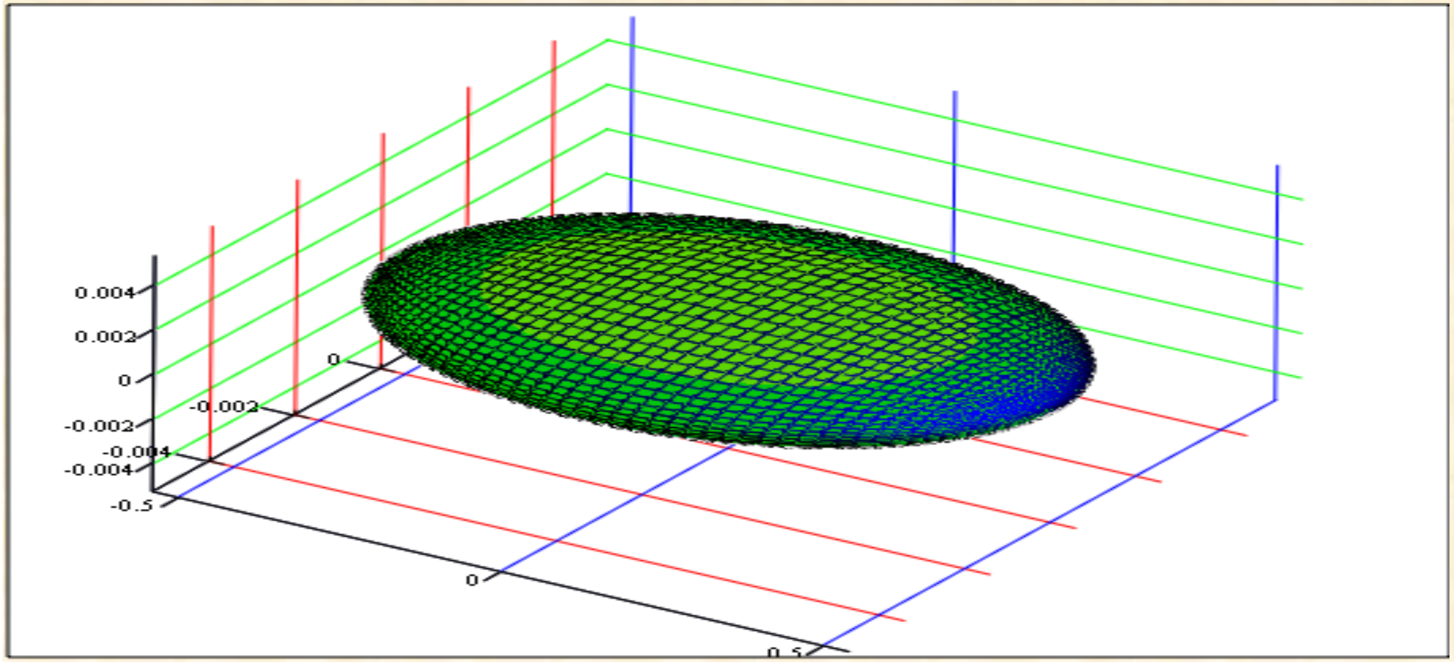
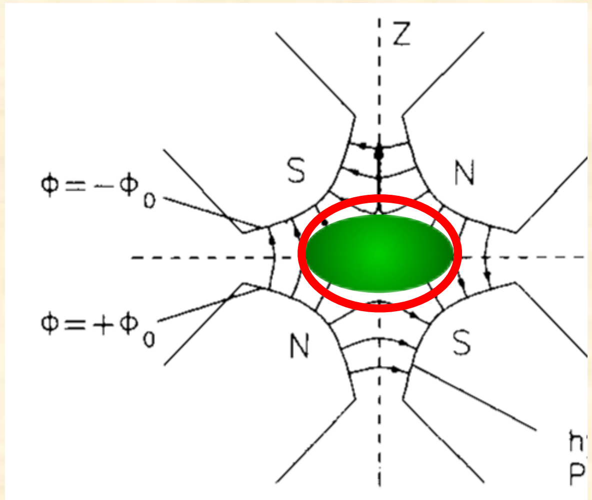
Introduction to Transverse Beam Optics

Bernhard Holzer,
CERN

II.) The Ideal World:
Particle Trajectories & Beams





Bunch in a storage ring

Reminder of Part I

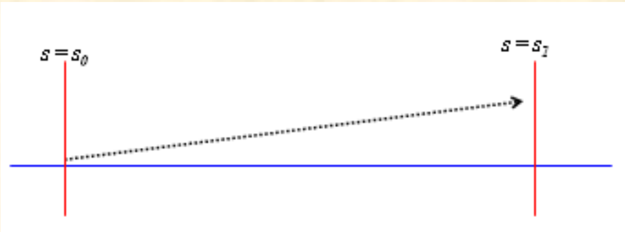
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \quad \text{... hor. plane:}$$

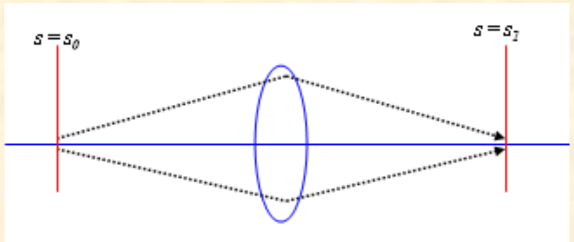
$$K = k \quad \text{... vert. Plane:}$$

Solution of Trajectory Equations

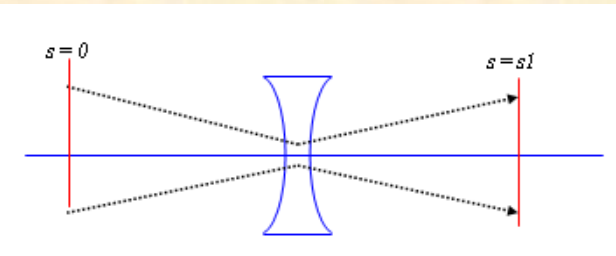
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



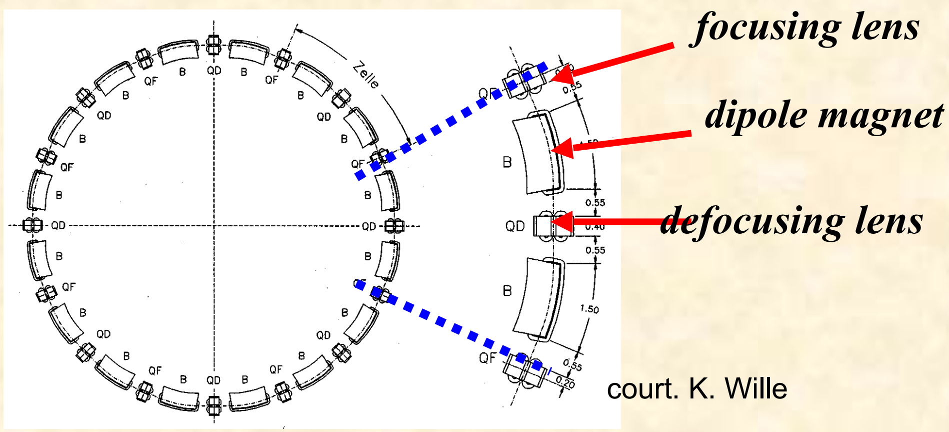
$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l \\ \sqrt{|K|}\sinh(\sqrt{|K|}l & \cosh(\sqrt{|K|}l \end{pmatrix}$$

Transformation through a system of lattice elements

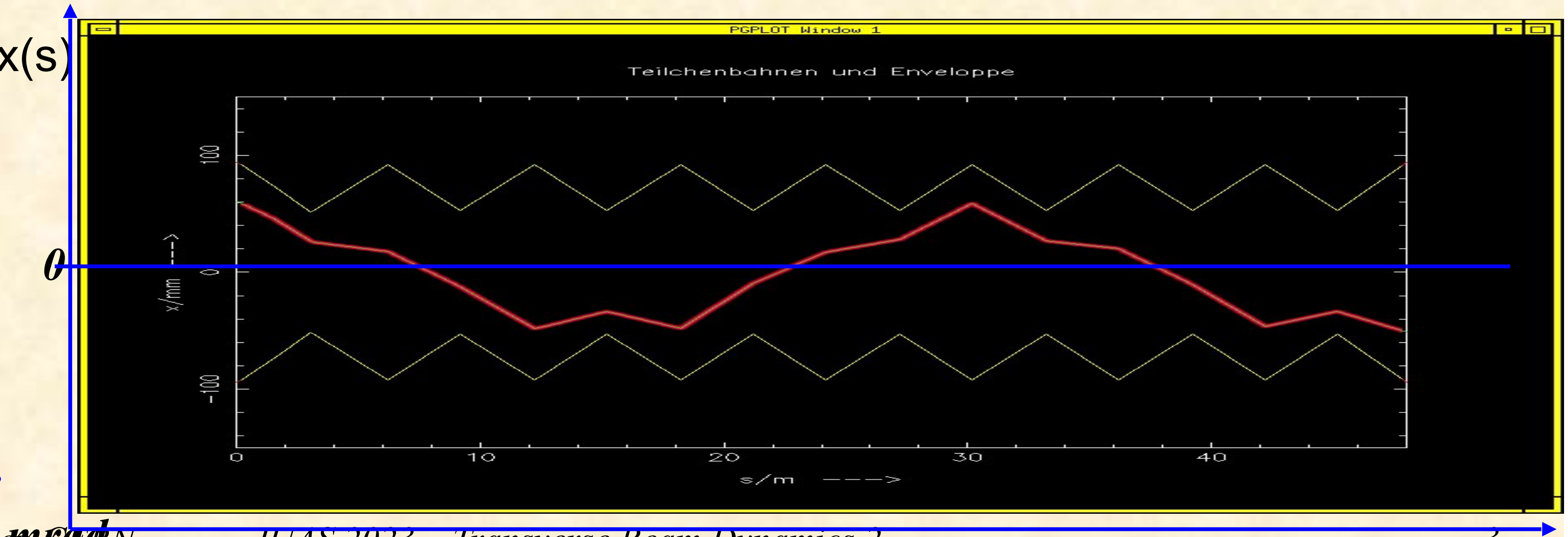
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M_{1->2} \begin{pmatrix} x \\ x' \end{pmatrix}_{s2}$$



in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:

$x \approx mm$, $x' \leq mrad$

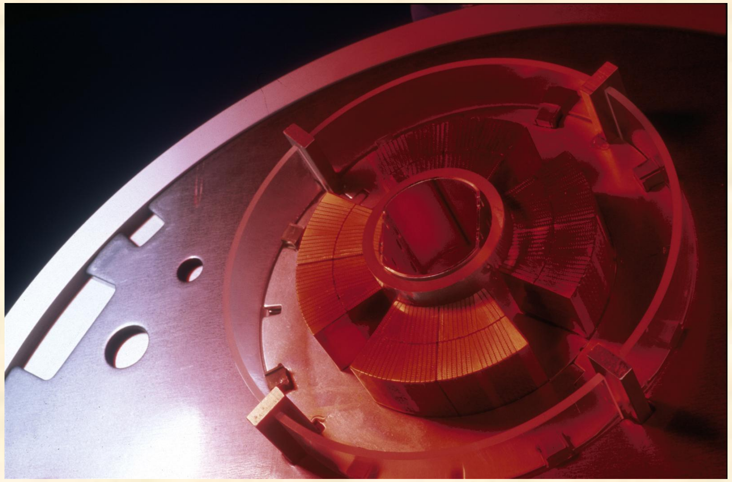
Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

linear increasing Lorentz force

linear increasing magnetic field

$$B_y = g\,x \qquad B_x = g\,y$$



normalised quadrupole field:

gradient of a quadrupole magnet: $g = \frac{dB_y}{dx}$

normalised gradient $k = \frac{g}{p/e}$

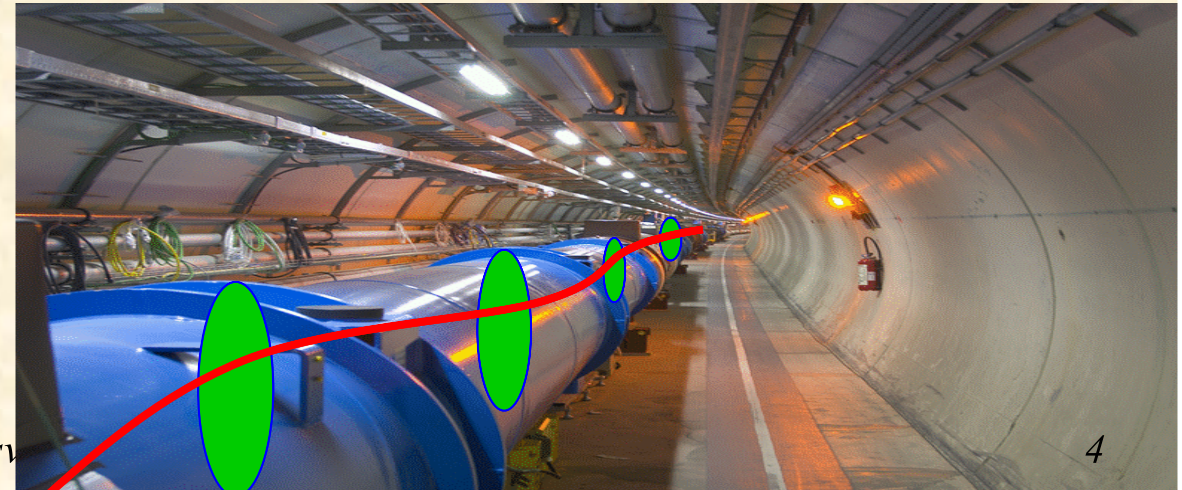
LHC main quadrupole magnet $g \approx 25 \ldots 220\ \ T/m$

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho} \qquad k := \frac{g}{p/q}$$

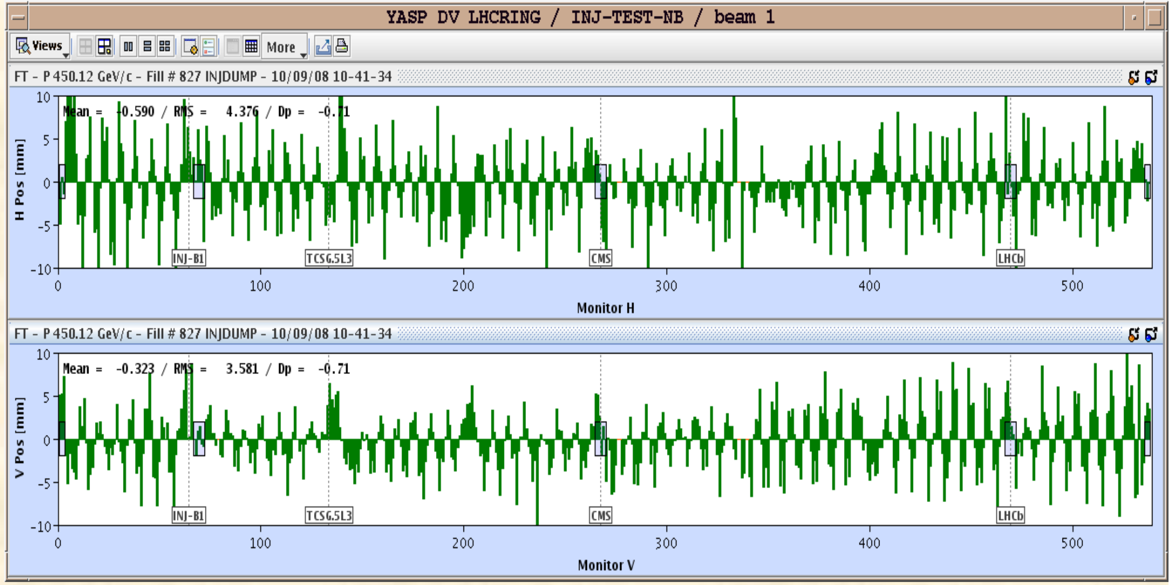
what about the vertical plane:
... Maxwell

$$\vec{\nabla}\times\vec{B} = \vec{j} + \frac{\partial\vec{E}}{\partial t} = 0$$

$$\Rightarrow \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$



5.) Orbit & Tune:



Tune: number of oscillations per turn

64.31

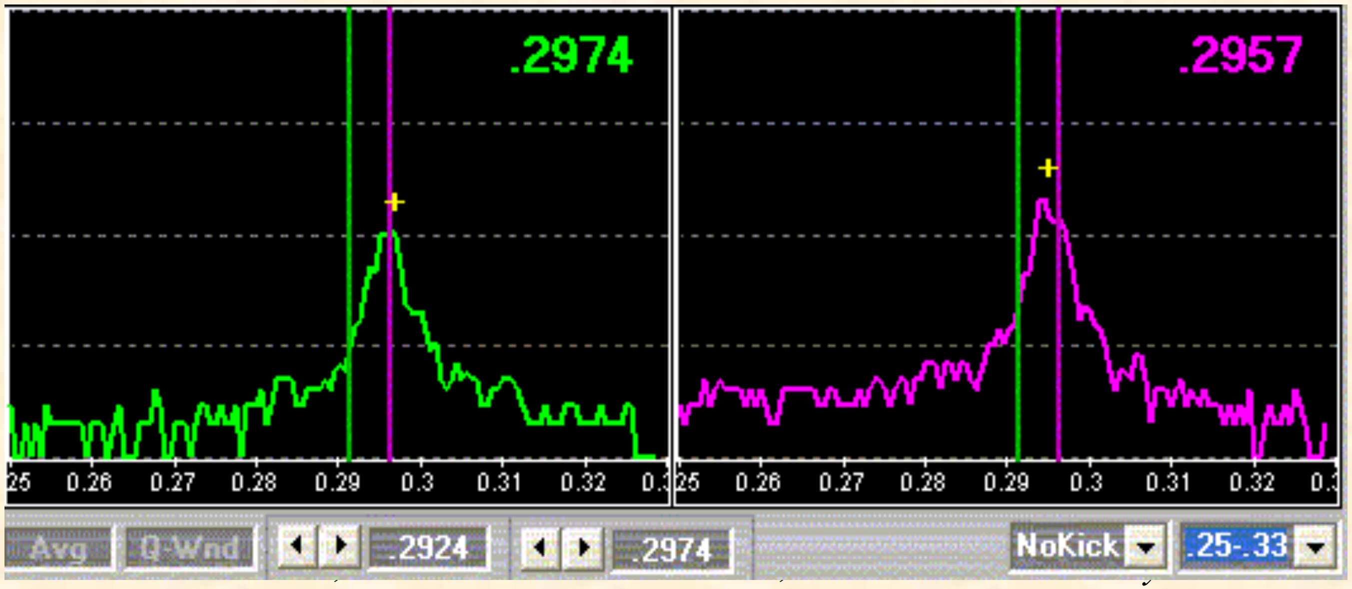
59.32

Relevant for beam stability:

non integer part

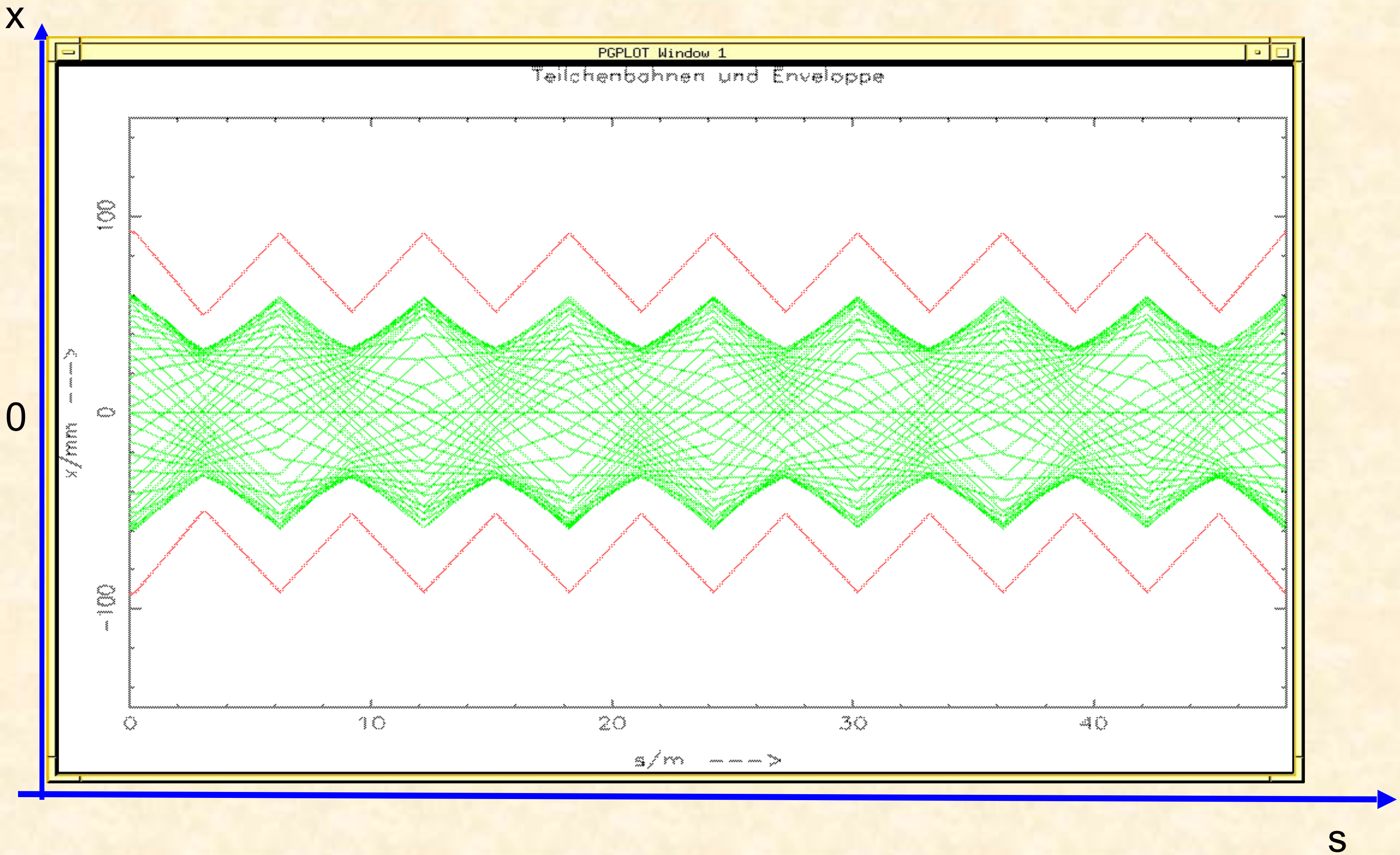
LHC revolution frequency: 11.3 kHz

$0.31 * 11.3 = 3.5 kHz$



Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with
periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$ ***Hill's equation"***

restoring force ≠ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

Hill's Equation:

Hill's equation: the origins

ON THE PART OF THE

MOTION OF THE LUNAR PERIGEE

WHICH IS A FUNCTION OF THE

MEAN MOTIONS OF THE SUN AND MOON

BY

G. W. HILL

in WASHINGTON.

Hill's original paper or orbital mechanics (1886)

9.) The Beta Function

General solution of Hill´s equation:

Ansatz: $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\cdot\cos(\psi(s)+\phi) \qquad (i)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

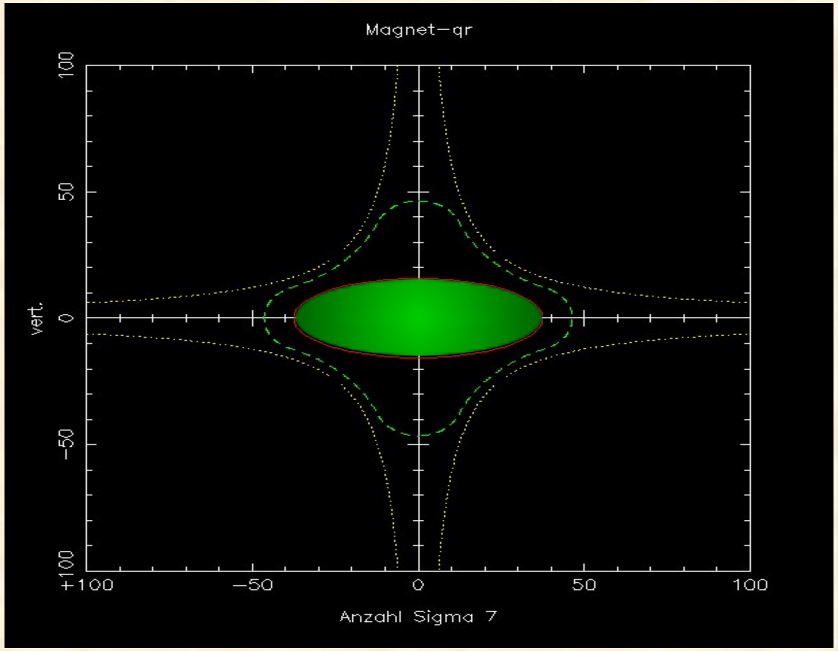
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice. For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi}\oint\frac{ds}{\beta(s)}$$

The Beta Function

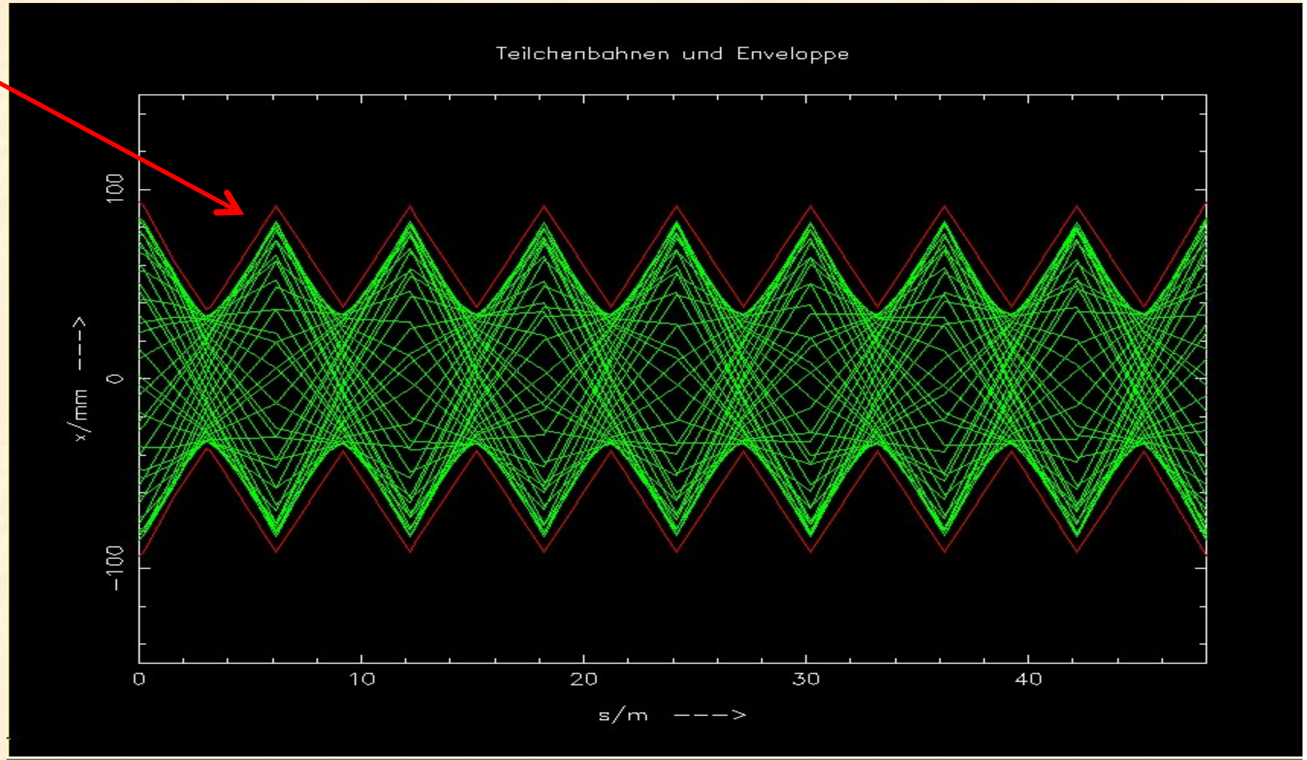


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} \cdot \sqrt{\beta(s)}\ cos(\psi(s) + \phi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

10.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

(1) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

(2) $$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

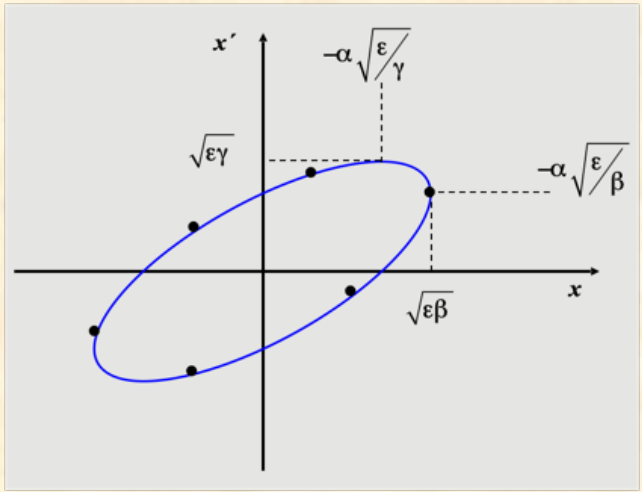
$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** ε is a constant of the motion … it is independent of „s“***
***** parametric representation of an ellipse in the x x‘ space***
***** shape and orientation of ellipse are given by α, β, γ***

Phase Space Ellipse



particle trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ *and solve for x´*

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

→ $x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$

* *A high β-function means a large beam size and a small beam divergence.* **!**
… et vice versa !!!

* *In the middle of a quadrupole β = maximum,*
α = zero
} $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

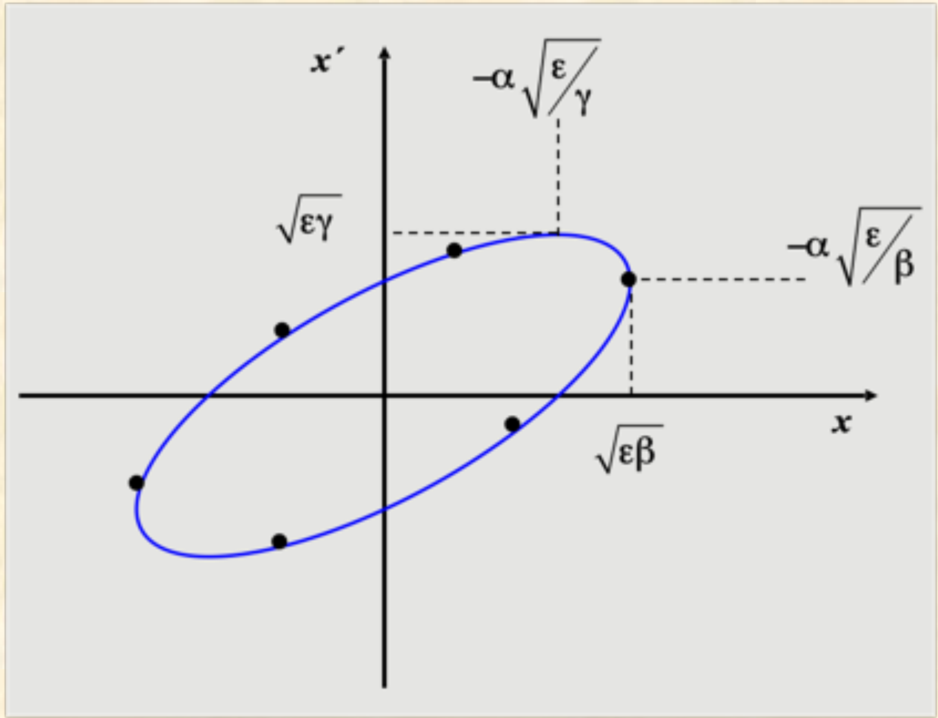
$$\rightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot x x' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ *via:* $\quad \dfrac{dx'}{dx} = 0$

$$\rightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

$$\rightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Liouville states that phase space is conserved.

Primarily, this refers to 6-dimensional phase space (x-x´, y-y´ and s-dp/p).

When the component phase spaces are uncoupled, the phase space is conserved within the 2- dimensional and/or 4-dimensional spaces.

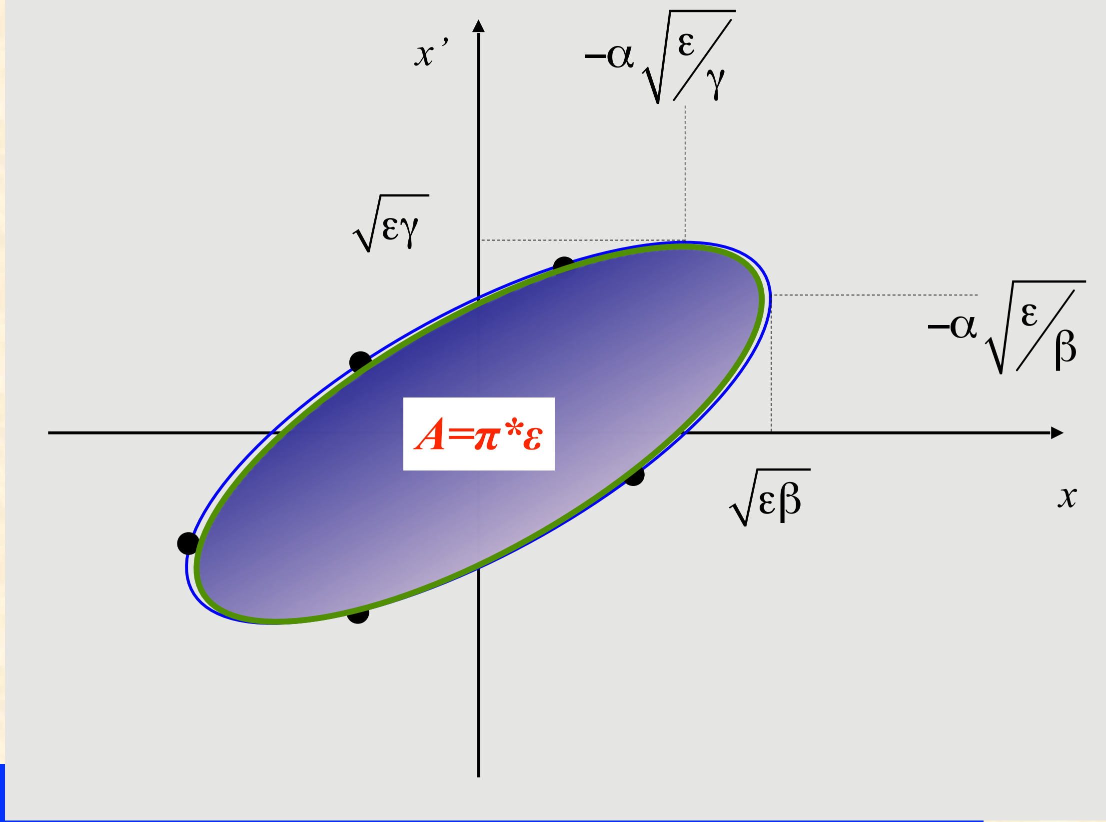
The invariant of the motion in the uncoupled x-x´or y-y´ spaces is another way of saying the phase space is conserved.

Phase space is not conserved if ions change, e.g. by stripping or nuclear fragmentation, or if non-Hamiltonian forces appear e.g. scattering or photon emission.

(Phil Bryant)

Phase Space Area & Emittance

shape and orientation of the phase space ellipse depend on the Optics parameters α, β, γ



The emittance of a beam is related to the phase-space area that it occupies and is therefore related to the motion invariants of the constituent ions. A practical definition of emittance requires a choice for the limiting ellipse that defines the phase-space area of the beam. Usually this is related to some number of standard deviations of the beam distribution, for example "the 1-sigma emittance is … " .

(Phil Bryant)

… just another harmonic oscillator

Harmonic oscillator is back

Restoring force:

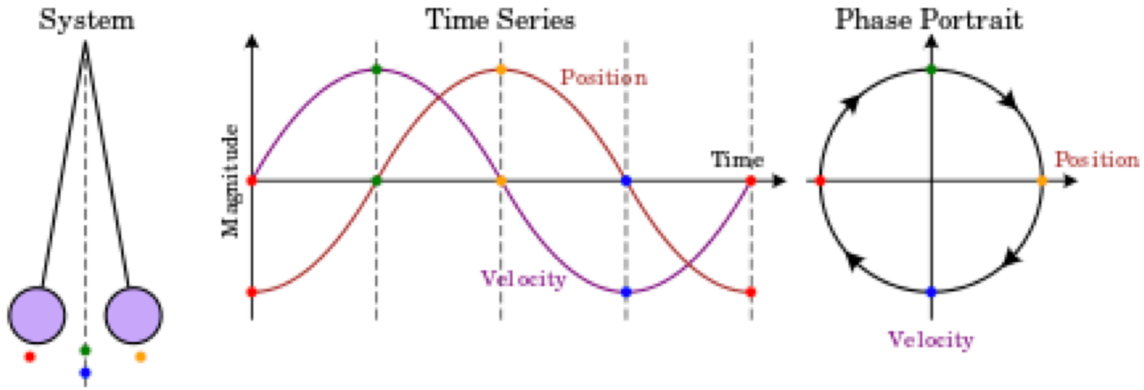
$$F = -ku \tag{9}$$

Equation of motion:

$$u'' = -\frac{k}{m}u \tag{10}$$

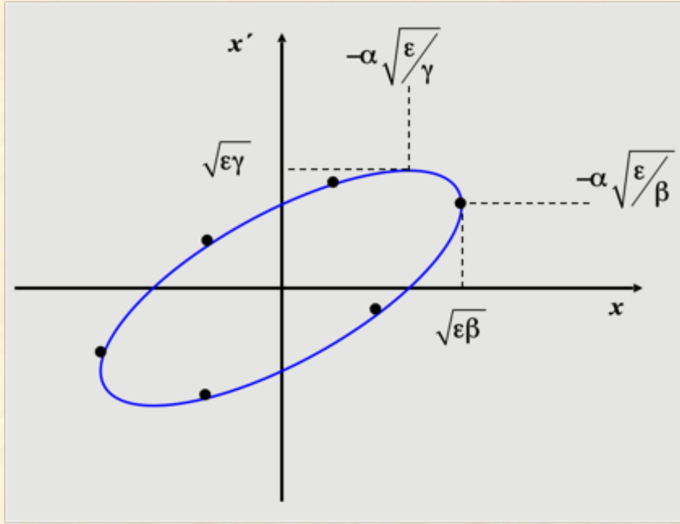
Solution:

$$u = a\cos(\omega t + \phi) \tag{11}$$



A storage ring is periodic !!

x'



shape and orientation of the phase space ellipse depend on the Optics parameters α, β, γ

x

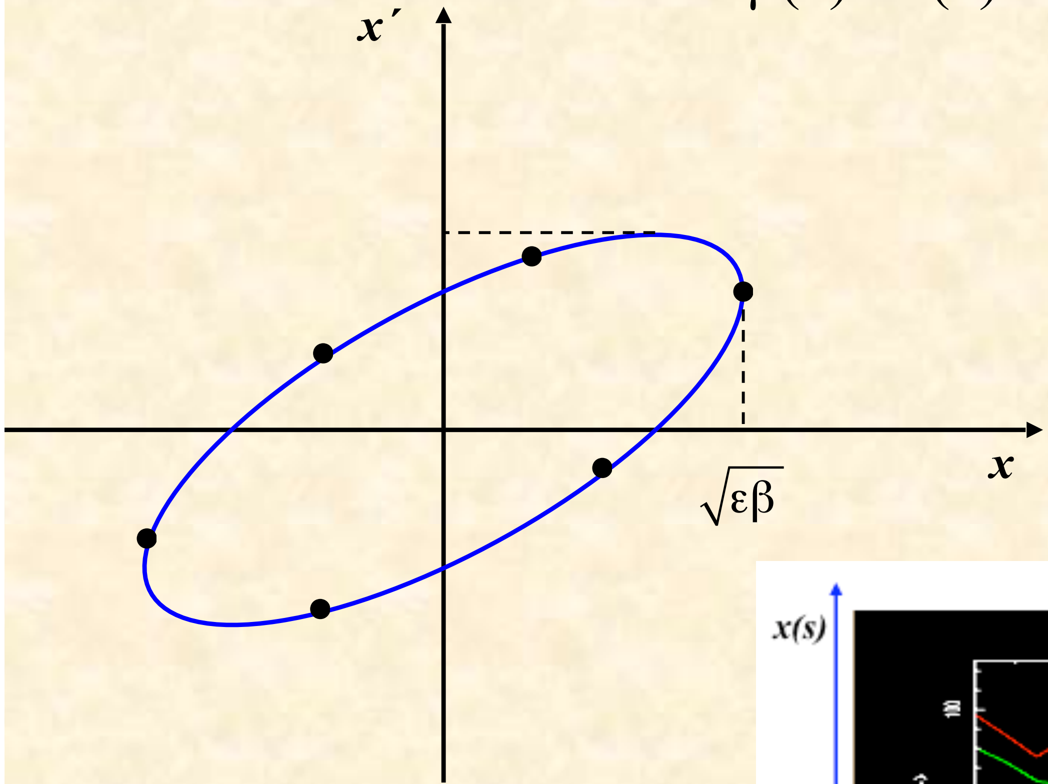
In the case of a ring or matched cell, the periodicity imposes equality on the input and output α and β values.

This means that the particle returns after each turn to the same ellipse but at phases $\mu_1 = b$, $\mu_2 = b+2\pi Q$, $\mu_3 = b+4\pi Q$, ..., $\mu_n = b + n2\pi$
and so on.

(Phil Bryant)

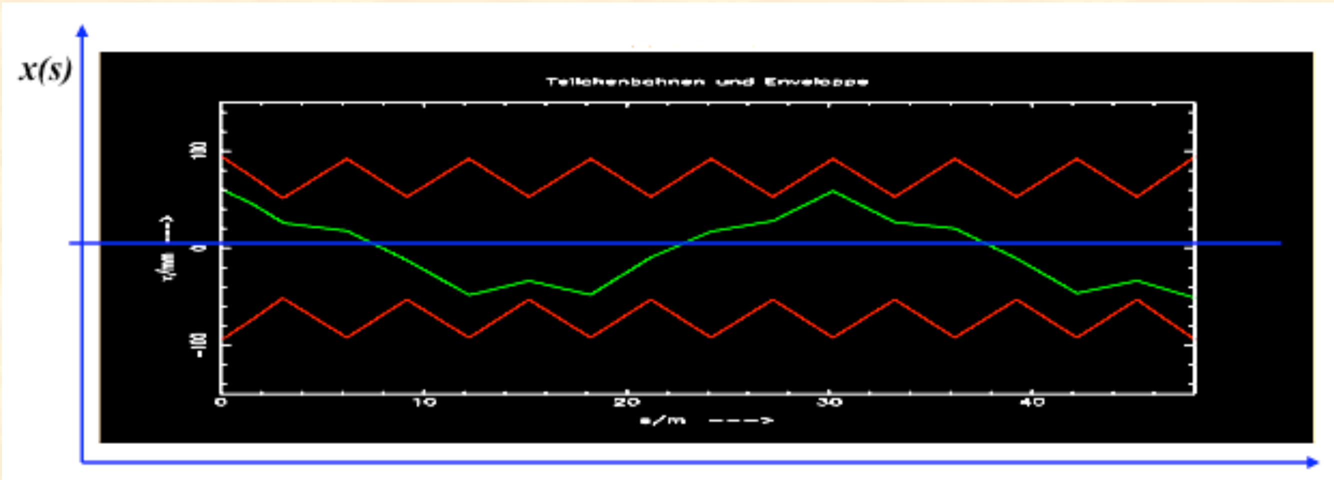
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

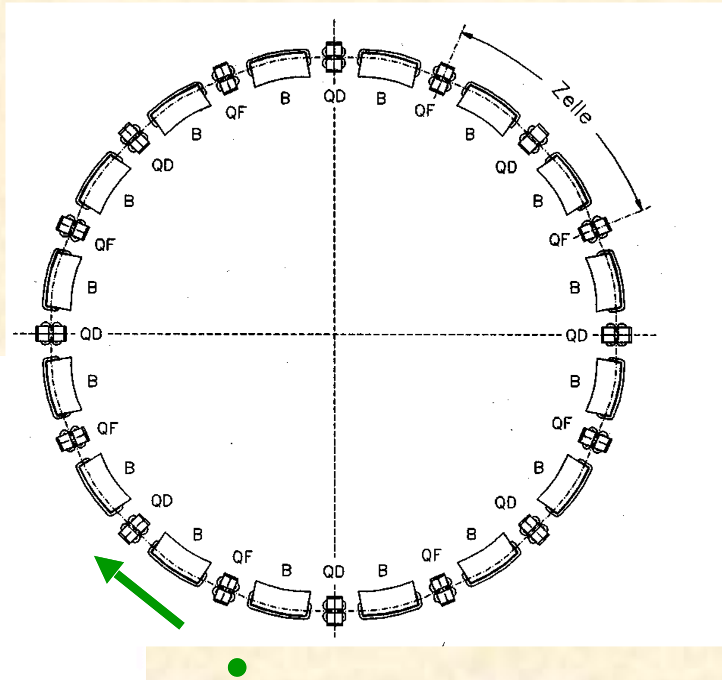


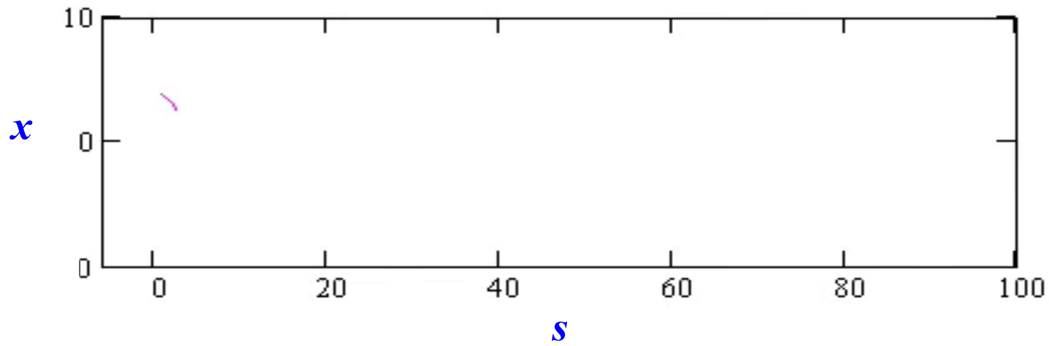
***ε beam emittance* = *woozilycity* of the particle ensemble, *intrinsic beam parameter*,**
cannot be changed by the foc. properties.
***Scientifiquely speaking:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

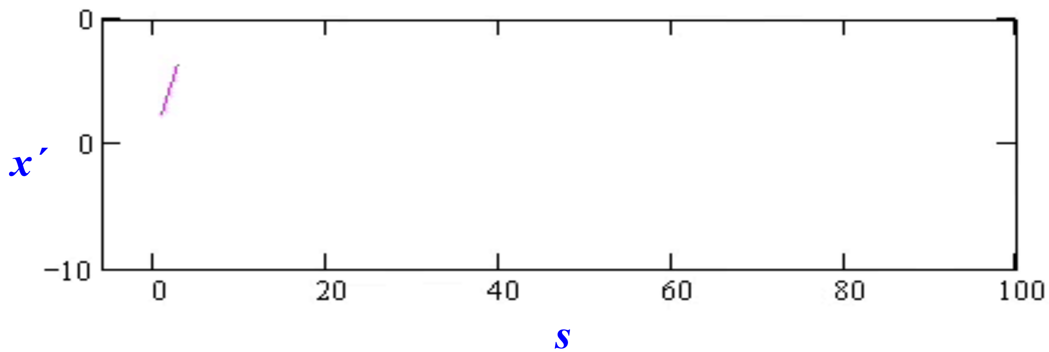
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s“



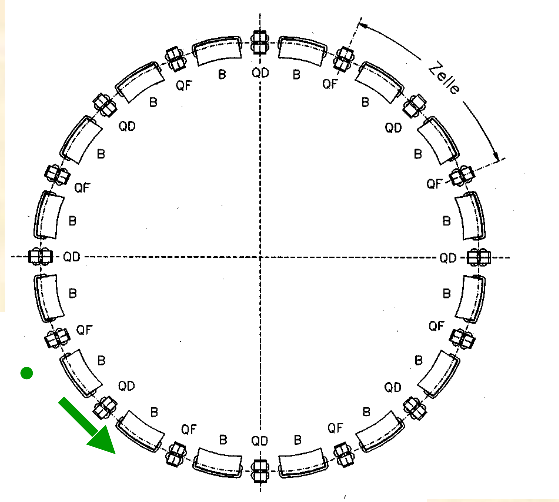


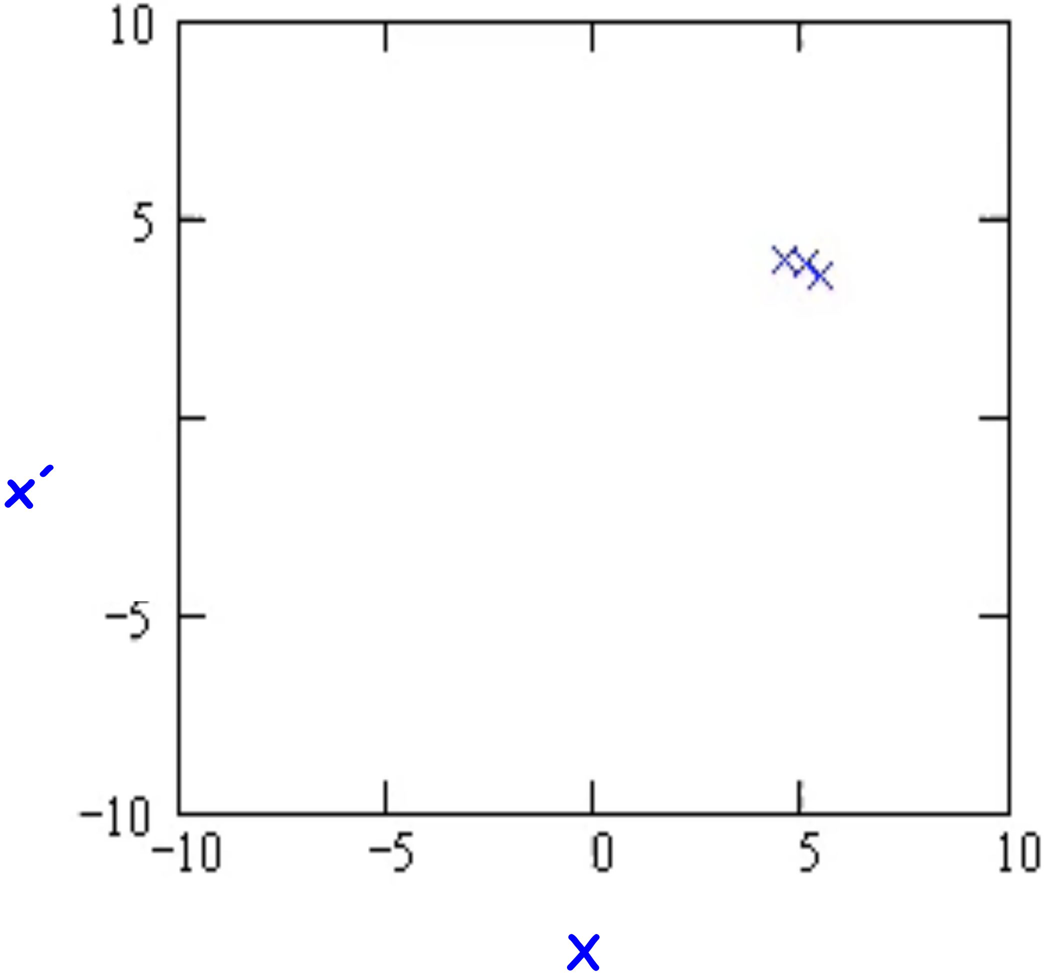


$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

… and now the ellipse:

note for each turn x, x´at a given position „s_1“ and plot in the phase space diagram





Particle Tracking in a Storage Ring

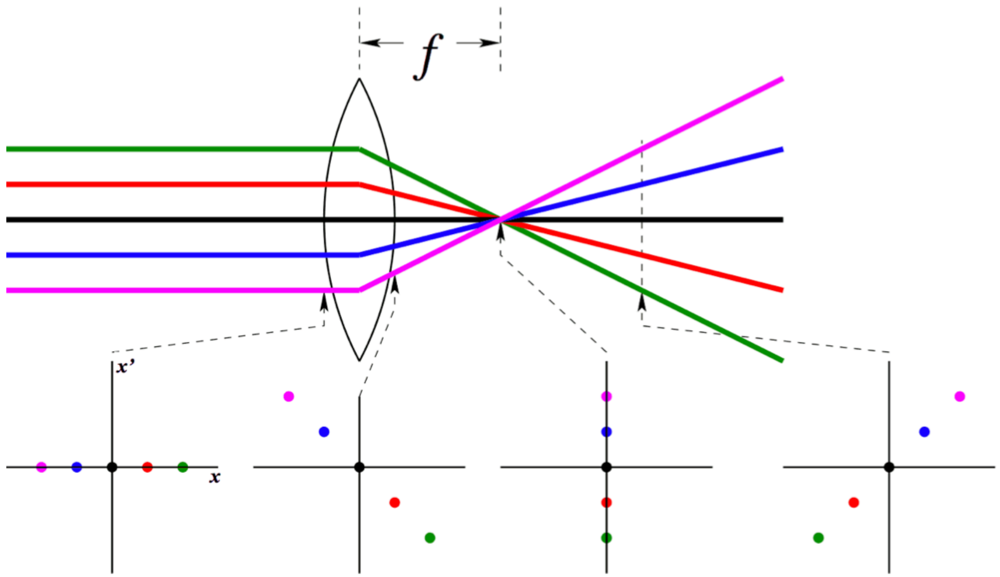
Calculate x, x´ for each accelerator element according to matrix formalism and plot x, x´at a given position „s“ turn by turn in the phase space diagram

Phase Space & Real Space

… don't worry: it takes some time to fully find your way in both worlds.

Focal length of a quadrupole

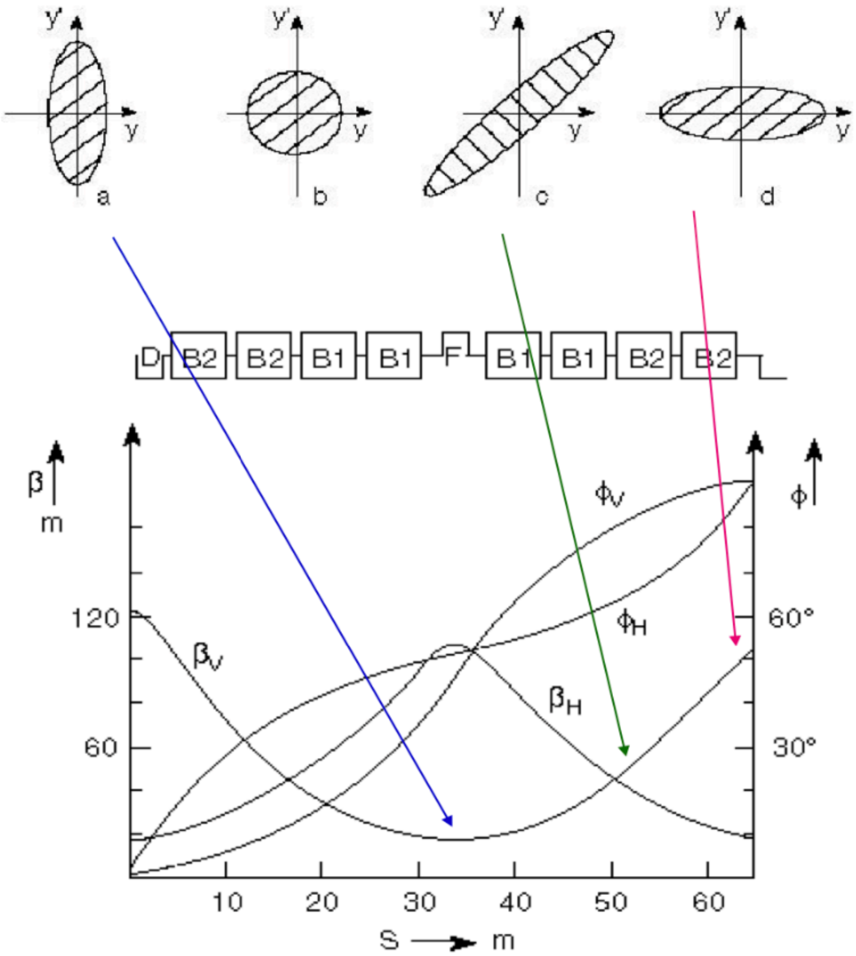
The focal length of a quadrupole is $f = \frac{1}{k \cdot L}$ [m], where L is the quadrupole length:



Shape & Orientation of Phase Space through a lattice

Let's repeat the remarks:

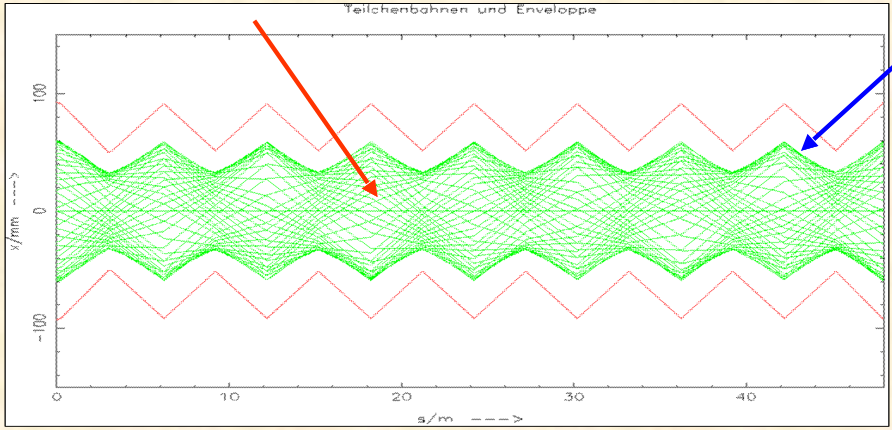
- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$



Emittance of the Particle Ensemble:

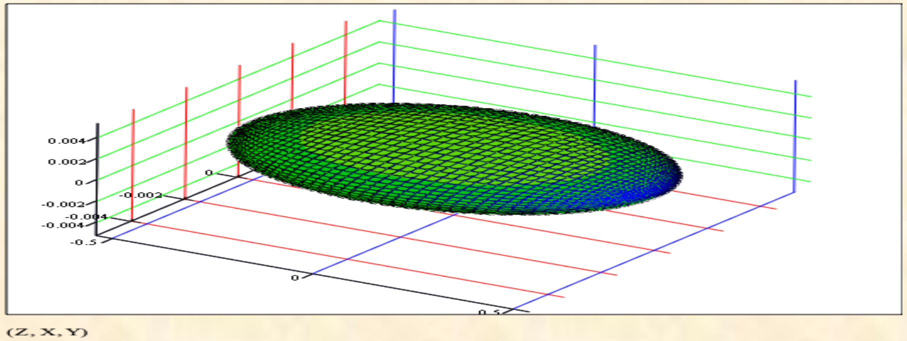
$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\cdot\cos(\Psi(s)+\phi)$$

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 11 per bunch

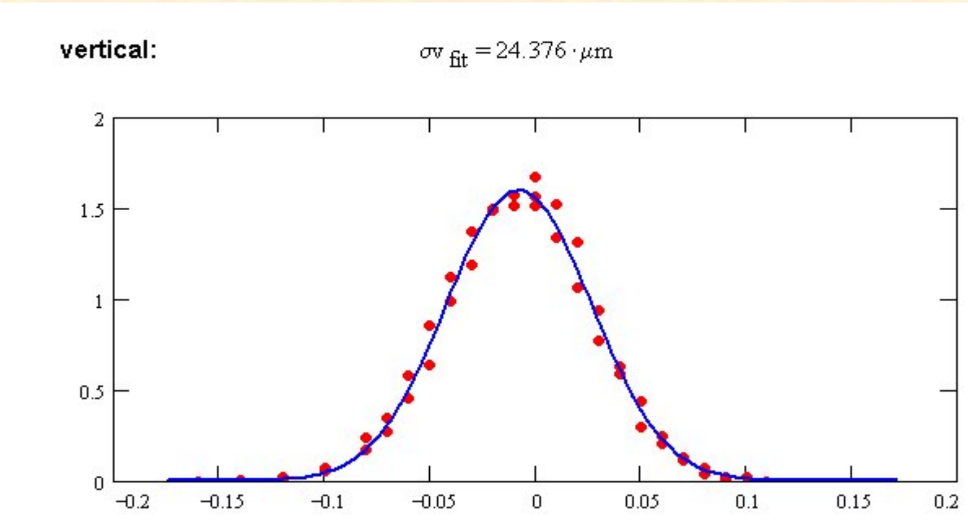
particle bunch

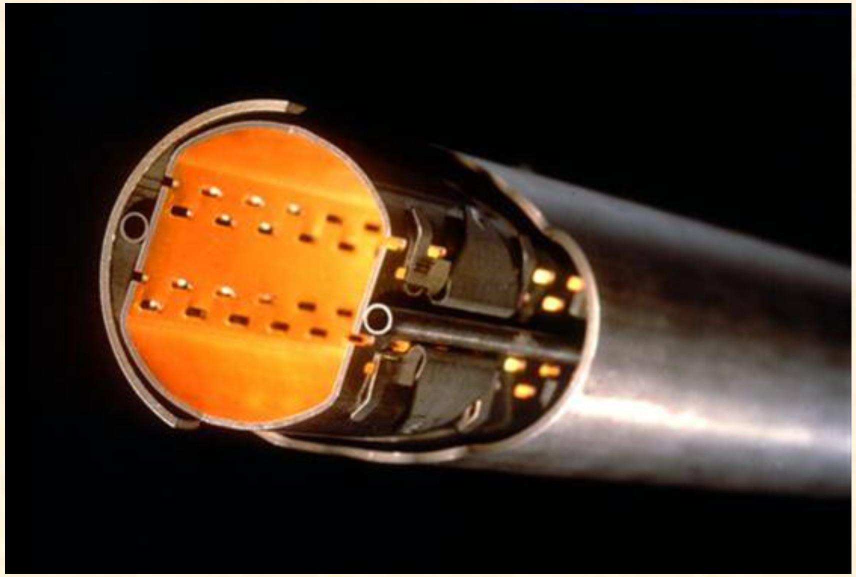


Gauß Particle Distribution:

$$\rho(x) = \frac{N\cdot e}{\sqrt{2\pi}\,\sigma_x}\cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles

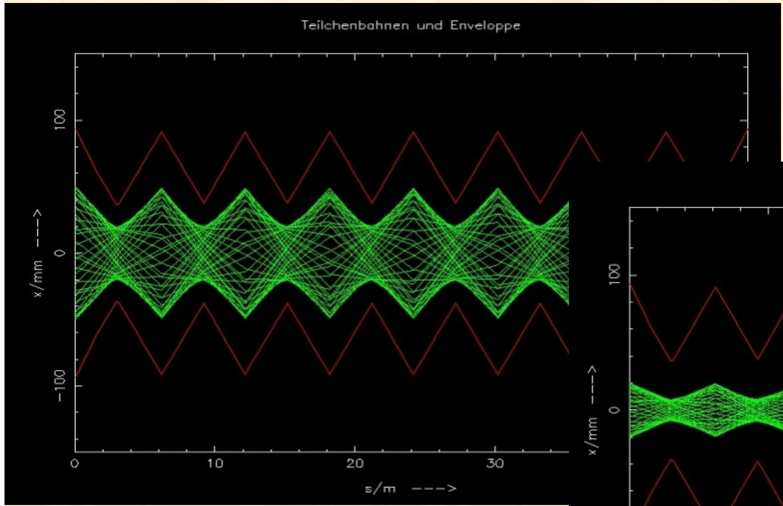


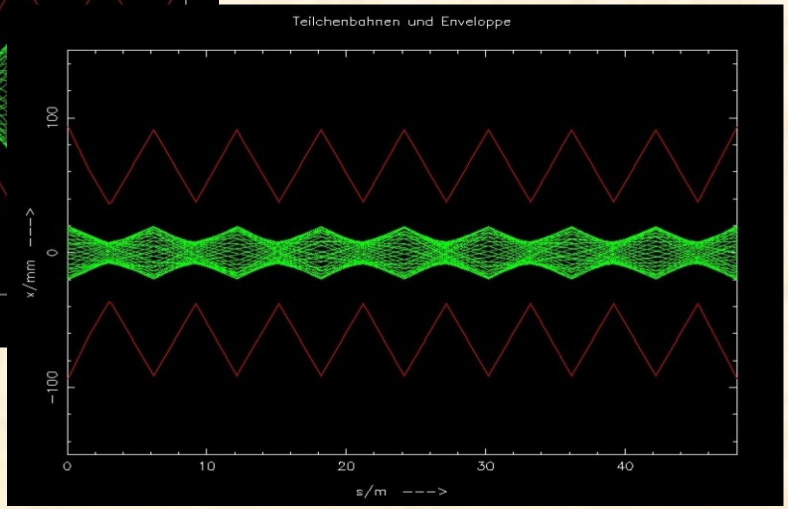


LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}\,m * 180\,m} = 0.3\ mm$

aperture requirements: $r_0 \geq 10 * \sigma$

Emittance of the Particle Ensemble:





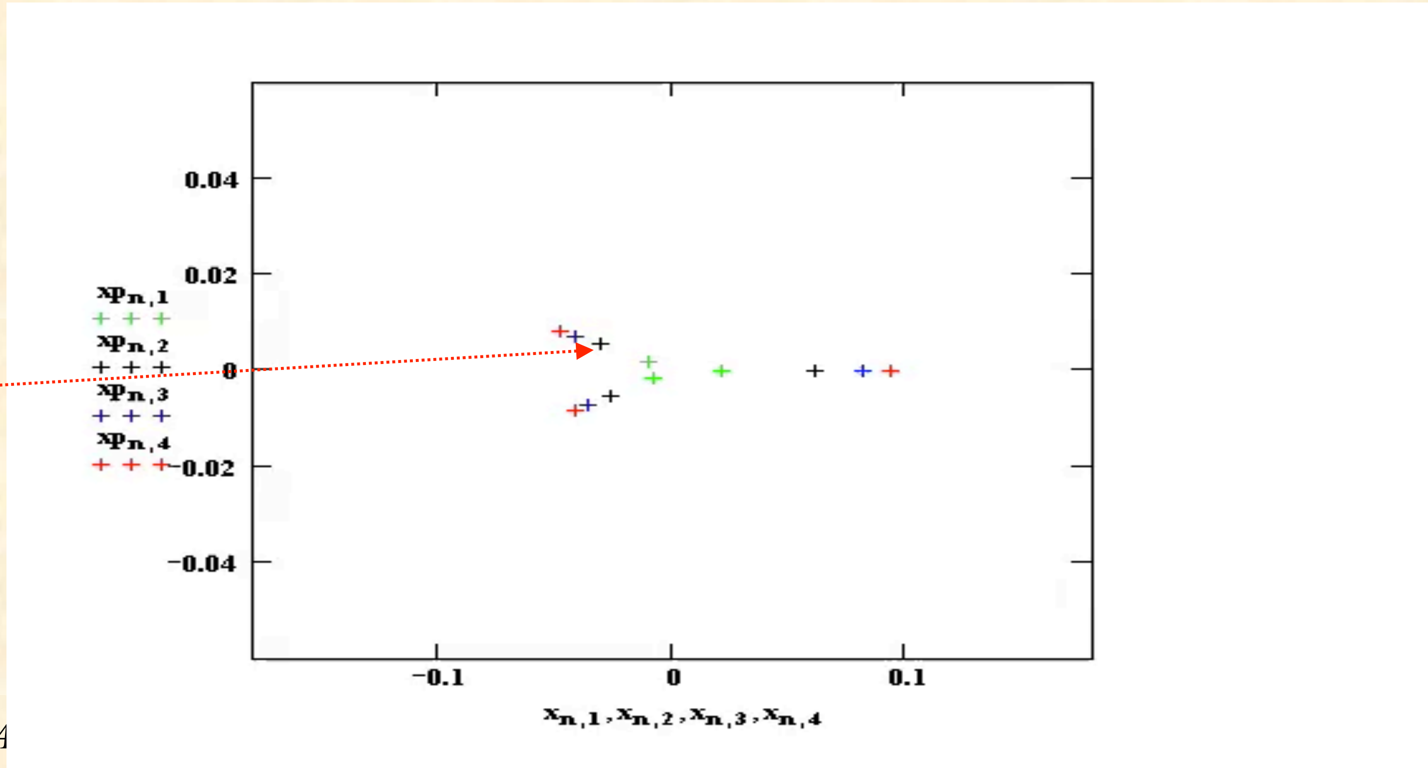
Example: LHC

beam parameters in the arc

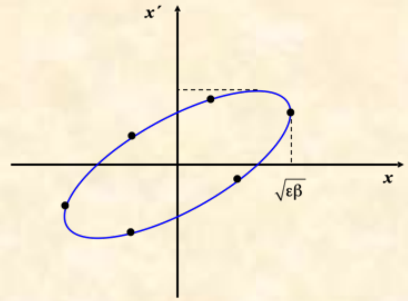
$\beta(x) \approx 180\ m$

$\varepsilon \approx 5*10^{-10}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$

$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\ mm$



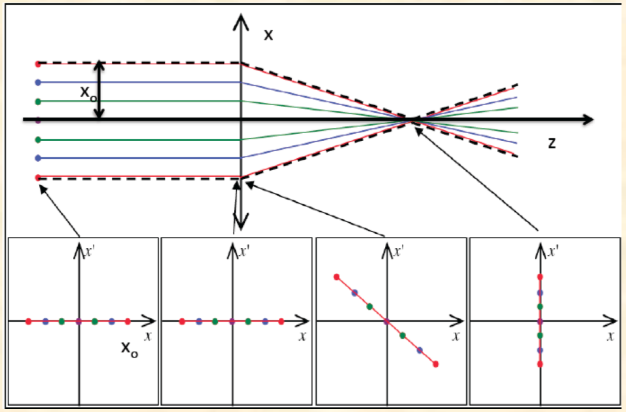
11.) Statistical Definition of Emittance:



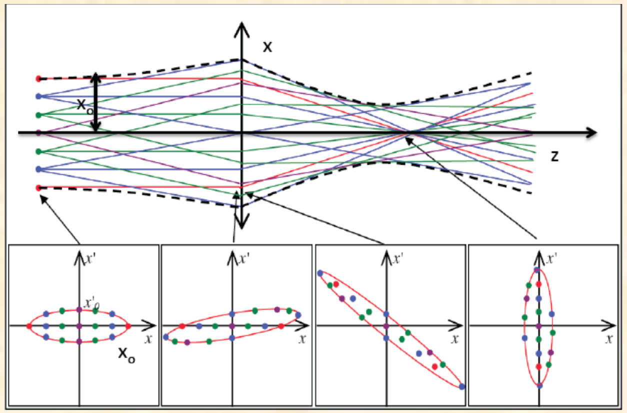
The emittance is the quality parameter of the particle distribution

the ideal case … *that never really exists …*

laminar ("LASER like) beam



the real case *… the non-laminar ("real") beam*



Maxwell distribution:

source temperature "T"

kinetic energy per degree of freedom:

$$E_{kin} = \frac{1}{2}kT$$

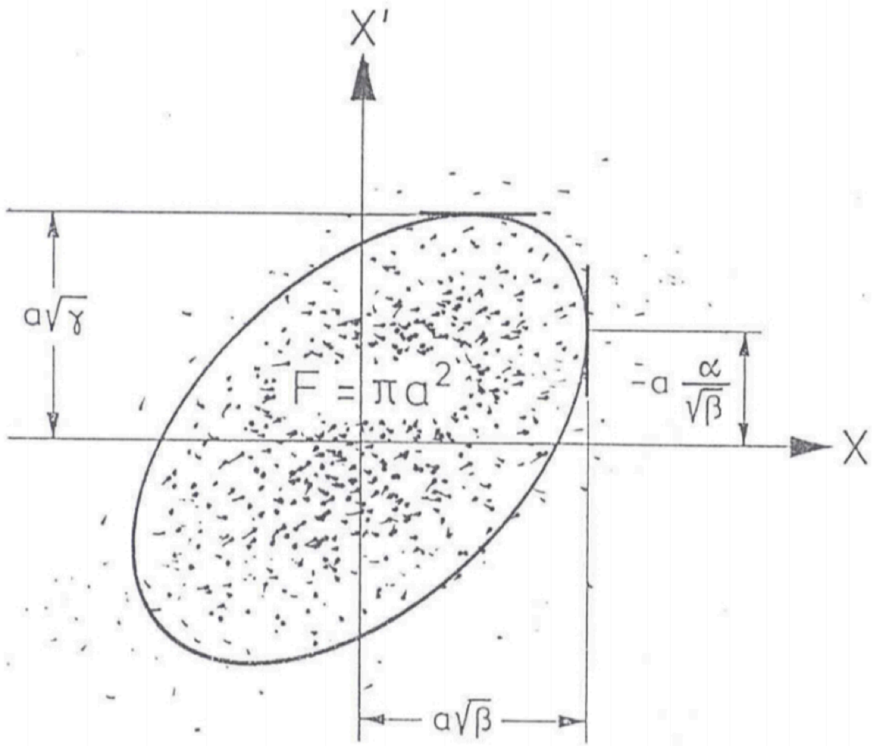
transverse momentum of the particles:

$$\frac{1}{2}mv_x^2 = \frac{p_x^2}{2m} = \frac{1}{2}kT \quad \longrightarrow \quad \sqrt{< p_x^2 >} = \sqrt{mkT}$$

the particles have an intrinsic (transverse) momentum distribution

Statistical Definition of Emittance:

The beam is composed of particles distributed in phase space.



Statistical emittance is defined by,

$$\epsilon_{\text{rms}} = \sqrt{\sigma_u^2 \sigma_{u'}^2 - \sigma_{uu'}^2} \qquad (77)$$

The rms emittance of a ring in phase space, i.e. particles uniformly distributed in phase ϕ at a fixed action J, is,

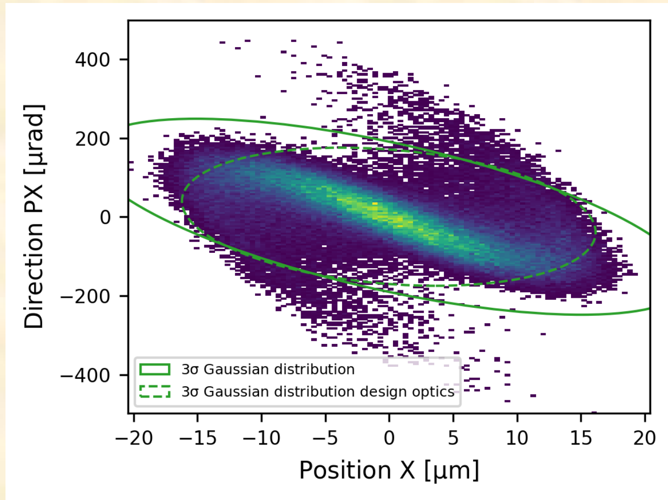
$$\epsilon_{\text{rms}} = J. \qquad (78)$$

If the accelerator is composed of linear elements, and no dissipative forces act ϵ_{rms} is invariant.

Statistical Definition of Emittance:

The r.m.s. emittance is a statistical definition of the amount of phase space covered by a beam.
If the beam is centred, (symmetric situation) (<x> = <x'> =0) we can write:

$$\varepsilon_{rms} = \frac{1}{N}\sqrt{\Sigma x^2 \Sigma x'^2 - (\Sigma x x')^2}$$



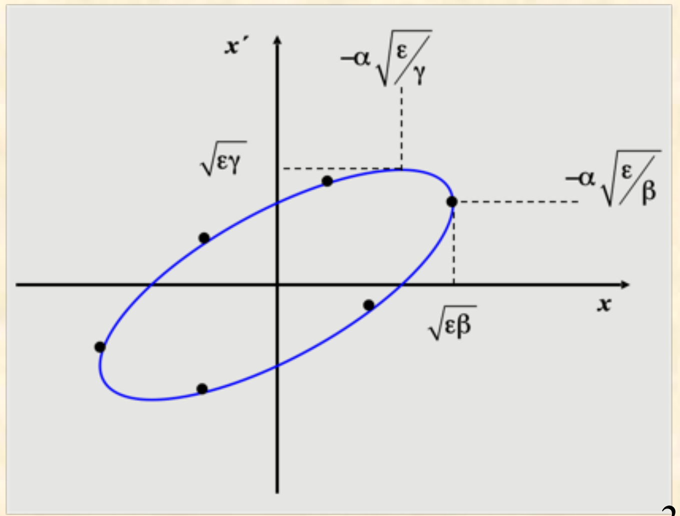
If we really refer to the actual particle distribution our emittance definition is much more precise.

We can translate into our Twiss language via:

$$\gamma_x \cdot \varepsilon_{rms} = < x'^2 >$$

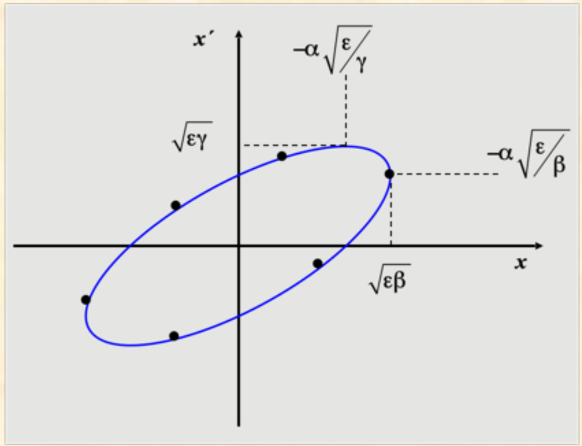
$$\beta_x \cdot \varepsilon_{rms} = < x^2 >$$

$$\alpha_x \cdot \varepsilon_{rms} = < x x' >$$



Emittance Dilution:

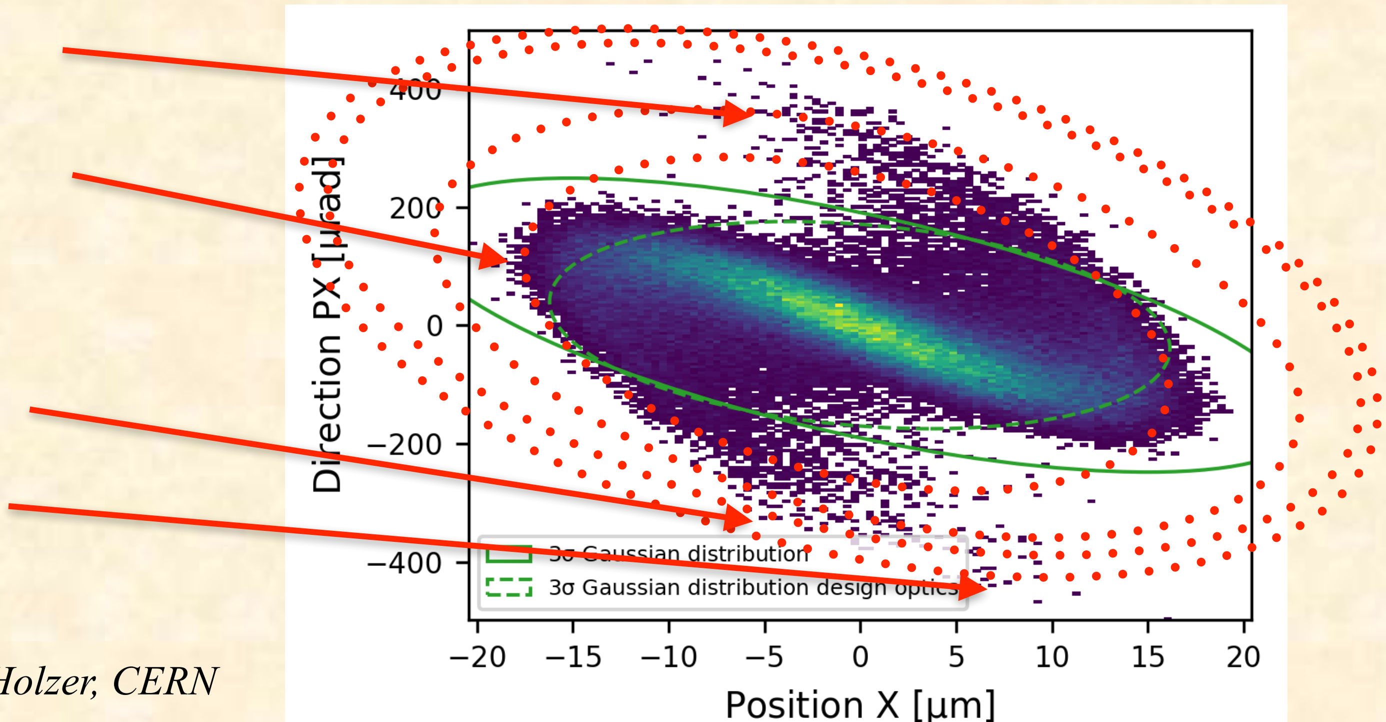
As soon as we inject the beam into an accelerator lattice, it is the actual Twiss parameters, that define the phase space ellipse in its shape and orientation.



We should optimise the α, β, γ to fit as much as possible to the actual distribution.

And we should keep ε as small as possible.

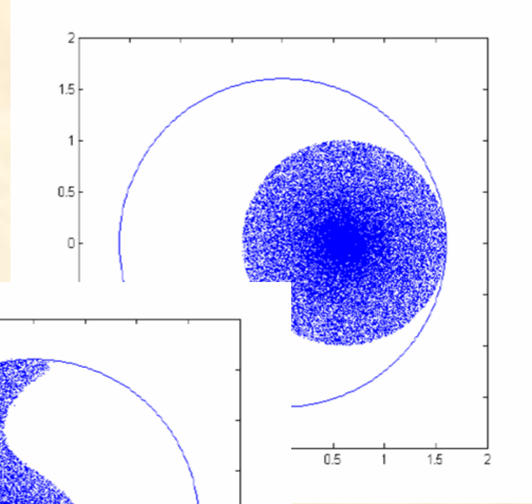
In the synchrotron each single particle will follow its phase space ellipse, that is defined by the ring optics.

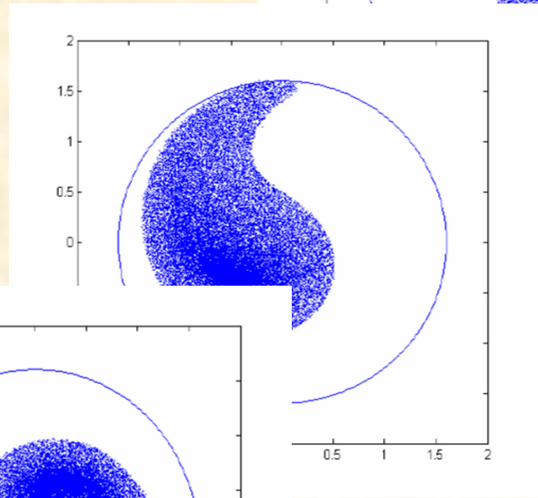


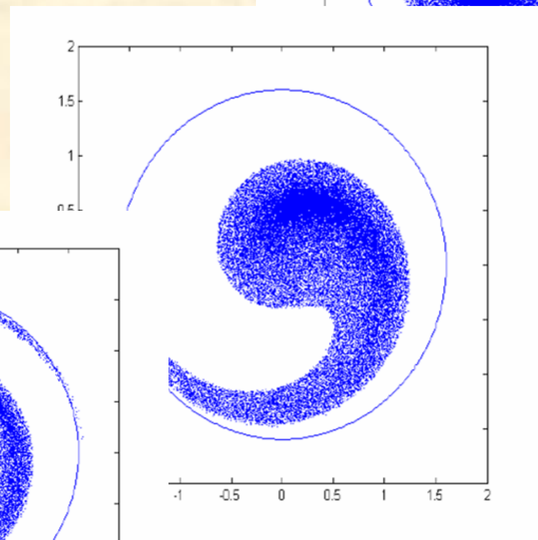
Filamentation

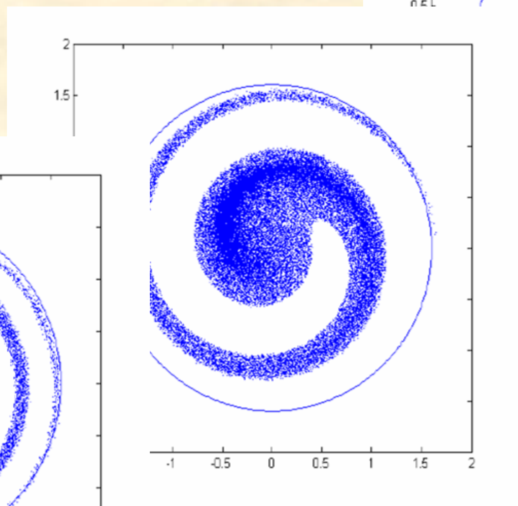
***Non-linear effects** (e.g. magnetic field multipoles) distort the harmonic oscillation and lead to **amplitude dependent effects** in the particle motion in phase space.*

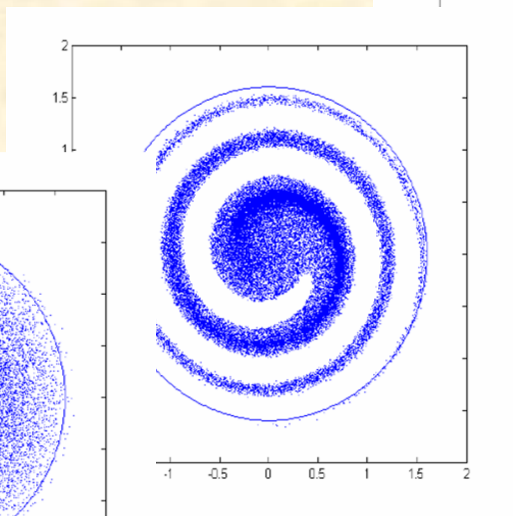
Over many turns, a non-ideal phase-space distribution is smeared out and transformed into an emittance increase.

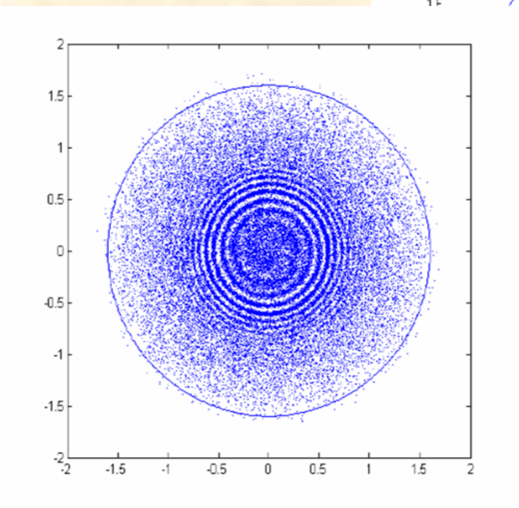












12.) Transfer Matrix M … *yes we had the topic already*

general solution of Hill´s equation

$$\left\{ \begin{aligned} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) &= \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{aligned} \right.$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left. \begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned} \right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\}x_0 + \left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form $\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

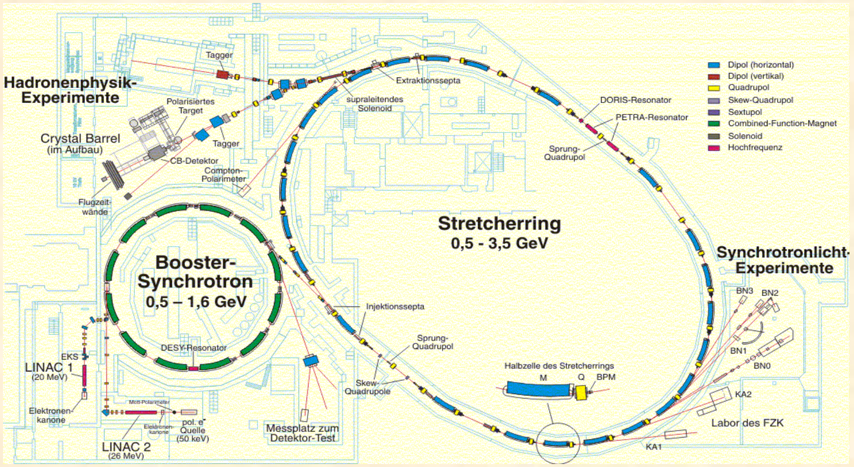
* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

* Äquivalenz der Matrizen

13.) Periodic Lattices

transfer matrix for particle trajectories as a function of the lattice parameters

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This** rather formidable looking matrix **simplifies considerably if we** consider one complete turn **…"

One Turn Matrix

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

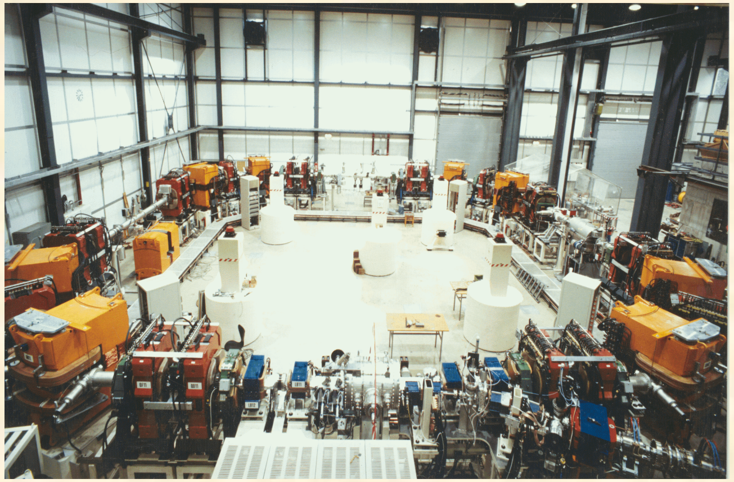
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

***Tune:* Phase advance per turn in units of 2π**

$$Q = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)}$$

Stability Criterion:



Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

Matrix for N turns:

$$\boldsymbol{M}^N = (\boldsymbol{I}\cos\psi + \boldsymbol{J}\sin\psi)^N = \boldsymbol{I}\cos N\psi + \boldsymbol{J}\sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| < 1 \quad \leftrightarrow \quad |Trace(\boldsymbol{M})| < 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn: $M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = \left(\boldsymbol{I} * \cos\psi_1 + \boldsymbol{J} * \sin\psi_1\right) * \left(\boldsymbol{I} * \cos\psi_2 + \boldsymbol{J} * \sin\psi_2\right)$$

$$= \boldsymbol{I}^2 * \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ} * \cos\psi_1 \sin\psi_2 + \boldsymbol{JI} * \sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\boldsymbol{I} * \boldsymbol{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{J} * \boldsymbol{I} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\Rightarrow \boldsymbol{I} * \boldsymbol{J} = \boldsymbol{J} * \boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I} * \cos(\psi_1 + \psi_2) + \boldsymbol{J} * \sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I} * \cos(2\psi) + \boldsymbol{J} * \sin(2\psi)$$

14.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

where ...

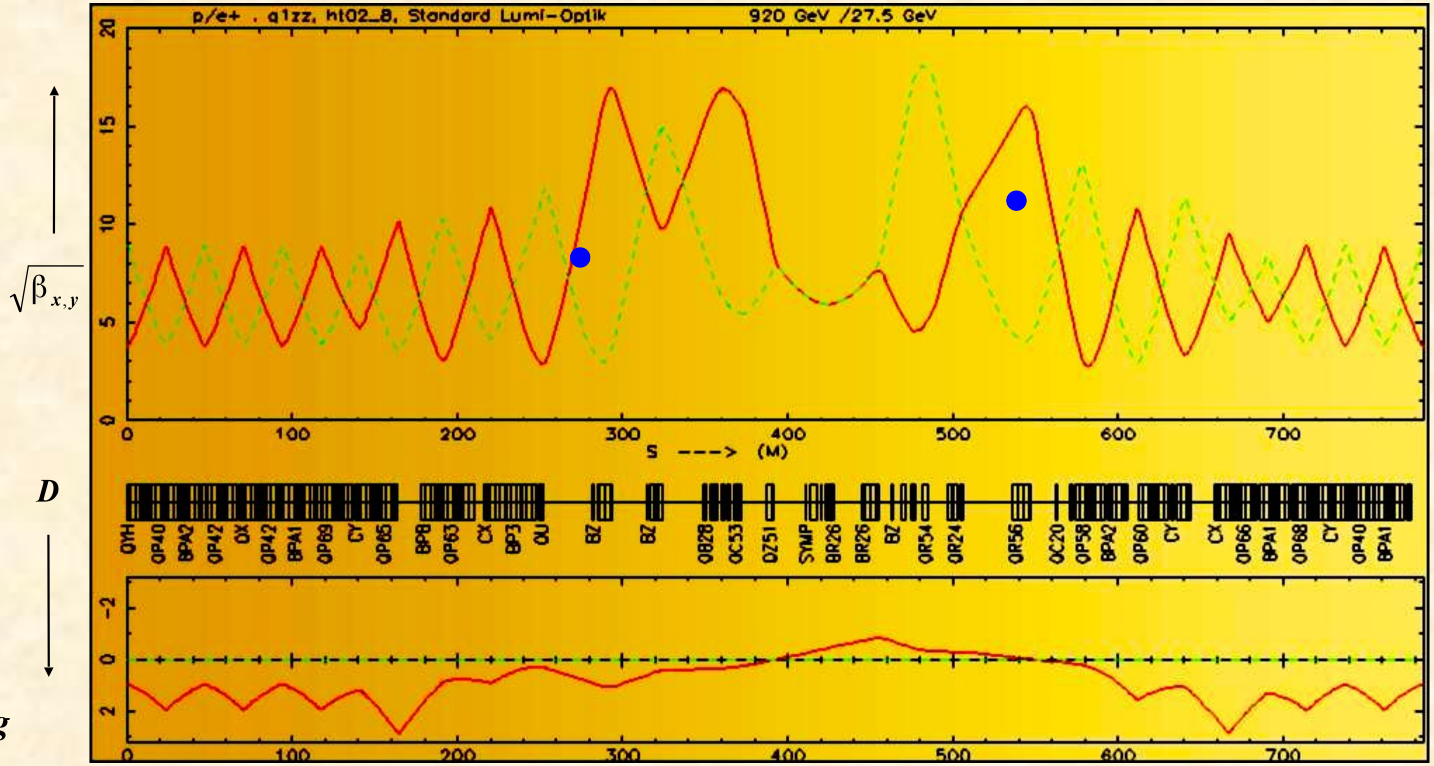
$$M = M_{QF} \cdot M_{QD} \cdot M_B \cdot M_{Drift} \cdot M_{QF} \cdot \ldots$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const (Liouville):

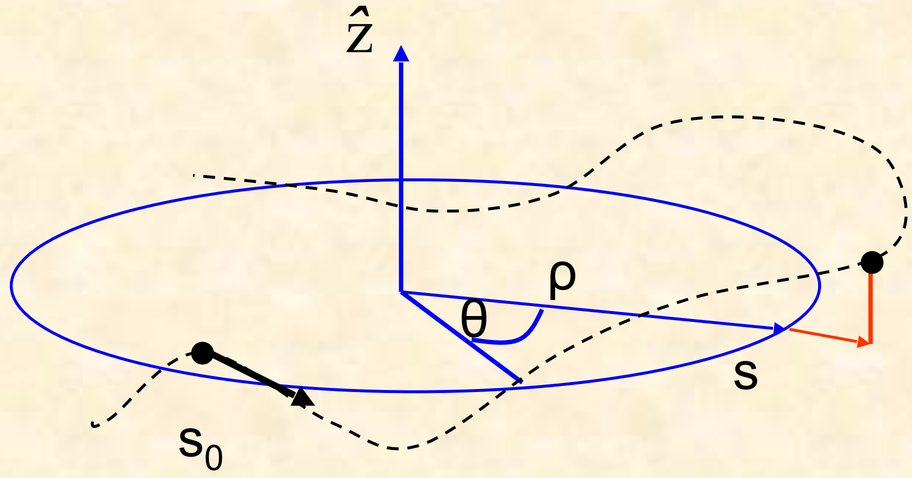
$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$



Beta function in a storage ring

express x_0, x'_0 as a function of x, x'.



$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

... remember W = CS′-SC′ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$\Rightarrow$

$$x_0 = S'x - Sx'$$
$$x'_0 = -C'x + Cx'$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x′and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

Résumé:

equation of motion:

$$x''(s) + K(s)\,x(s) = 0 \ , \qquad K = 1/\rho^2 - k$$

general solution of Hill´s equation:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

phase advance & tune:

$$\psi_{12}(s) = \int_{s1}^{s2} \frac{1}{\beta(s)} ds \ , \qquad Q(s) = \frac{1}{2\pi} \oint \frac{1}{\beta(s)} ds$$

emittance:

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

transfer matrix from s_1 —> s_2:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

matrix for 1 turn:

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

stability criterion:

$$\left|Trace(M)\right| < 2$$