Introduction to Transverse Beam Optics

Bernhard Holzer, CERN

IV.) Errors in Field and Gradient

The „ überhaupt nicht ideal world "

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

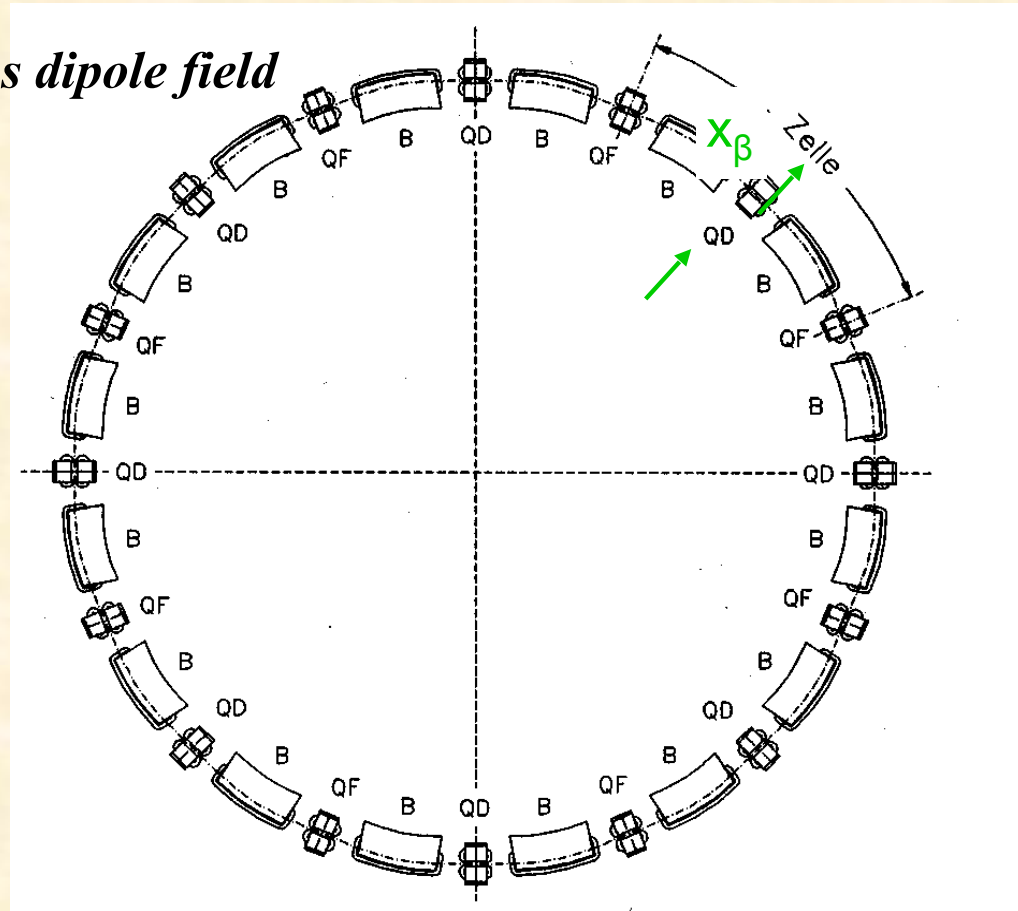$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

* *is that special orbit, an ideal particle would have for Δp/p = 1*

* *the orbit of any particle is the sum of the well known x_β and the dispersion*

* *as D(s) is just another orbit it will be subject to the focusing properties of the lattice*

Dispersion

Example: homogeneous dipole field

...bit for $\Delta p/p > 0$

$$= D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

HERA P-Ring, Lumi-A-Optik, 7/0.5 m, p/e+   920 GeV, 1999, qd997z, hts920e+8, A*8 Nomen

Example

$$x_\beta = 1 ... 2 \, mm$$

$$D(s) \approx 1 ... 2 \, m$$

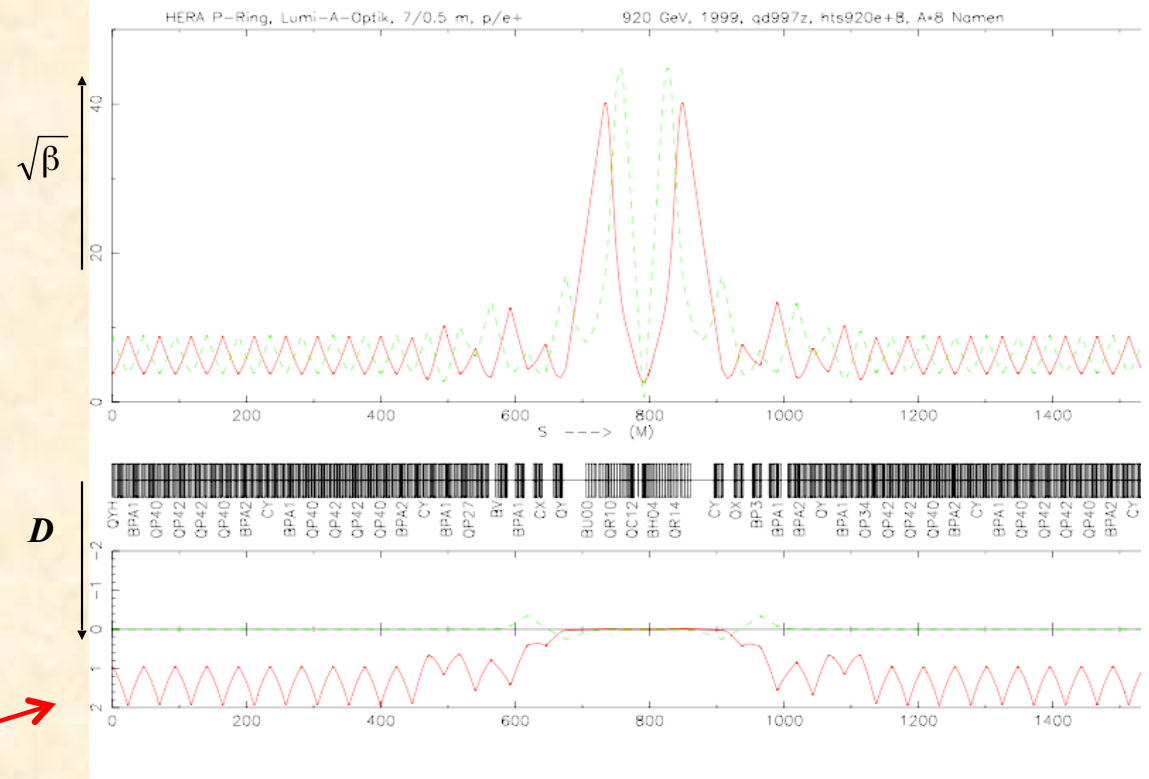$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation
contribution due to Dispersion ≈ beam size
—> Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

Dispersion and Beam Size:

Super-position of two Gaussian distributions

In this example from the HERA storage ring (DESY) we see the Twiss parameters and the dispersion near the interaction point. In the periodic region,
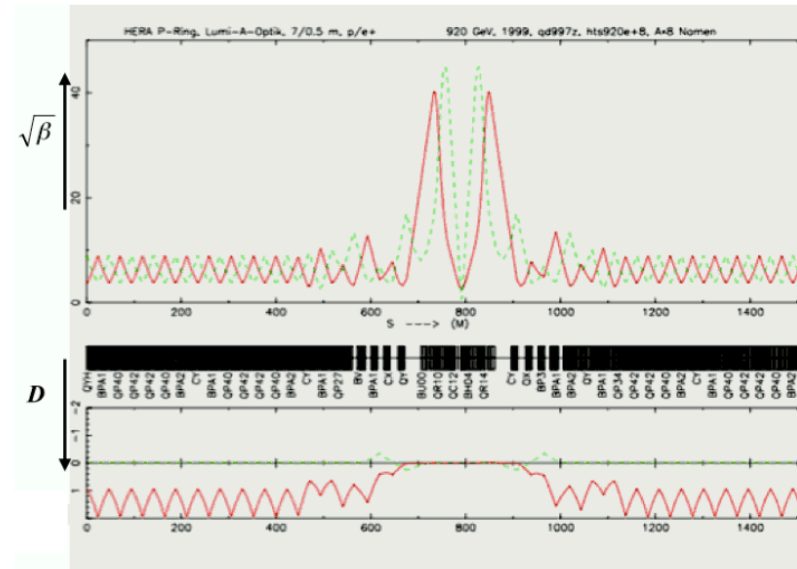
$$x_\beta(s) = 1 \dots 2 \text{ mm}$$

$$D(s) = 1 \dots 2 \text{ m}$$

$$\Delta P/P_0 \approx 1 \cdot 10^{-3}$$

Remember:

$$x(s) = x_\beta(s) + D(s) \frac{\Delta P}{P_0}$$

Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \text{std}\left(D \cdot \frac{\Delta P}{P_0}\right)^2} = \sqrt{\epsilon_{\text{geometric}} \cdot \beta + D^2 \cdot \frac{\sigma_P^2}{P_0^2}}$$

▶ We need to suppress the dispersion at the IP !

▶ We need a special insertion section: a *dispersion suppressor*

Dispersion:

Example: *Drift*

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

$$\underbrace{\qquad\qquad}_{= 0} \qquad \underbrace{\qquad\qquad}_{= 0}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: *Dispersion in a Sector Dipole Magnet*

Remember: *Matrix of a magnetic element*

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\ -\sqrt{|K|} \sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$

in general: $\quad K = k - \dfrac{1}{\rho^2}$

… but in a dipole, as k = 0 …

$$M_{foc} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\dfrac{l}{\rho} & \rho \sin\dfrac{l}{\rho} \\ -\dfrac{1}{\rho}\sin\dfrac{l}{\rho} & \cos\dfrac{l}{\rho} \end{pmatrix}$$

calculate the „D" elements for the marix a Sector Dipole Magnet

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

$$D(s) = (\rho \sin \frac{l}{\rho}) * \frac{1}{\rho} * (\rho \sin \frac{l}{\rho}) - \cos \frac{l}{\rho} * \frac{1}{\rho} * \rho \cdot (-\cos \frac{l}{\rho} + 1) * \rho$$

$$D(s) = \rho \sin^2 \frac{l}{\rho} + \rho \cos \frac{l}{\rho} * (\cos \frac{l}{\rho} - 1)$$

$$\boxed{D(s) = \rho \cdot (1 - \cos \frac{l}{\rho})} \quad , \quad \boxed{D'(s) = \sin \frac{l}{\rho}} \qquad \textbf{\textit{Dispersion elements in a sector dipole magnet}}$$

$$M_{dipole} = \begin{pmatrix} \cos \frac{l}{\rho} & \rho \sin \frac{l}{\rho} & D \\ -\frac{1}{\rho} \sin \frac{l}{\rho} & \cos \frac{l}{\rho} & D' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s2} = \begin{pmatrix} \cos \frac{l}{\rho} & \rho \sin \frac{l}{\rho} & \rho * (1 - \cos \frac{l}{\rho}) \\ -\frac{1}{\rho} \sin \frac{l}{\rho} & \cos \frac{l}{\rho} & \sin \frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s1}$$

Nota bene: even an ideal particle with x = x´= 0 will start to oscillate if it passes a dipole magnet and has a momentum error Δp/p.
A dispersion trajectory will obey the same focusing forces (i.e. will be transferred by the same matrices) as a normal betatron oscillation

Periodic Dispersion: η, η'

▶ The equation:

$$D(s) = S(s) \int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s) \int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

allows to compute **the dispersion inside a magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

▶ At the exit of a magnet of length L_m the dispersion reaches the value $D(L_\mathrm{m})$

▶ The dispersion (also indicated as η, with its derivative η') propagates from there, through the rest of the machine, just like any other particle:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_0$$

Periodic Dispersion: η, η'

In a periodic lattice, also the dispersion must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

Let's rewrite this in 2 × 2 form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1-C & -S \\ -C' & 1-S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1-C)(1-S') - C'S} \begin{pmatrix} 1-S' & S \\ C' & 1-C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Dispersion in a FoDo Cell:

QF   QD

l_D

L
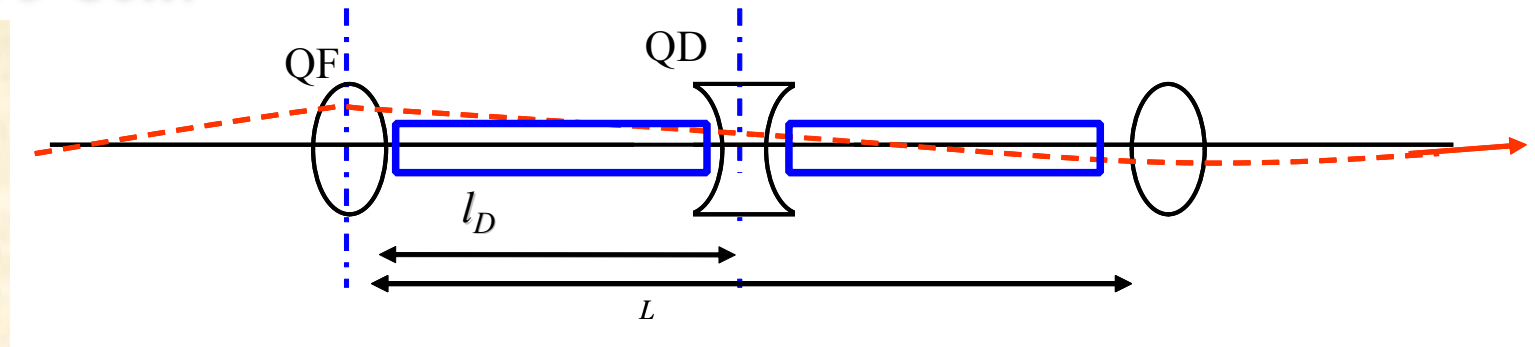
!! we have now introduced dipole magnets in the FoDo:
—> we *still neglect* the *weak focusing* contribution *$1/\rho^2$*
—> but *take into account $1/\rho$ for the dispersion effect*
assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \; \check{\beta}$

* *thin lens approximation:* $f = \dfrac{1}{k\ell_Q} \gg \ell_Q$

* *length of quad negligible* $\ell_Q \approx 0, \; \rightarrow \; \ell_D = \dfrac{1}{2}L$

* *start at half quadrupole* $\dfrac{1}{\tilde{f}} = \dfrac{1}{2f}$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{Half\ Cell} = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \dfrac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{Half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\ell}{\tilde{f}} & \ell \\ \dfrac{-\ell}{\tilde{f}^2} & 1 + \dfrac{\ell}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D′ from the matrix elements

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

$$D(\ell) = \ell * \frac{1}{\rho} * \int_0^\ell \left(1 - \frac{s}{\tilde{f}}\right) ds - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \int_0^\ell s \ ds$$

$$\underbrace{}_{S(s)} \quad \underbrace{}_{C(s)} \quad \underbrace{}_{C(s)} \quad \underbrace{}_{S(s)}$$

$$D(\ell) = \frac{\ell}{\rho}\left(\ell - \frac{\ell^2}{2\tilde{f}}\right) - \left(1 - \frac{\ell}{\tilde{f}}\right) * \frac{1}{\rho} * \frac{\ell^2}{2} \qquad = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$\boxed{D(\ell) = \frac{\ell^2}{2\rho}}$$

in full analogy on derives for D′:

$$\boxed{D'(s) = \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)}$$

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25 meter 180⁰ Arc based on 90⁰-FODO lattice

p = 3.8 GeV/c

BETA_X   BETA_Y   DISP_X   DISP_Y

2 × 'half-empty' cells   4 × 90⁰-FODO cells   2 × 'half-empty' cells

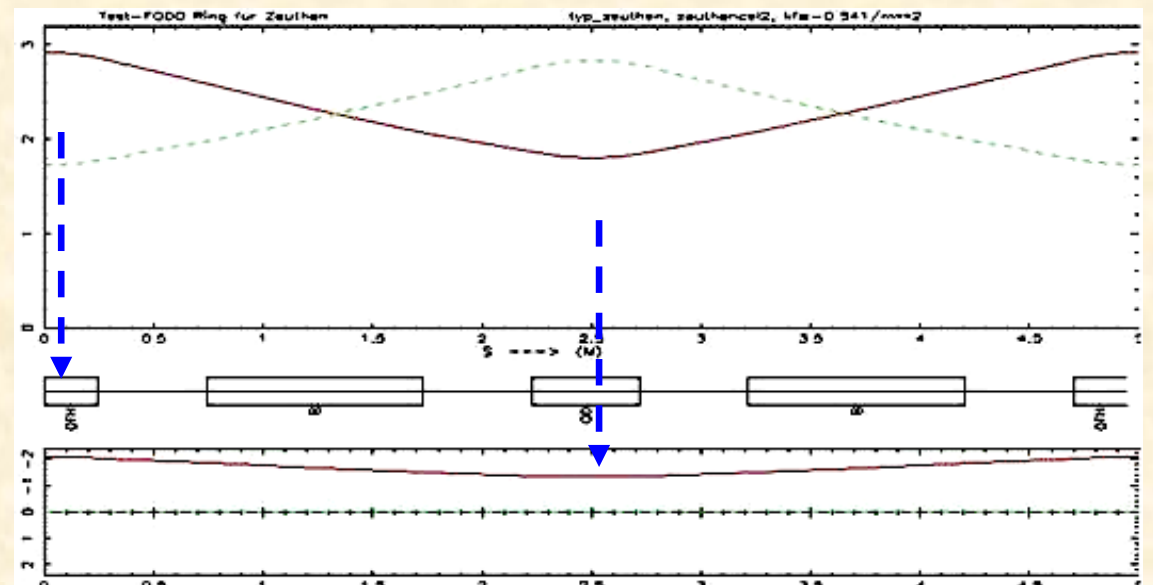*and we get the complete matrix including the dispersion terms D, D´*

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\ell}{\tilde{f}} & \ell & \dfrac{\ell^2}{2\rho} \\ \dfrac{-\ell}{\tilde{f}^2} & 1 + \dfrac{\ell}{\tilde{f}} & \dfrac{\ell}{\rho}\left(1 + \dfrac{\ell}{2\tilde{f}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

***boundary conditions** for the transfer from the center of the foc. to the center of the defoc. quadrupole*

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$

$\sqrt{\beta}$

D

Dispersion in a FoDo Cell

$$\rightarrow \check{D} = \hat{D}(1 - \frac{\ell}{\tilde{f}}) + \frac{\ell^2}{2\rho}$$

$$\rightarrow 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}(1 + \frac{\ell}{2\tilde{f}})$$

$$D_{max} = \frac{l^2}{r} \cdot \frac{1 + \frac{1}{2}sin\frac{\psi_{cell}}{2}}{sin^2 \frac{\psi_{cell}}{2}} \qquad D_{min} = \frac{l^2}{r} \cdot \frac{1 - \frac{1}{2}sin\frac{\psi_{cell}}{2}}{sin^2 \frac{\psi_{cell}}{2}}$$

*where ψ_{cell} denotes the phase
advance of the full cell
and $l/f = sin(\psi/2)$*

$D_{max}(\mu)$ ——

$D_{min}(\mu)$ ·······
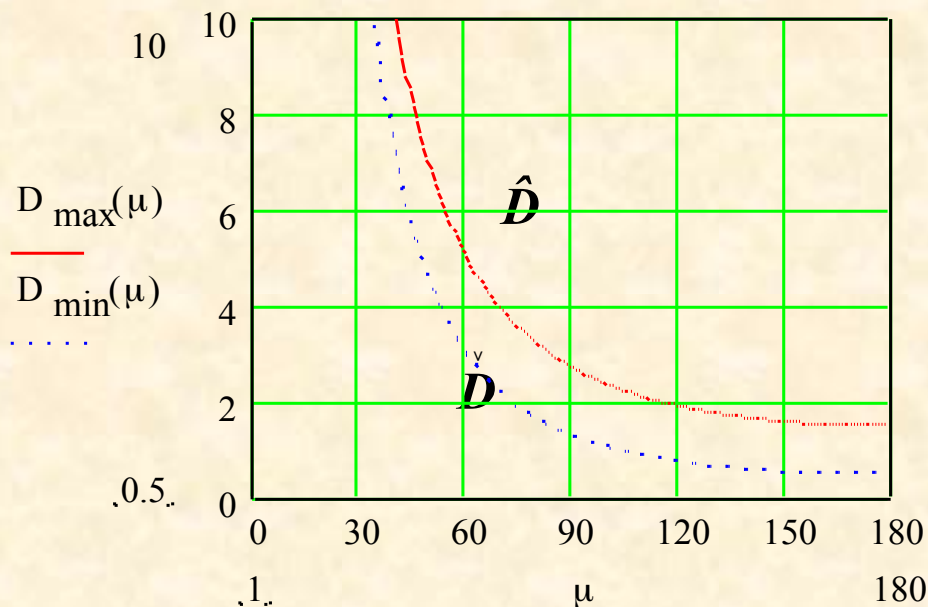
Nota bene:

! *small dispersion **needs** strong focusing
→ large phase advance*

!! ↔ *there is an **optimum phase for small β***

!!! *...do you remember the **stability criterion?**
½ trace = cos ψ ↔ ψ < 180°*

!!!! *... life is not easy*

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) * \frac{\Delta p}{p}$$
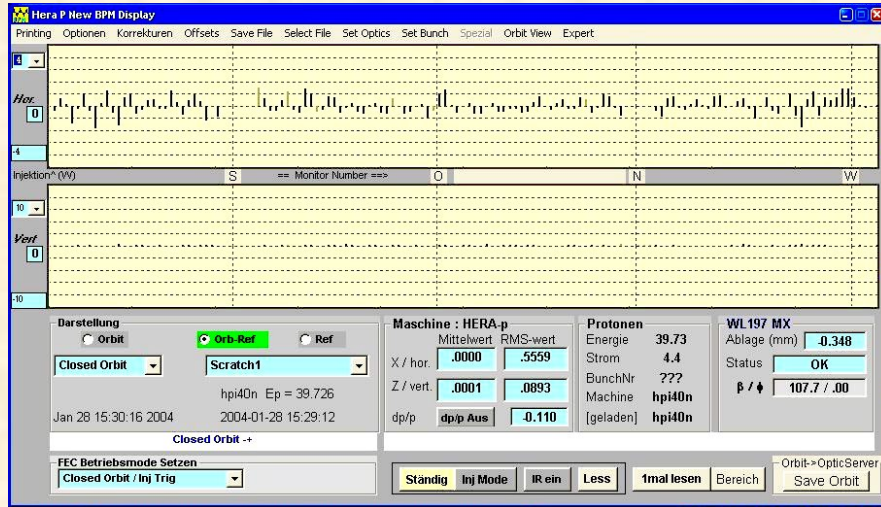
* D(s) is created by the dipole magnets

... and afterwards focused by the quadrupole fields
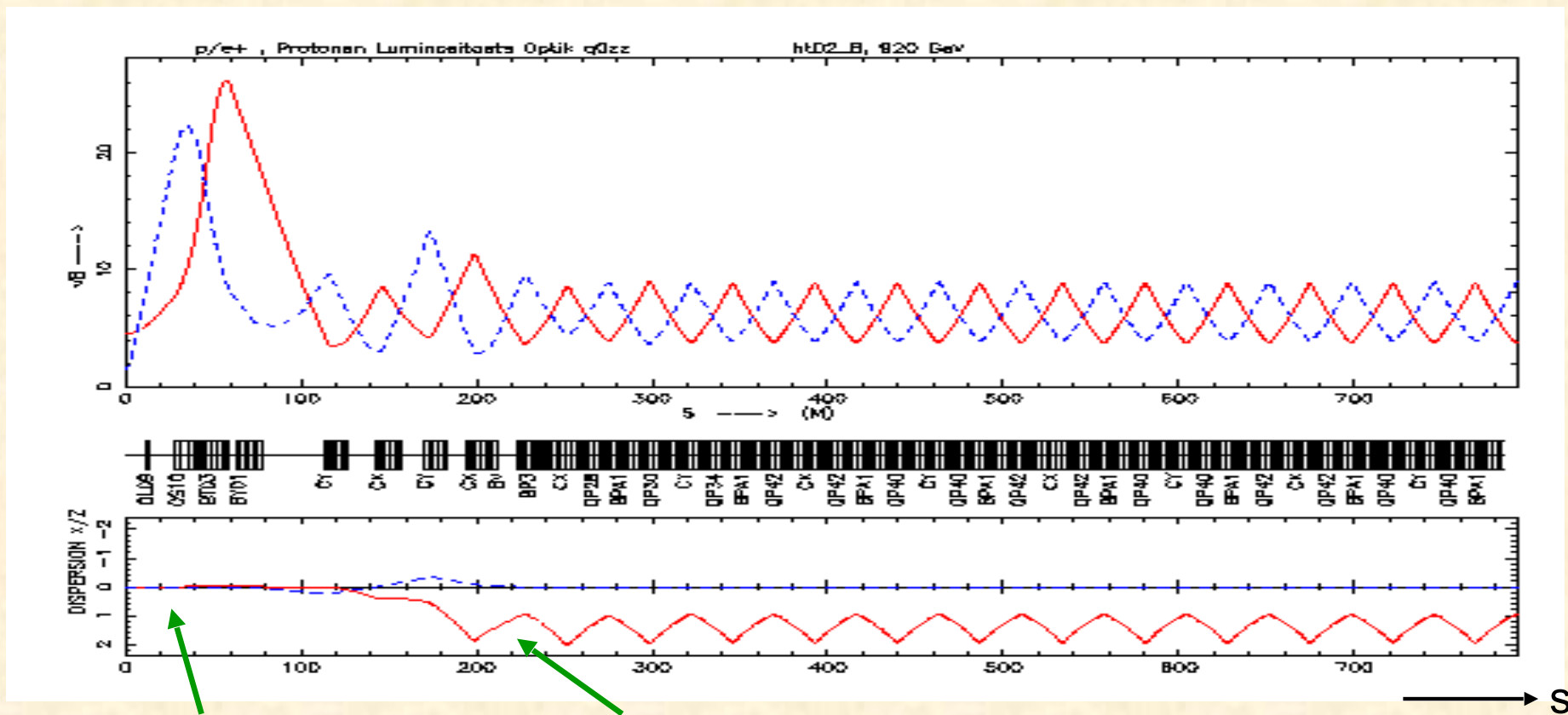
Mini Beta Section,
—> no dipoles !!!
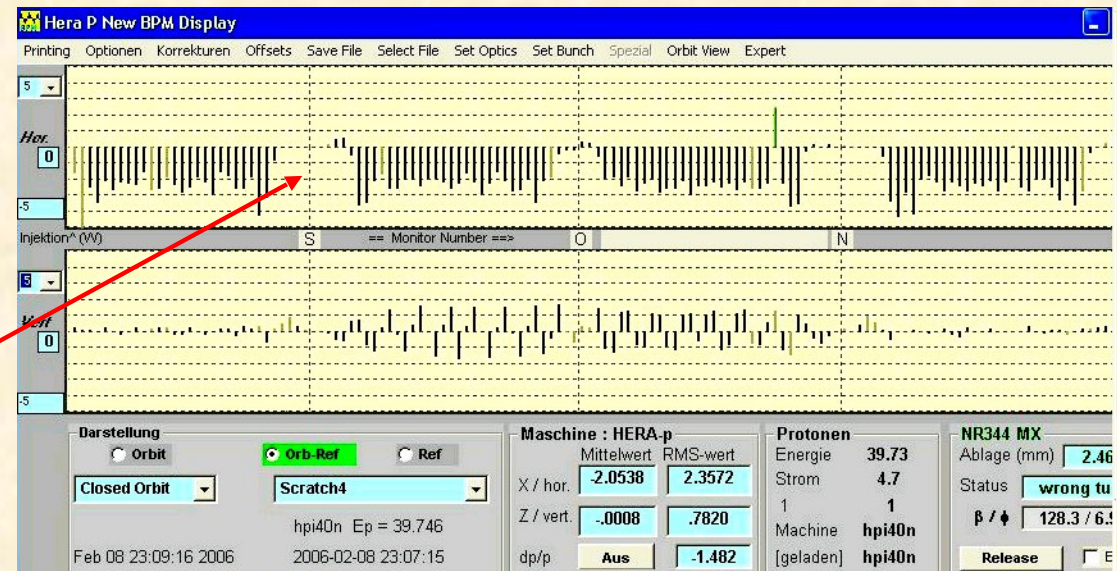
$D(s) \approx 1 \dots 2$ m

Dispersion is visible

HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

    —> closed orbit is moved to a

      dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$

Attention: at the Interaction Points
we require D=D´= 0

Periodic Dispersion:

"Sawtooth Effect" at LEP (CERN)

Electron course

BPM Number
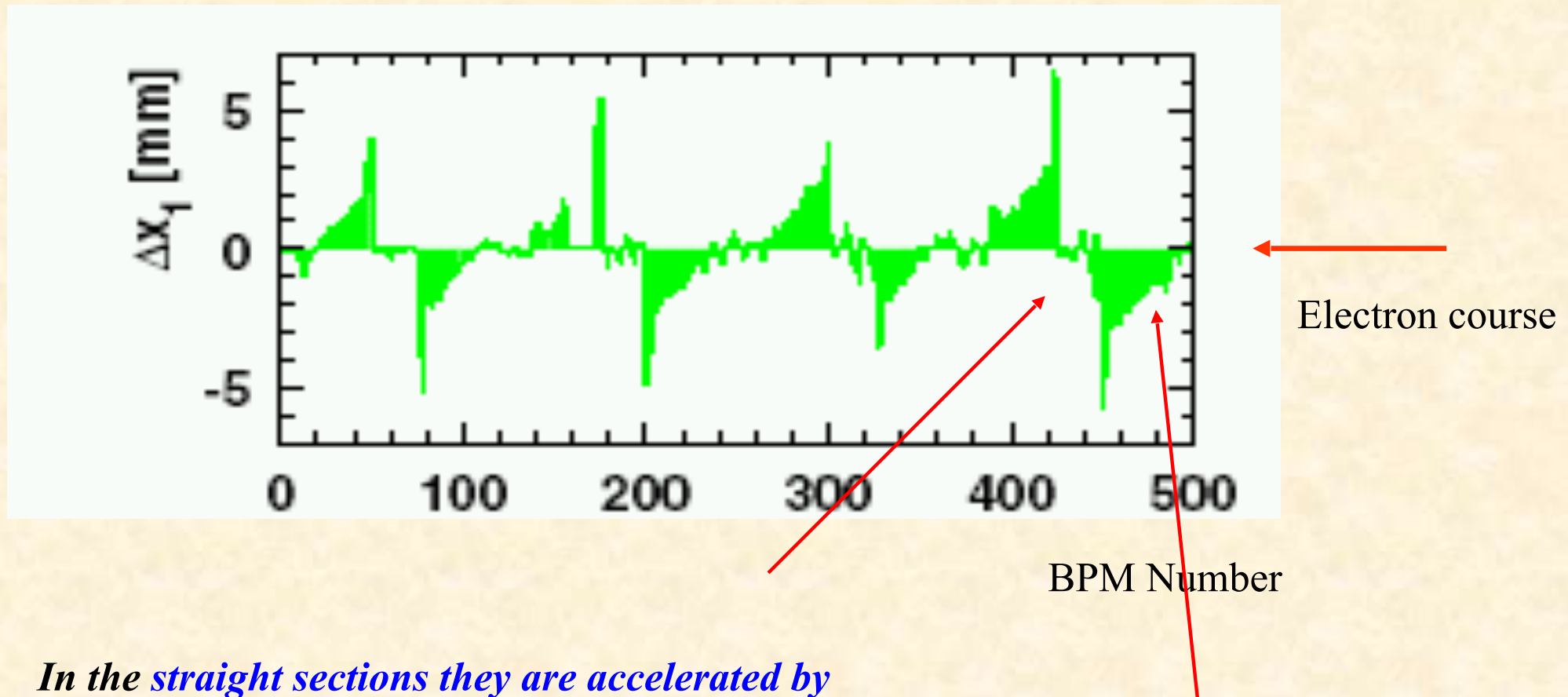
*In the **straight sections they are accelerated by the rf cavities** so much that they "overshoot" and reach nearly the **outer side of the vacuum chamber**.*

*In the **arc the electron beam loses so much energy** in ea octant that the particle are running more and more **on a dispersion trajectory.***
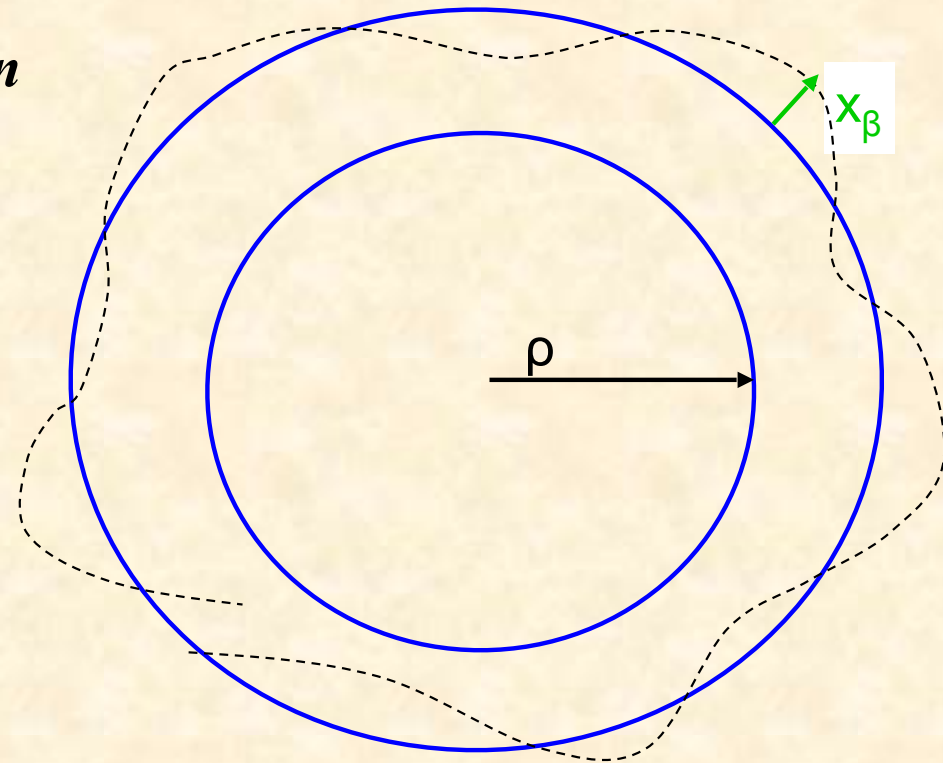
22.) Momentum Compaction Factor:

The *dispersion* function relates the *momentum error* of a particle to the horizontal orbit coordinate.

inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p}$$
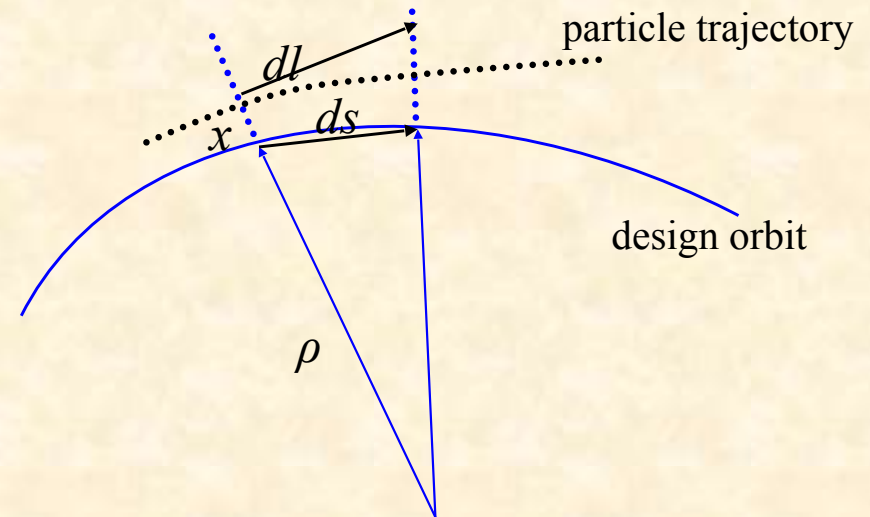
But it does much more:
it changes the length of the *off - energy - orbit* !!

Momentum Compaction Factor:

particle with a displacement x to the design orbit
has different path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

particle trajectory

design orbit

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p_0}$$

$$\delta l = l_{\Delta E} - l_0 = \frac{\Delta p}{p_0} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

* *The lengthening of the orbit for off-momentum*
particles is given by the dispersion function
and the bending radius.

Momentum Compaction Factor:

Definition:

$$\frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

$$\rightarrow \quad \alpha_p = \frac{1}{L_0} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume: $\dfrac{1}{\rho} = const$

$$\int_{dipoles} D(s)\, ds = \Sigma\, (l_{dipoles}) * \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L_0} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L_0} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L_0} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$
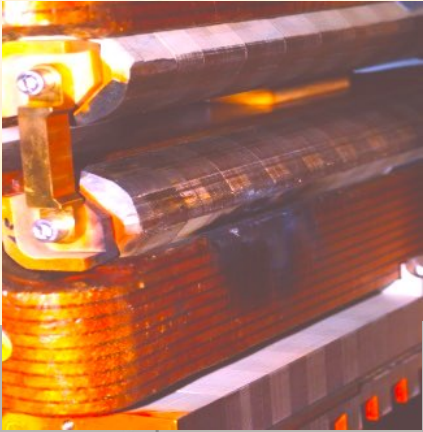
Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T_0} = \frac{\delta l}{L_0} = \alpha_p \frac{\Delta p}{p_0}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.
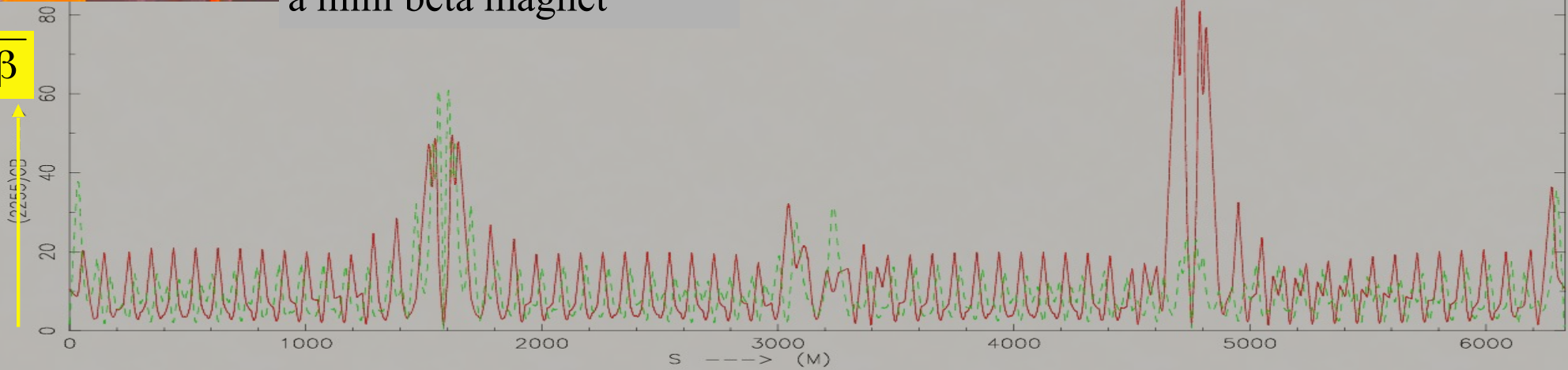
Introduction to Transverse Beam Optics

IV.) Errors in Field and Gradient
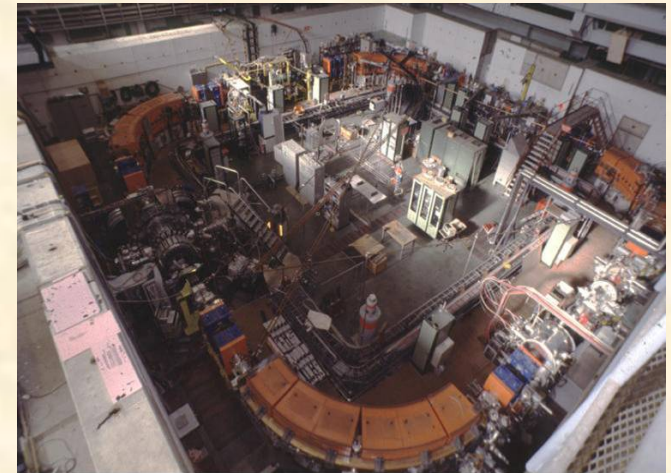

burned quadrupole coil in a mini beta magnet

23.) Errors in Field and Gradient
- life is not so easy -

The derivation of the equation of motion is based on the presumption that

… in our accelerator there are only linear magnetic fields ….

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k*x + \frac{1}{2!}mx^2 + \frac{1}{3!}nx^3 + \dots$$

$\underbrace{\phantom{\frac{1}{\rho}}}$ **dipole** $\underbrace{}$ **quadrupole**
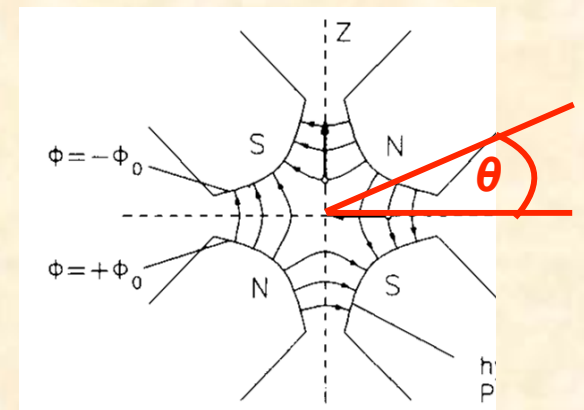
linear magnet structure of LEAR (CERN)

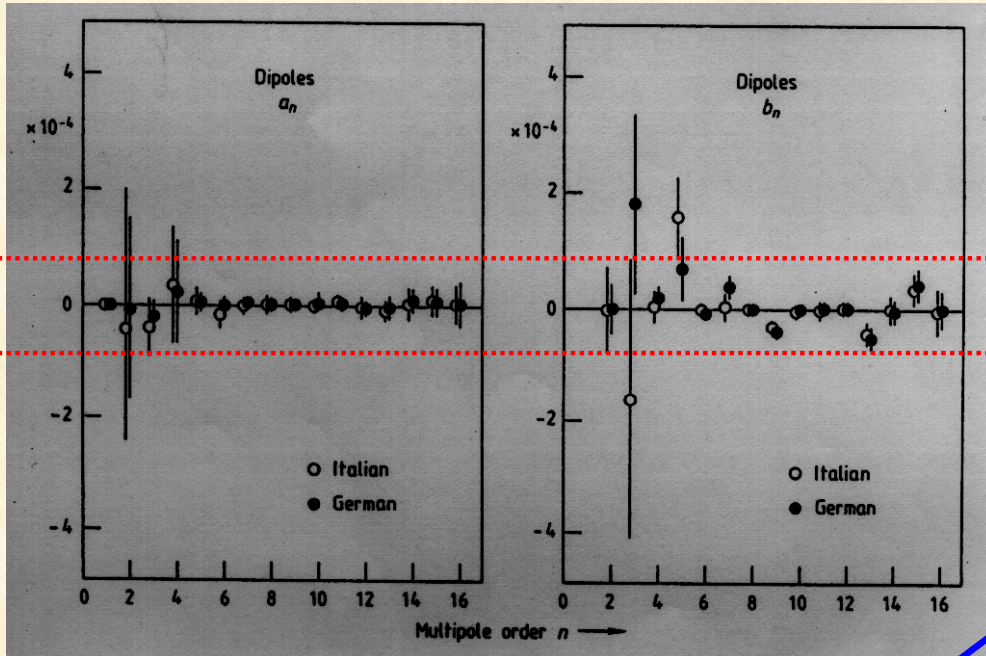**Multipole expansion of magnetic field:**

$$B_\theta(r,\theta) = B_{main} \cdot \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} [b_n \, cos(n\theta) + a_n \, sin(n\theta)]$$

example:
mid plane —> θ = 0,
radius = ref radius r₀

$$b_n = \frac{B_{multipole}}{B_{main}}$$

Example: HERA multipole coefficients of sc. dipole magnets

$$B_\theta(r,\theta) = B_{main} \cdot \sum_{n=1}^{\infty} (\frac{r}{r_0})^{n-1}[b_n \, cos(n\theta) + a_n \, sin(n\theta)]$$

$$b_n, a_n \approx 1...2*10^{-4}$$

CD_col := 0.0000

D_col := 0.0000 ; a1U_MQXCD_col := 0.0000 ; a1R_MQXCD_col := 0.0000 ;
D_col := 0.0000 ; a2U_MQXCD_col := 0.0000 ; a2R_MQXCD_col := 0.0000 ;
D_col := 0.0000 ; a3U_MQXCD_col := 0.8900 ; a3R_MQXCD_col := 0.8900 ;
a4M_MQXCD_col := 0.0000 ; a4U_MQXCD_col := 0.6400 ; a4R_MQXCD_col := 0.6400 ;

b5M_MQXCD_col := 0.0000 ; b5U_MQXCD_col := 0.4600 a5M_MQXCD_col := 0.0000 ; a5U_MQXCD_col := 0.4600 ; a5R_MQXCD_col := 0.4600 ;
;   a6M_MQXCD_col := 0.0000 ; a6U_MQXCD_col := 1.2700 ; a6R_MQXCD_col := 0.3300 ;
b6M_MQXCD_col := 0.0000 ; b6U_MQXCD_col := 1.7700 a7M_MQXCD_col := 0.0000 ; a7U_MQXCD_col := 0.2100 ; a7R_MQXCD_col := 0.2100 ;
;   a8M_MQXCD_col := 0.0000 ; a8U_MQXCD_col := 0.1600 ; a8R_MQXCD_col := 0.1600 ;
b7M_MQXCD_col := 0.0000 ; b7U_MQXCD_col := 0.2100   D_col := 0.0800 ;
;

b8M_MQXCD_col

Example: LHC multipole coefficients of sc. triplet quadrupoles

b9M_MQXCD_col := 0.0000 ; b9U_MQXCD_col := 0.0800 a11M_MQXCD_col := 0.0000 ; a11U_MQXCD_col := 0.0300 ; a11R_MQXCD_col := 0.0300 ;
;   a12M_MQXCD_col := 0.0000 ; a12U_MQXCD_col := 0.0200 ; a12R_MQXCD_col :=

b10M_MQX **general rule: multipole errors should be in the range of „some 10^{-4}"**
0.0600 ;
                                                     0.0100 ;
b11M_MQXCD_col := 0.0000 ; b11U_MQXCD_col := 0.0300 0.0100 ;
0.0300 ;   a14M_MQXCD_col := 0.0000 ; a14U_MQXCD_col := 0.0300 ; a14R_MQXCD_col :=
                                                     0.0100 ;
b12M_MQXCD_col := 0.0000 ; b12U_MQXCD_col := 0.020 0.0200 ;
0.0200 ;
                                                     a15M_MQXCD_col := 0.0000 ; a15U_MQXCD_col := 0.0000 ; a15R_MQXCD_col :=
b13M_MQXCD_col := 0.0000 ; b13U_MQXCD_col := 0.020 0.0000 ;

Sources of field errors

1.) power supply errors:

dipole error: **remember from lecture N° 1:**

$$B = \frac{\mu_0 \, n \, I}{h}$$

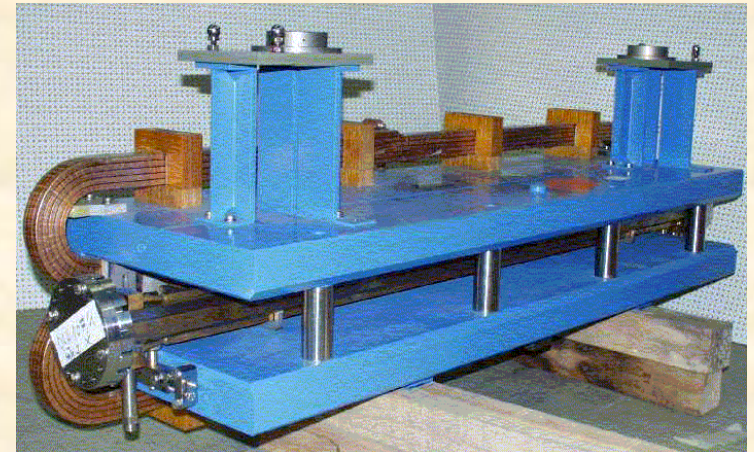**2.) error in dipole strength: the gap**

$$B = \frac{\mu_0 \, n \, I}{h}$$

Yoke production: laminations, made by stamping out of steel sheet.
variations of gap „h" by wear out of die or use of multiple dies
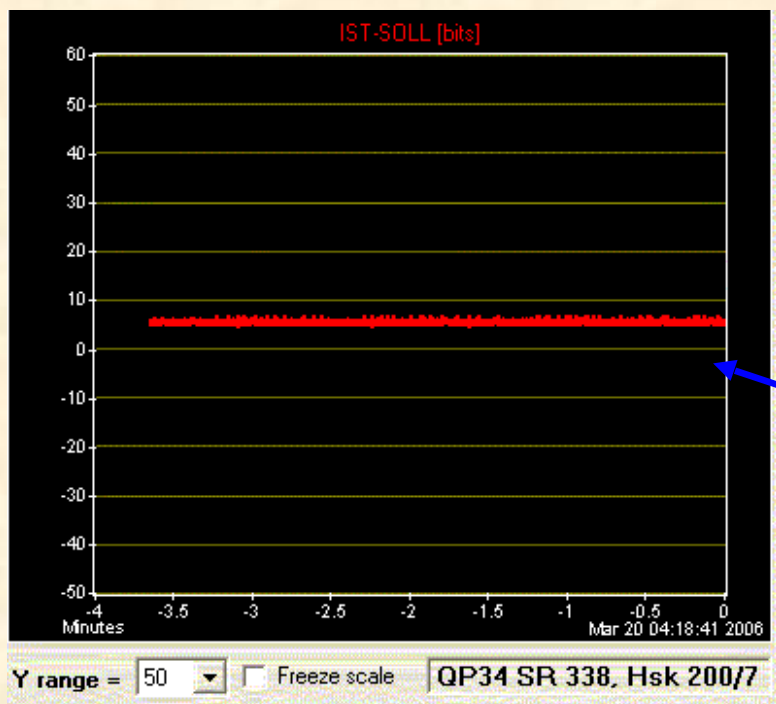
Tolerance:

$h = 5 \, cm$

$\Delta h = 25 \, \mu m$

$$\frac{\Delta B}{B} = \left| \frac{\Delta h}{h} \right| = \frac{25 \mu m}{5 cm} = 5 * 10^{-4}$$

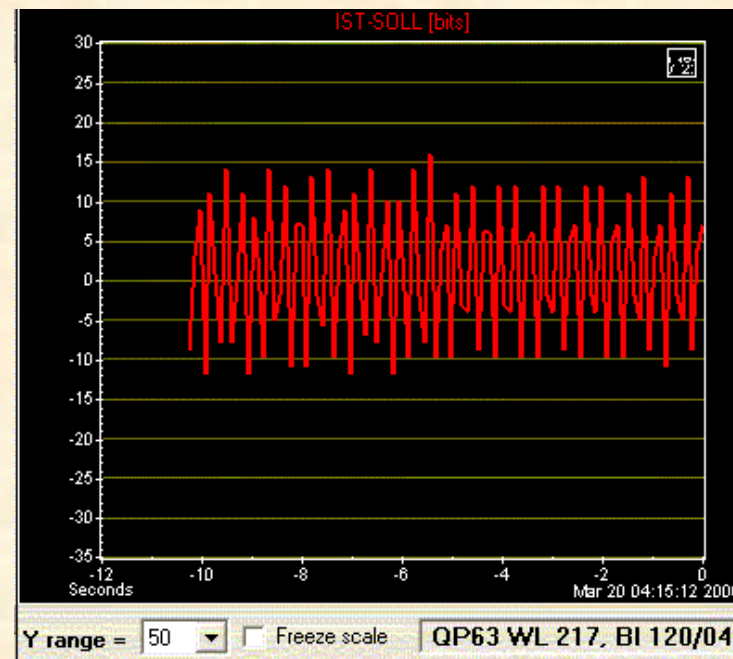# *Sources of field errors*

power supply stability:

16 bit digital electronic for current control and stabilisation

survey of power supply electronics: bit stability

$$2^{16} = 65536 \qquad \frac{1 bit}{2^{16}} \leftrightarrow 1.5 * 10^{-5}$$

require $\quad \dfrac{\Delta I}{I} \leq 5 * 10^{-5}$

$$\Delta I \approx \pm 12 \quad bit$$

$$\frac{\Delta I}{I} \approx 1 ... 2 * 10^{-4}$$

24) Dipole Magnet errors: closed orbit distortion

*The **sum of all dipole magnets** in a ring defines a curve that we call closed orbit.*
perfect situation ↔ design orbit

normalised effect on the beam:

$$\int \frac{Bdl}{B\rho} = \frac{L_0}{\rho} = \alpha = 2\pi$$

effect of single dipole magnet error:

$$\int \frac{(Bds)}{B\rho} = \int \frac{1}{\rho} ds$$

*A **dipole error** will cause a **distortion of the closed orbit**, that will „run around" the storage ring, being observable everywhere ... but – if small enough – still will lead to a **closed orbit !!***
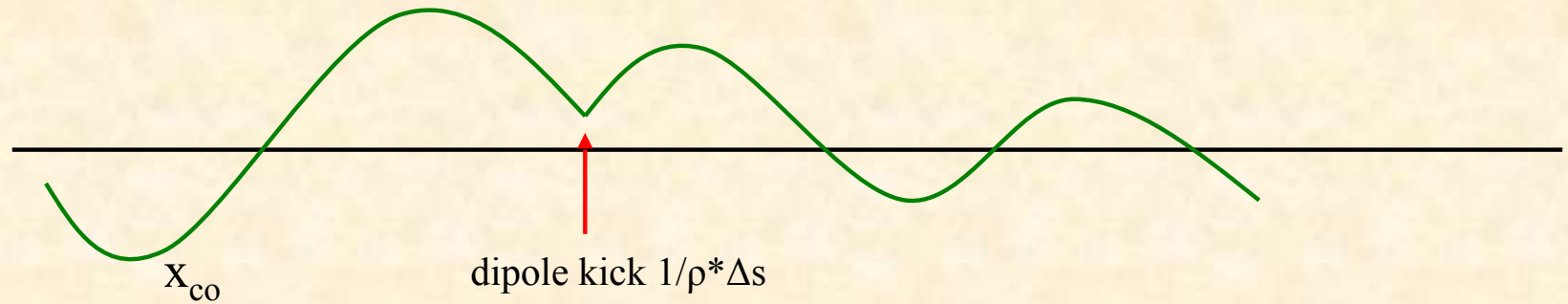
*Assume one **single dipole error** in a linac,*

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{lattice} * \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_{s0}$$

Overall amplitude of a single particle trajectory: $x = x_{co}(s) + x_\beta(s) + x_D(s)$

Calculation of Orbit Distortion in a circular machine:

x_co   dipole kick $1/\rho * \Delta s$

periodicity condition still has to be fulfilled: we still get (!) a closed orbit

in any case: distorted orbit will be a betatron oscillation.

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

a = orbit amplitude, φ = initial phase

put starting conditions:
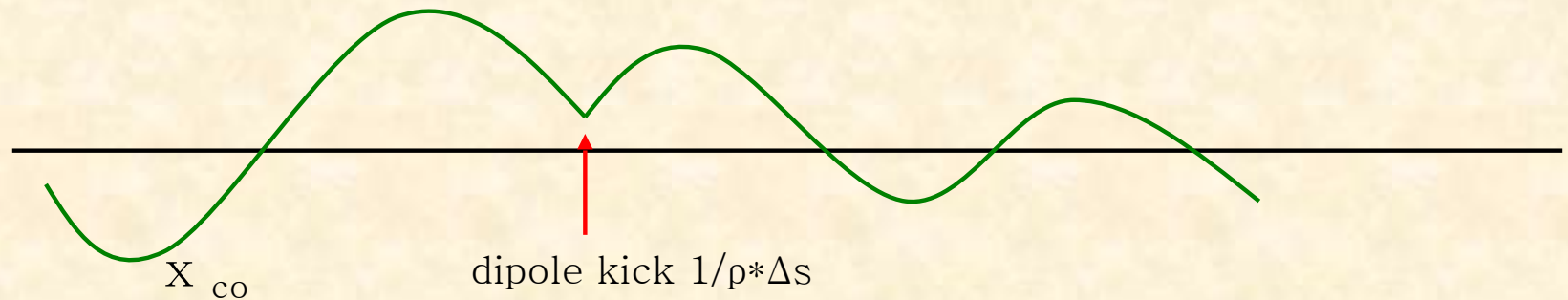
$$s = 0 \ , \quad \psi(s) = 0$$

boundary condition (1):

$$x_d(s + L) = x_d(s)$$

boundary condition (2):

$$x'_d(s + L) + \frac{\Delta s}{\rho} = x'_d(s)$$

periodic closed orbit at the place of the distortion, (s = 0, ψ = 0)

Closed Orbit Distortion

Calculation of Orbit Distortion in a circular machine:

X $_{co}$ dipole kick $1/\rho * \Delta s$

periodicity condition still has to be fulfilled: **we still get (!) a closed orbit**

in any case: distorted orbit will be a betatron oscillation.

$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi) \qquad \textit{a = orbit amplitude, } \varphi \textit{ = initial phase}$$

boundary condition (1): $x_d(s + L) = x_d(s)$ *periodic closed orbit*

$$a\sqrt{\beta(s+L)} * \cos(\psi(s) + 2\pi Q - \varphi) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

$$\varphi = \pi Q$$

Calculation of Orbit Distortion:

angle x´:
$$x_d(s) = a\sqrt{\beta(s)} * \cos(\psi(s) - \varphi)$$

$$x'_d(s) = -a\sqrt{\beta} * \sin(\psi(s) - \varphi) * \psi'(s) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

remember:
$$\psi'(s) = \frac{1}{\beta}$$

$$x'_d(s) = \frac{-a}{\sqrt{\beta}} \sin(\psi(s) - \varphi) + \frac{\beta'}{2\sqrt{\beta}} a * \cos(\psi(s) - \varphi)$$

boundary condition (2): $\quad x'_d(s+L) + \dfrac{\Delta s}{\rho} = x'_d(s)$ $\qquad$ **at the place of the distortion, s = 0, ψ = 0**

$$\frac{-a}{\sqrt{\beta(s+L)}} \sin(2\pi Q - \varphi) + \frac{\beta'(s+L)}{2\sqrt{\beta(s+L)}} a * \cos(2\pi Q - \varphi) + \frac{\Delta s}{\rho} =$$

$$= \frac{-a}{\sqrt{\beta(s)}} \sin(-\varphi) + \frac{\beta'(s)}{2\sqrt{\beta(s)}} a * \cos(-\varphi)$$

periodicity: $\quad \beta(s) = \beta(s+L), \quad \varphi = \pi Q$

$$\frac{-a}{\sqrt{\beta}}\sin(\pi Q) + \frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q) + \frac{\Delta s}{\rho} = \frac{-a}{\sqrt{\beta}}\sin(-\pi Q) + \frac{\beta'}{2\sqrt{\beta}}a*\cos(-\pi Q)$$

remember: $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$

$$\frac{-a}{\sqrt{\beta}}\sin(\pi Q) + \cancel{\frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q)} + \frac{\Delta s}{\rho} = \frac{a}{\sqrt{\beta}}\sin(\pi Q) + \cancel{\frac{\beta'}{2\sqrt{\beta}}a*\cos(\pi Q)}$$

$$\frac{\Delta s}{\rho} = \frac{2a}{\sqrt{\beta}}\sin(\pi Q) \qquad \longrightarrow \qquad a = \frac{\frac{\Delta s}{\rho}\sqrt{\beta}}{2\sin(\pi Q)}$$

put into orbit equation:

$$x_d(s) = a\sqrt{\beta(s)}*\cos(\psi(s) - \pi Q) = \frac{\delta_1*\sqrt{\beta(s)\beta_1}}{2\sin(\pi Q)}*\cos(\psi(s) - \pi Q)$$

where $\delta = \frac{\Delta s}{\rho}$
denotes the orbit kick

$$x_{co}(s) = \frac{\sqrt{\beta(s)}*\int\frac{1}{\rho_{s1}}\sqrt{\beta_{s1}}*\cos(|\psi_{s1} - \psi_s| - \pi Q)\,ds}{2\sin\pi Q}$$

Nota bene: * *orbit distortion is visible at any position „s" in the ring,*
*… even if the dipole error is located at one single point „s1".*

* *the β function describes the sensitivity of the beam to external fields*

* *the β function acts as amplification factor for the orbit amplitude at the given observation point*

* *in any case we (clearly …) will obtain a cosine-like orbit travelling around the ring … but being closed !!! after one turn.*

* *there is a resonance denominator*

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_{s}| - \pi Q)\, ds}{2\sin \pi Q}$$

closed orbit after quad offset
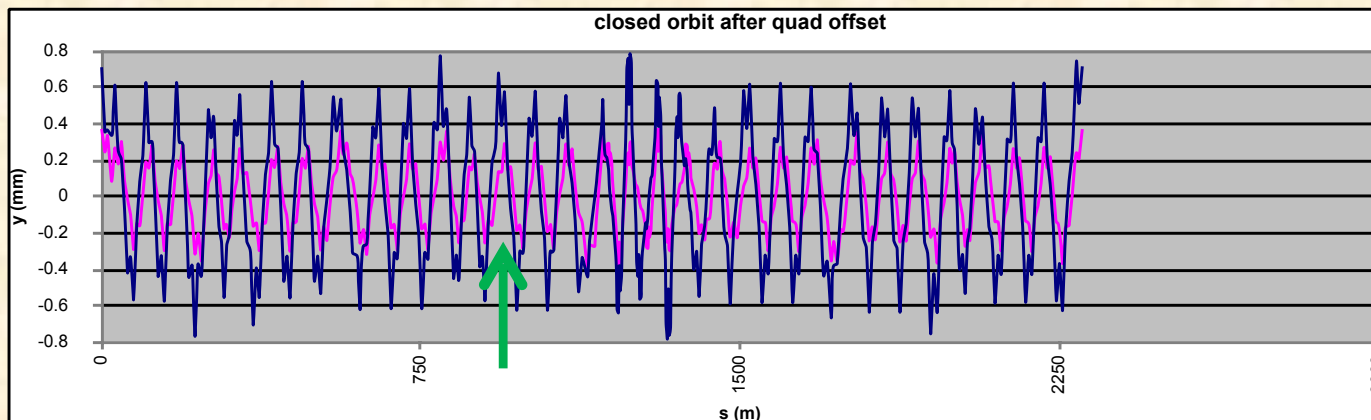
PETRA III Light Source:

closed orbit error after offset of 0.3mm in 2 quadrupole magnets
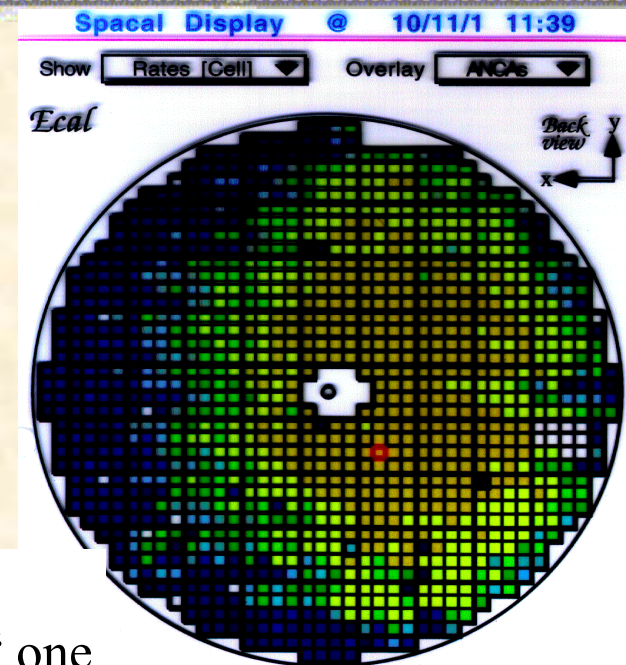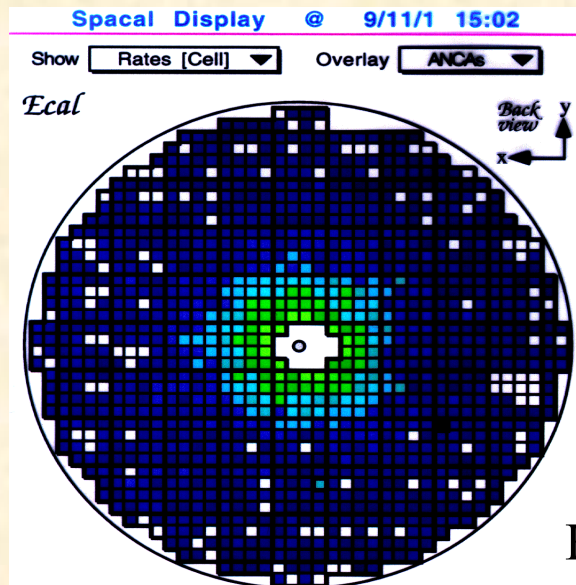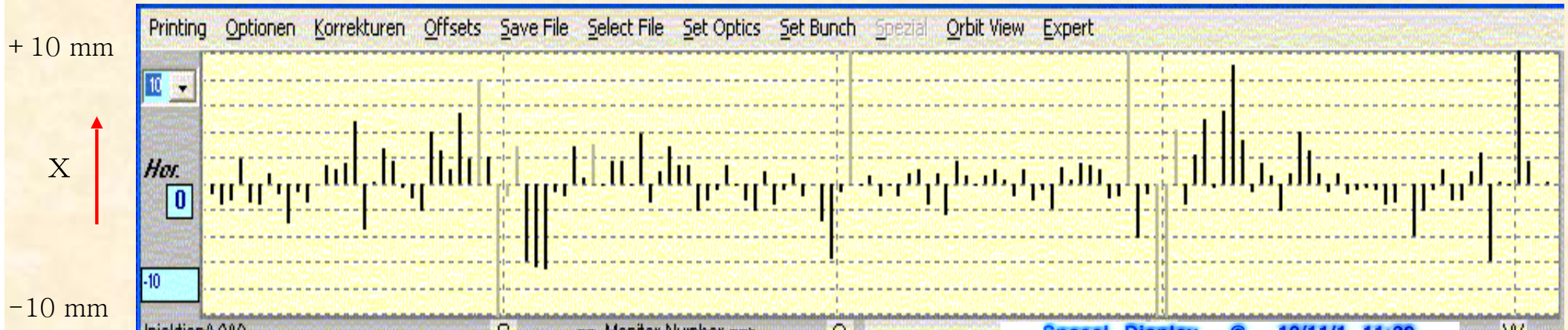
Example: „*bad orbit*", *i.e. closed orbit that contains* *large oscillation amplitudes*

→ *eats up available magnet* *aperture*

$$x(s) = x_\beta(s) + x_D(s) + x_{co}(s)$$

—> *particle trajectories pass* *nonlinear field regions*
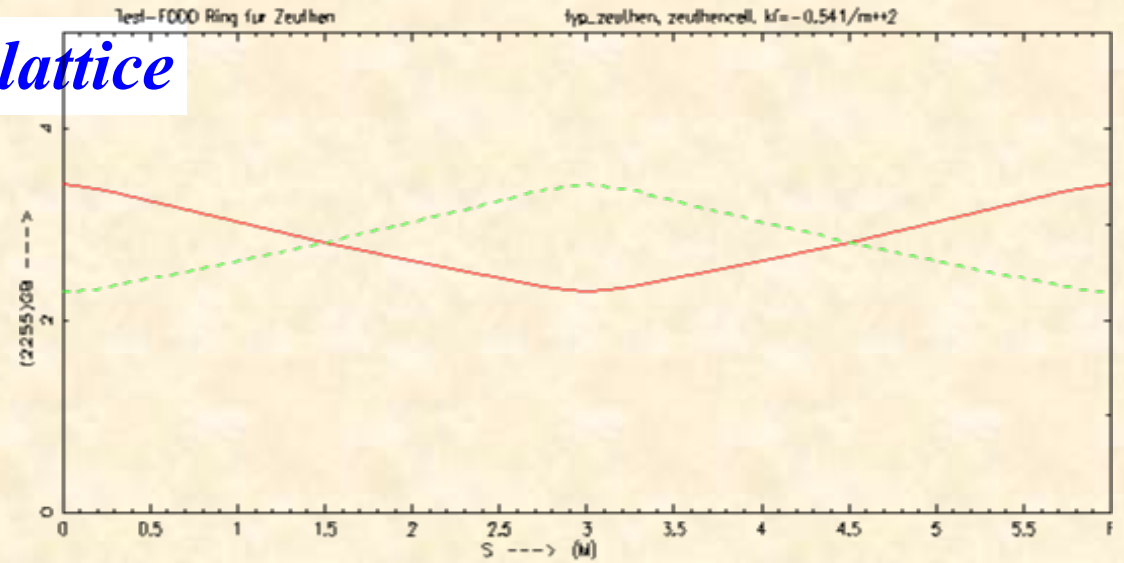—> *detector components suffer from* *beam halo particles & light*

H1 drift chamber during a good run ... or a „bad" one

Orbit distortions in a periodic lattice

field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q} \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\phi(\tilde{s}) - \phi(s)| - \pi Q) d\tilde{s}$$

* the orbit amplitude will be large if the β function at the location of the kick is large β (s̃) *indicates the sensitivity of the beam —> here orbit correctors should be placed in the lattice*

* the orbit amplitude will be large at places where in the lattice β(s) is large —> here beam position monitors should be installed

25.) Finally: .... Resonances

closed orbit distortion:

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\,ds}{2\sin\pi Q}$$

remember from lecture 1:

μ = phase advance per revolution
in general measured and expressed in
units of 2π … and called „Tune" Q

$$Q = \frac{\mu}{2\pi}$$

… and it depends on the focusing strength of the lattice cells.
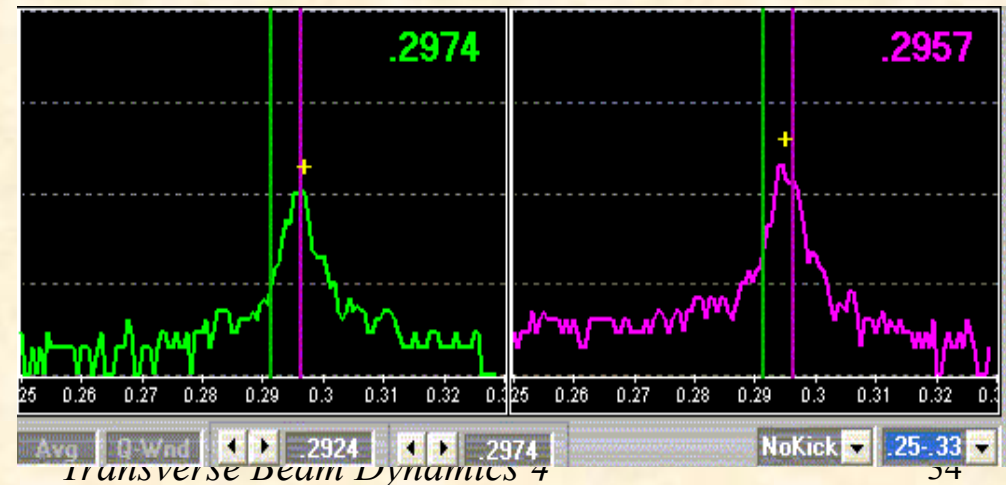
Tune: number of oscillations per turn

31.292
32.297

Relevant for beam stability:
non integer part

HERA revolution frequency: 47.3 kHz

$0.292 * 47.3 \ kHz = 13.81 \ kHz$

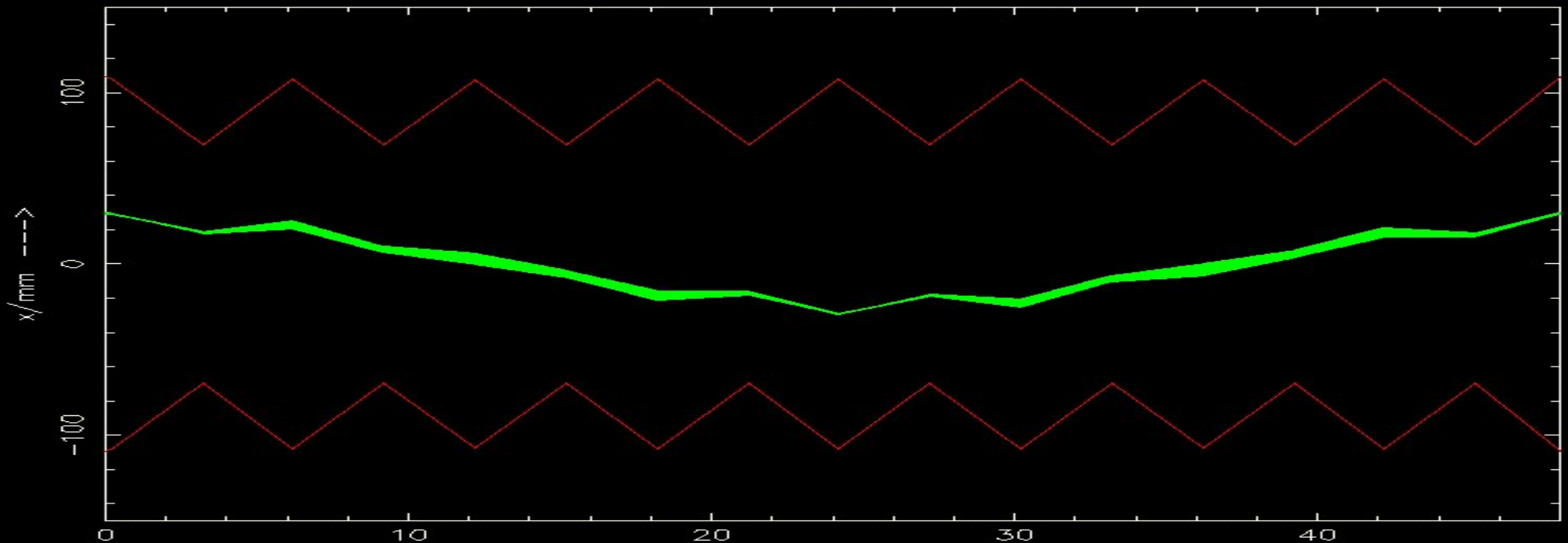permanent tune measurement …and control
in both planes

Resonances

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(\psi_{s1} - \psi_s - \pi Q)\, ds}{2\sin\pi Q}$$

Assume: Tune = integer $Q = 1 \quad \rightarrow \quad 2\sin\frac{\mu}{2} = 2\sin\pi = 0$

Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.

Qualitatively spoken:

Teilchenbahnen und Enveloppe

Tune & Resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in

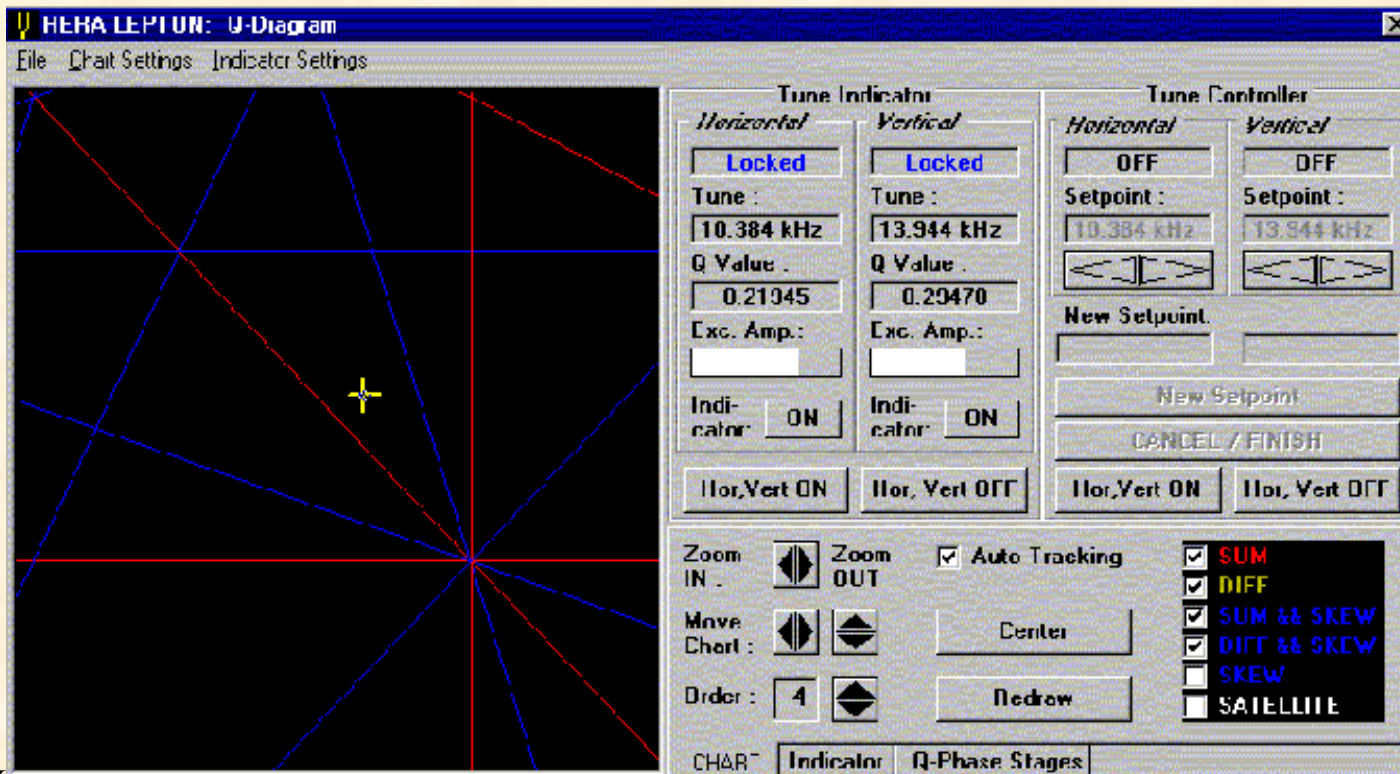*case of resonant tunes to infinite high amplitudes.*
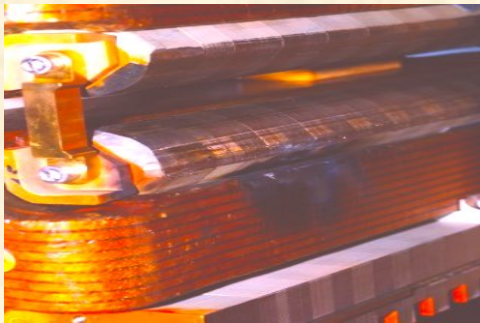*—> particle loss within a short number of turns.*

—> avoid large magnet errors
—> avoid forbidden tune values in both planes
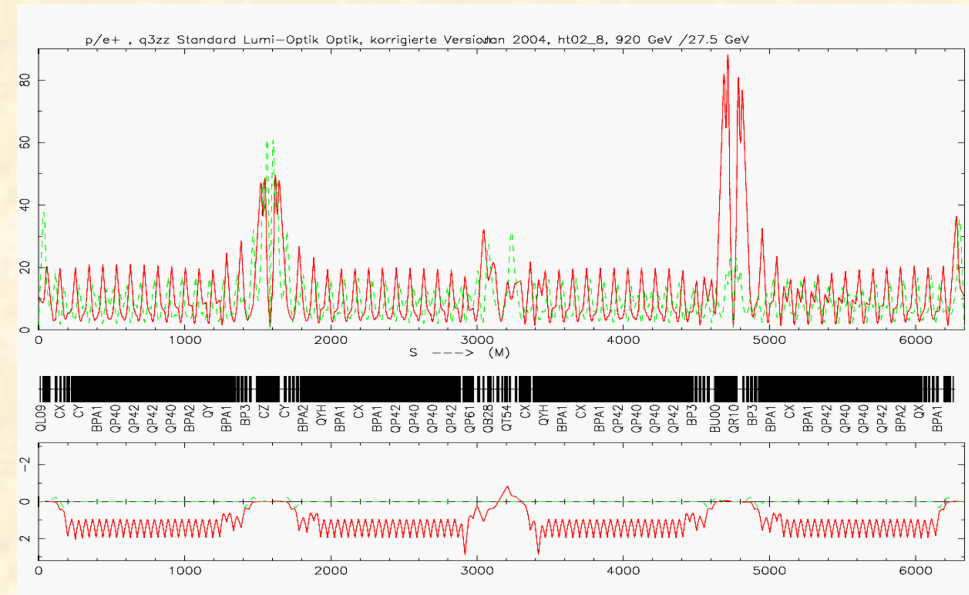
$$m * Q_X + n * Q_Y = p$$ *n,m,p = integer numbers*

26.) Quadrupole Errors:

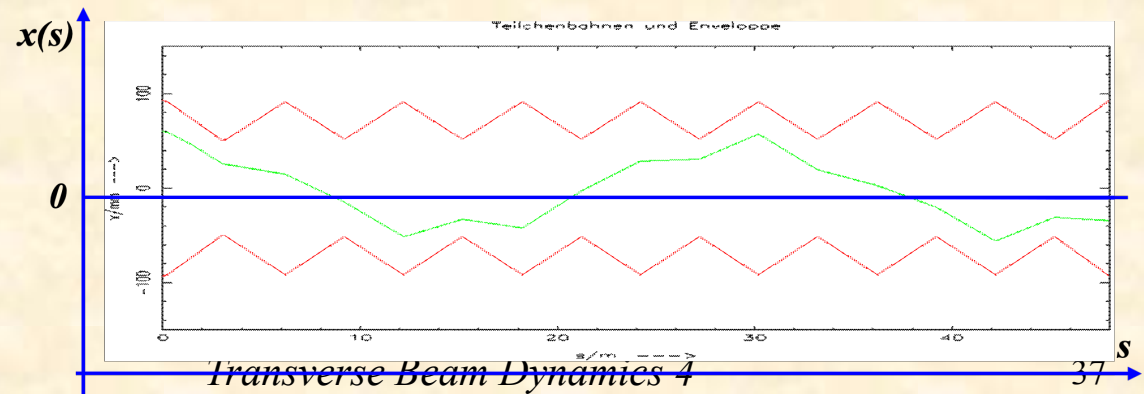**go back to Lecture I, page 1**

single particle trajectory

Solution of equation of motion

$$x = x_0 * \cos(\sqrt{k*l}) + x_0' * \frac{1}{\sqrt{k}}\sin(\sqrt{k*l})$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad M_{QF} = \begin{pmatrix} \cos(\sqrt{k}*l) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}*l) \\ -\sqrt{k}\sin(\sqrt{k}*l) & \cos(\sqrt{k}*l) \end{pmatrix}$$

Definition: *phase advance*
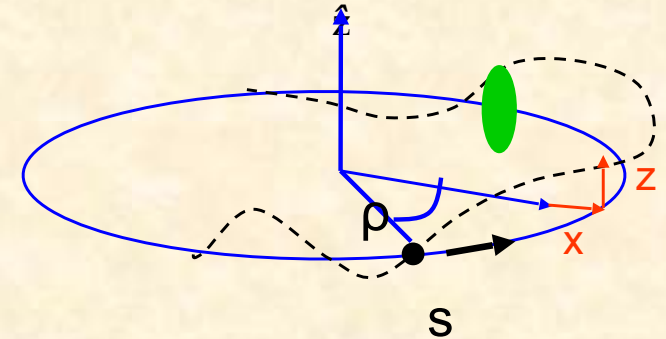***of the particle oscillation***
per revolution in units of 2π
***is called* tune**

$$Q = \frac{\Delta\psi_{turn}}{2\pi} = \frac{\mu}{2\pi}$$

B. J. Holzer, CERN JUAS 2023 *Transverse Beam Dynamics 4* 37

Quadrupole Error in the Lattice

optics perturbation described by thin lens quadrupole

$$M(s) = \begin{pmatrix} cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$M_{dist} = M_{\Delta k} \cdot M_0 = \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} \cdot \begin{pmatrix} cos\psi_{turn} + \alpha sin\psi_{turn} & \beta sin\psi_{turn} \\ -\gamma sin\psi_{turn} & cos\psi_{turn} - \alpha sin\psi_{turn} \end{pmatrix}$$

$$\underbrace{}_{\textit{quad}} \qquad \underbrace{}_{\textit{ideal storage}}$$

$$M_{dist} = \begin{pmatrix} cos\psi_0 + \alpha \cdot sin\psi_0 & \beta \cdot sin\psi_0 \\ -\Delta k ds \cdot (cos\psi_0 + \alpha sin\psi_0) - \gamma \cdot sin\psi_0 & -\Delta k ds \cdot \beta sin\psi_0 + cos\psi_0 - \alpha sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2cos\psi = 2cos\psi_0 - \Delta k ds \beta \sin\psi_0$$

Quadrupole error —> Tune Shift

$$\psi = \psi_0 + \Delta\psi \qquad \longrightarrow \qquad cos\psi = cos(\psi_0 + \Delta\psi) = cos\psi_0 - \frac{\Delta k \ ds \ \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$cos\psi_0 \cdot cos\Delta\psi) - sin\psi_0 \cdot sin\Delta\psi) = cos\psi_0 - \frac{\Delta k ds \beta \sin\psi_0}{2}$$

$$\underbrace{\qquad}_{\approx 1} \qquad \underbrace{\qquad}_{\approx \Delta\psi}$$

$$\Delta\psi = \frac{kds \ \beta}{2}$$

and referring to Q instead of ψ:

!   **the tune shift is proportional to the β-function at the quadrupole**

!!   **field quality, power supply tolerances etc are much tighter at places where β is large**

$$\psi = 2\pi Q$$

!!!   **mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m**

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}$$

!!!!   **β is a measure for the sensitivity of the beam**

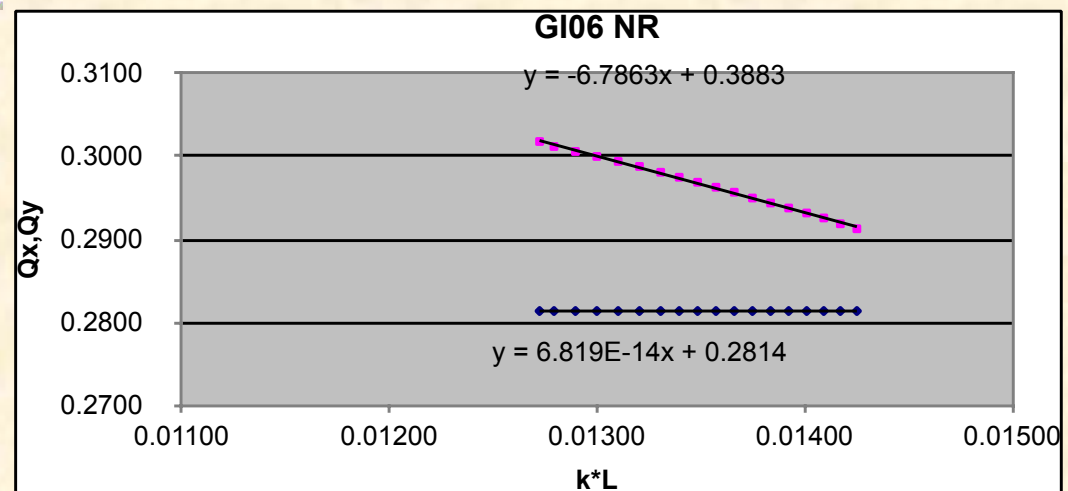**Example:** *deliberate change of quadrupole strength in a synchrotron:*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta K(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta K(s) * l_{quad} * \overline{\beta}}{4\pi}$$
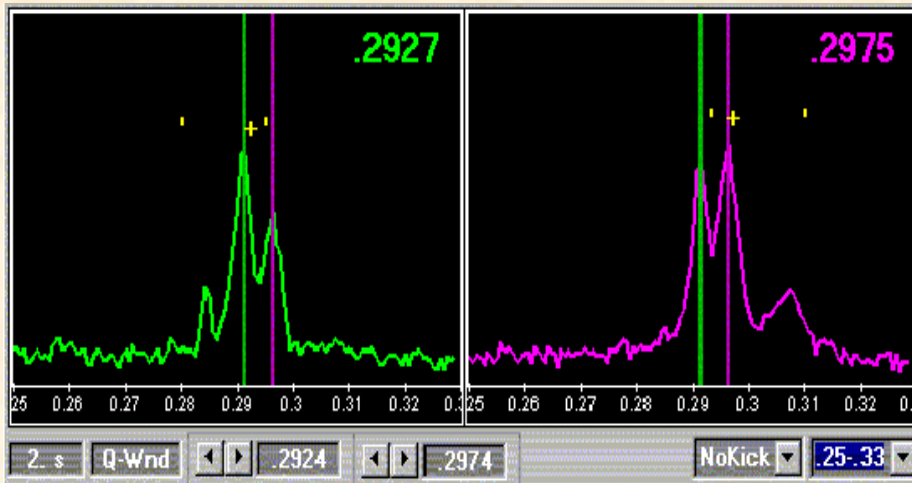
tune spectrum ...

... for heaven's sake:

   why do we get three peaks ????

tune shift as a function gradient change

Clearly there is another problem:

       ***a focussing error* at any location in the machine**

           ... will shift the tune

           ... and *distort the optics

           ... at any place in the ring

Example GA quadrupole:
      burned quadrupole coil

Quadrupole error —> Beta Beat

$$\Delta\beta(s_0) = -\frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1}-\psi_{s0})-2\pi Q\right)ds$$

$\sqrt{\beta}$

orbit is not affected to first order !

Many small quadrupole errors (gradient tolerances) add up

Tolerance limit: $\dfrac{\Delta\beta}{\beta} < 20\,\%$

Quadrupole error —> Beta Beat

A series of quadrupole errors Δk_i cause distortion of the β-function at s,

$$\frac{\Delta\beta}{\beta}(s) = \frac{1}{2\sin 2\pi Q}\sum_i \beta_i \Delta k_i \cos\left(2\pi Q - 2(\mu_i - \mu_s)\right) \qquad (132)$$

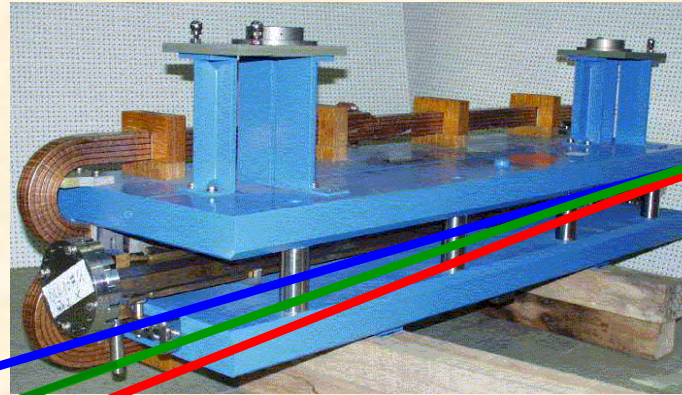Unstable motion if Q is a half integer!

LHCB1 4TeV β^*=0.6m
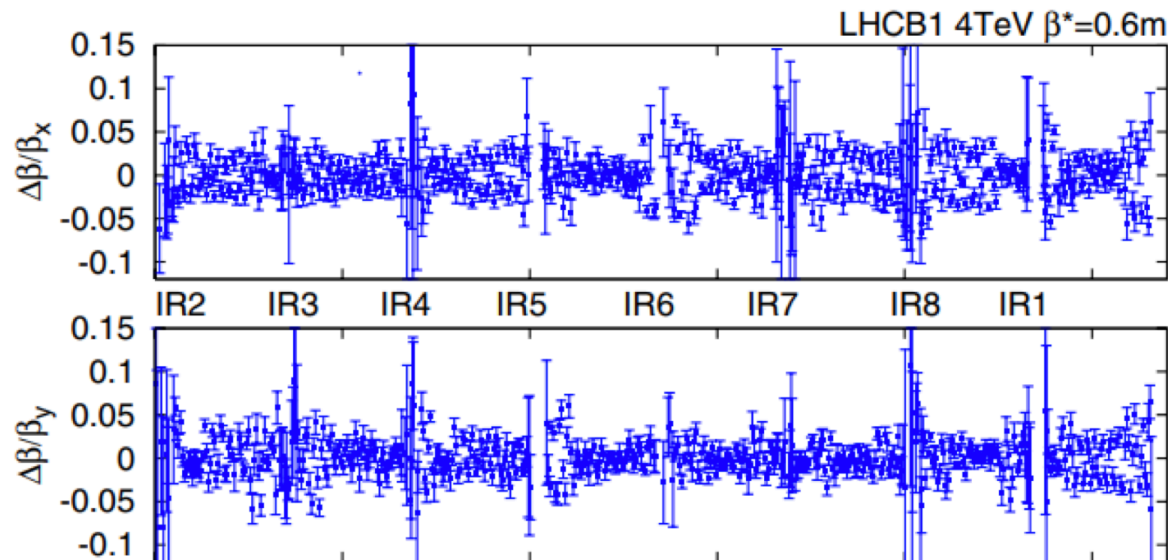
27.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*
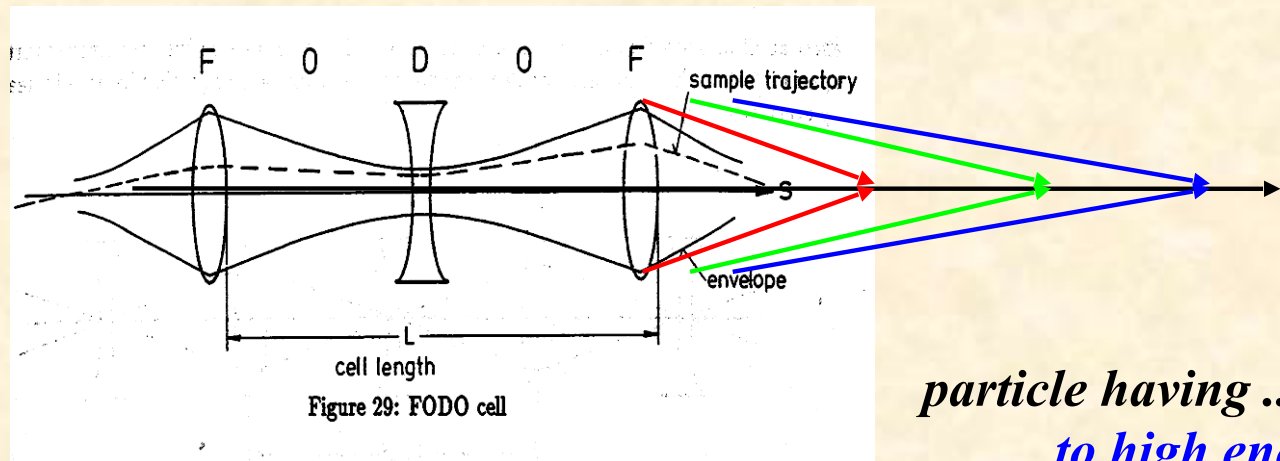
dipole magnet
$$\alpha = \frac{\int B \, dl}{p/e}$$

$$x_D(s) = D(s)\frac{\Delta p}{p}$$

focusing lens
$$k = \frac{g}{p/e}$$

Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q' *(... sometimes aka ... "ξ")*

$$k = \frac{g}{p/e} \qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})\, g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which **acts like a quadrupole error** *in the machine and* **leads to a tune spread:**

$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s)\, ds$$

definition of chromaticity:

$$\Delta Q = Q'\ \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)\, ds$$

Where is the Problem ?

... *what is wrong about Chromaticity:*

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
  —> it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi} \oint k(s)\beta(s)ds$$

k = quadrupole strength
β = betafunction indicates the beam size ... and even more the sensitivity of
*   the beam to external fields*

Example: LHC

Q' = 250
Δ p/p = +/- 0.2 *10⁻³
Δ Q = 0.256 ... 0.36

→*Some particles get very close to*
   resonances and are lost

in other words: the tune is not a point
                    it is a pancake

Measurement of Betatron Tune: Qx, Qy

▶ Remember that the tune is related to the amount of transverse oscillations around the ring.

▶ To extract the betatron tune we have to perform a frequency analysis of the transverse oscillations.

$$x(n) = \sum_{j=1}^{N} \psi(Q_j) \exp(2\pi i n Q_i) \tag{133}$$

▶ Where the coefficients $\psi(Q_i)$ are related to the amplitude of the different oscillation frequencies Q_i.

▶ Applying FFT to $x(n)$ we can obtain (ψ_i, Q_i).

▶ In principle, the frequency with largest amplitude is associated to the tune.

Betatron Tune: Qx, Qy

… to improve the signal strength we can - carefully - excite the beam with an oscillating dipole corrector following a frequency sweep.
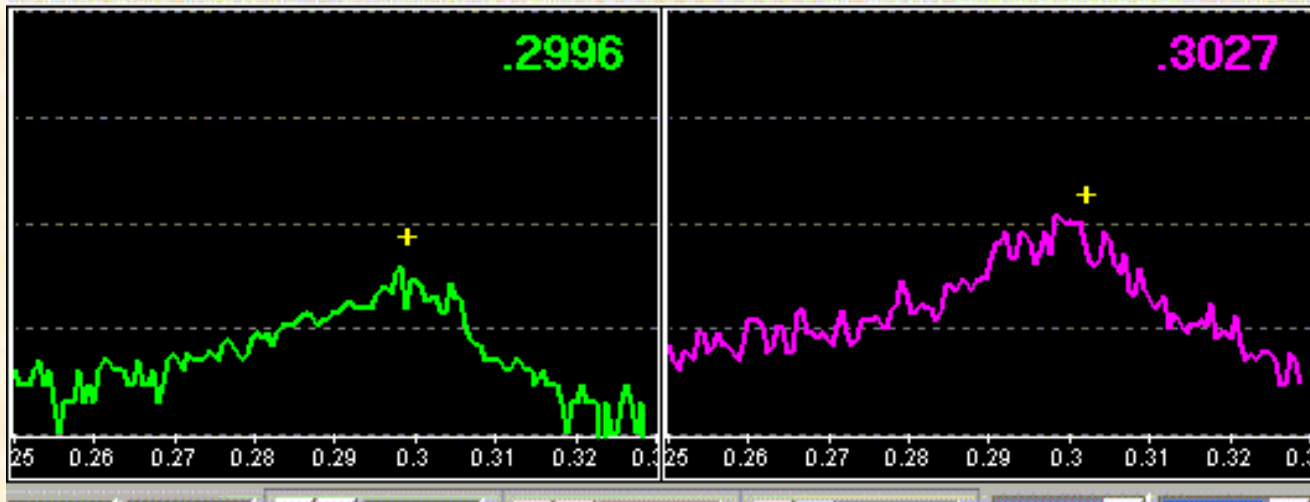
—> the amplitude response will be highest at the tune frequency
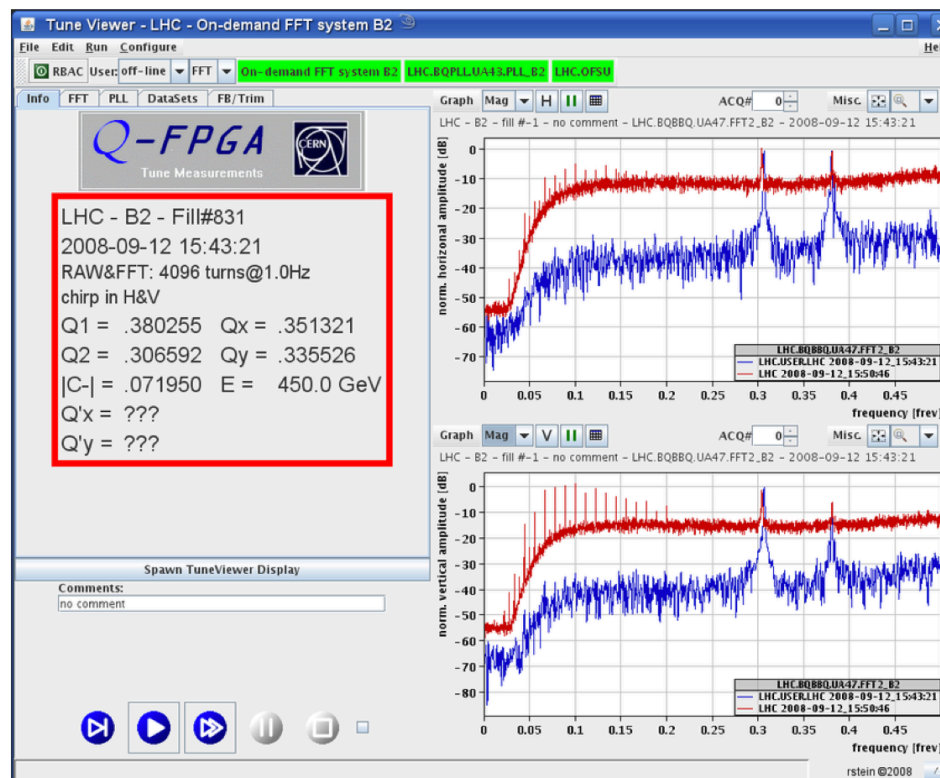
… and as the beam is oscillating in the whole ring, we can pick up the signal at any BPM.
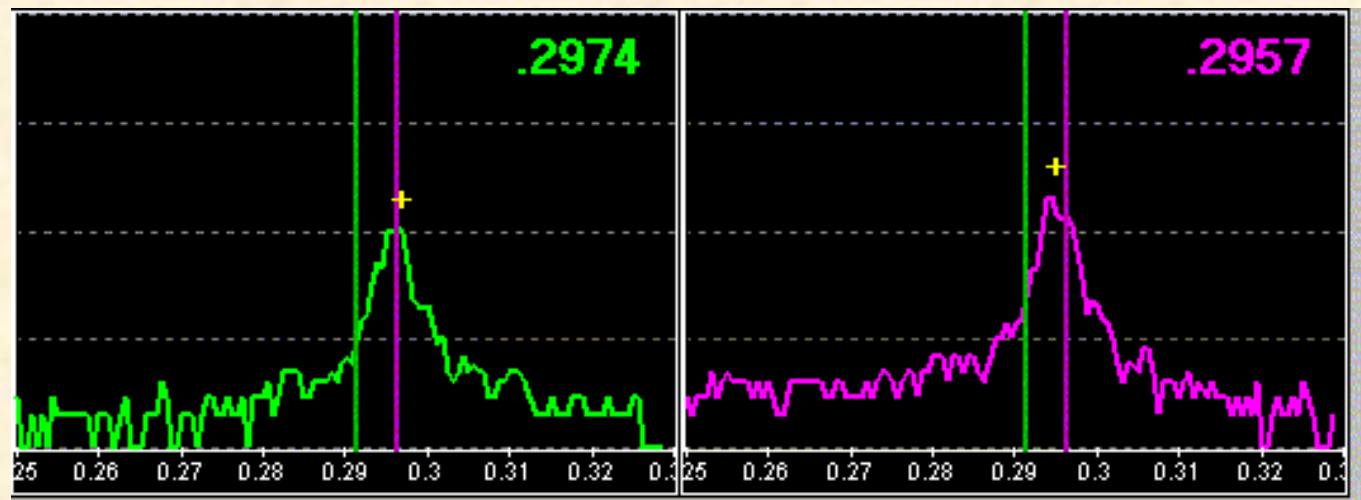
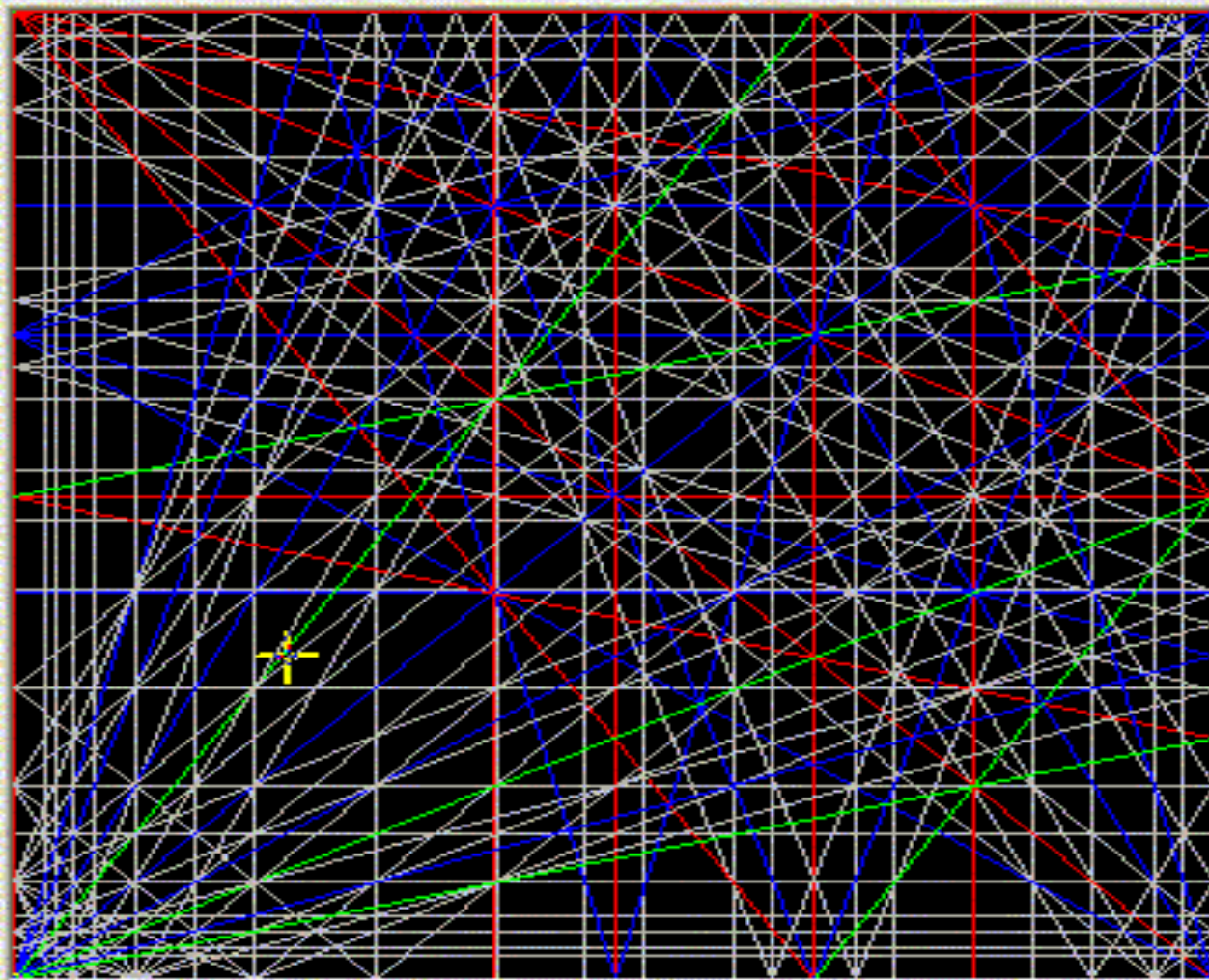# *Betatron Tune: Qx, Qy*

Effect of Chromaticity

Tune signal for a nearly uncompensated cromaticity
(Q' ≈ 20)

Ideal situation:
cromaticity well corrected,
(Q' ≈ 1)

Once more: Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$


HERA e Tune diagram up to 3rd order

… and up to 7th order

Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

Correction of Q':

Need: additional quadrupole strength for each momentum deviation Δp/p

Sextupoles, through a non-linear magnetic field, correct the effect of energy spread and focuses particles at a single location.
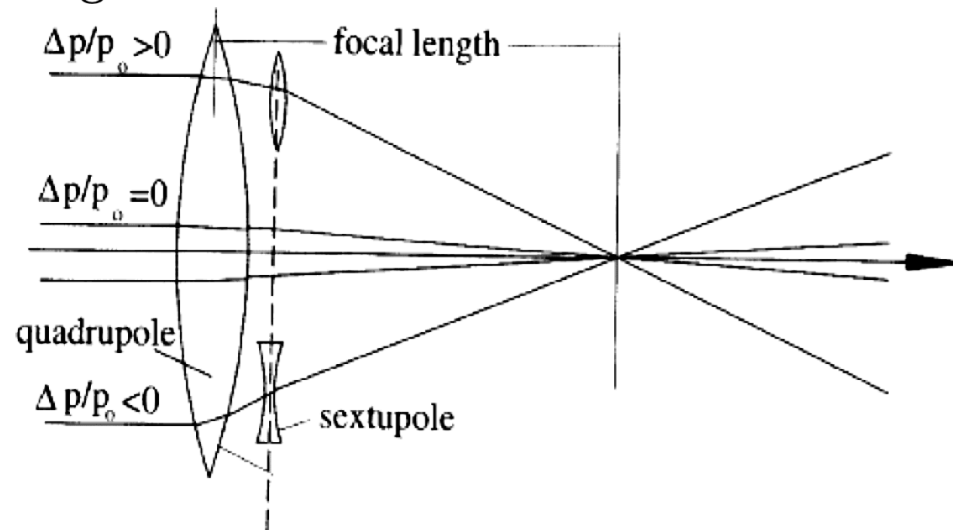
- ► Located in dispersive regions.
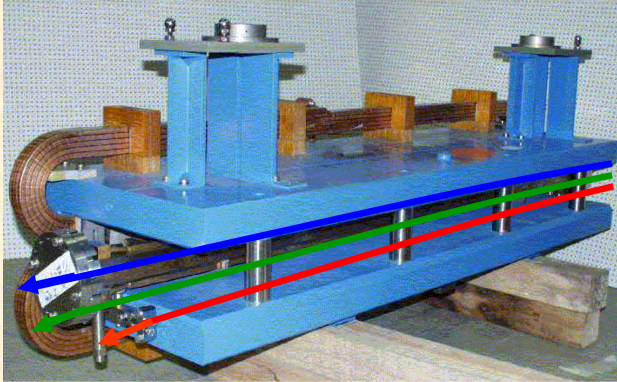- ► Usually in arcs.
- ► Sextupole families.

Now is when the party starts

- ► Sextupoles introduce non-linear fields.
- ► ...i.e. they induce non-linear motion.
- ► resonances, tune shifts, chaotic motion.

Correction of Q':

Need: additional quadrupole strength for each momentum deviation $\Delta p/p$

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$

… using the dispersion function
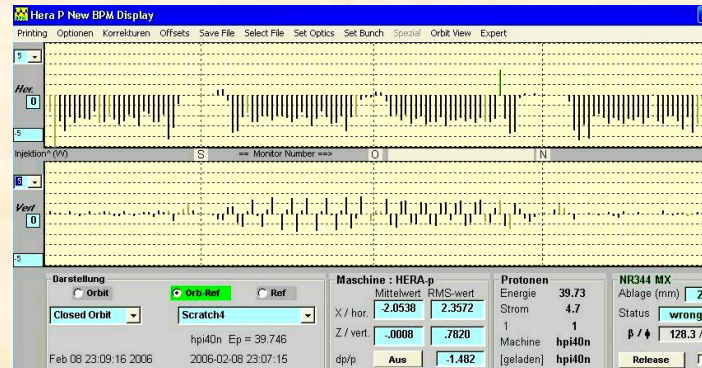
2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$B_x = \tilde{g}xz$$

$$B_z = \frac{1}{2}\tilde{g}(x^2 - z^2)$$

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient":

Correction of Q':

Sextupole Magnets:
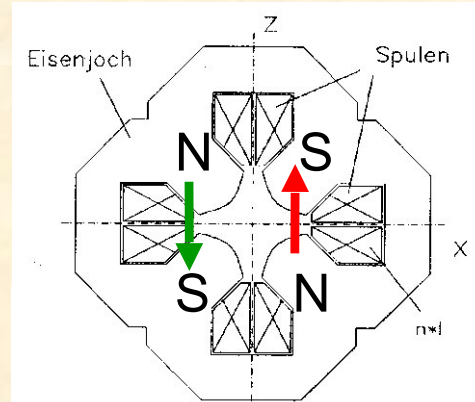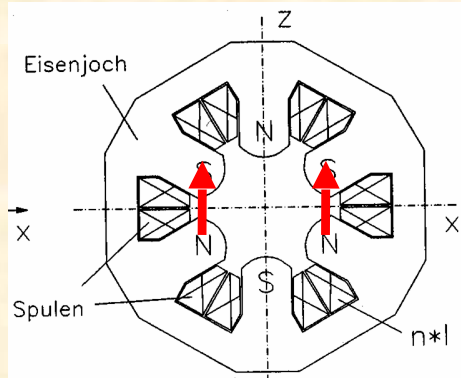

k_1 *normalised quadrupole strength*

k_2 *normalised sextupole strength*

$$k_1(sext) = \frac{\tilde{g} \, x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$
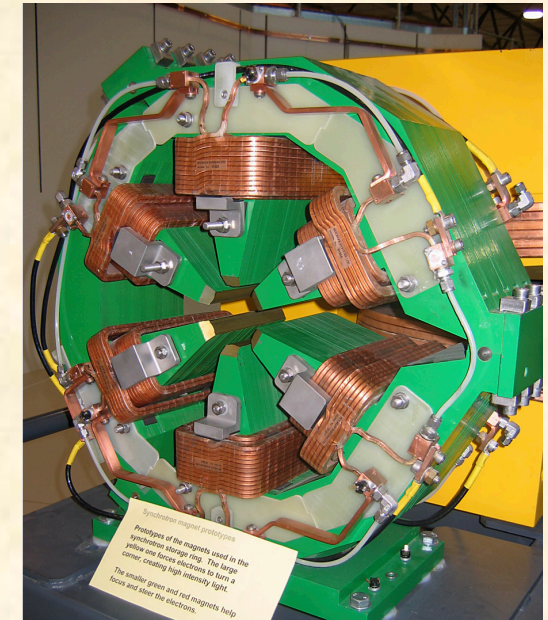
corrected chromaticity

counter acting effect in the two planes

$$Q'_x = -\frac{1}{4\pi} \oint \beta_x(s) \, [+k_q(s) - S_F D_x(s) + S_D D_x(s)] \, ds$$

$$Q'_y = -\frac{1}{4\pi} \oint \beta_y(s) \, [-k_q(s) + S_F D_x(s) - S_D D_x(s)] \, ds \,, \qquad \text{with} \quad S_F = k_2^F \cdot l_{sext} \,, \quad S_D = k_2^D \cdot l_{sext}$$

"natural" chromaticity   *sextupole correction of chromaticity*

Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\!\left(2(\psi_{s1}-\psi_{s0}) - 2\pi Q\right) ds$$

chromaticity $\Delta Q = Q' * \dfrac{\Delta p}{p}$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)\,ds$$

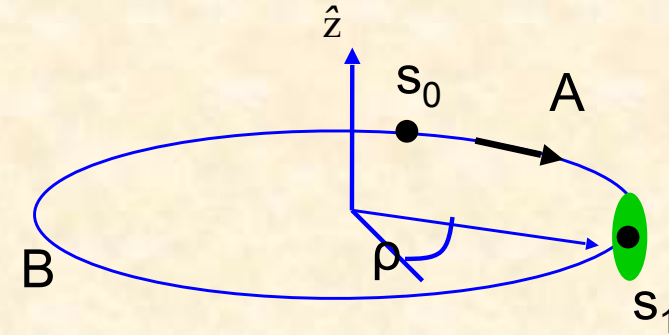*in a FoDo* $Q'_{cell} = -\dfrac{1}{\pi}\tan\dfrac{\mu}{2}$

corrected chromaticty

$$Q'_x = \frac{-1}{4\pi} * \oint k_1(s)\beta(s)\,ds + \frac{1}{4\pi}\sum_{F\,sext} k_2^F\, l_{sext}\, D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D\, l_{sext}\, D_x^D \beta_x^D$$

Appendix: Quadrupole Errors and Beta Function

**a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"**

$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

distorted matrix $\quad M_{dist} = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} A$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta k ds\, a_{11} + a_{12} & -\Delta k ds\, a_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11} a_{12} + b_{12}(-\Delta k ds\, a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12", which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^* = \underbrace{b_{11}a_{12} + b_{12}a_{22}} - b_{12}a_{12}\Delta k ds$$

$$m_{12} = \beta_0 \sin 2\pi Q$$

(1) $$m_{12}^* = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M* is still a matrix for one complete turn we still can express the element m_{12} in twiss form:

(2) $$m_{12}^* = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}$$

$$\approx 1 \qquad\qquad \approx 2\pi dQ$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = \beta_0 \sin 2\pi Q + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + d\beta_0 2\pi dQ \cos 2\pi Q$$

ignoring second order terms

$$- a_{12}b_{12}\Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $\quad dQ = \dfrac{\Delta k \beta_1 ds}{4\pi}$

(index „1" refers to location of the error)

$$- a_{12}b_{12}\Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{ 2 a_{12}b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\}\Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\[4mm] \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1\cos 2\pi Q\right\}\Delta k ds$$

$$a_{12} = \sqrt{\beta_0\beta_1}\,\sin\Delta\psi_{0\to1}$$

$$b_{12} = \sqrt{\beta_0\beta_1}\,\sin(2\pi Q - \Delta\psi_{0\to1})$$

$$\boldsymbol{d}\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi\boldsymbol{Q}}\left\{2\sin\Delta\psi_{01}\sin(2\pi\boldsymbol{Q} - \Delta\psi_{01}) + \cos 2\pi\boldsymbol{Q}\right\}\Delta\boldsymbol{k}\boldsymbol{ds}$$

... after some TLC transformations .. $= \cos(2\Delta\psi_{01} - 2\pi Q)$

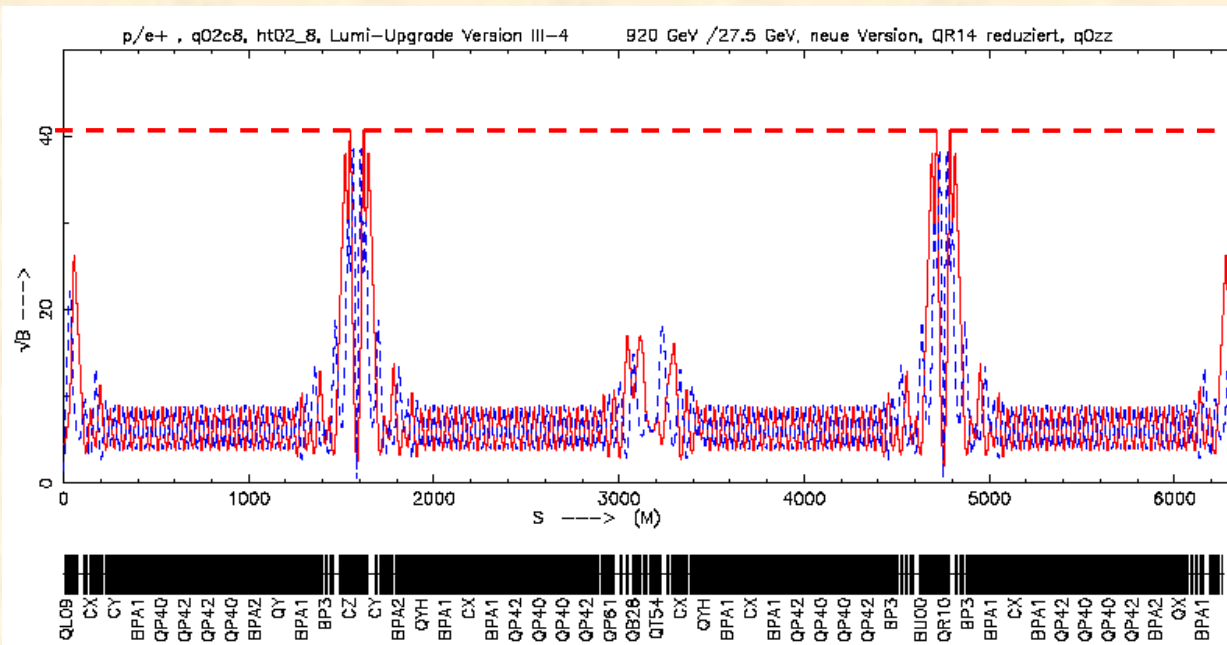$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\big(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\big)ds$$

Nota bene: **!**   the beta beat is *proportional to the strength of the error Δk*

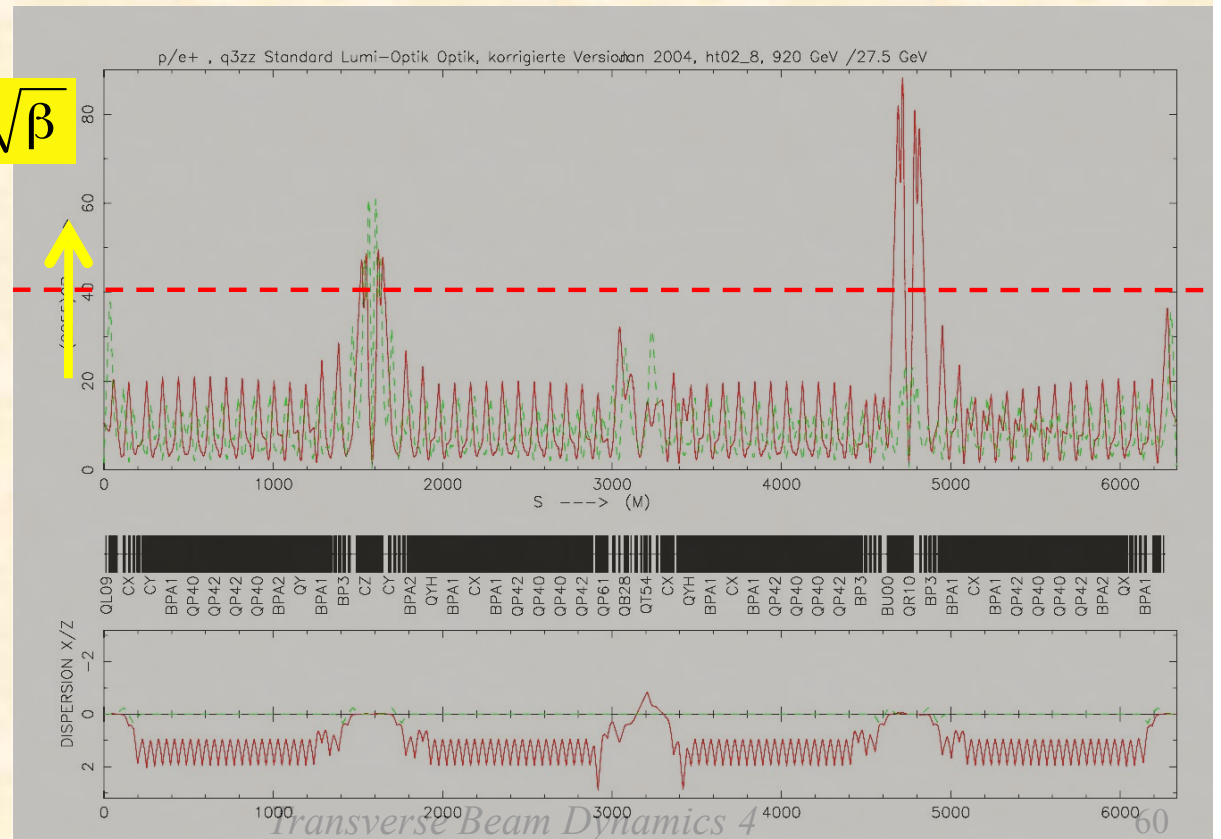**!!** and to the *β function* at the place of the error ,

!!! and to the *β function* at the observation point,
     (... remember orbit distortion !!!)

!!!! there is a *resonance denominator*

ideal i.e. unperturbed beam optics for luminosity operation

$\sqrt{\beta}$

perturbed bam optics due to broken quadrupole winding

B. J. Holzer, CERN   JUAS 2023

Appendix: Dispersion

Solution of the inhomogenious equation of motion

Ansatz:
$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C \, dt + S \frac{1}{\rho} C - C' * \int \frac{1}{\rho} S \, dt - C \frac{1}{\rho} S$$

$$D'(s) = S' * \int \frac{C}{\rho} \, dt \; - \; C' * \int \frac{S}{\rho} \, dt$$

$$D''(s) = S'' * \int \frac{C}{\rho} \, d\tilde{s} + S' \frac{C}{\rho} \; - \; C'' * \int \frac{S}{\rho} \, d\tilde{s} \; - \; C' \frac{S}{\rho}$$

$$= S'' * \int \frac{C}{\rho} \, d\tilde{s} - C'' * \int \frac{S}{\rho} \, d\tilde{s} + \frac{1}{\rho} \underbrace{(CS' \quad - S \, C')}_{= det \, M = 1}$$

----

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent
of the variable „s"

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital
conditions that we had chosen …

$$C_0 = 1, \quad C_0' = 0$$
$$S_0 = 0, \quad S_0' = 1$$

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:
$$S'' + K * S = 0$$
$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\} + \frac{1}{\rho}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{=D(s)}$$

$$D'' = -K * D + \frac{1}{\rho} \qquad \textbf{… or} \qquad D'' + K * D = \frac{1}{\rho} \qquad \textbf{qed}$$

Dispersion

the dispersion function D(s) is (...obviously) defined by the focusing properties of the lattice and is given by:

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

> *! weak dipoles —> large bending radius —> small dispersion*

Example: Drift

$$M_D = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \qquad D(s) = S(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} C(\tilde{s}) d\tilde{s} - C(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} S(\tilde{s}) d\tilde{s}$$

$$\rightarrow M_D = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...in similar way for quadrupole matrices,
!!! in a quite different way for dipole matrix (see appendix)