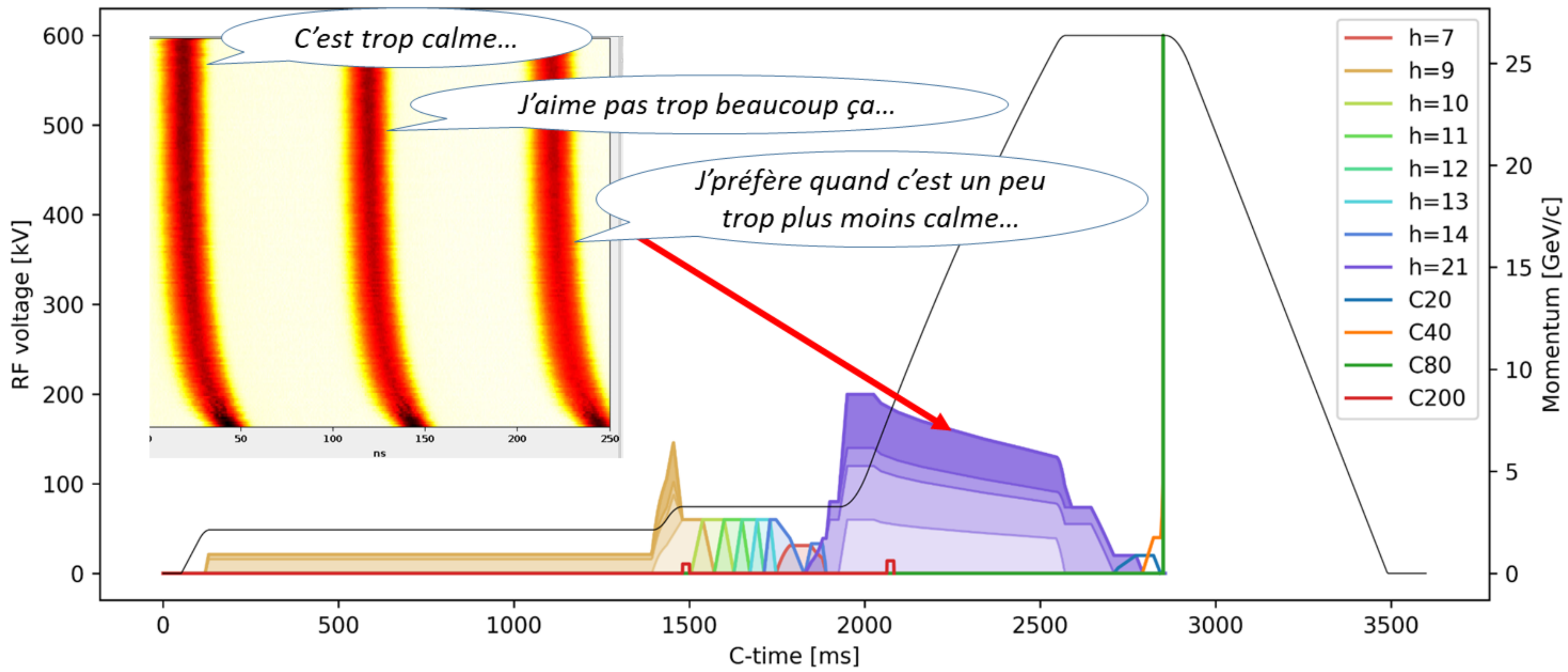
The separatrices are reprensented in red (several inner/outer separatrices, including intensity effects).

DURING THE ACCELERATION RAMP



The acceleration ramp is the moment when the bunch is manipulated the least, the bunches are accelerated smoothly till reaching top energy.

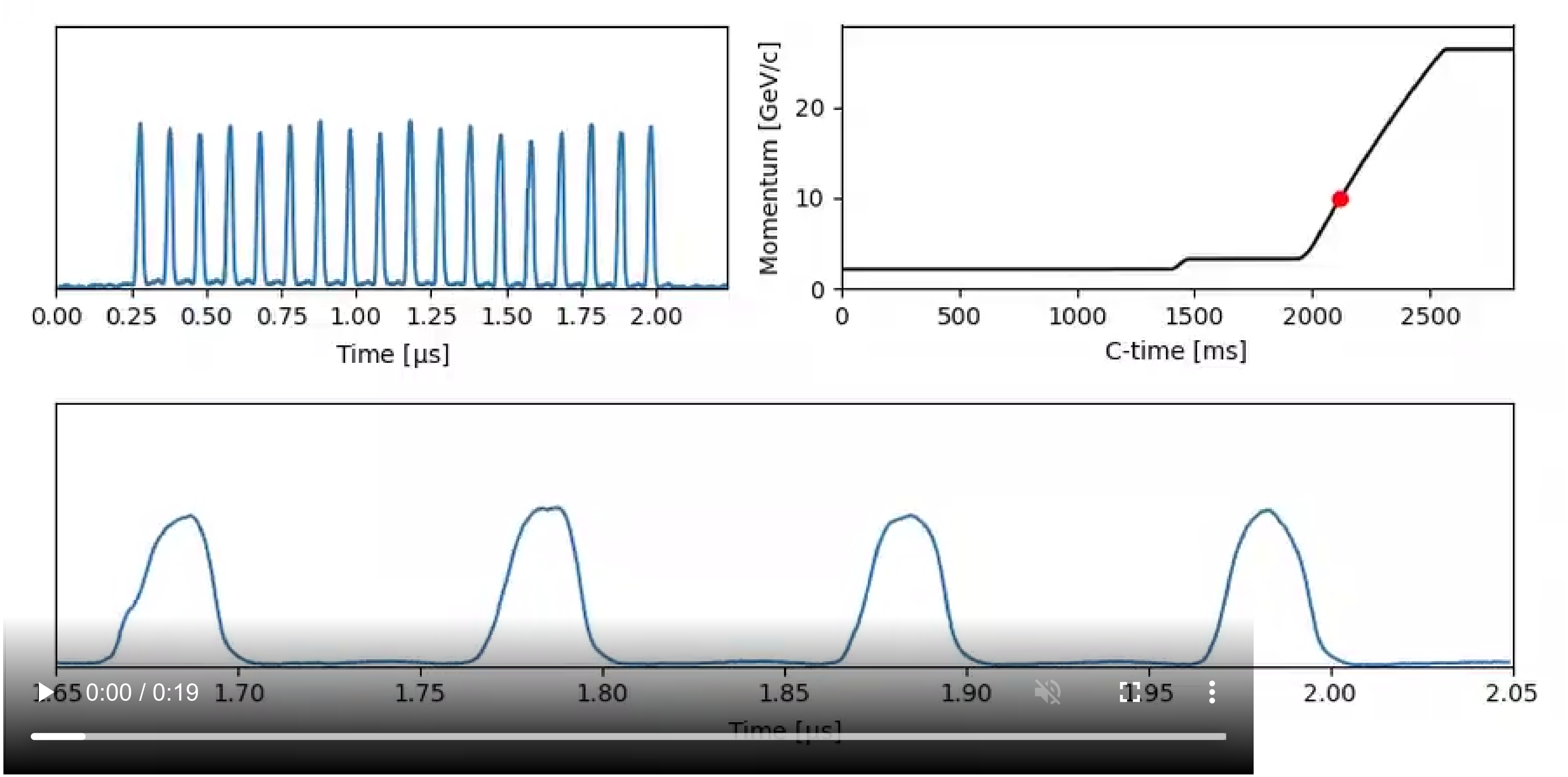
DURING THE ACCELERATION RAMP



The acceleration ramp is the moment when the bunch is manipulated the least, the bunches are accelerated smoothly till reaching top energy.

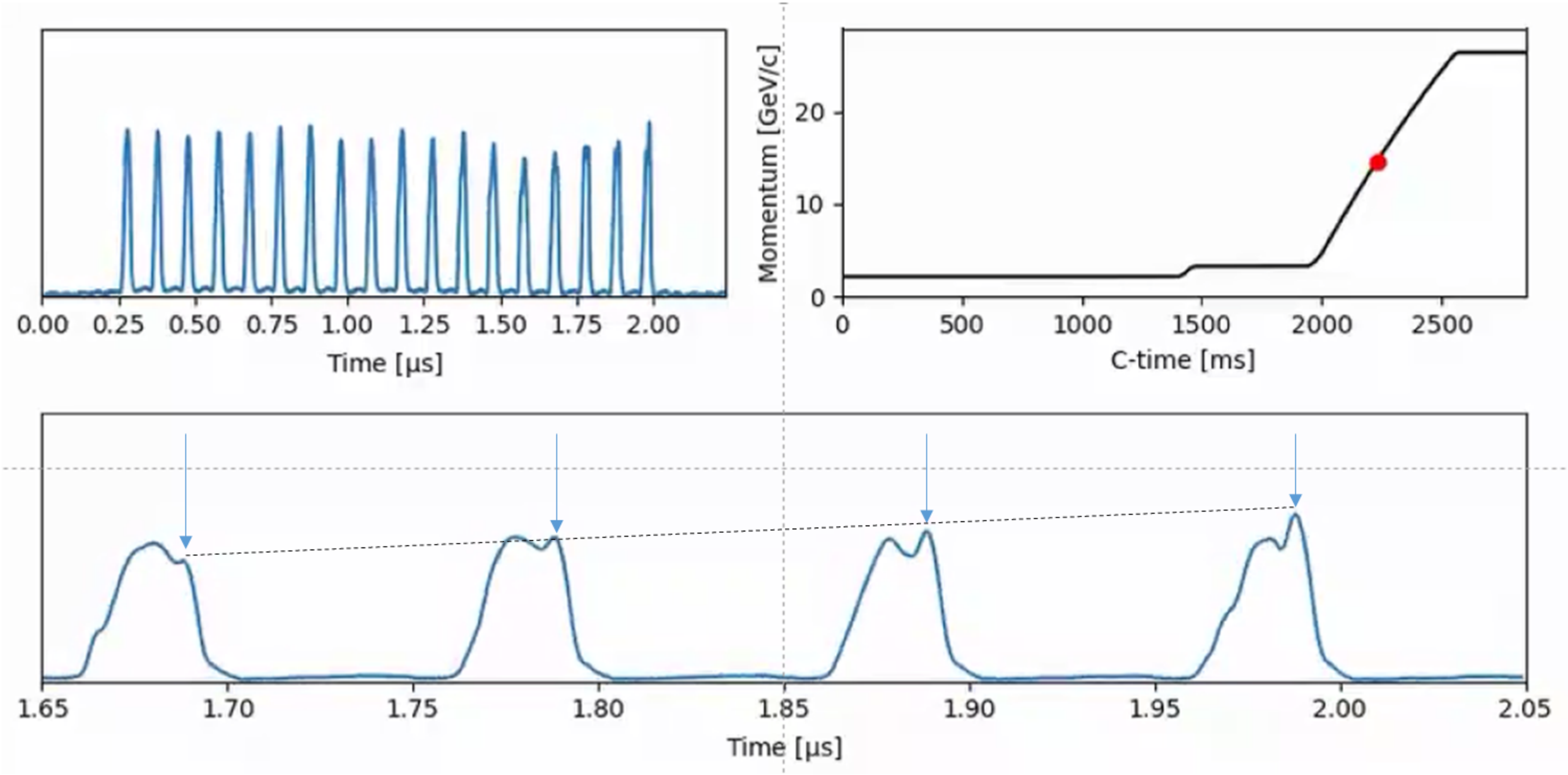
Or so it seems...

COUPLED BUNCH INSTABILITIES



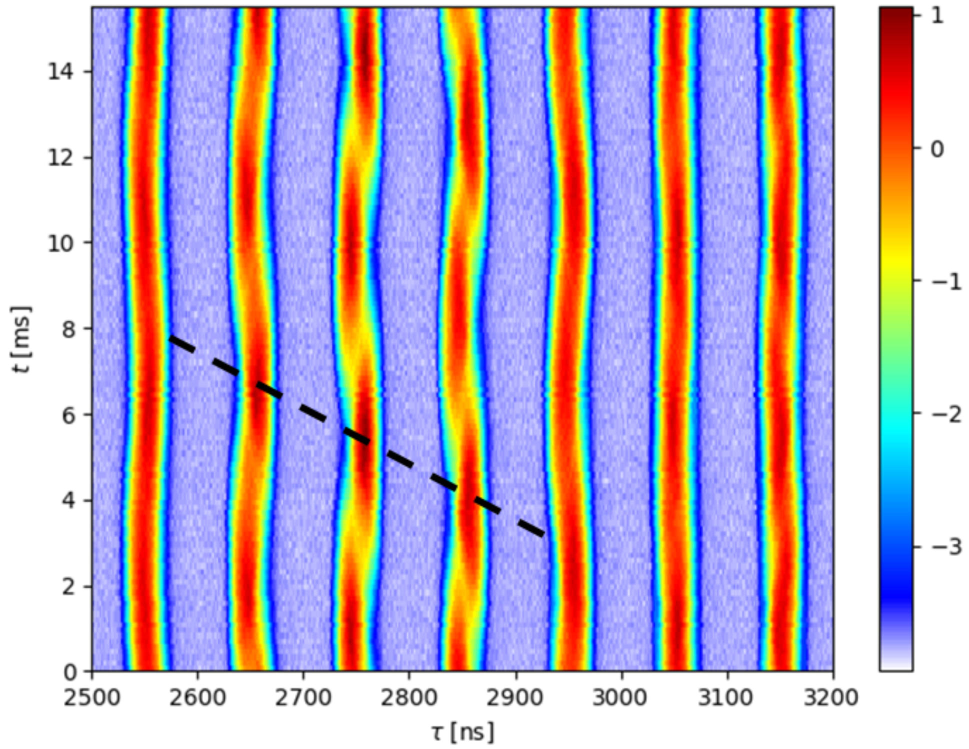
Bunches start to oscillate during the ramp at very high beam intensity (wakefields and instabilities!!)

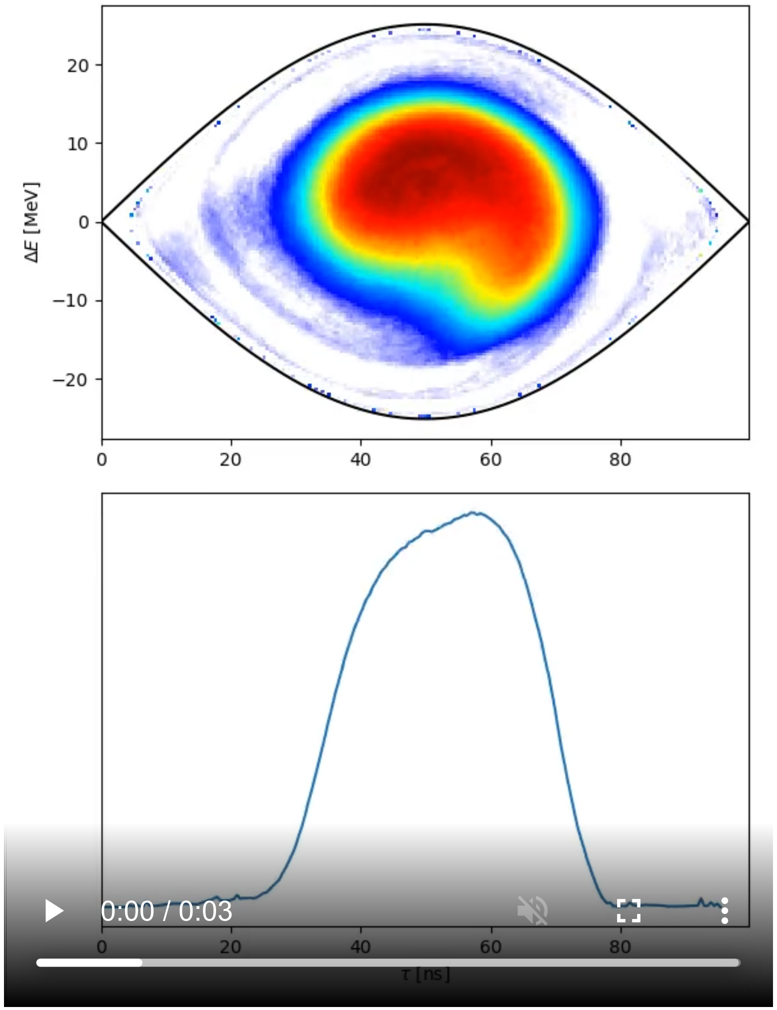
COUPLED BUNCH INSTABILITIES



Coupling between the bunches, phase advance from one bunch to the next in phase space.

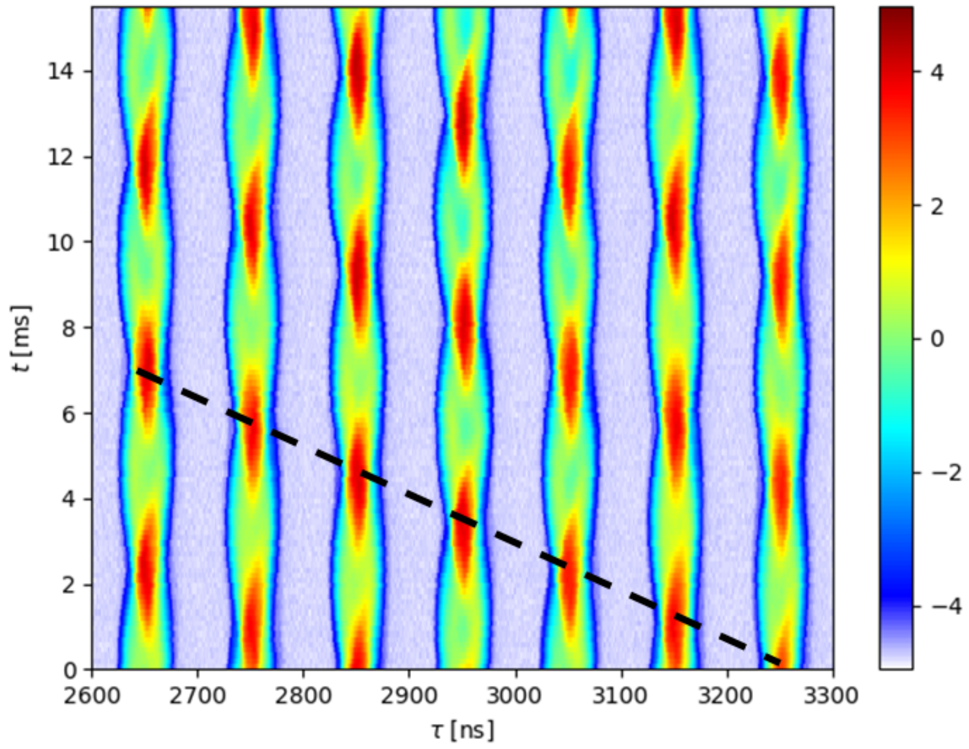
DIPOLE MODE OF OSCILLATIONS

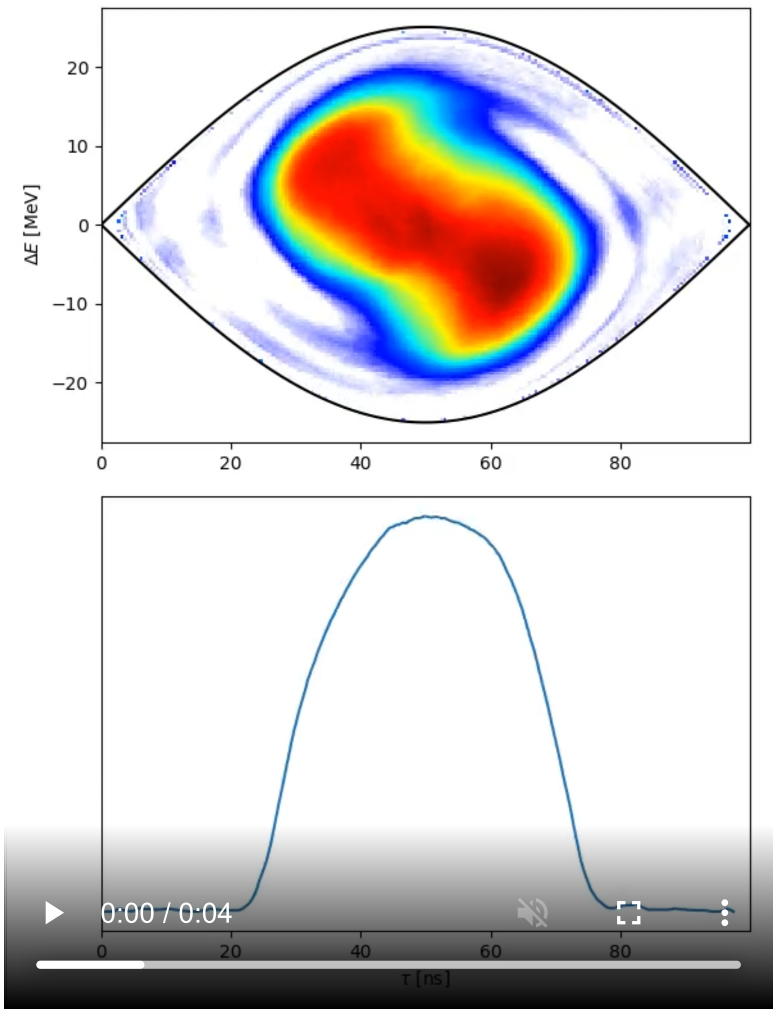




- Dipole mode of instability.
- Phase oscillations of the bunch, single node.
- Oscillates at 1 x fs0.

QUADRUPOLE MODE OF OSCILLATIONS





- Quadrupole mode of instability.
- Oscillations of the bunch length, two node.
- Oscillates at 2 x fs0.

THE END