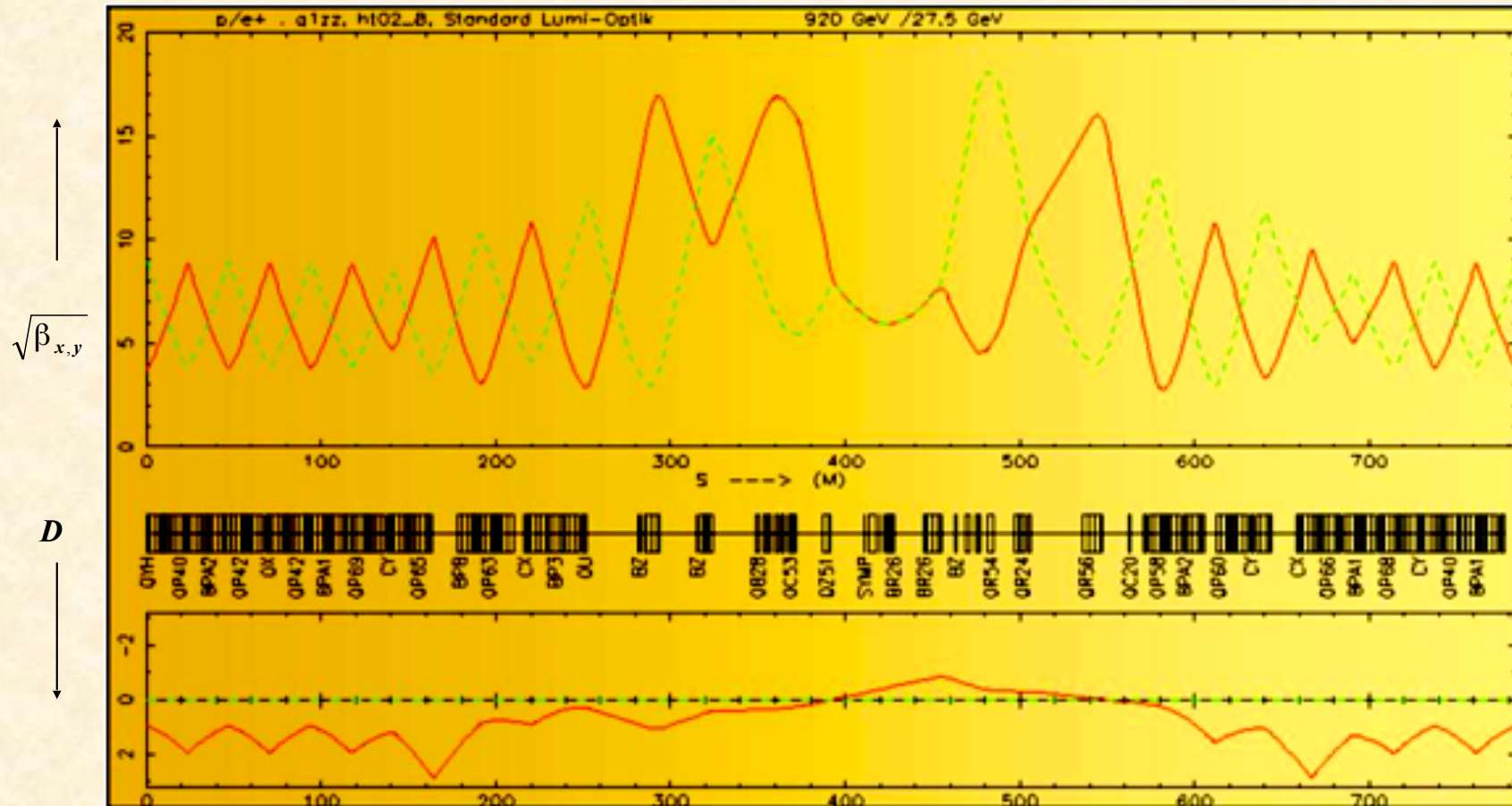


III.) The „not so ideal“ World Lattice Design in Particle Accelerators

Bernhard Holzer, CERN



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

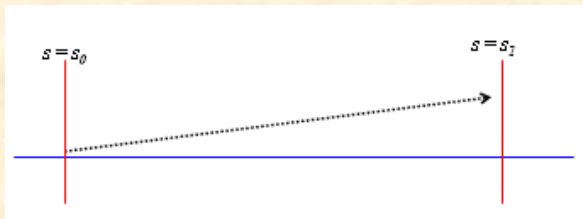
Equation of Motion:

$$\mathbf{x}'' + \mathbf{K} \mathbf{x} = 0 \quad K = 1/\rho^2 - k \quad \dots \text{ hor. plane:}$$

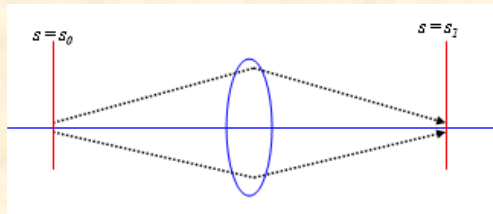
$$K = k \quad \dots \text{ vert. Plane:}$$

Solution of Trajectory Equations

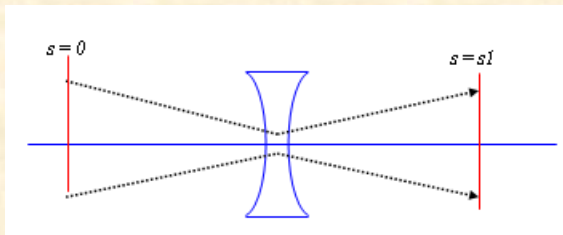
$$\begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}_{s_1} = \mathbf{M} * \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix}_{s_0}$$



$$\mathbf{M}_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$\mathbf{M}_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\ -\sqrt{|K|} \sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



$$\mathbf{M}_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}l) \\ \sqrt{|K|} \sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix}$$

$$\mathbf{M}_{total} = \mathbf{M}_{QF} * \mathbf{M}_D * \mathbf{M}_B * \mathbf{M}_D * \mathbf{M}_{QD} * \mathbf{M}_D * \dots$$

Recapitulation: ...and for the complete particle ensemble the betas and epsilons !!!

general solution of Hill equation

$$\left\{ \begin{array}{l} (1) \quad x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi) \\ (2) \quad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \{ \alpha(s) * \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \} \end{array} \right.$$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

Insert into (2) and solve for ε

$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

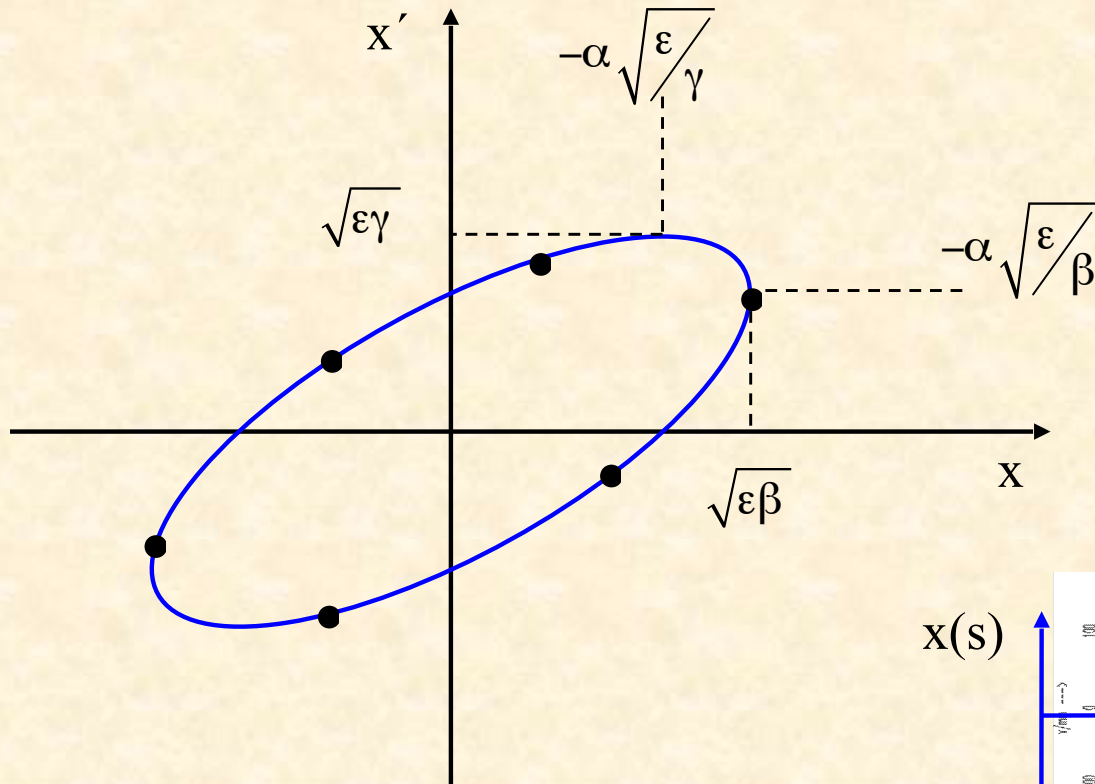
$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

- * ε is a **constant** of the motion ... it is independent of „s“
- * parametric representation of an **ellipse** in the $x x'$ space
- * shape and orientation of ellipse are given by α, β, γ

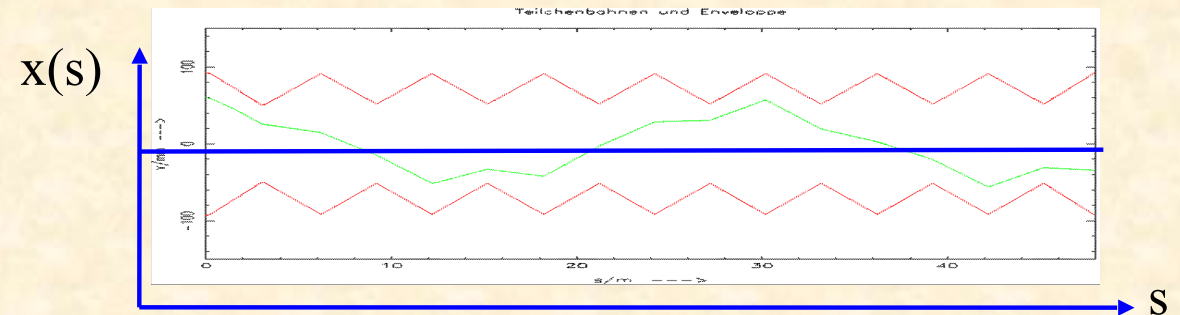
Beam Emittance and Phase Space Ellipse

$$\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$



Liouville: in reasonable storage rings area in phase space is constant.

$$A = \pi * \epsilon = \text{const}$$



ϵ beam emittance = **woozilycity** of the particle ensemble, **intrinsic beam parameter**, cannot be changed by the foc. properties.

Scientifiquely speaking: area covered in transverse x, x' phase space ... and it is constant !!!

Transfer Matrix M

Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

using matrix notation in magnet parameters:

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \dots \quad M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\ -\sqrt{|K|} \sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

using matrix notation in Twiss form:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos\psi_s - (1 + \alpha_0 \alpha_s) \sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$

Transformation of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

III. Lattice Design in Particle Accelerators

„... how to build a storage ring“

High energy accelerators → **circular machines**

somewhere in the lattice we need a number of **dipole magnets**,
that are bending the design orbit to a **closed ring**

Geometry of the ring:

centrifugal force = Lorentz force



Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

$$e * v * B = \frac{mv^2}{\rho}$$
$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$

p = momentum of the particle,
 ρ = curvature radius

$B\rho$ = beam rigidity

15.) Lattice Design:

$$B \rho = p / q$$

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

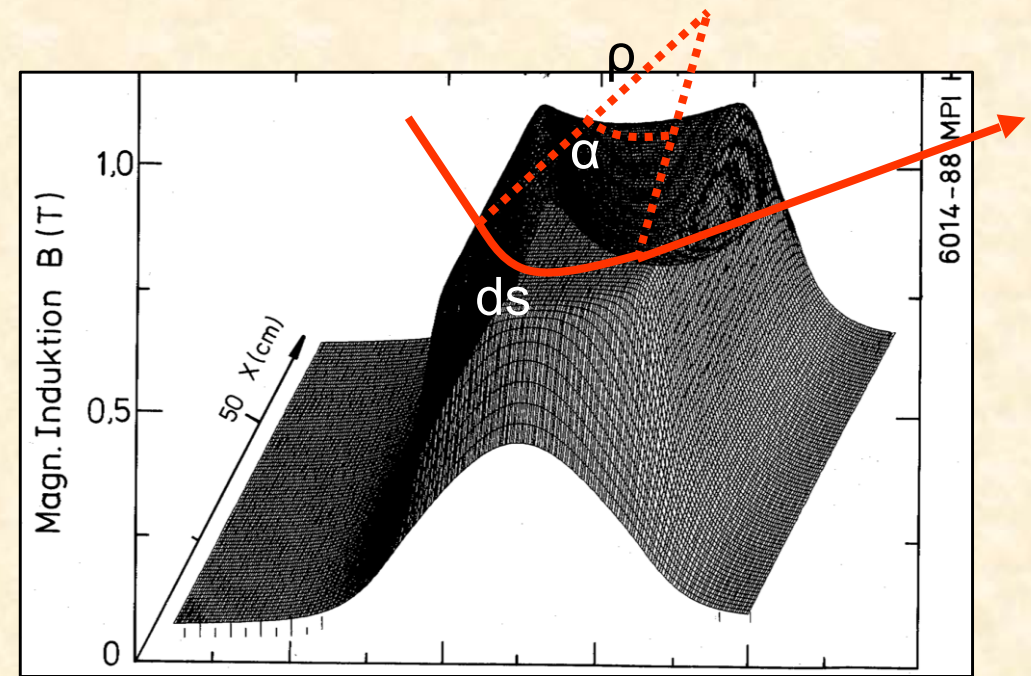
The angle run out in one revolution must be 2π , so

... for a full circle

$$\alpha = \frac{\int Bdl}{B\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi \frac{p}{q}$$

... defines the integrated dipole field around the machine.

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ is usually required !!



field map of a storage ring dipole magnet



7000 GeV Proton storage ring
 dipole magnets $N = 1232$
 $l = 15 \text{ m}$
 $q = +1 e$

$$\int \mathbf{B} \, dl \approx N \, l \, B = 2\pi \, p / e$$

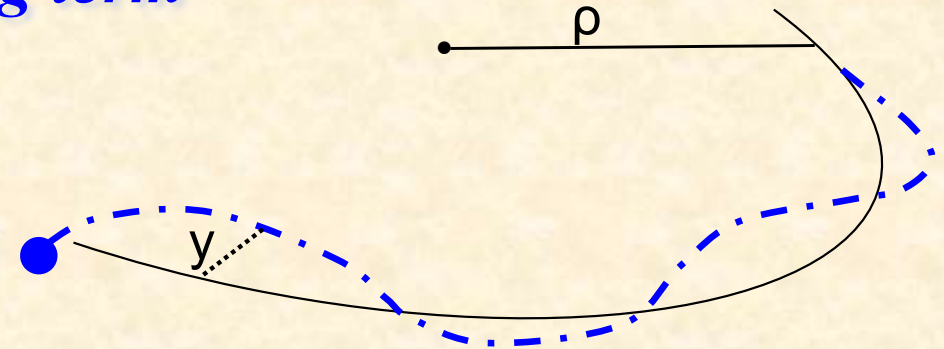
$$B \approx \frac{2\pi \, 7000 \, 10^9 \, eV}{1232 \, 15 \, m \, 3 \, 10^8 \, \frac{m}{s} \, e} = \underline{\underline{8.3 \, \text{Tesla}}}$$

16.) Focusing forces ... weak focusing term

$$x'' + K * x = 0$$

$$K = -k + 1/\rho^2 \quad \text{hor. plane}$$

$$K = k \quad \text{vert. plane}$$



<i>dipole magnet</i>	$\frac{1}{\rho} = \frac{B}{p/q}$	}
<i>quadrupole magnet</i>	$k = \frac{g}{p/q}$	

Example: LHC Ring:

Bending radius: $\rho = 2.8 \text{ km}$

Quadrupole Gradient: $g = 220 \text{ T/m}$

$$k = 9.4 * 10^{-3} / \text{m}^2$$

$$1/\rho^2 = 1.3 * 10^{-7} / \text{m}^2$$

For estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected

Solution for a focusing magnet

$$x(s) = x_0 * \cos(\sqrt{K} * s) + \frac{x'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$x'(s) = -x_0 \sqrt{K} * \sin(\sqrt{K} * s) + x'_0 * \cos(\sqrt{K} * s)$$

The Twiss parameters α , β , γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

Question: „ What does that mean ????? “

Most simple example: *drift space*

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

→

$$\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}$$

transformation of twiss parameters:

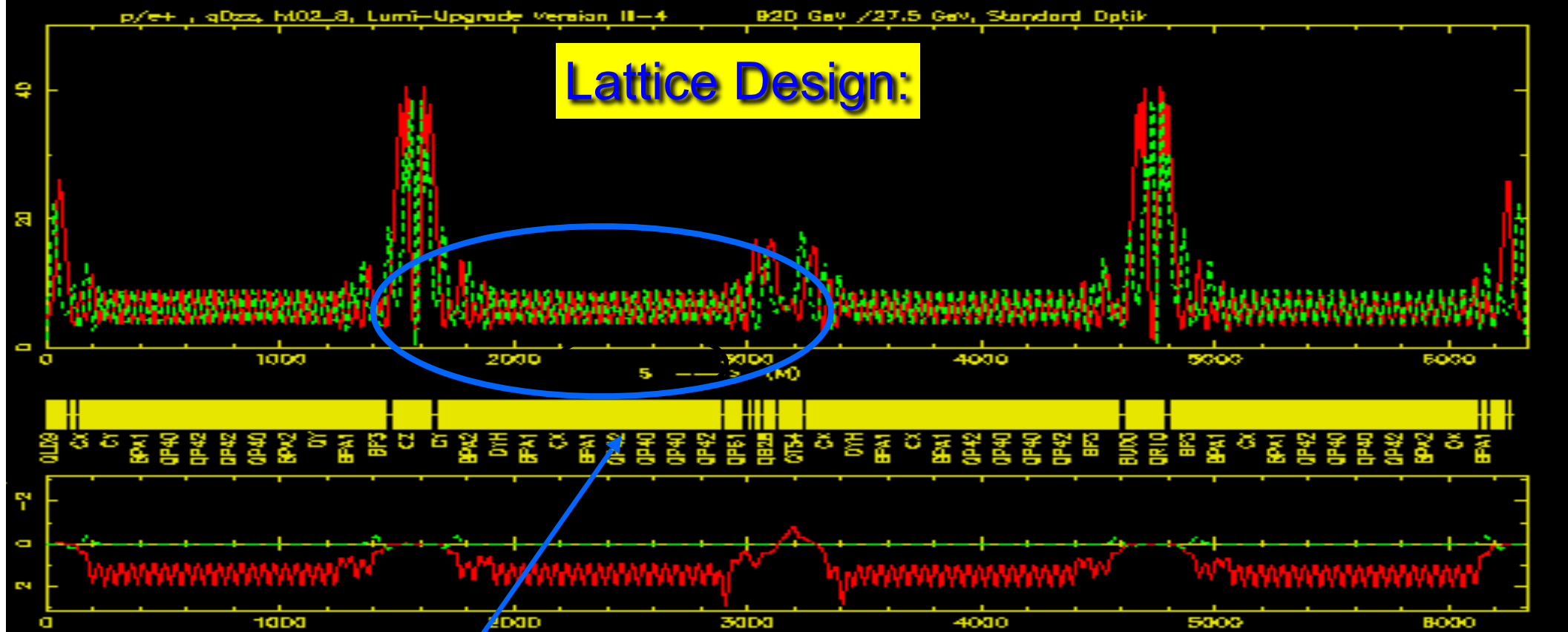
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$\text{trace}(M) = 1 + 1 = 2$$

→ *A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.*



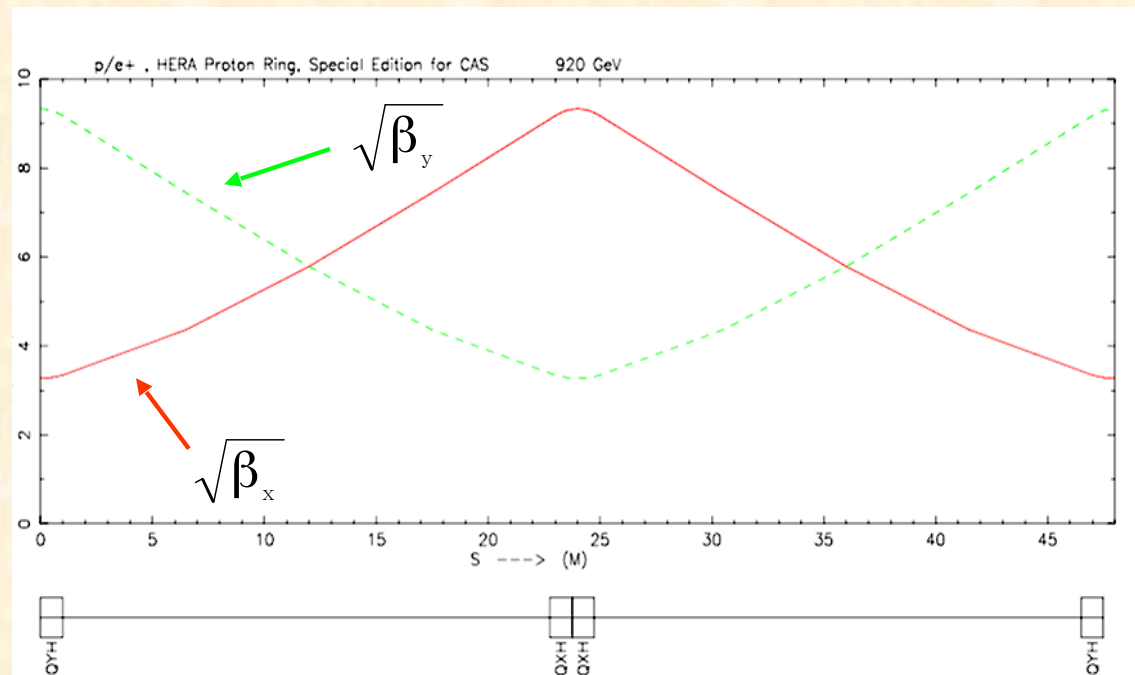
Arc: regular (periodic) magnet structure:

bending magnets to define the energy of the ring
 main focusing & tune control, chromaticity correction,
 multipoles for higher order corrections

Straight sections: drift spaces for injection, dispersion suppressors,
 low beta insertions, RF cavities, etc....

... and the high energy experiments if they cannot be avoided

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	Type	Length m	Strength 1/m ²	β_x m	α_x	φ_x 1/2 π	β_y m	α_y	φ_y 1/2 π
0	Marker	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	Marker	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

Qx= 0,125 Qy= 0,125

$$0.125 * 2\pi = 45^\circ$$

Can we understand what the optics code is doing ?

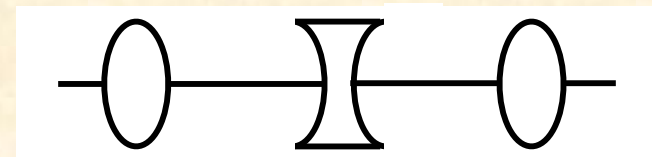
matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \quad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

$$K = +/- 0.54102 \text{ m}^{-2}$$
$$l_q = 0.5 \text{ m}$$
$$l_d = 2.5 \text{ m}$$

The matrix for the **complete cell** is obtained by multiplication of the element matrices



$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and **multiplying out** ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?
trace < 2

$$\text{trace}(M_{FoDo}) = 1.415 \rightarrow \text{😊}$$

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos(\Psi_{cell}) = \frac{1}{2} \text{Trace}(M) = 0.707$$

$$\rightarrow \Psi_{cell} = \cos^{-1} \left\{ \frac{1}{2} \text{Trace}(M) \right\} = \underline{\underline{45^\circ}}$$

3.) hor β -function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = \underline{\underline{11.611 \text{ m}}}$$

4.) hor α -function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = \underline{\underline{0}}$$

Can we do a bit easier ?

We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the focal length f is much larger than the length of the quadrupole magnet,

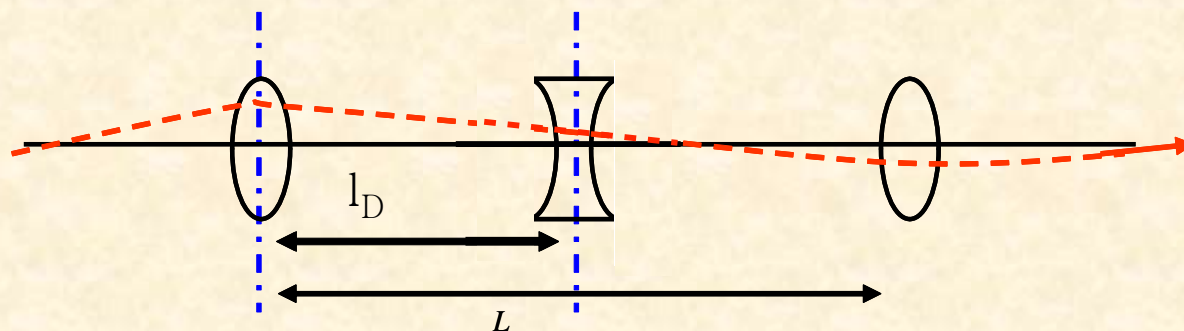
$$f = 1/kl_q \gg l_q$$

the transfer matrix can be approximated using

$$kl_q = \text{const}, l_q \rightarrow 0$$

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \quad M_{QD} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

18.) FoDo in thin lens approximation



$$l_D = L/2$$

$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{ID} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: \tilde{f} denotes the focusing strength of half a quadrupole, so

$$\tilde{f} = 2f$$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set $f \rightarrow -f$

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1 + \frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1 - \frac{l_D}{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1 - \frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1 + \frac{l_D}{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D \left(1 + \frac{l_D}{\tilde{f}}\right) \\ 2\left(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}\right) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos(\Psi_{cell}) = \frac{1}{2} \text{trace}(M) = \frac{1}{2} \cdot \left(2 - \frac{4l_D^2}{\tilde{f}^2}\right) = 1 - \frac{2l_D^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2 \cdot \sin^2(x/2)$$

we can calculate the phase advance as a function of the FoDo parameter ...

$$\cos(\Psi_{cell}) = 1 - 2 \cdot \sin^2(\Psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2} \rightarrow \sin(\Psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

*phase advance of a FoDo cell:
(in thin lens approx)*

$$\sin(\Psi_{cell}/2) = \frac{L_{cell}}{4f}$$

*Example:
45-degree Cell*

$$\begin{aligned} L_{Cell} &= l_{QF} + l_D + l_{QD} + l_D \\ &= 0.5m + 2.5m + 0.5m + 2.5m = 6m \end{aligned}$$

$$1/f = k \cdot l_Q = 0.5m \cdot 0.541 \text{ m}^{-2} = 0.27 \text{ m}^{-1} \rightarrow f = 3.7 \text{ m}$$

$$\sin(\Psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \psi_{cell} = 47.8^\circ$$

$$\rightarrow \beta = 11.4 \text{ m}$$

Remember:

Exact calculation yields:

$$\rightarrow \psi_{cell} = 45^\circ$$

$$\rightarrow \beta = 11.6 \text{ m}$$

Stability in a FoDo structure



SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D \left(1 + \frac{l_D}{\tilde{f}}\right) \\ 2\left(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}\right) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|\text{Trace}(M)| < 2 \quad |\text{Trace}(M)| = \left| 2 - \frac{4l_D^2}{\tilde{f}^2} \right| < 2$$

$$\rightarrow f > \frac{L_{\text{cell}}}{4}$$

For stability the focal length has to be larger than a quarter of the FoDo cell length ... don't focus too strong !

Example:

45-degree Cell

$$\begin{aligned} L_{\text{Cell}} &= l_{QF} + l_D + l_{QD} + l_D \\ &= 0.5\text{m} + 2.5\text{m} + 0.5\text{m} + 2.5\text{m} = 6\text{m} \end{aligned}$$

$$\rightarrow L_{\text{cell}}/4 = 1.5 \text{ m}$$

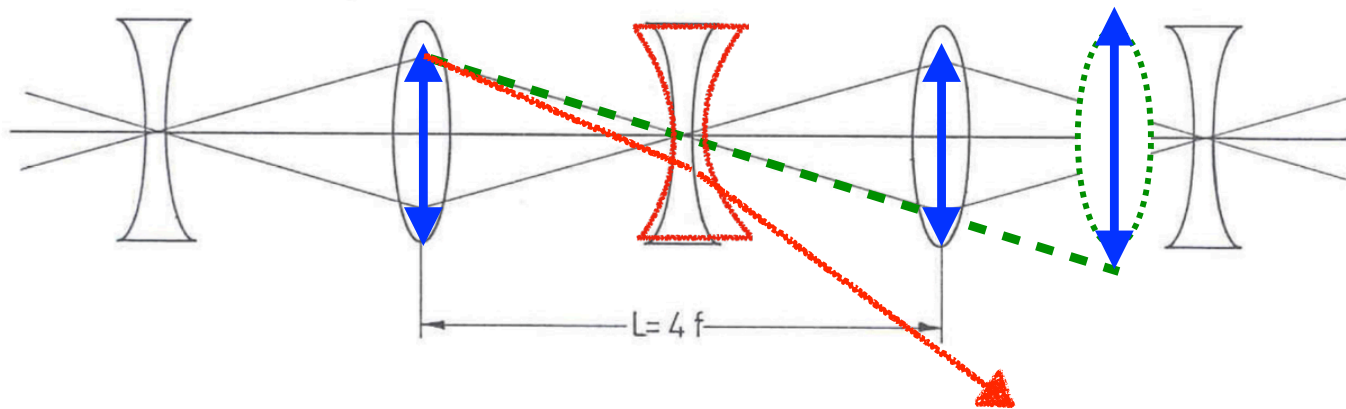
$$1/f = k \cdot l_Q = 0.5\text{m} \cdot 0.541 \text{ m}^{-2} = 0.27 \text{ m}^{-1} \quad \rightarrow f = 3.7 \text{ m}$$

o.k.

The FODO cell

Stability condition $4f \geq L$, has a simple interpretation: **—> do not focus too strong !!**

- ▶ It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$

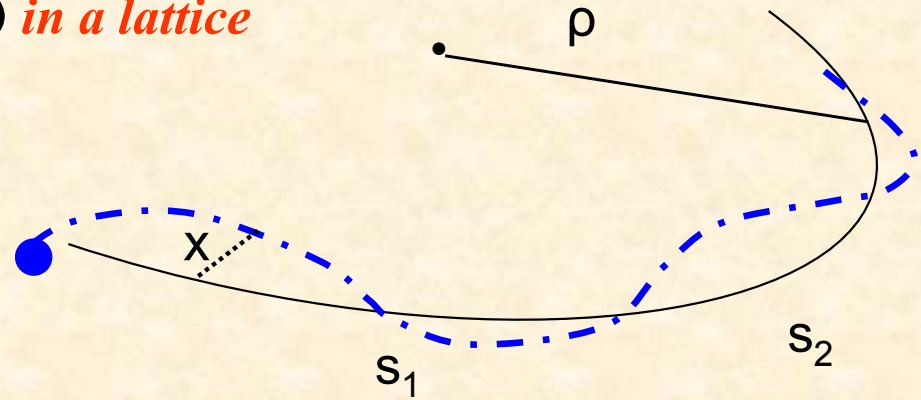


- ▶ The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- ▶ If however the lens system is moved further apart ($L > 4f$), this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

Optics parameters of a FoDo:

Transformation of the coordinate vector (x, x') in a lattice

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s_1, s_2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

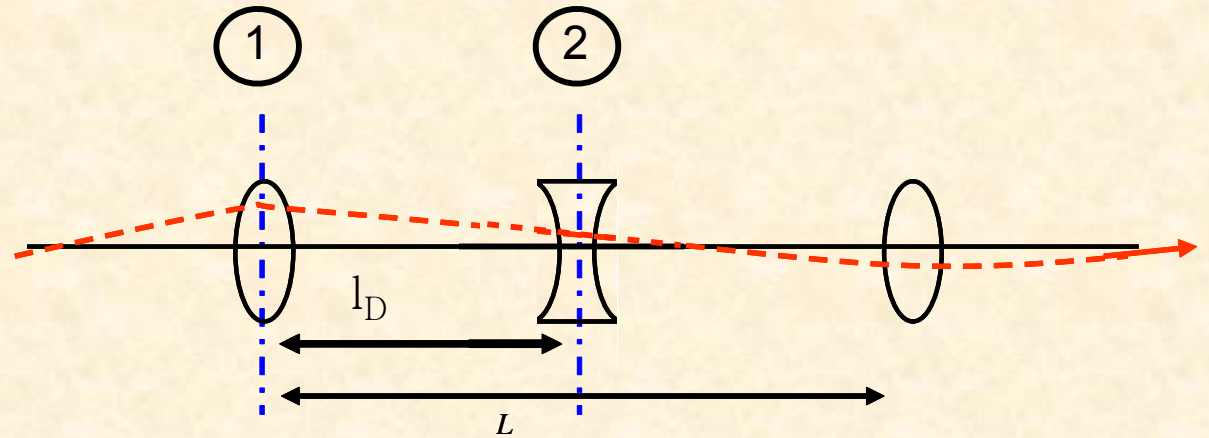


Transformation of the coordinate vector (x, x') expressed as a function of the Optics parameters

$$M_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \psi_{12} + \alpha_1 \sin \psi_{12}) & \sqrt{\beta_1 \beta_2} \sin \psi_{12} \\ \frac{(\alpha_1 - \alpha_2) \cos \psi_{12} - (1 + \alpha_1 \alpha_2) \sin \psi_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \psi_{12} - \alpha_2 \sin \psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$\mathbf{M}_{\text{halfcell}} = \begin{pmatrix} 1 - l_D / \tilde{f} & l_D \\ -l_D / \tilde{f}^2 & 1 + l_D / \tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$\mathbf{M}_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \psi_{12} + \alpha_1 \sin \psi_{12}) & \sqrt{\beta_1 \beta_2} \sin \psi_{12} \\ \frac{(\alpha_1 - \alpha_2) \cos \psi_{12} - (1 + \alpha_1 \alpha_2) \sin \psi_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \psi_{12} - \alpha_2 \sin \psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we always have $\alpha = 0$, and the half cell will lead us from β_{\max} to β_{\min}

$$\begin{aligned} \beta_1 &= \beta_{\max} \\ \beta_2 &= \beta_{\min} \end{aligned} \quad \mathbf{M} = \begin{pmatrix} \sqrt{\frac{\beta_{\min}}{\beta_{\max}}} \cos \frac{\psi_{\text{cell}}}{2} & \sqrt{\beta_{\min} \beta_{\max}} \sin \frac{\psi_{\text{cell}}}{2} \\ \frac{-1}{\sqrt{\beta_{\min} \beta_{\max}}} \sin \frac{\psi_{\text{cell}}}{2} & \sqrt{\frac{\beta_{\max}}{\beta_{\min}}} \cos \frac{\psi_{\text{cell}}}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that

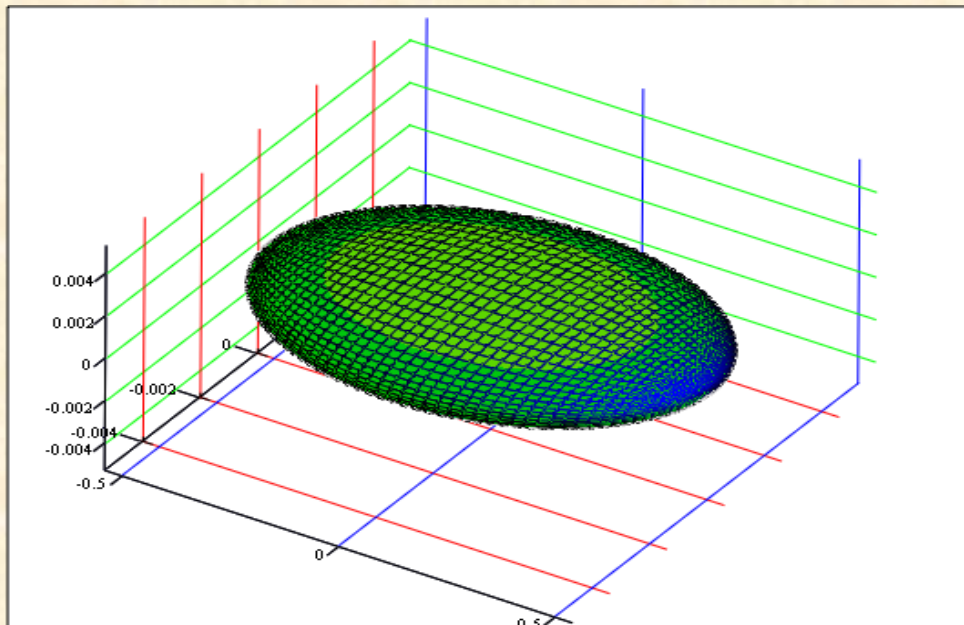
$$\sin \frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$$

$$\frac{m_{22}}{m_{11}} = \frac{\beta_{max}}{\beta_{min}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1 + \sin \frac{\psi_{cell}}{2}}{1 - \sin \frac{\psi_{cell}}{2}}$$

$$\frac{m_{12}}{m_{21}} = \beta_{max} \cdot \beta_{min} = \tilde{f}^2 = \frac{l_d^2}{\sin^2 \frac{\psi_{cell}}{2}}$$

$$\beta_{max} = \frac{(1 + \sin \frac{\psi_{cell}}{2}) L}{\sin(\psi_{cell})} !$$

$$\beta_{min} = \frac{(1 - \sin \frac{\psi_{cell}}{2}) L}{\sin(\psi_{cell})} !$$



The maximum and minimum values of the β -function are solely determined by the phase advance and the length of the cell.

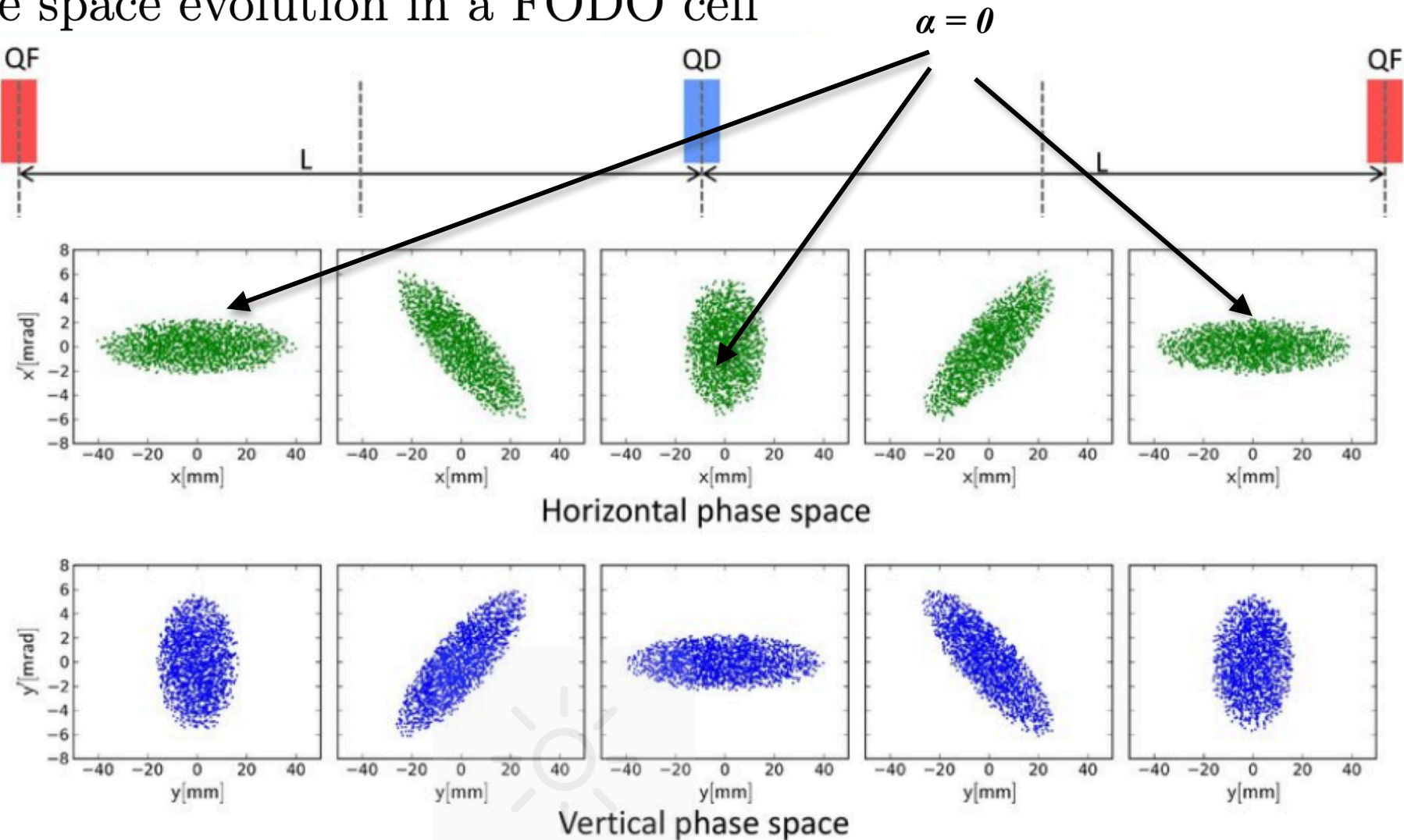
Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell

Phase Space Ellipse in a FoDo

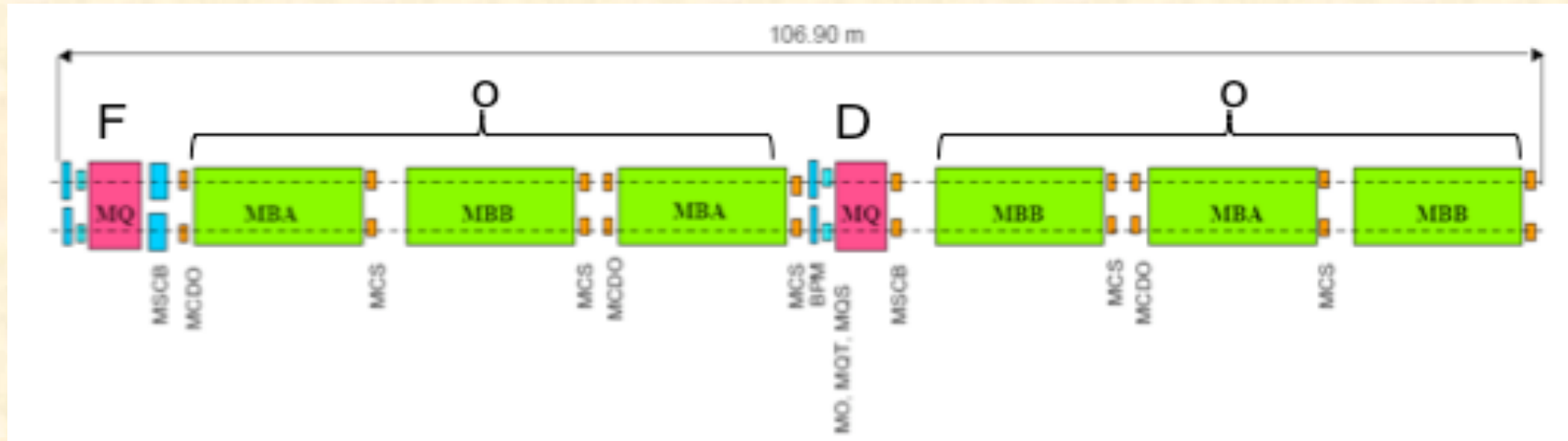
The Twiss family α, β, γ determines shape & orientation of the ellipse

Phase space evolution in a FODO cell



and now in reality ...

LHC: Arc structure based on 90° FoDo in both planes



equipped with additional corrector coils

- MB: main dipole*
- MQ: main quadrupole*
- MQT: Trim quadrupole*
- MQS: Skew trim quadrupole*
- MO: Lattice octupole (Landau damping)*
- MSCB: Skew sextupole*
- Orbit corrector dipoles*
- MCS: Spool piece sextupole*
- MCDO: Spool piece 8 / 10 pole*
- BPM: Beam position monitor + diagnostics*

One Word about non-periodic Lattices:

- ▶ In the previous sections the Twiss parameters α , β , γ , and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- ▶ Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- ▶ In a circular accelerator α , β , and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- ▶ In a transfer line, α , β , and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

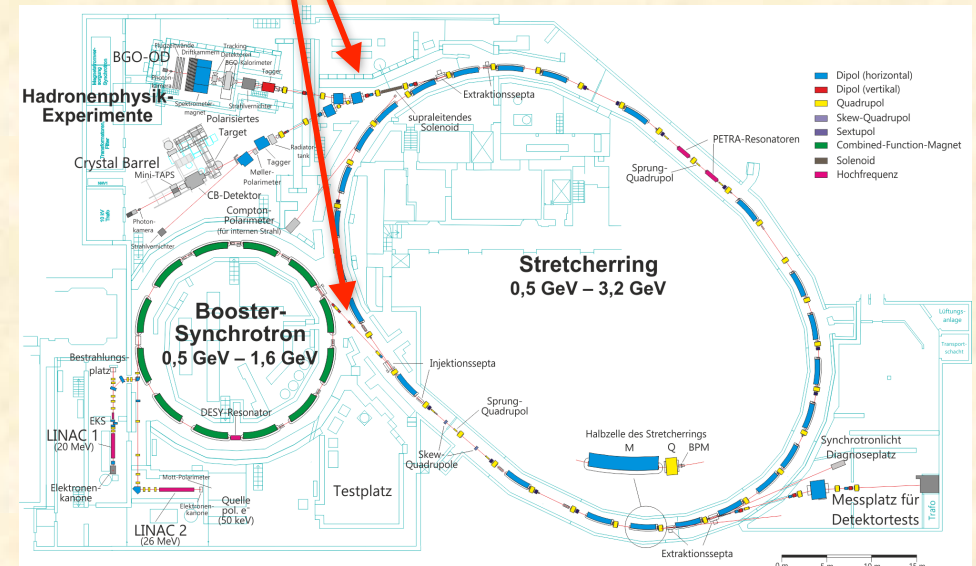
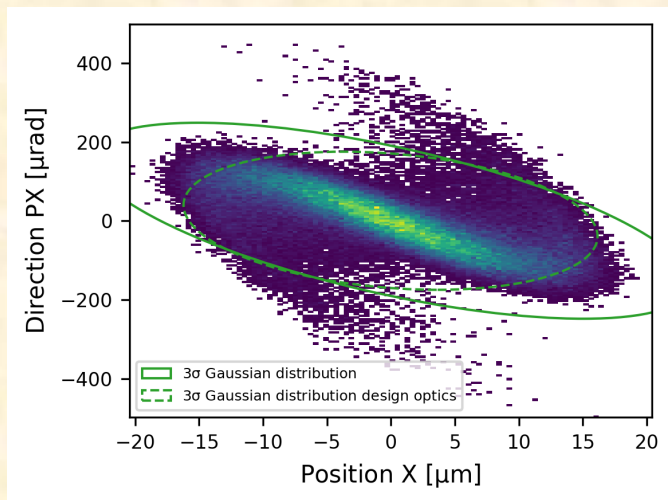
Transferring the Twiss family:

Rule for a non-periodic situation

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

However we have to **determine (!)** the initial Twisses

ELSA storage ring: start with the (uniquely) defined periodic Twiss.



LHeC: energy recovery linac after the collision
 —> fit ... and guess: β, α, ϵ

III.) The „not so ideal world“

Acceleration and Momentum Spread

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $x''(s) - k(s) x(s) = 0$

general solution of Hills equation: $x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \varphi)$

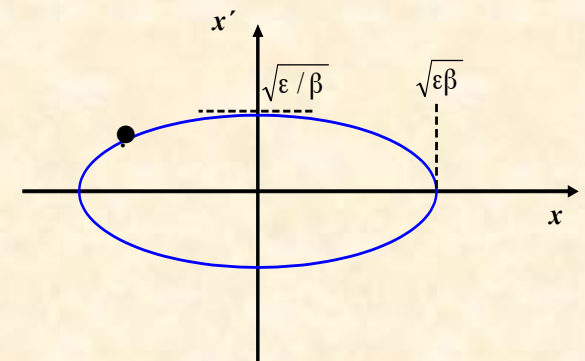
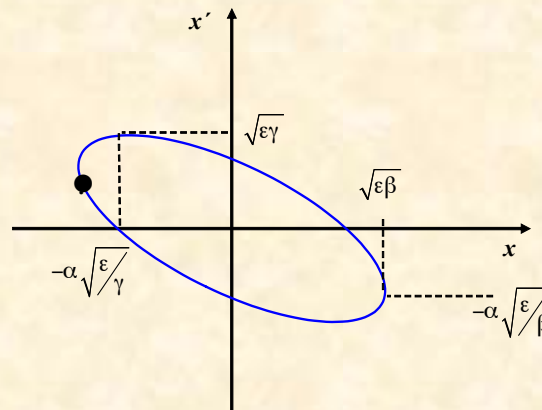
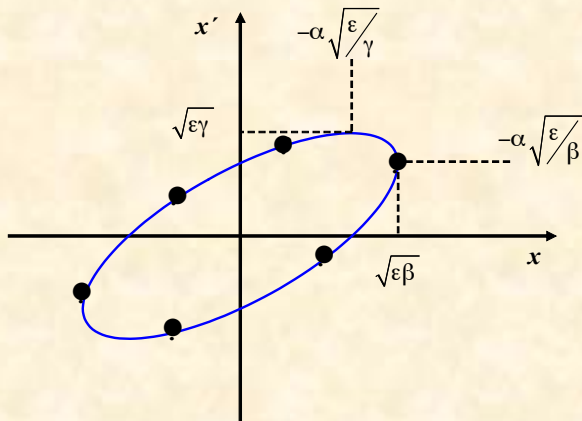
beam size: $\sigma = \sqrt{\varepsilon \beta} \approx \text{"mm"}$

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

- * ε is a **constant of the motion** ... it is independent of „s“
- * parametric representation of an **ellipse in the $x x'$ space**
- * shape and orientation of ellipse are given by α, β, γ

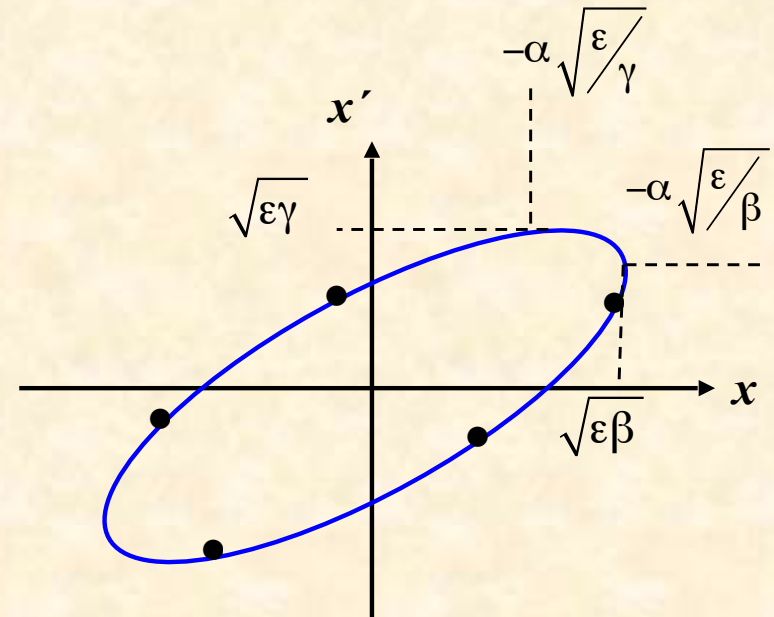


19.) Liouville during Acceleration

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

Beam Emittance corresponds to the area covered in the x, x' Phase Space Ellipse

Liouville: Area in phase space is constant.



But so sorry ... $\varepsilon \neq \text{const}$!

Classical Mechanics:

*phase space = diagram of the two canonical variables
position & momentum*

x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = \text{kin. Energy} - \text{pot. Energy}$$

*According to Hamiltonian mechanics:
phase space diagram relates the variables q and p*

Liouville's Theorem:

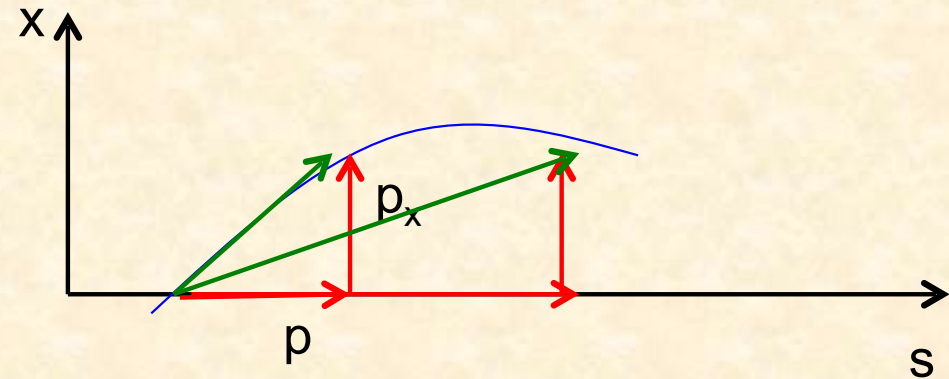
$$\int pdq = \text{const}$$

$$\int p_x dx = \text{const}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$



$$\underbrace{\int x' dx}_{\varepsilon} = \frac{p_x dx}{p} \propto \frac{\text{const}}{m_0 c \gamma \beta}$$

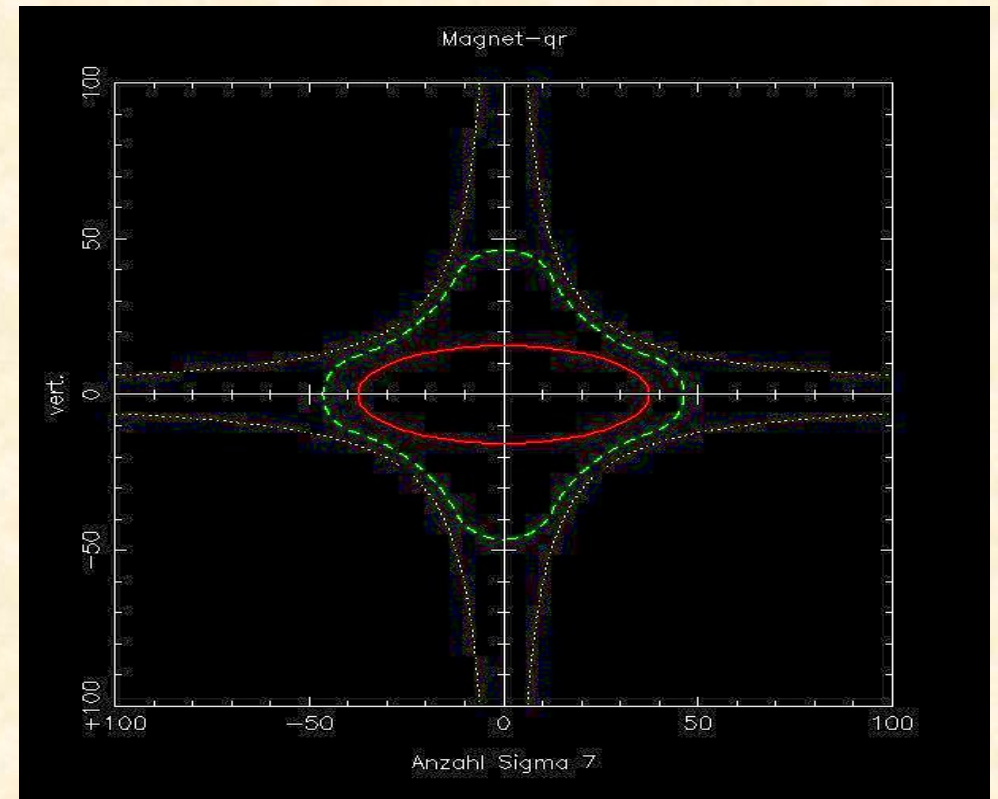
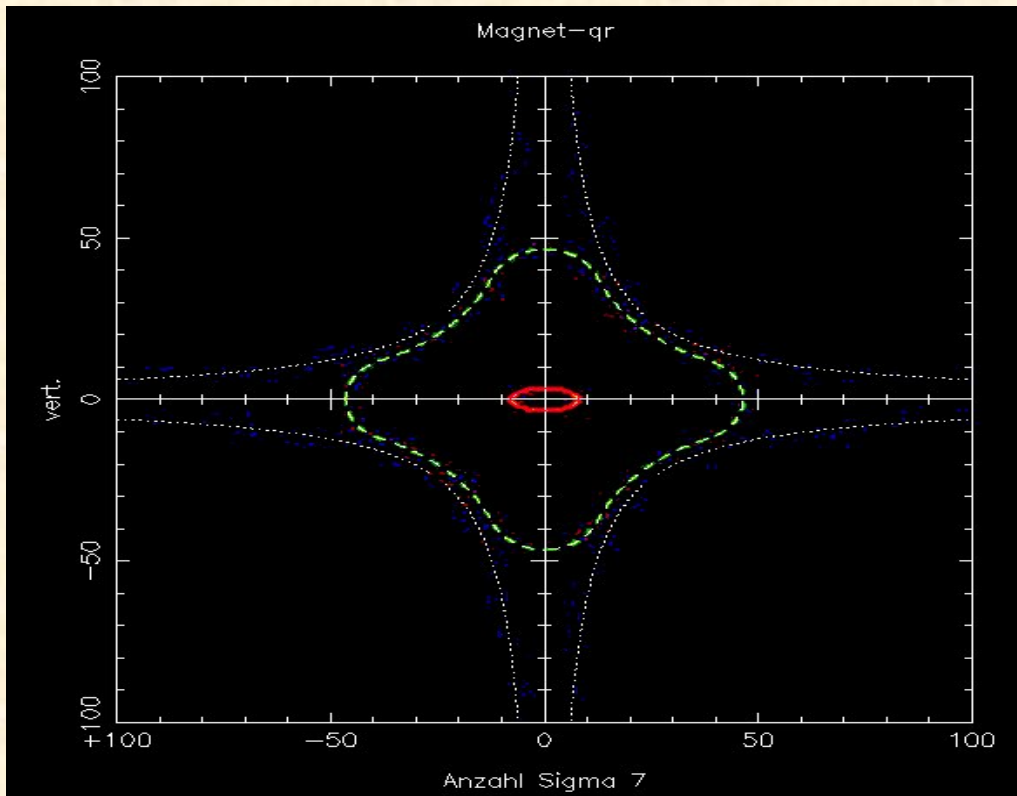
$$\Rightarrow \varepsilon = \int x' dx \propto \frac{1}{\beta \gamma}$$

*the beam emittance
shrinks during
acceleration $\varepsilon \sim 1/\gamma$*

Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε (40 GeV) = $1.2 * 10^{-7} \text{ m rad}$
 ε (920 GeV) = $5.1 * 10^{-9} \text{ m rad}$



7 σ beam envelope at E = 40 GeV

... and at E = 920 GeV

Nota bene:

1.) A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!

as soon as we start to accelerate the **beam size shrinks as $\gamma^{-1/2}$** in both planes.

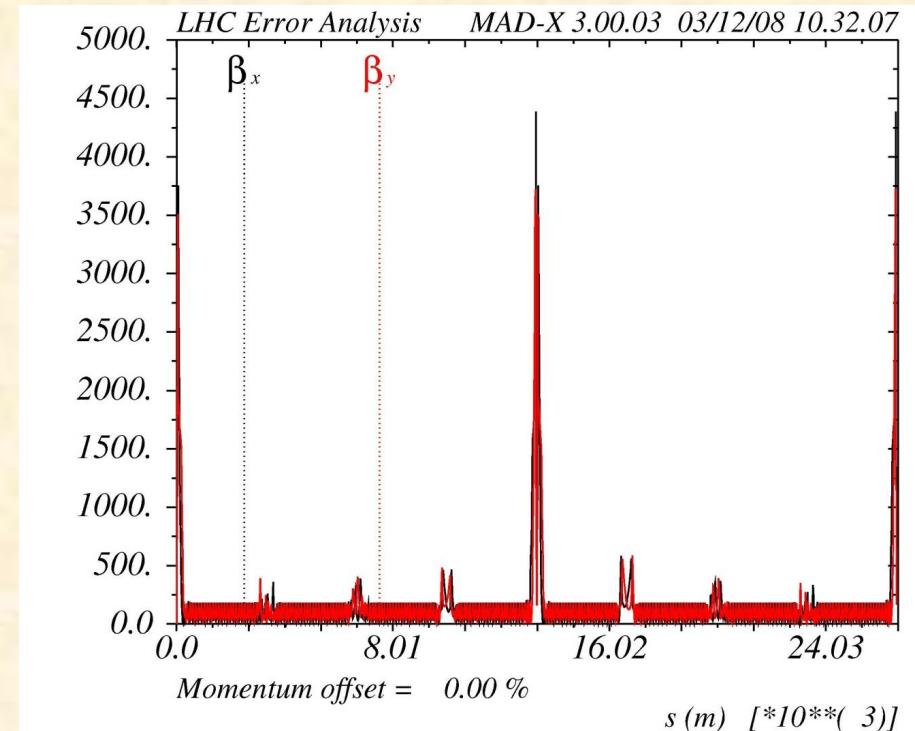
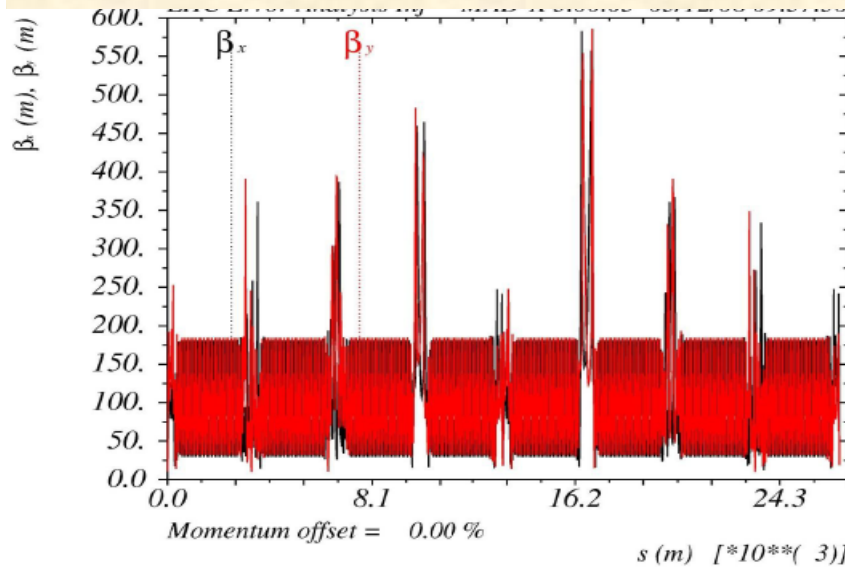
$$\sigma = \sqrt{\epsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,

—> here we have to **minimise $\hat{\beta}$**

3.) we need **different beam optics** adopted to the energy:

A Mini Beta concept will only be adequate at flat top where the beam emittance is small !!



LHC mini beta
optics at 7000 GeV

LHC injection
optics at 450 GeV

Liouville during Acceleration

Protons

... shrink during acceleration

ATTENTION !!!

Electron beams *in a storage ring are determined by light emission and behave completely different.*

... they grow.

The „ not so ideal world “

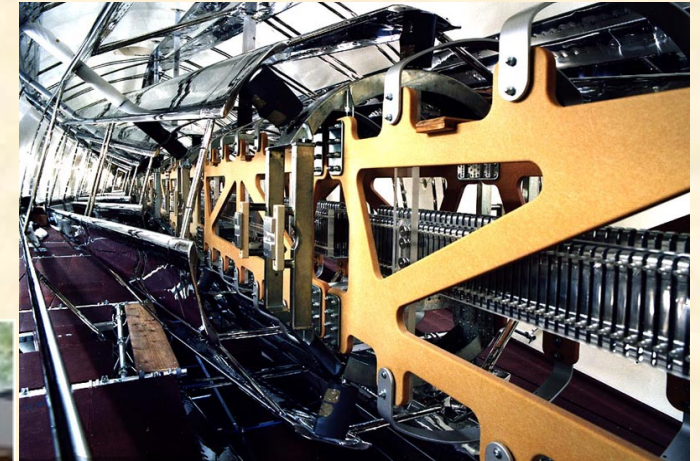
20.) The „ $\Delta p / p \neq 0$ “ Problem

ideal accelerator: all particles will see the same accelerating voltage.

$$\rightarrow \Delta p / p = 0$$

„nearly ideal“ accelerator: Cockroft Walton or van de Graaf

$$\Delta p / p \approx 10^{-5}$$



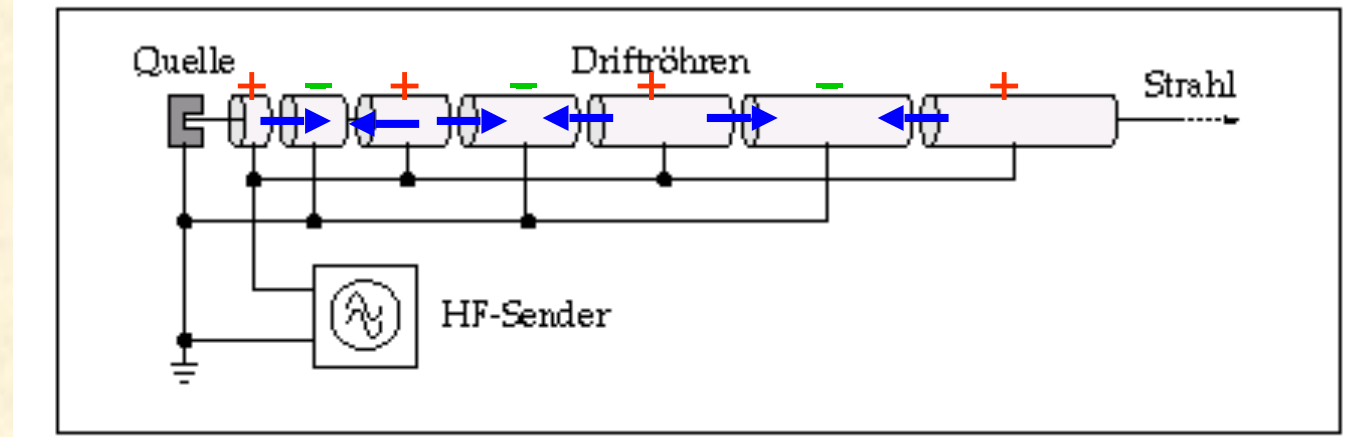
Vivitron, Straßbourg, inner structure of the acc. section

Linear Accelerator

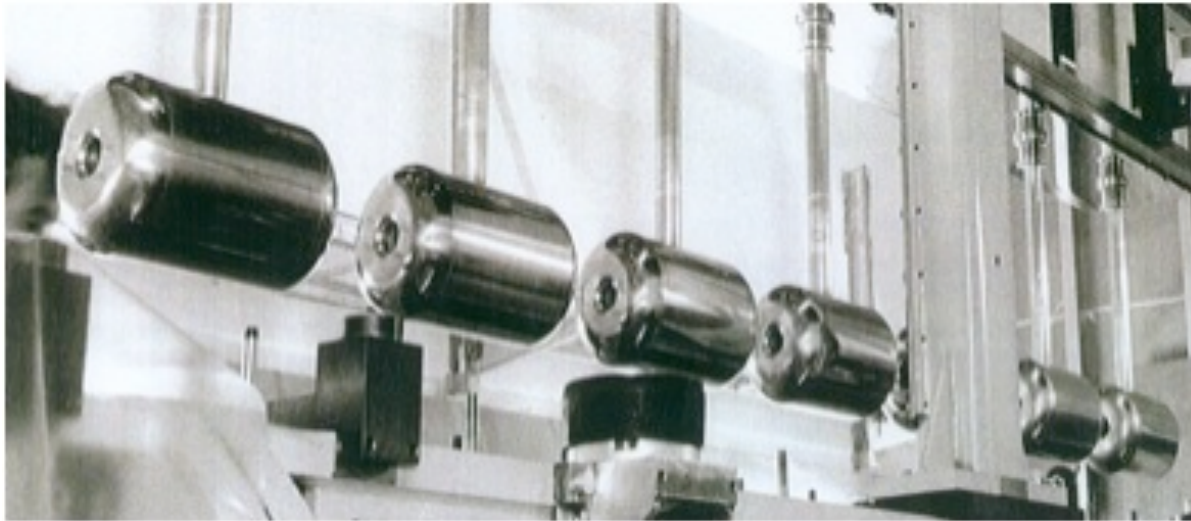
1928, Wideroe: schematic Layout:

Energy Gain per „Gap“:

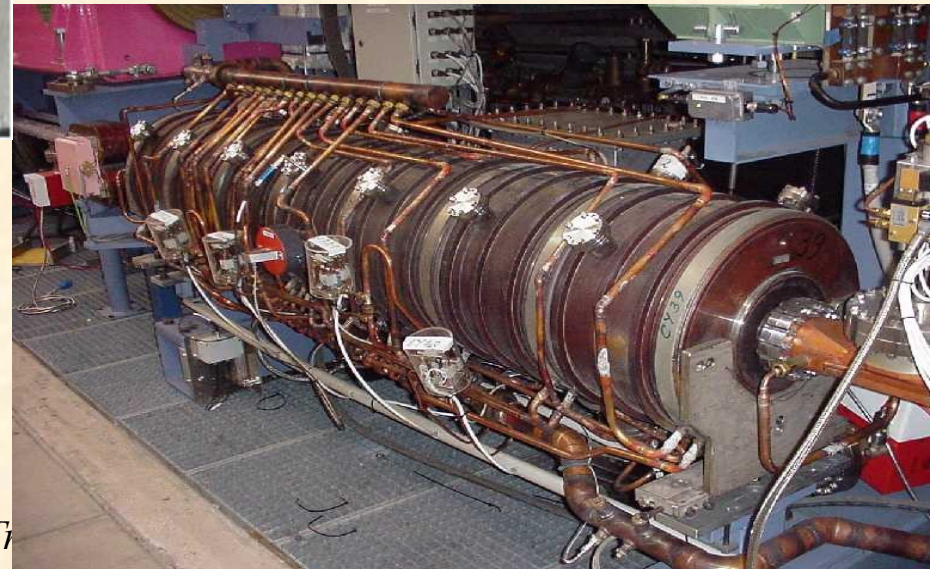
$$W = q U_0 \sin \omega_{RF} t$$



drift tube structure at a proton linac



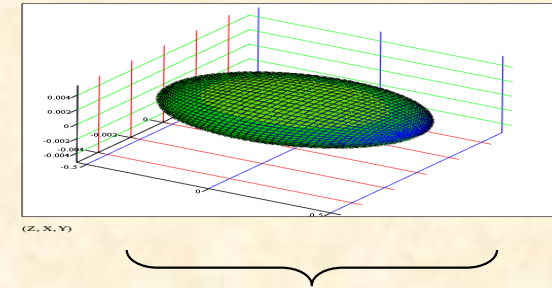
500 MHz cavities in an electron storage ring



* **RF Acceleration:** multiple application of the same acceleration voltage; brilliant idea to gain higher energies

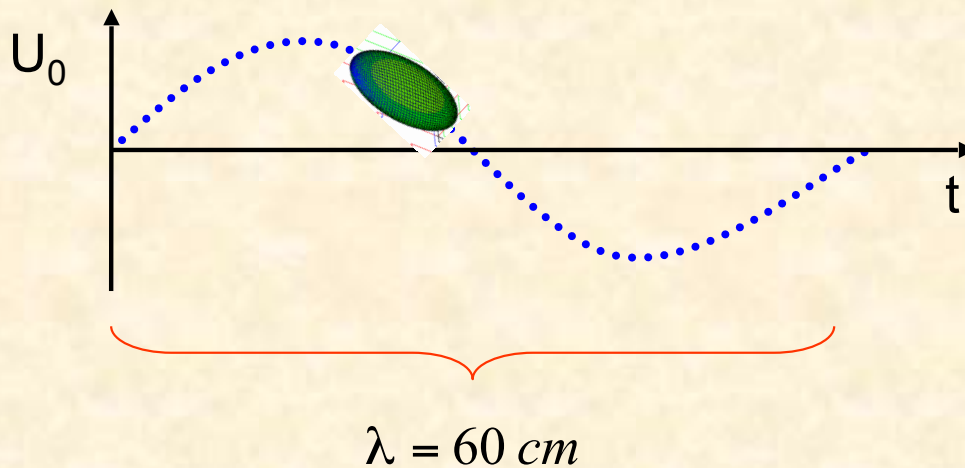
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons $\approx 1\text{cm}$

Example: HERA RF:



$$\left. \begin{aligned} \nu &= 500 \text{ MHz} \\ c &= \lambda \nu \end{aligned} \right\} \lambda = 60 \text{ cm}$$

$$\sin(90^\circ) = 1$$

$$\sin(84^\circ) = 0.994$$

$$\frac{\Delta U}{U} = 6.0 \cdot 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0 \cdot 10^{-3}$$

21.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? ... sure you do

Force acting on the particle

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v$$

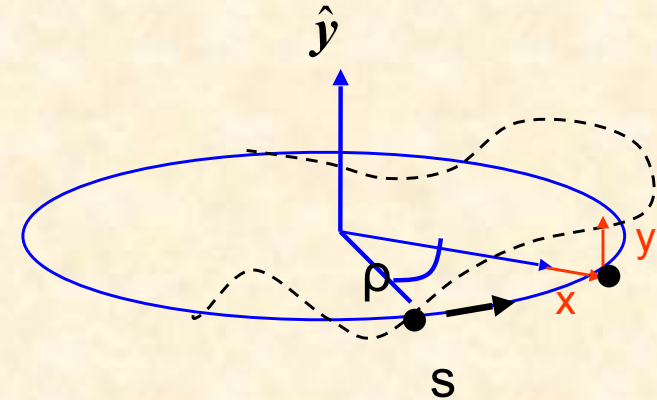
remember: $x \approx mm$, $\rho \approx m$... \rightarrow develop for small x

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = e B_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$ $B_y = B_0 + x \frac{\partial B_y}{\partial x}$

$$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{e B_0}{mv} + \frac{e x g}{mv}$$

$$p = p_0 + \Delta p$$



... but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} e B_0 + \underbrace{\frac{x e g}{p_0}}_{k * x} - \underbrace{x e g \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-e B_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \longrightarrow \quad x'' + x \left(\frac{1}{\rho^2} - k \right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
 —> **inhomogeneous differential equation.**

Dispersion: trajectories for $\Delta p / p \neq 0$

Dispersion

What if particles in a bunch have different momenta?

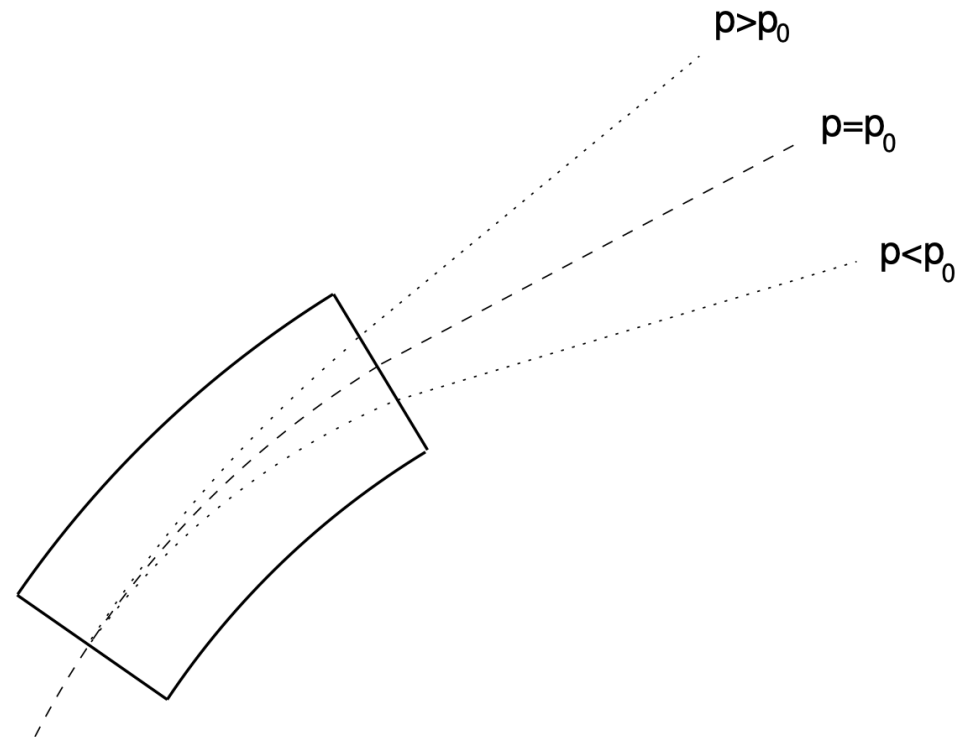
Remember beam rigidity:

$$B\rho = \frac{P}{q} \quad (100)$$

Orbit:

$$x(s) = D(s) \frac{\Delta P}{P_0} \quad (101)$$

where $D(s)$ is the dispersion function, an intrinsic property of dipole magnets.



Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to $\Delta p/p$:

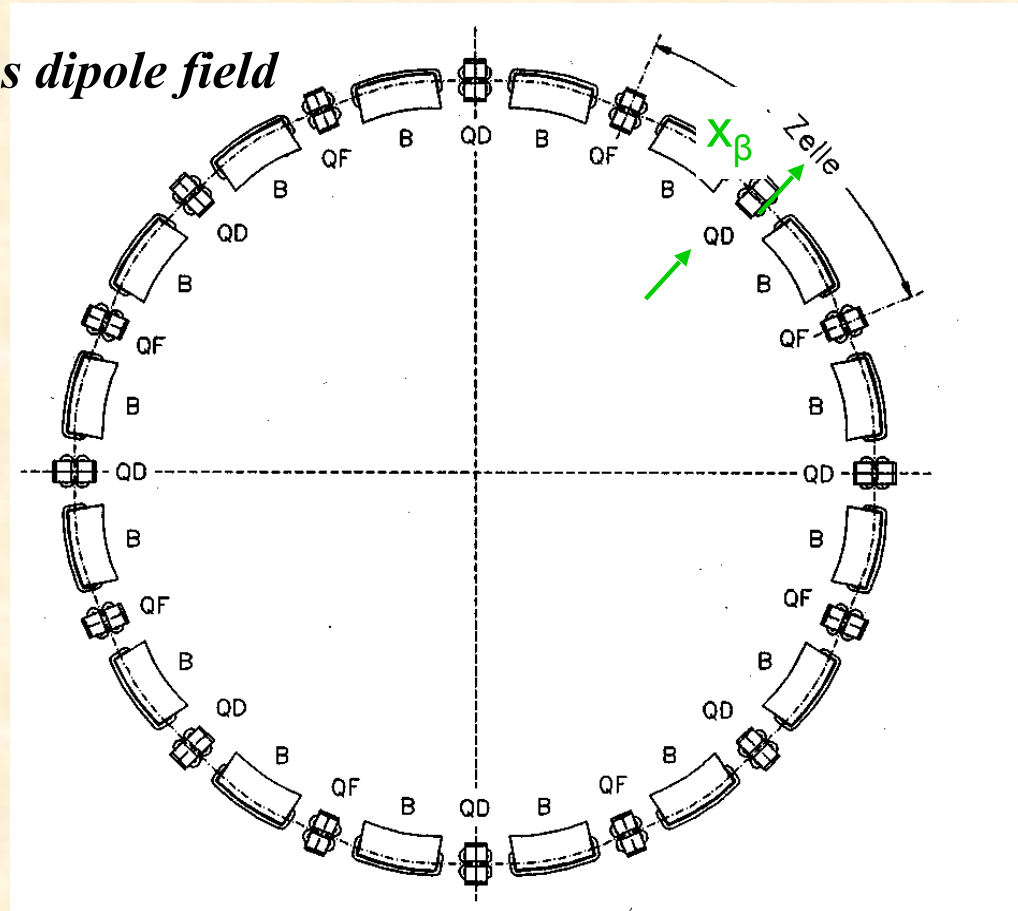
$$D(s) = \frac{x_i(s)}{\frac{\Delta p}{p}}$$

Dispersion function $D(s)$

- * is that **special orbit**, an **ideal particle** would have for $\Delta p/p = 1$
- * the **orbit of any particle** is the **sum** of the well known x_β and the **dispersion**
- * as **$D(s)$** is just **another orbit** it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field



for $\Delta p/p > 0$

$$D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Resume':

beam emittance

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

... and for $\alpha = 0$

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

*particle trajectory for $\Delta p/p \neq 0$
inhomogeneous equation*

$$\mathbf{x}'' + \mathbf{x} \left(\frac{1}{\rho^2} - k \right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

... and its solution

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$