1. To calculate the sextupole field gradient S we need Bρ

$$B\rho \;\; [\mathrm{T\ m}] = \frac{1}{0.2998}\beta_r E \;\; [\mathrm{GeV}]$$

the total energy is $E = T + E_0 = 0.160 + 0.938\ \mathrm{GeV} = 1.098\ \mathrm{GeV}$

the relativistic gamma and beta follow

$$\gamma_r = \frac{E}{E_0} = \frac{1.098}{0.938} = 1.17 \qquad\qquad \beta_r = \sqrt{1 - 1/\gamma_r^2} = 0.52$$

and thus $B\rho = \dfrac{0.52 \times 1.098}{0.2988}\ \mathrm{Tm} = 1.9\ \mathrm{Tm}$

finally $B_2 = \dfrac{2\ \mathrm{m}^{-3} \times 1.9\ \mathrm{Tm}}{2} = \underline{\underline{1.9\ \mathrm{T/m}^2}}$

2. The integrated normalized quadrupole gradient is given by

$$\delta K l = 0.008 \times 0.3\ \mathrm{m}^{-1} = \underline{0.0024\ \mathrm{m}^{-1}}$$

The feed-down in the sextupole is

$$B_y(y=0) = B_2 \bar{x}^2 = B_2 (x + \delta x)^2 = B_2 x^2 + \underbrace{2 B_2 \delta x}_{\delta K \cdot B\rho}\ x + \underbrace{B_2 (\delta x)^2}_{\delta B}$$

therefore the displacement of the sextupole is

$$\delta x = \frac{\delta K}{2 B_2 / B\rho} = \frac{0.008\ \mathrm{m}^{-2}}{2\ \mathrm{m}^{-3}} = 0.004\ \mathrm{m} = \underline{4\ \mathrm{mm}}$$

the integrated normalized dipole kick is

$$\theta = l \frac{\delta B}{B\rho} = l \frac{B_2 (\delta x)^2}{B\rho} = 0.3\ \mathrm{m} \frac{2\ \mathrm{m}^{-3}}{2} (0.004\ \mathrm{m})^2 = \underline{4.8\ \mu\mathrm{rad}}$$

… these values do not depend on the energy (they are normalised!)

Exercises on linear imperfections: solution

3. The tune shifts induced by the quadrupole error are given by

$$\delta Q_x = \frac{1}{4\pi}\delta K\, l\, \beta_x = \frac{1}{4\pi} 0.008 \times 0.3 \times 6 = \underline{0.0011}$$

the maximum beta-beating in the sextupole locations can be estimated as

$$\frac{\delta\hat{\beta}}{\beta} = \frac{1}{2\sin(2\pi Q)}\beta\, \delta K\, l$$

and therefore

$$\frac{\delta\hat{\beta}_x}{\beta_x} = \frac{1}{2\sin(2\pi \times 4.25)} 6 \times 0.008 \times 0.3 \approx \underline{0.7\%}$$

$$\frac{\delta\hat{\beta}_y}{\beta_y} = \frac{1}{2\sin(2\pi \times 4.45)} 12 \times 0.008 \times 0.3 \approx \underline{5\%}$$

BONUS:

from question 4 we know that θ = 4.8 μrad. The maximum closed orbit distortion in the sextupole locations is therefore estimated as (assuming that the cosine term reaches 1 in one of the sextupole locations)

$$\hat{x}_{\mathrm{CO}} = \frac{\sqrt{\beta_x}\sqrt{\beta_x}\theta}{2\sin(\pi Q)} = \frac{\sqrt{6} \times \sqrt{6} \times 4.8\text{e-}6}{2\sin(\pi 4.25)} \approx \underline{20\ \mu\mathrm{m}}$$

there is no orbit distortion in the vertical plane …

Exercises on linear imperfections: solution

4. The horizontal displacement of the sextupole creates a (normal) quadrupole field which affects both the horizontal and the vertical tunes.
5. Q_x = 4.25 is a tune which is far away from both integer and half integer resonances. If the tune is moved towards the half integer, the effect on the orbit may be reduced, but the effect of quadrupole errors (beta-distortion) is amplified. On the other hand, close to the integer both beta distortion and orbit error is amplified.

6. A closed three corrector bump is created when the three kicks satisfy

$$\frac{\sqrt{\beta_1}}{\sin\psi_{23}}\theta_1 = \frac{\sqrt{\beta_2}}{\sin\psi_{31}}\theta_2 = \frac{\sqrt{\beta_3}}{\sin\psi_{12}}\theta_3$$

here the beta functions are all the same. Since the machine has 16 identical cells, the phase advance between sextupoles is obtained from the tune:

$$\Delta\psi_{12} = \frac{2\pi \cdot Q}{16} \approx 0.26 \cdot 2\pi$$

$$\Delta\psi_{12} = \Delta\psi_{23} = -\Delta\psi_{31}/2$$

and $\theta_2 = \theta_1 \dfrac{\sin(\psi_{31})}{\sin(\psi_{12})}$

$\theta_2 = \underline{0.13 \text{ mrad}}$ $\theta_3 = \theta_1 = \underline{0.66 \text{ mrad}}$

BONUS:

The closed orbit in the central sextupole is calculated using the transport matrix so that

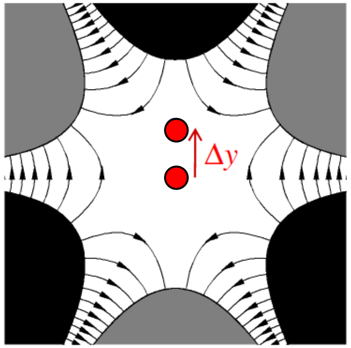
$$\begin{aligned}\delta x_2 &= \sqrt{\beta_1 \beta_2}\sin(\psi_{12})\theta_1 \\ &= \sqrt{6\ \mathrm{m} \times 6\ \mathrm{m}} \times 1 \times 0.66\text{e-}3 \approx 0.004\ \mathrm{m} = 4\ \mathrm{mm}\end{aligned}$$

Exercises on linear imperfections: solution

7. By shifting vertically the sextupole, there will be a horizontal field which depends linearly in the horizontal position and a vertical field that depends linearly on the vertical position, i.e.

$$B_x = 2B_2 x\bar{y} = 2B_2 xy + \underbrace{2B_2 x\delta y}_{\text{skew quadrupole}}$$

$$B_y = B_2(x^2 - \bar{y}^2) = \overbrace{-2B_2 y\delta y} + B_2(x^2 - y^2) - B_2(\delta y)^2$$



These two components correspond to an equivalent skew quadrupole, i.e. they induce coupling. The coupling resonance $Q_x - Q_y = N$ should be avoided.

- **Conceptual questions:**

1. Considering the transverse horizontal motion only, a single particle tracked turn by turn over a linear lattice draws an ellipse in the phase space (x, x') observed at a chosen location in the synchrotron. The closed orbit represents the center of such an ellipse, and corresponds the the coordinates of a particle that, after one turn, comes back to the exact same location. In absence of dipole errors/kicks, in an ideal machine the closed orbit coincides with the reference orbit which has (0,0) coordinates.

2. At least two correctors with a phase advance difference of multiple of pi are needed to introduce a closed orbit bump. When this condition cannot be met, one has to use at least 3 closed orbit correctors. Additional orbit correctors allow to add flexibility on amplitude and angle of the bump at a given location inside the bump.

3. Sources for closed orbit errors are all dipole kicks coming from any source such as dipoles themselves (strength error), dipole roll (resulting in mostly vertical kick), feed-down from displaced quadrupoles or even from feed-down in higher order multipoles (although weaker). Possible correction schemes include local correction with closed orbit bumps, or global corrections using algorithms like MICADO, SVD, …; Closed orbit correctors are preferably installed in locations with high beta functions (because their impact on the closed orbit scales with the square-root of the beta function)

4. Quadrupole errors result in beta-beating (distortion of the optics around the machine) and in a tune shift.