Problem 1 solution

Derive an expression for the resulting magnetic field components (Bx and By) when the closed orbit in a normal sextupole is horizontally displaced by -δx from its reference position. Do the same for an octupole.

The field generated by a sextupole is

$$B_y(x,y) = B_2(x^2 - y^2)$$
$$B_x(x,y) = B_2(2xy)$$

For a **horizontally displaced closed orbit in** a sextupole:

$$x \mapsto x + \delta x$$

$$B_y(x+\delta x, y) = B_2\left((x+\delta x)^2 - y^2\right) = B_2 x^2 + 2B_2 x\delta x + B_2(\delta x)^2 - B_2 y^2 =$$
$$= \underbrace{B_2(x^2 - y^2)}_{\text{sextupole}} + \underbrace{2B_2(\delta x)x}_{\text{quadrupole}} + \underbrace{B_2(\delta x)^2}_{\text{dipole}}$$

$$B_x(x+\delta x, y) = \overbrace{B_2(2xy)}^{\text{sextupole}} + \overbrace{2B_2(\delta x)y}^{\text{quadrupole}}$$

Problem 1 solution

The field generated by an octupole is:

$$B_y(x,y) = B_3(x^3 - 3xy^2)$$
$$B_x(x,y) = B_3(-y^3 + 3x^2y)$$

- For a horizontally displaced closed orbit in an octupole:

$$x \mapsto x + \delta x$$

$$\begin{aligned} B_y(x+\delta x, y) &= B_3\left(x^3 + 2(\delta x)x^2 + 3(\delta x)^2 x + (\delta x)^3 - 3xy^2 - 3(\delta x)y^2\right) = \\ &= B_3(x^3 - 3xy^2) + 3B_3(\delta x)(x^2 - y^2) + 3B_3(\delta x)^2 x + B_3(\delta x)^3 \\ B_x(x+\delta x, y) &= B_3\left(-y^3 + 3(x^2 + 2(\delta x)x + (\delta x)^2)y\right) = \\ &= B_3(-y^3 + 3x^2y) + 3B_3(\delta x)2xy + 3B_3(\delta x)^2 y \end{aligned}$$

- For direct comparison:

$$B_y(x+\delta x, y) = \underbrace{B_3(x^3 - 3xy^2)}_{\text{octupole}} + \underbrace{3B_3(\delta x)(x^2 - y^2)}_{\text{sextupole}} + \underbrace{3B_3(\delta x)^2 x}_{\text{quadrupole}} + \underbrace{B_3(\delta x)^3}_{\text{dipole}}$$

$$B_x(x+\delta x, y) = \overbrace{B_3(-y^3 + 3x^2y)}^{\text{octupole}} + \overbrace{3B_3(\delta x)2xy}^{\text{sextupole}} + \overbrace{3B_3(\delta x)^2 y}^{\text{quadrupole}}$$

Problem 2 solution

SNS: A **proton** ring with kinetic energy of 1 GeV and a **circumference of 248 m** has **18, 1 m-long** focusing quads with **gradient of 5 T/m**. In one of the quads, the horizontal and vertical **beta function** are **12 m** and **2 m** respectively. The **rms beta** function in both planes on the focusing quads is **8 m**. With a horizontal tune of **6.23** and a vertical of **6.2**, compute the expected horizontal and vertical orbit distortions on a single focusing quad given by **horizontal** and **vertical misalignments of 1 mm in all the quads**. What happens to the horizontal and vertical orbit distortions if the horizontal tune drops to **6.1** and **6.01**?

- The rms orbit distortion is given by $u_{\text{rms}}(s) = \frac{\sqrt{N\beta(s)\beta_{\text{rms}}}}{2\sqrt{2}|\sin(\pi Q)|}\theta_{\text{rms}}$
- We need to determine the rms kick angle, which in the case of a quadrupole displacement is given by

$$\theta_{\text{rms}} = \frac{Gl}{B\rho}(\delta u)_{\text{rms}}$$

- The magnetic rigidity is $B\rho\,[\text{Tm}] = 3.3356\,\beta_r E\,[\text{GeV}]$

Problem 2 solution

- We need to compute the total energy which is $E = T + E_0 = 1.938 \text{ GeV}$

- Now we need to compute the relativistic beta. First we compute the relativistic gamma

$$\gamma_r = \frac{E}{E_0} = 2.07$$ and the relativistic beta is $$\beta_r = \sqrt{1 - 1/\gamma_r^2} = 0.875$$

- The magnetic rigidity is then $B\rho = 5.657 \text{ Tm}$ and the rms angle in both planes is $\theta_{\text{rms}} = 8.8 \times 10^{-4} \text{ rad}$

- Now we can calculate the rms orbit distortion on the single focusing quad

$$x_{\text{rms}}(s) = \frac{\sqrt{N\beta_x(s)\beta_{x\text{rms}}}}{2\sqrt{2}|\sin(\pi Q_x)|}\theta_{x\text{rms}} = \frac{\sqrt{18 \times 12 \times 8}}{2\sqrt{2}|\sin(6.23\pi)|} 8.8 \times 10^{-4} = 19.6\text{mm}$$

- The vertical is

$$y_{\text{rms}}(s) = \frac{\sqrt{N\beta_y(s)\beta_{y\text{rms}}}}{2\sqrt{2}|\sin(\pi Q_y)|}\theta_{y\text{rms}} = \frac{\sqrt{18 \times 2 \times 8}}{2\sqrt{2}|\sin(6.20\pi)|} 8.8 \times 10^{-4} = 9\text{mm}$$

- For $Q_x = 6.1$ the horizontal orbit distortion becomes $x_{\text{rms}}(s) = 41.9\text{mm}$
- For $Q_x = 6.01$ we have $x_{\text{rms}}(s) = 0.41 \text{ m}$
- The vertical remains unchanged…

Problem 3 solution

Three correctors are placed at locations with phase advance of $\pi/4$ between them and beta functions of 12, 2 and 12 m. How are the corrector kicks related to each other in order to achieve a closed 3-corrector bump (i.e. what is the relative strength between the three kicks)?

- The relations for achieving a 3-bump are

$$\frac{\sqrt{\beta_1}}{\sin\psi_{23}}\theta_1 = \frac{\sqrt{\beta_2}}{\sin\psi_{31}}\theta_2 = \frac{\sqrt{\beta_3}}{\sin\psi_{12}}\theta_3$$

- The phase advances are $\psi_{12} = \psi_{23} = \pi/4$ and $\psi_{13} = \psi_{12} + \psi_{23} = \pi/2$

  which gives $\psi_{31} = -\pi/2$

- So $\theta_1 = \theta_3$ and $\theta_2 = -\theta_1\sqrt{12}$

Problem 4 solution

The SPS is a **400 GeV proton synchrotron** with a FODO lattice consisting of **108** focusing and **108** defocusing quadrupoles of length **3.22 m** and a gradient of **15 T/m**, with a **horizontal and vertical beta of 108 m and 30 m** in the focusing quads (**30 m and 108 m** for the defocusing ones). The tunes are **Q_x=20.13** and **Q_y=20.18**. Due to a mechanical problem, a focusing quadrupole was sinking down in 2016, resulting in an increasing closed orbit distortion compared to a reference taken earlier in the year.

- By how much the quadrupole had shifted down when the maximum vertical closed orbit distortion amplitude in defocusing quadrupoles reached 4 mm?
- Why was there no change of the horizontal orbit measured?
- How big would have been the maximum closed orbit distortion amplitude if it would have been a defocusing quadrupole?

- The magnetic rigidity is $B\rho \ \ [\mathrm{T\ m}] = \dfrac{1}{0.2998}\beta_r E \ \ [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is $B\rho = 1334 \ \ \mathrm{T\ m}$
- The focusing normalized gradient is $K_F = \dfrac{G_F}{B\rho} = \dfrac{15}{1334} = 0.011\,\mathrm{m}^{-2}$
- The defocusing one is just the same with opposite sign $K_D = -0.011\,\mathrm{m}^{-2}$

Problem 4 solution

- The closed orbit distortion from a single dipole error is given by

$$u(s) = \theta \frac{\sqrt{\beta(s)\beta_0}}{2\sin(\pi Q)} \cos(\pi Q - |\psi(s) - \psi_0|)$$

- We are interested in the peak orbit distortion

$$\hat{y} = \theta \frac{\sqrt{\hat{\beta}_y \beta_0}}{2\sin(\pi Q)}$$

- From this we can calculate the required kick

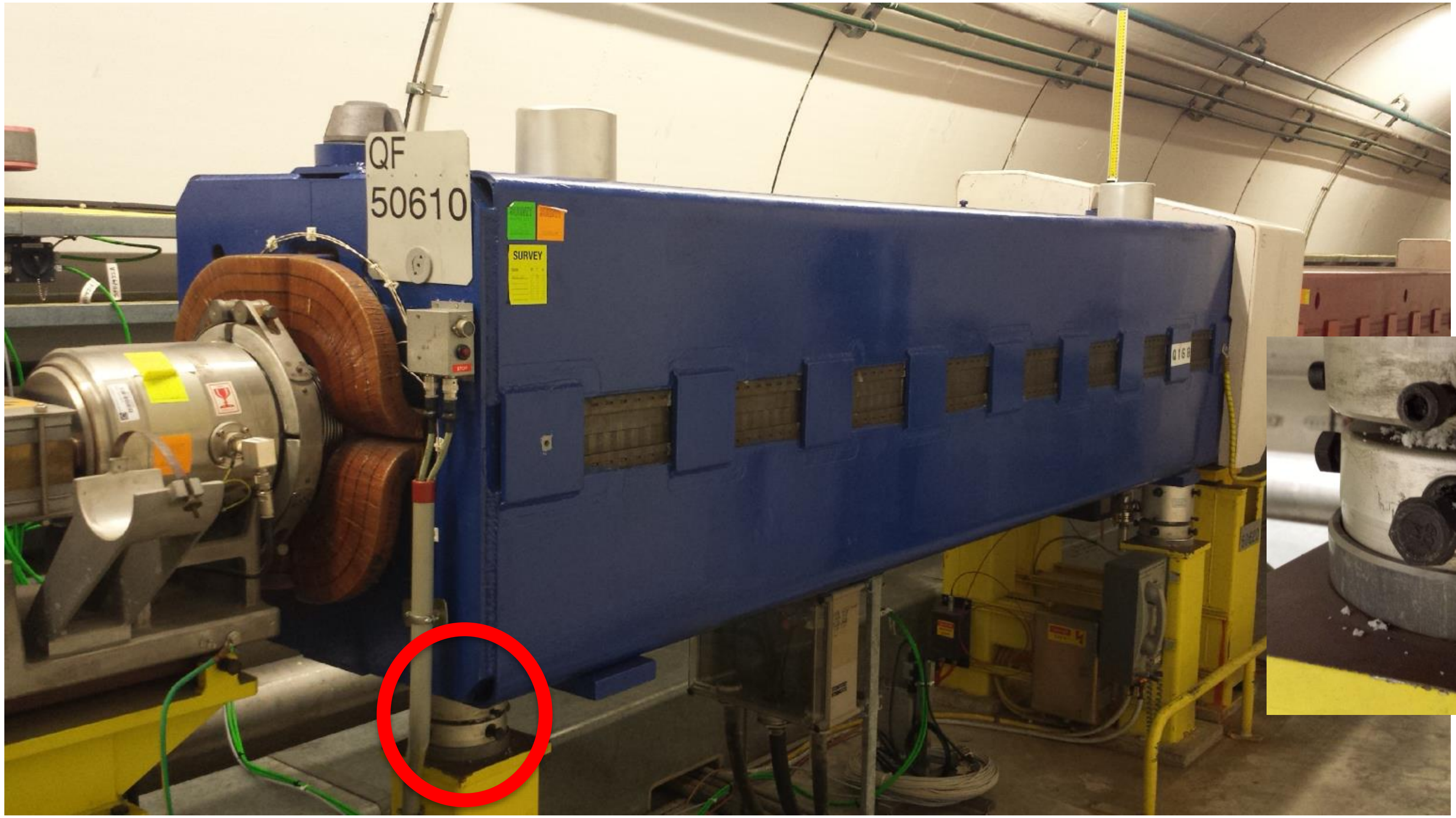
$$\theta = \frac{\hat{y} 2\sin(\pi Q)}{\sqrt{\hat{\beta}_y \beta_0}} = \frac{0.004 \times 2\sin(\pi 20.18)}{\sqrt{108 \times 30}} = 75\ \mu\text{rad}$$

- And finally the required quadrupole displacement to produce this deflection

$$\theta = \frac{Gl\delta y}{B\rho} = K_F l_F \delta y \qquad \delta y = \frac{\theta}{K_F l_F} = \frac{75 \times 10^{-6}}{0.011 \times 3.22}\text{m} = 2\,\text{mm}$$

Problem 4 solution

- In fact what was found in the machine was that one of the supports of the quadrupole was broken and on that end it was shifted down by 4 mm, resulting in 2 mm average shift



Problem 4 solution

- No horizontal orbit change was observed, because the quadrupole shifted only in the vertical plane resulting in a pure vertical kick.
- If it would have been a defocusing quadrupole, the kick would have been the same but with opposite sign. However, the impact on the closed orbit would have been bigger since the vertical β-function is bigger in the defocusing quadupole, such that the peak orbit distortion would reach

$$\hat{y} = \theta \frac{\sqrt{\hat{\beta}_y \beta_0}}{2\sin(\pi Q)} = \theta \frac{\sqrt{\hat{\beta}_y \hat{\beta}_y}}{2\sin(\pi Q)} = 75 \times 10^{-6} * \frac{\sqrt{108 \times 108}}{2\sin(\pi 20.18)}\,\mathrm{m} = 7.5\,\mathrm{mm}$$

Problem 5 solution

The SPS is a **400 GeV proton synchrotron** with a FODO lattice consisting of **108** focusing and **108** defocusing quadrupoles of length **3.22 m** and a gradient of **15 T/m**, with a **horizontal and vertical beta of 108 m and 30 m** in the focusing quads (**30 m and 108 m** for the defocusing ones). The tunes are **Q_x=20.13** and **Q_y=20.18**.

1. Find the tune shift for **systematic** gradient errors of 1% in the focusing and 0.5% in the defocusing quads
2. Find the $\boldsymbol{\beta}_x$ and $\boldsymbol{\beta}_y$ rms beating for **rms** gradient errors of 1% in both focusing and defocusing quads

- The magnetic rigidity is $B\rho \;\; [\mathrm{T\ m}] = \frac{1}{0.2998} \beta_r E \;\; [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is $B\rho = 1334 \;\; \mathrm{T\ m}$
- The focusing normalized gradient is $K_F = \frac{G_F}{B\rho} = \frac{15}{1334} = 0.011\,\mathrm{m}^{-2}$
- The defocusing one is just the same with opposite sign $K_D = -0.011\,\mathrm{m}^{-2}$
- Now, the total tune change is given by $\delta Q_u = \frac{1}{4\pi} \sum_i \beta_u K_i \left(\frac{\delta K}{K}\right)_i l_i$
- The rms beta beating is given by $\left(\frac{\delta\beta}{\beta_0}\right)_{\mathrm{rms}} = \frac{1}{2\sqrt{2}|\sin(2\pi Q)|} \left(\sum_i \delta k_i^2 \beta_i^2\right)^{1/2}$

Problem 5 solution

- By splitting the focusing and defocusing quads, we have

$$\delta Q_u = \frac{1}{4\pi}\left(N_F \beta_u^F K_F \left(\frac{\delta K}{K}\right)_F l_F + N_D \beta_u^D K_D \left(\frac{\delta K}{K}\right)_D l_D\right)$$

- As $N_F = N_D = N$, $l_F = l_D = l$ and $K_F = -K_D = K$ the tune shift can be rewritten as

$$\delta Q_{x,y} = \frac{1}{4\pi} N l K \left(\pm \beta_{x,y}^F \left(\frac{\delta K}{K}\right)_F \mp \beta_{x,y}^D \left(\frac{\delta K}{K}\right)_D\right)$$

- This gives a horizontal and vertical tune shift of

$$\delta Q_x = \frac{108 \times 3.22 \times 0.011}{4\pi}(+108 \times 0.01 - 30 \times 0.005) = 0.3 \qquad \delta Q_y = \frac{108 \times 3.22 \times 0.011}{4\pi}(-30 \times 0.01 + 108 \times 0.005) = 0.07$$

- In a similar way, one can compute the **rms beta beating** in case of rms error of 1% for both focusing and defocusing quadrupoles (note that in this case, all signs are positive…):

$$\left(\frac{\delta\beta_{x,y}}{\beta_{0x,y}}\right)_{\rm rms} = \frac{1}{2\sqrt{2}|\sin(2\pi Q_{x,y})|}\sqrt{N}\left|lK\left(\frac{\delta K}{K}\right)\right|\left[\left(\beta_{x,y}^F\right)^2 + \left(\beta_{x,y}^D\right)^2\right]^{1/2}$$

$$= \frac{1}{2\sqrt{2}|\sin(2\pi Q_{x,y})|}\sqrt{108} \times 3.22 \times 0.011 \times 0.01 \times \left(108^2 + 30^2\right)^{1/2} \approx \frac{0.15}{|\sin(2\pi Q_{x,y})|}$$

$$\left\{ \begin{array}{l} \left(\frac{\delta\beta_x}{\beta_{0x}}\right)_{\rm rms} = 20\% \\ \left(\frac{\delta\beta_y}{\beta_{0y}}\right)_{\rm rms} = 16\% \end{array} \right.$$

Problem 6 solution

The SPS is a **400 GeV proton synchrotron** with a FODO lattice consisting of **108** focusing and **108** defocusing quadrupoles of length **3.22 m** and a gradient of **15 T/m**, with a **horizontal and vertical beta of 108 m and 30 m** in the focusing quads (**30 m and 108 m** for the defocusing ones). The tunes are **Q_x=20.13** and **Q_y=20.18**.
In order to correct its natural chromaticity, several **0.42 m long sextupoles** are installed next to focusing and defocusing quadrupoles at locations with high dispersion.
Assume that **one** of those sextupoles installed next to a **focusing quad** has a gradient of **60.3 T/m^2** and it is **vertically misaligned by δy=10 mm.** Assume that the **beta functions at the sextupole** are equal to the one at the **nearby quadrupole**

1. What is the **normalized sextuple strength**?
2. Compute the impact of the vertical misalignment on: **tune**, **max beta beating**, **minimum tune separation**, **max closed orbit** deviation
(neglect next order effect of such an orbit on transverse optics due to other machine sextupoles…)
3. Repeat for the case in which the **sextupole is displaced horizontally**
4. What would be the **maximum closed orbit deviation** if only one **focusing quadrupole** would be **vertically displaced by -δy=10 mm**? Qualitatively, would you expect some effect on coupling or tune or beta-beating in such a case?

Problem 6 solution

- For a vertical offset in a normal sextupole, the resulting magnetic field can be computed as:

$$y \mapsto y + \delta y \qquad B_y(x, y+\delta y) = B_2\left(x^2 - (y+\delta y)^2\right) = B_2 x^2 - B_2 y^2 - 2B_2 y\delta y - B_2(\delta y)^2 =$$

$$= B_2(x^2 - y^2) - 2B_2(\delta y)y - B_2(\delta y)^2$$

normal sextupole — skew quadrupole — normal dipole

$$B_x(x, y+\delta y) = B_2(2xy) + 2B_2(\delta y)x$$

  - i.e. a skew quadrupole with $\bar{A}_1 = 2B_2\delta y$ and normal dipole with $\bar{B}_0 = -B_2(\delta y)^2$ where $B_2 = 60.3\,[\mathrm{T/m^2}]$
  - Hence, $\bar{A}_1 = 2 \times 60.3 \times 0.01 = 1.21\,\mathrm{T/m}$ and $\bar{B}_0 = -60.3 \times (0.01)^2 = -0.006\,[\mathrm{T}]$
- The magnetic rigidity is as usual $B\rho\ [\mathrm{T\ m}] = \frac{1}{0.2998}\beta_r E\ [\mathrm{GeV}]$
- For 400GeV, the relativistic beta is almost 1 and then the magnetic rigidity is $B\rho = 1334\ \mathrm{T\ m}$
- The **normalised** sextupole **strength** is $K2 = \frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}(60.3x^2) = 2 \times 60.3/1334 \approx 0.09\,[\mathrm{m^{-3}}]$
- The skew quadrupole normalized gradient is $\delta K_S = 1.21/1334 \approx 0.0009\,[\mathrm{m^{-2}}]$
- The horizontal dipole normalized field is $-0.006/1334 \approx -4.5 \times 10^{-6}\,[\mathrm{m^{-1}}]$

Problem 6 solution

- Due to the skew quadrupole, we expect that minimum tune separation:

$$\Delta Q_{\min} \approx \frac{\sqrt{\beta_x \beta_y}}{2\pi} |\delta K_s| l_{\text{sext}} = \frac{\sqrt{108 \times 30}}{2\pi} \times 0.0009 \times 0.42 = 0.0034$$

- In principle, no beta beating or tune shift is expected because a skew quadrupole does not affect, at a first order, beta functions (in reality, in presence of (strong) coupling the concept of beta functions should be reviewed)

- The kick induced by the normal dipole feed-down is $\theta_x = -4.5 \times 10^{-6} \times 0.42 \approx 1.9 \times 10^{-6}\ [\text{rad}]$

- The maximum horizontal displacement is $\Delta x|_{\max} = |\theta_{s_0}| \frac{\sqrt{\beta_x|_{\max}\ \beta_x|_{s_0}}}{2\sin(\pi Q_x)} = 1.9 \times 10^{-6} \frac{\sqrt{108 \times 108}}{2 \times \sin(\pi 20.13)} \approx 2.6 \times 10^{-4}\ [\text{m}]$

- In case the offset is horizontal, one obtains a normal quadrupole and normal dipoles components, hence:
  - The **maximum horizontal closed orbit** would be the same as before (note that it will be on the **same plane**! As the misalignment, and that the dipole kick will have positive sign, but no change in absolute term)
  - No coupling will be introduced, as no skew components will be generated
  - On the other hand, there will be a tune shift $\delta Q_{x,y} = \pm \frac{1}{4\pi} l_{\text{sext}} \delta K \beta^F_{x,y}$

$$\begin{cases} \delta Q_x = \dfrac{0.42 \times 0.0009 \times 108}{4\pi} = 0.0032 \\ \delta Q_y = -\dfrac{0.42 \times 0.0009 \times 30}{4\pi} = -0.0009 \end{cases}$$

Problem 6 solution

- There will also be a beta-beating

$$\left(\frac{\delta\beta_{x,y}}{\beta_{0x,y}}\right)_{\max} = \frac{1}{2|\sin(2\pi Q_{x,y})|}\beta^F_{x,y} l_{\text{sext}} \delta K$$

$$\begin{cases} \left(\frac{\delta\beta_x}{\beta_{0x}}\right)_{\max} = \frac{108 \times 0.42 \times 0.0009}{2|\sin(2\pi \times 20.13)|} \approx 2.8\% \\ \left(\frac{\delta\beta_y}{\beta_{0y}}\right)_{\max} = \frac{30 \times 0.42 \times 0.0009}{2|\sin(2\pi \times 20.18)|} \approx 0.6\% \end{cases}$$

- If one **focusing quadrupole** would be vertically displaced, a **vertical closed orbit kick** will appear:

$$\theta_y = \frac{G_F l_{\text{quad}}}{B\rho}\delta y = \frac{15 \times 3.22}{1334} 0.01 \approx 3.6 \times 10^{-4}\,[\text{rad}]$$

  - The maximum vertical closed orbit deviation (at defocusing quadrupoles) would be :

$$\Delta y|_{\max} = \theta_{s_0} \frac{\sqrt{\beta_y|_{\max}\, \beta_y|_{s_0}}}{2\sin(\pi Q_y)} = 3.6 \times 10^{-4} \frac{\sqrt{108 \times 30}}{2 \times \sin(\pi 20.18)} \approx 20\,[\text{mm}]$$

  - Such a **vertical** orbit will also be present in the sextupoles of the machine! Such a big orbit, double than the misalignment considered earlier, will certainly induce **strong coupling**! but in principle no beta-beating or tune shift.