EXERCISE 1: TRANSIT TIME FACTOR

- Derive the general expression of the transit time factor of an accelerating gap of length L, with constant accelerating field, in which the field is oscillating at f_{RF} and that accelerate particles with relativistic factor β.

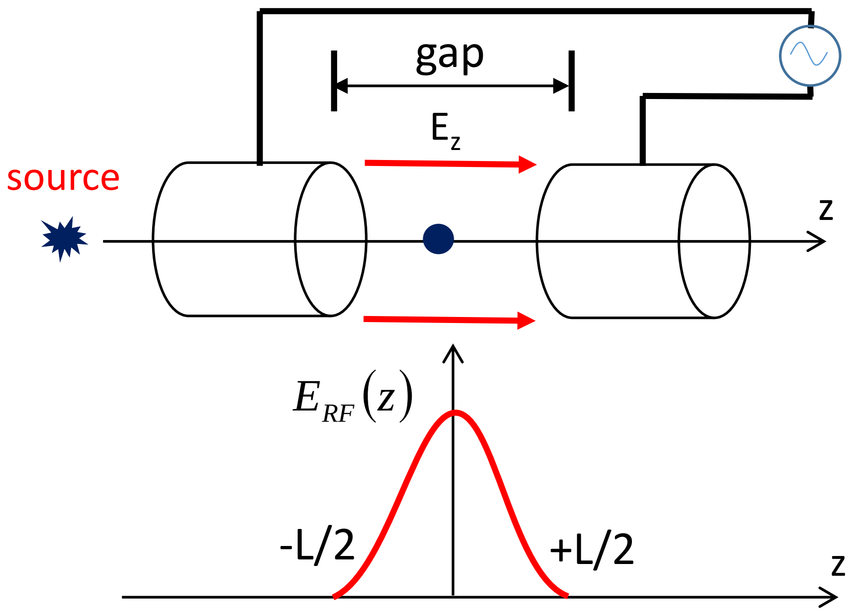
- Remembering that the light wavelength in free space is given by $\lambda_{RF}=c/f_{RF}$, for which value of the accelerating gap length L, T is equal to zero?

- Calculate the numerical value of T for L=10 cm, f_{RF}=1 GHz and ultra-relativistic electrons (β=1).

- Calculate the accelerating voltage as a function of the gap length L assuming an injection phase on crest (ϕ_{inj}=0)

RF ACCELERATION: ENERGY GAIN

We consider now the acceleration between two electrodes fed by an RF generator



$$\Delta V = V_{RF}\cos(\omega_{RF} t) \qquad \omega_{RF} = 2\pi f_{RF} = \frac{2\pi}{T_{RF}}$$

$$V_{RF} = \int_{gap} E_{RF}(z)dz \qquad E_z(z,t) = E_{RF}(z)\cos(\omega_{RF} t)$$

$$E_z(z,t)\Big|_{\substack{seen\\by\\particle}} = E_{RF}(z)\cos[\omega_{RF}(t+t_{inj})] = E_{RF}(z)\cos(\omega_{RF} t + \phi_{inj})$$

$$\phi_{inj} = \omega_{RF} t_{inj}$$

t=z/v

Hyp. of symmetric accelerating field

$$\Delta E = q\int_{gap} E_z(z,t)\Big|_{\substack{seen\\by\\particle}} dz = q\int_{-L/2}^{+L/2} E_{RF}(z)\cos\left(\omega_{RF}\frac{z}{v}+\phi_{inj}\right)dz = q\int_{gap} E_{RF}(z)dz \; \frac{\int_{gap} E_{RF}(z)\cos\left(\omega_{RF}\frac{z}{v}\right)dz}{\int_{gap} E_{RF}(z)dz}\;\cos(\phi_{inj})$$

V_{RF} Peak gap voltage

T<1 Transit time factor

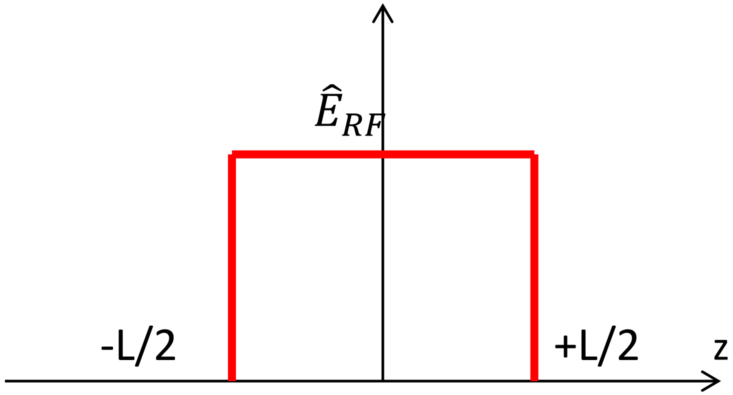
Injection phase

$$\Delta E = q\underbrace{V_{RF}T}_{\hat{V}_{acc}}\cos(\phi_{inj}) = q\underbrace{\hat{V}_{acc}\cos(\phi_{inj})}_{V_{acc}}$$

$\hat{E}_{acc} = \hat{V}_{acc}/L$ Average accelerating field in the gap

$E_{acc} = V_{acc}/L$ Average accelerating field seen by the particle

EXERCISE 1: SOLUTION



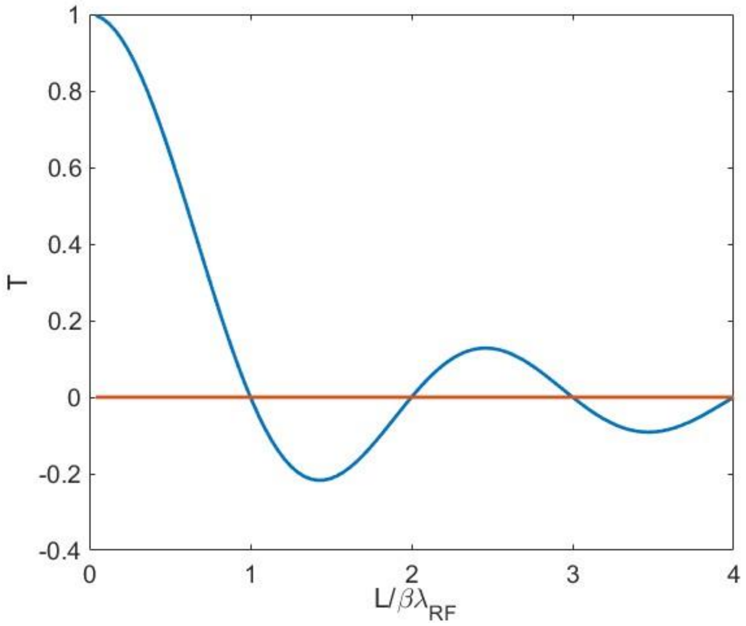
$$T = \frac{\int_{-L/2}^{L/2} E_{RF}(z)\cos\left(\omega_{RF}\frac{z}{v}\right)dz}{\int_{-L/2}^{L/2} E_{RF}(z)dz} = \frac{\int_{-L/2}^{L/2}\cos\left(\omega_{RF}\frac{z}{\beta c}\right)dz}{L} = \frac{\sin\left(\omega_{RF}\frac{L}{2\beta c}\right)}{\frac{\omega_{RF}L}{2\beta c}} = \frac{\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)}{\frac{\pi L}{\beta\lambda_{RF}}}$$

It is equal to zero when

$$2\pi f_{RF}\frac{L}{2\beta c} = \frac{\pi L}{\beta\lambda_{RF}} = \pi \Rightarrow L = \beta\lambda_{RF}$$

Numerical calculation

$$T = \frac{\sin\left(\pi 1\cdot 10^9\frac{0.1}{3\cdot 10^8}\right)}{\pi 1\cdot 10^9\frac{0.1}{3\cdot 10^8}} \cong 0.8268$$

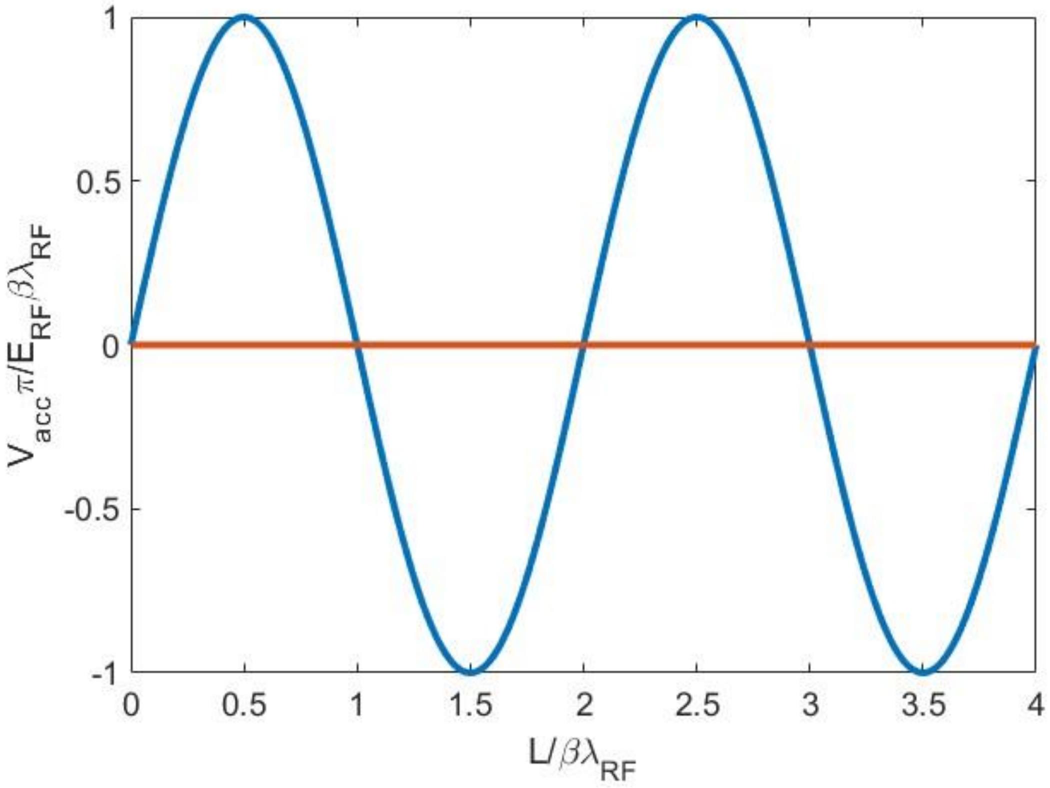


Total accelerating voltage

$$V_{acc} = \int_{-L/2}^{L/2} E_{RF}(z)dz \cdot \left[\frac{\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)}{\frac{\pi L}{\beta\lambda_{RF}}}\right] = \hat{E}_{RF} L \frac{\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)}{\frac{\pi L}{\beta\lambda_{RF}}} = \hat{E}_{RF} \frac{\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)}{\frac{\pi}{\beta\lambda_{RF}}}$$

$$V_{acc}\Big|_{max} \rightarrow \frac{L}{\beta\lambda_{RF}} = \frac{1}{2} \Rightarrow L = \frac{\beta\lambda_{RF}}{2}$$

Very small accelerating gaps for low beta particles



EXERCISE 2: ALVAREZ STRUCTURES

A proton beam is injected into a DTL (Alvarez structure) working at f_{RF}=300 MHz, with a kinetic energy W_{in}=4 MeV. Calculate:

1) the distance between the first two centers of the accelerating gaps (L_{gaps}) assuming a constant velocity of the proton beam between the first two gaps and a negligible increase of the velocity due to the accelerating field;

2) if the structure is composed by 40 accelerating gaps (N_{gaps}) and the average accelerating voltage per gap is V_{acc}=0.5 MV, calculate final proton beam kinetic energy.

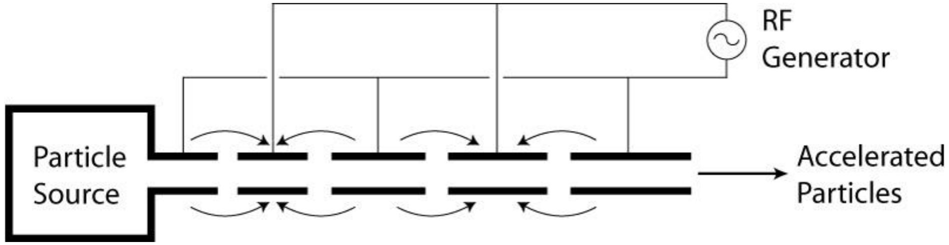
REMEMBER

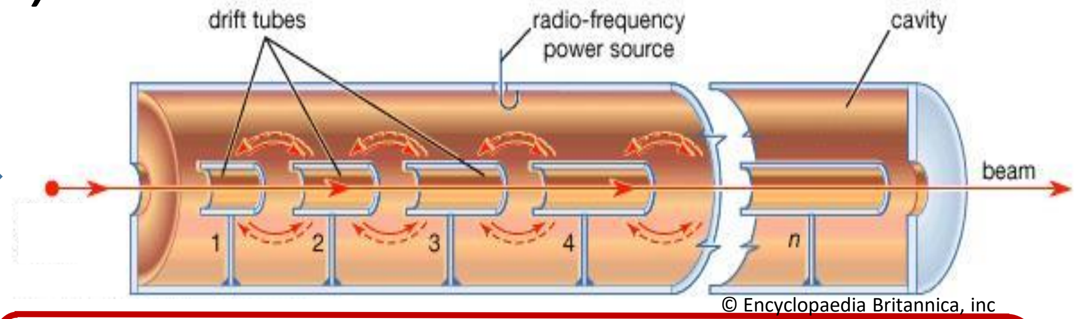
proton rest energy $m_{0\_p}c^2=E_{0\_p}=938$ *MeV*
velocity of light $c=2.998e8$

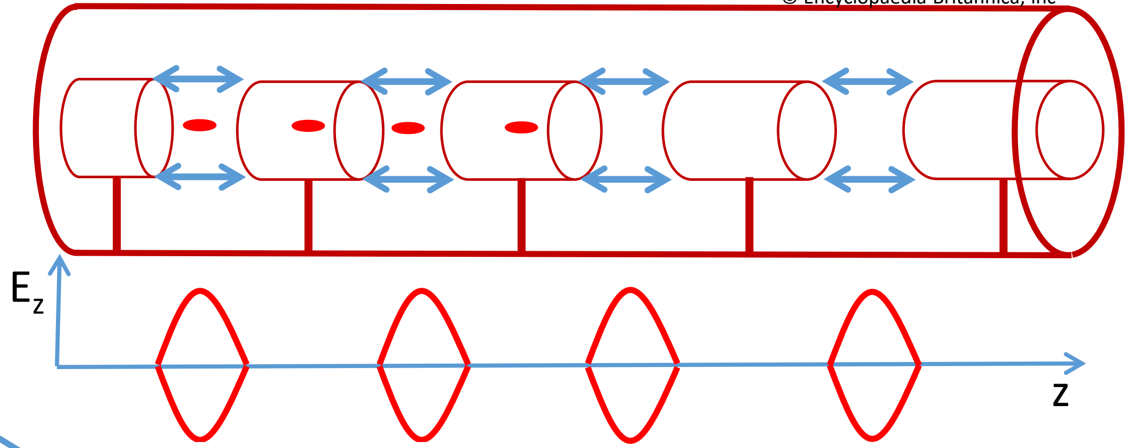
ALVAREZ STRUCTURES

(protons and ions)

Alvarez's structure can be described as a special DTL in which the electrodes are part of a **resonant macrostructure.**

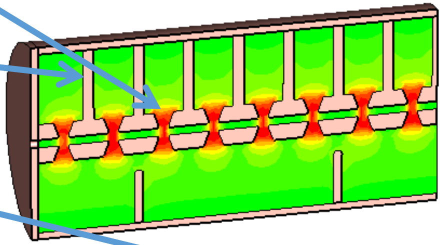






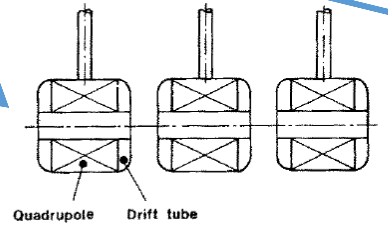
⇒The DTL operates in **0 mode** for **protons and ions** in the range β=0.05-0.5 (f_{RF}=50-400 MHz, λ_{RF}=6-0.7 m) 1-100 MeV;

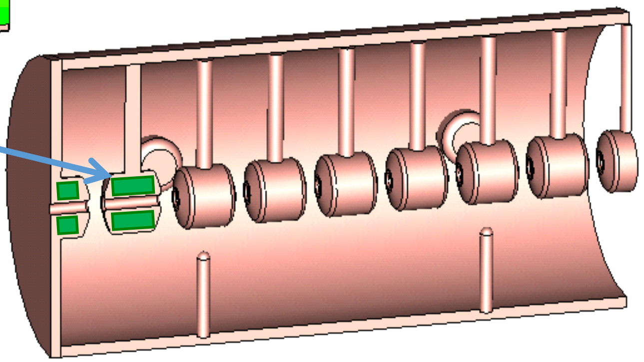
⇒The beam is inside the "**drift tubes**" when the electric field is decelerating. The **electric field** is concentrated between gaps;



⇒The drift tubes are suspended by **stems**;

⇒**Quadrupole** (for transverse focusing) can fit inside the drift tubes.





⇒In order to be synchronous with the accelerating field at each gap the **length of the n-th drift tube** L_n has to be:

$$L_n = \overline{\beta}_n \lambda_{RF}$$

EXERCISE 2: SOLUTIONS

$L_n = \bar{\beta}_n \lambda_{RF}$ → $L_{gaps}=\beta_{in}\lambda_{RF}$=92 mm

$\lambda_{RF}=c/f_{RF}$=0.9993m

$\gamma_{in}=(W_{in}+E_{0\_p})/E_{0\_p}$=1.0043

β_{in}=sqrt(1-1/γ_{in}^2)= 0.0921

$W_{fin} = W_{in} + qN_{gaps} * V_{acc}$=24 MeV

EXERCISE 3: ALVAREZ STRUCTURES AND TRANSIT TIME FACTOR

Particles at β=0.5 are accelerated through an ideal DTL operating at f_{RF}=400 MHz. Assuming a uniform accelerating RF field (E_{RF}) along the gap, calculate the accelerating gap length (L) that maximize the energy gain of the accelerated particles.

EXERCISE 3: SOLUTIONS

$$\Delta E = q \int_{-L/2}^{L/2} E_{RF}(z) \cos\left(\omega_{RF} \frac{z}{v}\right) dz = qE_{RF} \frac{2v}{\omega_{RF}} \sin\left(\omega_{RF} \frac{L}{2v}\right) = qE_{RF} \frac{\beta\lambda_{RF}}{\pi} \underbrace{\sin\left(\frac{\pi L}{\beta\lambda_{RF}}\right)}$$

The energy gain as a function of L oscillates

Maximum energy gain when:

$$\omega_{RF} \frac{L}{2v} = \frac{\pi}{2} \Rightarrow L = \frac{\beta c}{2f_{RF}} = 18.7cm$$

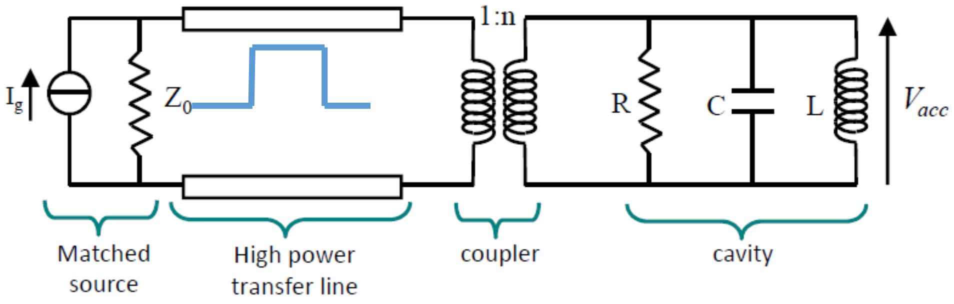
$$\omega_{RF} \frac{L}{2v} = \frac{\pi}{2} + 2n\pi \Rightarrow L = \frac{\beta c}{f_{RF}} \left(\frac{1}{2} + 2n\right) = \beta\lambda_{RF} \left(\frac{1}{2} + 2n\right) = 18.7cm + n \cdot 75cm$$

EXERCISE 4: FILLING TIME

A SW cavity is feed by an RF generator with constant power, calculate after how much time the field in the cavity is 90% the field at full regime supposing that the cavity operate at f_{RF}=1.3 GHz and has an equivalent Q factor of 10000.

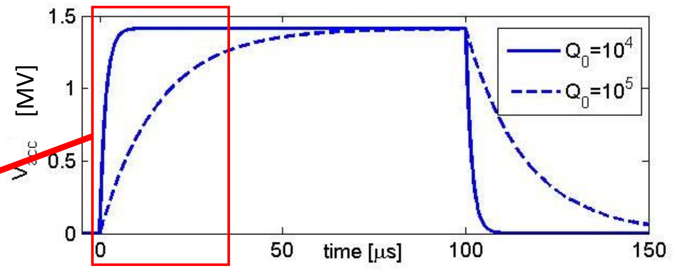
SW CAVITIES : FILLING TIME AND DISSIPATED POWER

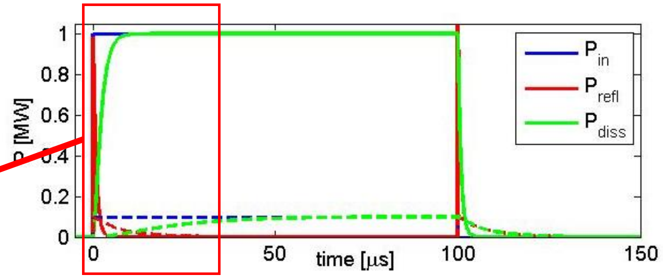
Let us now consider the case of a cavity powered by a source (klystron) in **pulsed mode** at a frequency $f_{RF}=f_{res}$. The accelerating voltage has an exponential behavior and reach the steady state regime asymptotically with a certain filling time (**τ_F**) that is proportional to the quality factor of the resonator.



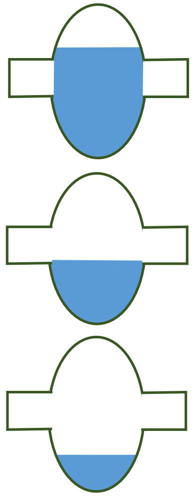
$$V_{acc}(t)=\hat{V}_{acc}\left(1-e^{-\frac{t}{\tau_F}}\right)$$

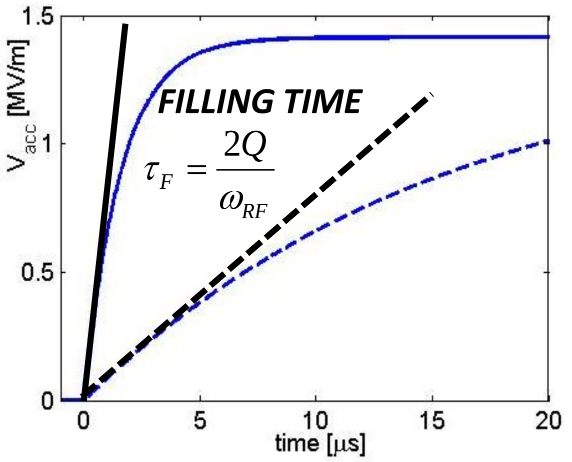
Time domain

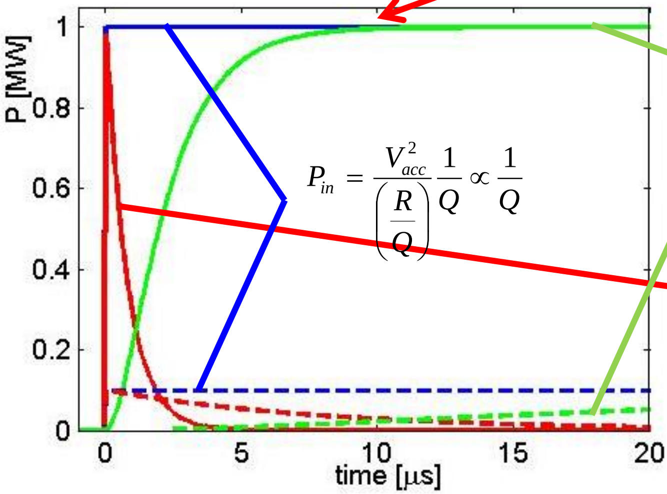




One needs several filling times to completely fill the cavity







$P_{diss} \propto 1/Q$

To reach a given voltage the required power is inversely proportional to the Q-factor

P_{refl} to the generator (it that has to be protected!)

The reachable V_{acc} for a given power is proportional to $\sqrt{Q}$ but, on the other hand, the filling time is $\propto Q$

$$\tau_F\big|_{NC}\approx\mu s$$

$$\tau_F\big|_{SC}>100ms$$

Example:
Q~10000
f_{RF}=1 GHz
τ_F=3.1 µs

EXERCISE 4: SOLUTION

$$V_{acc}(t_{90\%}) = \hat{V}_{acc}\left(1 - e^{-\frac{t_{90\%}}{\tau_F}}\right) = 0.9\hat{V}_{acc}$$

$$\tau_F = \frac{2Q}{\omega_{RF}} = \frac{2 \cdot 10000}{2\pi 1.3 \cdot 10^9} = 2.45\mu s$$

$$t_{90\%} = -\tau_F \ln(1 - 0.9) = 5.64\mu s$$

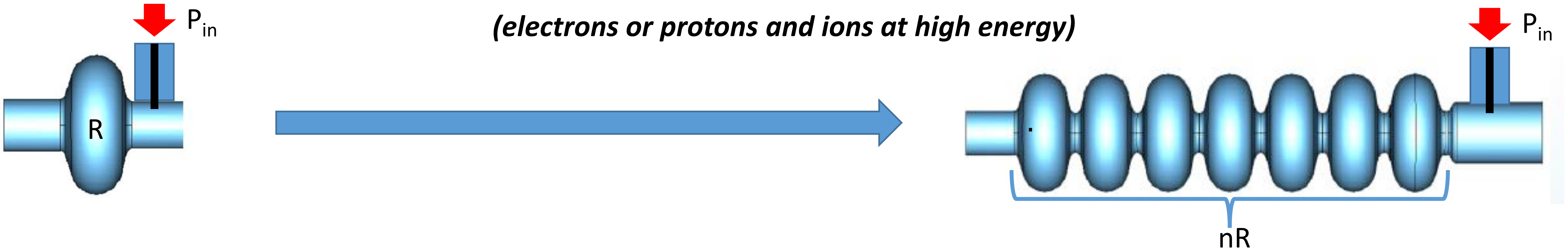
EXERCISE 5: π MODE STRUCTURES

A π-mode structure, operating at f_{RF}=400 MHz, is made up of N=8 cells and is used to accelerate protons. Each single cell has a shunt impedance R=3 MΩ and a length L_{cell}=15 cm. Calculate:

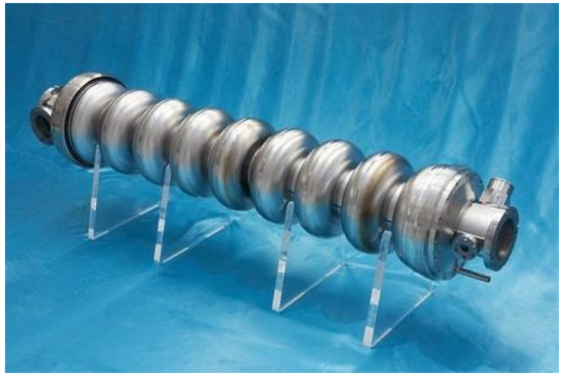
1) the total shunt impedance of the π-mode structure;
2) the accelerating voltage if the total dissipated power into the cavity is P_{diss}=1 MW;
3) the average accelerating field;
4) the average β of the proton beam accelerated by this structure.

MULTI-CELL SW CAVITIES

(electrons or protons and ions at high energy)



- In a multi-cell structure there is **one RF input coupler**. As a consequence the **total number of RF sources is reduced**, with a **simplification of the layout and reduction of the costs;**



- The **shunt impedance is n time** the impedance of a single cavity

- They are **more complicated** to fabricate than single cell cavities;

- The fields of adjacent cells couple through the cell **irises** and/or through properly designed coupling **slots**.

SW CAVITIES PARAMETERS: R, Q, R/Q

ACCELERATING VOLTAGE (V_{acc}) ***DISSIPATED POWER (P_{diss})*** ***STORED ENERGY (W)***

SHUNT IMPEDANCE

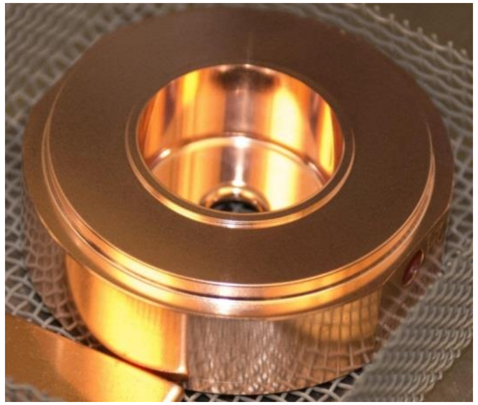
The shunt impedance is the parameter that qualifies the **efficiency of an accelerating mode**. The highest is its value, the larger is the obtainable accelerating voltage for a given power. Traditionally, it is the quantity to optimize in order to **maximize the accelerating field for a given dissipated power**:

$$R = \frac{\hat{V}_{acc}^2}{P_{diss}} \quad [\Omega]$$

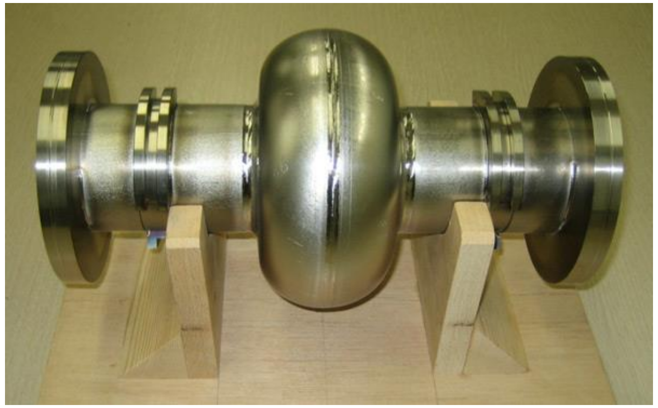
SHUNT IMPEDANCE PER UNIT LENGTH

$$r = \frac{\left(\hat{V}_{acc}/L\right)^2}{P_{diss}/L} = \frac{\hat{E}_{acc}^2}{p_{diss}} \quad [\Omega/m]$$

NC cavity R~1MΩ



SC cavity R~1TΩ



QUALITY FACTOR

$$Q = \omega_{RF}\frac{W}{P_{diss}}$$

NC cavity Q~10^4

SC cavity Q~10^{10}

$$\frac{R}{Q} = \frac{\hat{V}_{acc}^2}{\omega_{RF} W}$$

The R/Q is a **pure geometric qualification factor**. It does not depend on the cavity wall conductivity. R/Q of a single cell is of the order of 100.

Example:

R~1MΩ

P_{diss}=1 MW

V_{acc}=1MV

For a cavity working at 1 GHz with a structure length of 10 cm we have an average accelerating field of 10 MV/m

MULTI-CELL SW CAVITIES: π MODE STRUCTURES

(electrons or protons and ions at high energy)

- The N-cell structure behaves like a system composed by **N coupled oscillators** with **N coupled multi-cell resonant modes**.

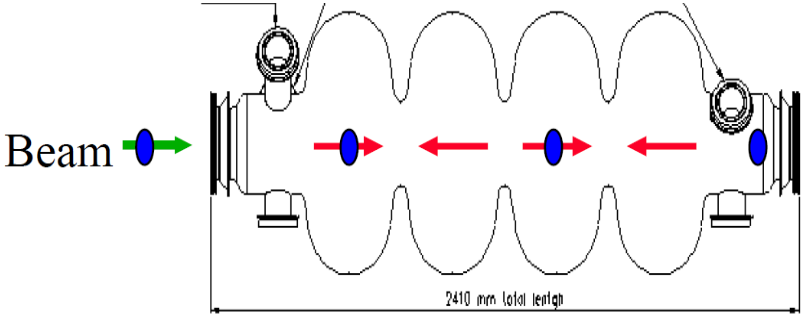
- The modes are characterized by a cell-to-cell phase advance given by:

$$\Delta\phi_n = \frac{n\pi}{N-1} \qquad n = 0,1,\ldots,N-1$$

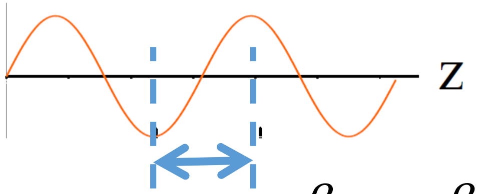
- The multi cell mode generally used for acceleration is the **π, π/2 and 0 mode** (DTL as example operate in the 0 mode).

- The cell lengths have to be chosen in order to synchronize the accelerating field with the particle traveling into the structure at a certain velocity

EXAMPLE: 4 cell cavity operating on the π-mode



Electric field (at time t_0)

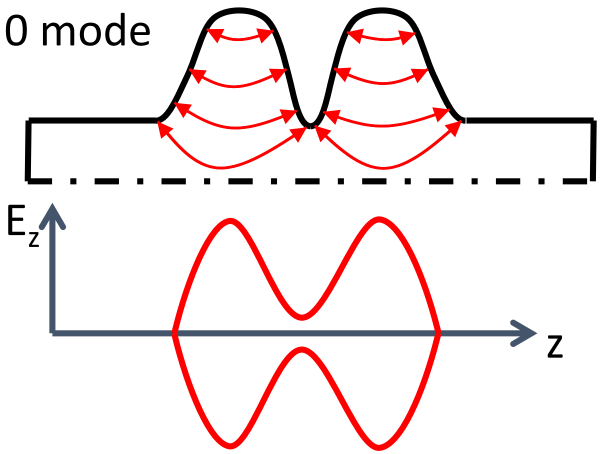


$$d = \frac{\beta c}{2 f_{RF}} = \frac{\beta\lambda_{RF}}{2}$$

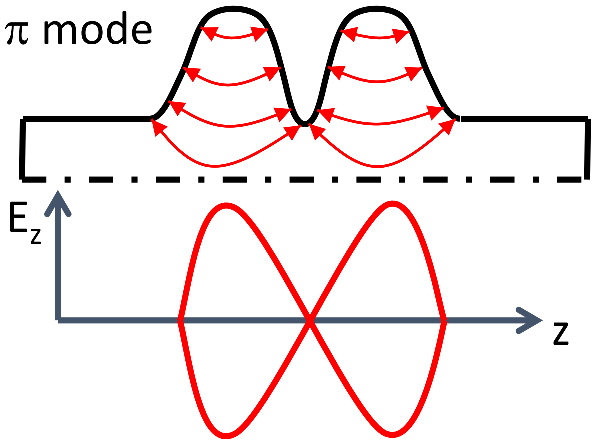
⇒For **ions and protons** the cell lengths have to be increased along the linac that will be a sequence of different accelerating structures matched to the ion/proton velocity.

⇒For **electron**, β=1, $d=\lambda_{RF}/2$ and the linac will be made of an injector followed by a series of identical accelerating structures, with cells all the same length.

0 mode



π mode



EXERCISE 5: SOLUTION

1) $R_{TOT}=N*R_{cell}=24\ M\Omega$

2) $V_{acc}=sqrt(R_{TOT}*P_{diss})=4.9$ MV

3) $E_{acc}=V_{acc}/(L_{cell}*N)=4.08$ MV/m

4) $\beta_{in} =2*L_{cell} /\lambda_{RF}=0.4$ ⇐

$$d = \frac{\beta c}{2 f_{RF}} = \frac{\beta \lambda_{RF}}{2}$$

being

$\lambda_{RF}=c/f_{RF}=0.7495$ m

EXERCISE 6: TW STRUCTURES

A) Demonstrate that if we define the **attenuation constant** as: $\alpha = \frac{p_{diss}}{2P_F}$

the **power flow along the structure scales as**: $P_F(z) = P_{IN} e^{-2\alpha z}$

B) Demonstrate that if we define the **shunt impedance per unit length** as: $r = \frac{\hat{E}_{acc}^2}{p_{diss}}$

the accelerating field "seen" by an ultrarelativistic particle (z=ct) along the structure can be expressed as:

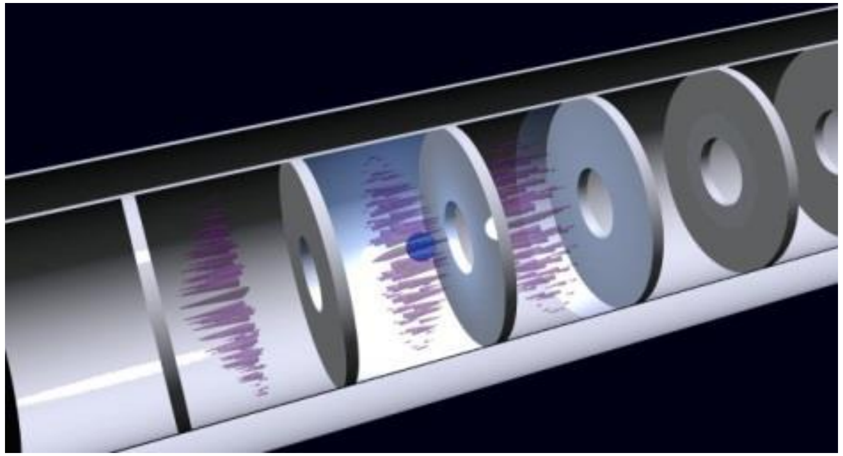
$$E_{acc}(z) = \sqrt{2\alpha r P_{IN}}\, e^{-\alpha z} = E_{IN} e^{-\alpha_0 z}$$

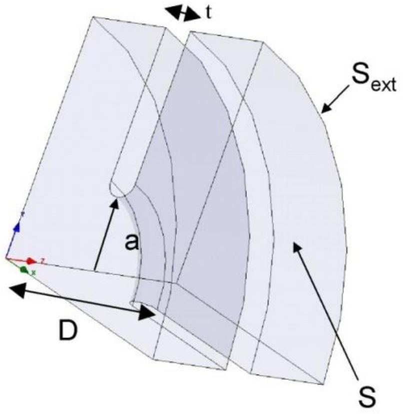
C) Demonstrate that the **total accelerating voltage** is given by:

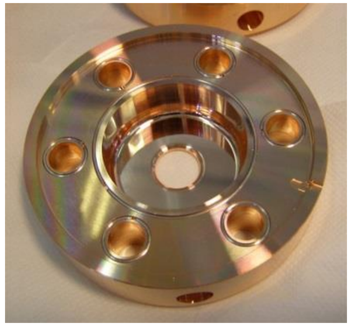
$$V_{acc} = E_{IN} \frac{1 - e^{-\alpha L}}{\alpha}$$

TW CAVITIES PARAMETERS: r, α, v_g

Similarly to the SW cavities it is possible to define some figure of merit for the TW structures







$$\hat{V}_{acc} = \left|\int_0^D E_{acc} \cdot e^{j\omega_{RF}\frac{z}{c}} dz\right|$$ single cell accelerating voltage

$$\hat{E}_{acc} = \frac{\hat{V}_{acc}}{D}$$ average accelerating field in the cell

$$P_F = \int_{Section} \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)\cdot\hat{z}dS$$ flux power

$$P_{diss} = \frac{1}{2}R_s \int_{\substack{cavity\\wall}} \left|H_{tan}\right|^2 dS$$ average dissipated power in the cell

$$p_{diss} = \frac{P_{diss}}{D}$$ average dissipated power per unit length

$$W = \int_{\substack{cavity\\volume}} \overbrace{\left(\frac{1}{4}\varepsilon\left|\vec{E}\right|^2 + \frac{1}{4}\mu\left|\vec{H}\right|^2\right)}^{\text{energy density}} dV$$ stored energy in the cell

$$w = \frac{W}{D}$$ average stored energy per unit length

$$r = \frac{\hat{E}_{acc}^2}{p_{diss}}$$

$$\alpha = \frac{p_{diss}}{2P_F}$$

$$v_g = \frac{P_F}{w}$$

$$Q = \omega_{RF}\frac{w}{p_{diss}}$$

$$\Delta\phi = kD$$

Shunt impedance per unit length [Ω/m]. Similarly to SW structures the higher is r, the higher the available accelerating field for a given RF power.

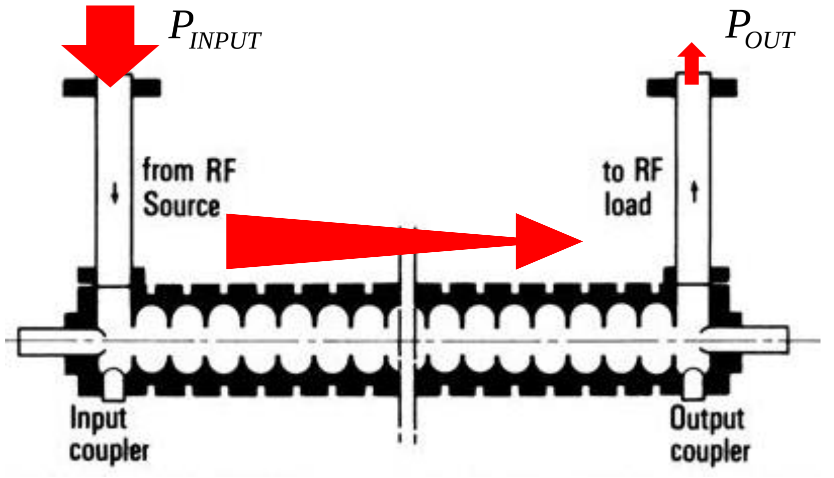
Field attenuation constant [1/m]: because of the wall dissipation, the RF power flux and the accelerating field decrease along the structure.

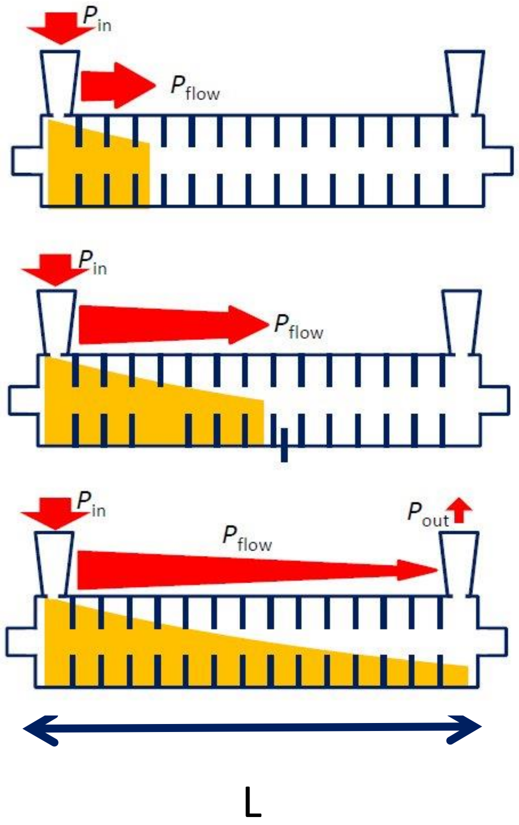
Group velocity [m/s]: the velocity of the energy flow in the structure (~1-2% of c).

Working mode [rad]: defined as the phase advance over a period *D*. For several reasons the most common mode is the **2π/3**

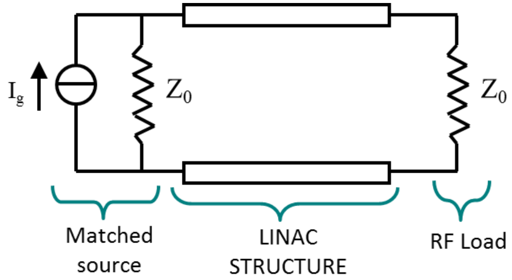
TW CAVITIES: EQUIVALENT CIRCUIT AND FILLING TIME

In a TW structure, the **RF power enters** into the cavity through an **input coupler**, flows (travels) through the cavity in the same direction as the beam and an **output coupler at the end** of the structure is connected to a **matched power load**.
If there is no beam, the input power, reduced by the cavity losses, goes to the power load where it is dissipated.
In the presence of a large beam current, however, a fraction of the TW power is transferred to the beam.





In a purely periodic structure, made by a sequence of **identical cells** (also called "**constant impedance structure**"), α does not depend on z and both the RF power flux and the intensity of the accelerating field decay exponentially along the structure :



$$E_{acc}(z,t) = \underbrace{E_P(r,z)}_{\text{periodic function with period D}} \cos(\omega_{RF}t - k_z^* z) e^{-\alpha z} \approx E_{IN} \cos(\omega_{RF}t - k_z^* z) e^{-\alpha z}$$

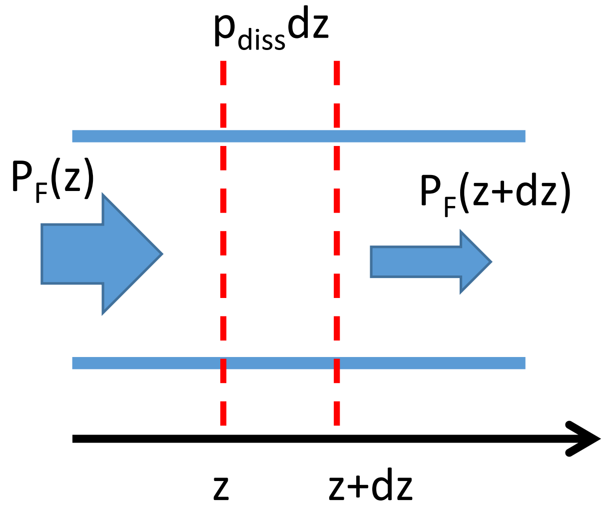
$$P_F(z) = P_{IN} e^{-2\alpha z} \qquad P_{OUT} = P_{IN} e^{-2\alpha L} \qquad E_{IN} = \sqrt{2\alpha r P_{IN}} \qquad V_{acc} = E_{IN} \frac{1 - e^{-\alpha L}}{\alpha}$$

The **filling time** is the time necessary to propagate the RF wave-front from the input to the end of the section of length L is:

$$\tau_F = \frac{L}{v_g}$$

Differently from SW cavities after **one filling time the cavity is completely full of energy**

EXERCISE 6: SOLUTION



A) $$dP_F = -p_{diss}dz \underbrace{\Rightarrow}_{\alpha=\frac{p_{diss}}{2P_F}} \frac{dP_F}{dz} = -2\alpha P_F \Rightarrow \frac{1}{P_F}\frac{dP_F}{dz} = -2\alpha \underbrace{\Rightarrow}_{\substack{\text{integrating} \\ \text{and assuming} \\ P(0)=P_{IN}}} P_F(z) = P_{IN}e^{-2\alpha z}$$

B) $$E_{acc}(z) \underbrace{=}_{r=\frac{E_{acc}^2}{p_{diss}}} \sqrt{rp_{diss}} \underbrace{=}_{\alpha=\frac{p_{diss}}{2P_F}} \sqrt{2\alpha r P_F(z)} \underbrace{=}_{P_F(z)=P_{IN}e^{-2\alpha z}} \underbrace{\sqrt{2\alpha r P_{IN}}}_{E_{IN}} e^{-\alpha z} = E_{IN}e^{-\alpha z}$$

$$P_F(z+dz) = P_F(z) - p_{diss}dz \Rightarrow$$
$$P_F(z+dz) - P_F(z) = -p_{diss}dz \Rightarrow$$
$$dP_F = -p_{diss}dz$$

C) $$V_{acc} = \int_0^L E_{IN}e^{-\alpha z}dz = E_{IN}\frac{1-e^{-\alpha L}}{\alpha}$$

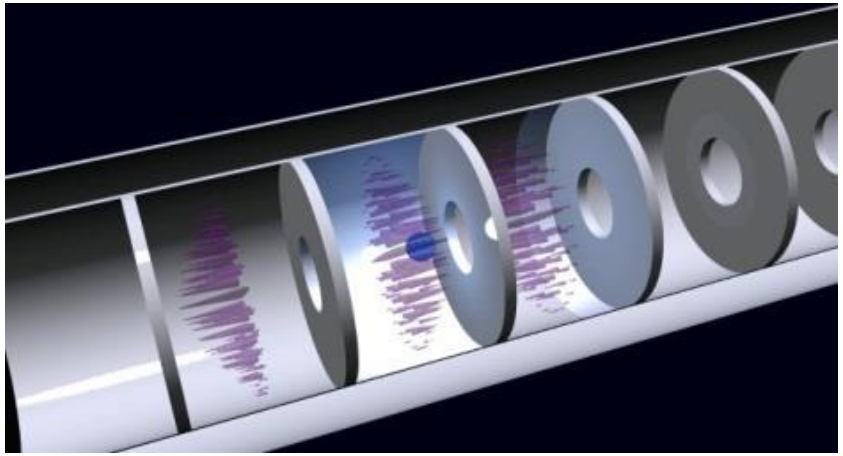
EXERCISE 7: TW STRUCTURES

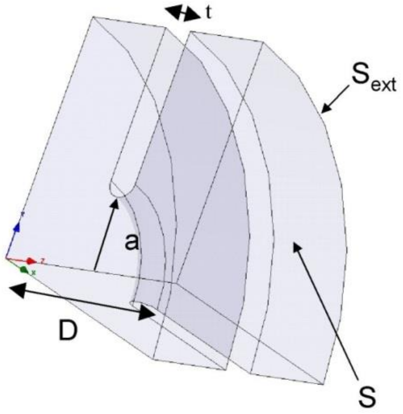
A **SLAC-type TW structure** accelerate ultra-relativistic electrons. The structure length is L=3m and it can be simplified as a structure with a group velocity is **v_g=1.1%** the velocity of light. Calculate:

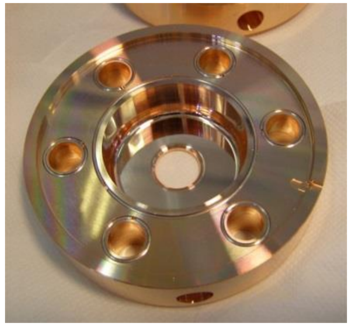
1) the **filling time**;
2) if we suppose that the structure has a field attenuation constant α=0.2 m^{-1}, calculate the total **accelerating voltage** if the accelerating field at the beginning of the structure is E_{INPUT}=20 MV/m;
3) Calculate the **average accelerating field**
4) if the average dissipated power per unit length in the structure, corresponding to the previous value of the accelerating field, is p_{diss}= 4 MW/m calculate the **shunt impedance per unit length**.

TW CAVITIES PARAMETERS: r, α, v_g

Similarly to the SW cavities it is possible to define some figure of merit for the TW structures







$$\hat{V}_{acc} = \left| \int_0^D E_{acc} \cdot e^{j\omega_{RF}\frac{z}{c}} dz \right|$$ single cell accelerating voltage

$$\hat{E}_{acc} = \frac{\hat{V}_{acc}}{D}$$ average accelerating field in the cell

$$P_F = \int_{Section} \frac{1}{2} \mathrm{Re}\left(\vec{E} \times \vec{H}^* \right) \cdot \hat{z} dS$$ flux power

$$P_{diss} = \frac{1}{2} R_s \int_{cavity\ wall} \left| H_{tan} \right|^2 dS$$ average dissipated power in the cell

$$p_{diss} = \frac{P_{diss}}{D}$$ average dissipated power per unit length

$$W = \int_{cavity\ volume} \overbrace{\left(\frac{1}{4}\varepsilon \left|\vec{E}\right|^2 + \frac{1}{4}\mu \left|\vec{H}\right|^2 \right)}^{\text{energy density}} dV$$ stored energy in the cell

$$w = \frac{W}{D}$$ average stored energy per unit length

$$r = \frac{\hat{E}_{acc}^2}{p_{diss}}$$

Shunt impedance per unit length [Ω/m]. Similarly to SW structures the higher is r, the higher the available accelerating field for a given RF power.

$$\alpha = \frac{p_{diss}}{2P_F}$$

Field attenuation constant [1/m]: because of the wall dissipation, the RF power flux and the accelerating field decrease along the structure.

$$v_g = \frac{P_F}{w}$$

Group velocity [m/s]: the velocity of the energy flow in the structure (~1-2% of c).

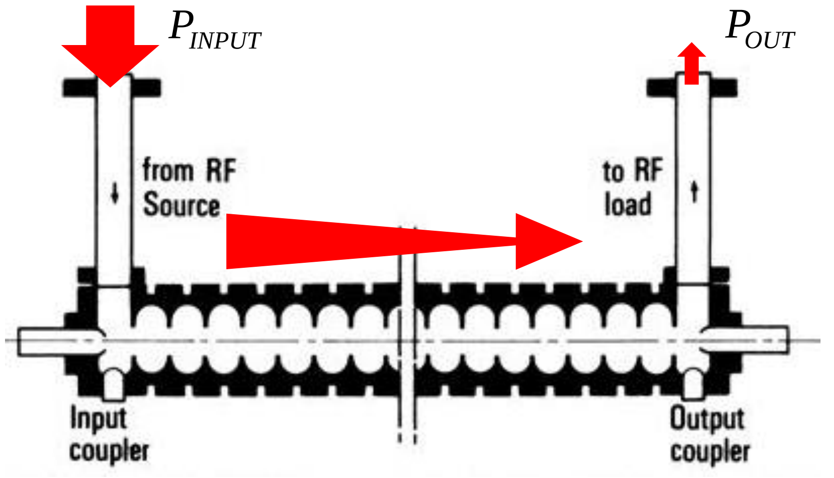
$$Q = \omega_{RF} \frac{w}{p_{diss}}$$

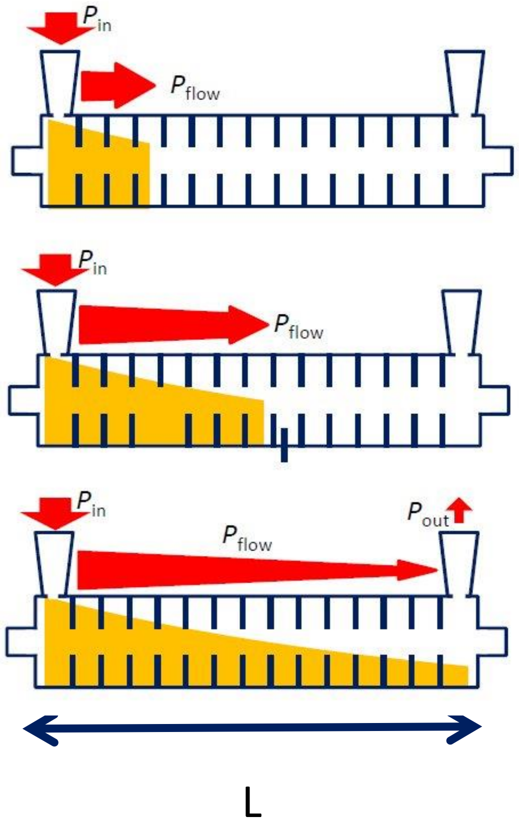
$$\Delta\phi = kD$$

Working mode [rad]: defined as the phase advance over a period *D*. For several reasons the most common mode is the **2π/3**

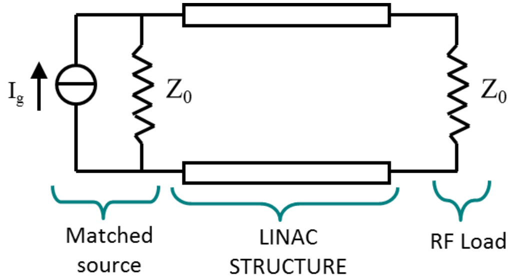
TW CAVITIES: EQUIVALENT CIRCUIT AND FILLING TIME

In a TW structure, the **RF power enters** into the cavity through an **input coupler**, flows (travels) through the cavity in the same direction as the beam and an **output coupler at the end** of the structure is connected to a **matched power load**.
If there is no beam, the input power, reduced by the cavity losses, goes to the power load where it is dissipated.
In the presence of a large beam current, however, a fraction of the TW power is transferred to the beam.





In a purely periodic structure, made by a sequence of **identical cells** (also called "**constant impedance structure**"), α does not depend on z and both the RF power flux and the intensity of the accelerating field decay exponentially along the structure :



$$E_{acc}(z,t) = \underbrace{E_P(r,z)}_{\text{periodic function with period D}} \cos\left(\omega_{RF} t - k_z^* z\right) e^{-\alpha z} \approx E_{IN} \cos\left(\omega_{RF} t - k_z^* z\right) e^{-\alpha z}$$

$$P_F(z) = P_{IN} e^{-2\alpha z} \qquad P_{OUT} = P_{IN} e^{-2\alpha L} \qquad E_{IN} = \sqrt{2\alpha r P_{IN}} \qquad V_{acc} = E_{IN} \frac{1 - e^{-\alpha L}}{\alpha}$$

The **filling time** is the time necessary to propagate the RF wave-front from the input to the end of the section of length L is:

$$\tau_F = \frac{L}{v_g}$$

Differently from SW cavities after **one filling time the cavity is completely full of energy**

EXERCISE 7: SOLUTION

1) L=3; $\Rightarrow$ $t_F = L/v_g = 909.7$ ns

2) $$V_{acc} = \int_0^L E_{INPUT} e^{-\alpha z} dz = E_{INPUT} \frac{1-e^{-\alpha L}}{\alpha} = 45 MV$$

3) $E_{acc} = V_{acc}/L = 15$ MV/m

4) $r = E_{acc}^{2}/p_{diss} = 56.5$ MΩ/m

EXERCISE 8: TW STRUCTURES

A **constant impedance TW structure**, accelerates ultra-relativistic electrons ($\beta=1$). The cavity has the following parameters: $\alpha=0.25\ m^{-1}$; shunt impedance r=65 MOhm/m and a total length of 2 m. Calculate:

1) the **input power** to have an energy gain of the particles of 60 MeV
2) if the group velocity v_g is 1% the speed of light, which is the **filling time** of the structure?

EXERCISE 8: SOLUTION

1) $$V_{acc}=E_{IN}\frac{1-e^{-\alpha L}}{\alpha}=\sqrt{2\alpha rP_{IN}}\frac{1-e^{-\alpha L}}{\alpha}\Rightarrow P_{IN}=\frac{1}{2\alpha r}\left(\frac{V_{acc}\alpha}{1-e^{-\alpha L}}\right)^2=44.7MW$$

2) $t_F=L/v_g=667$ ns

EXERCISE 9: π MODE STRUCTURES AND DUTY CYCLE

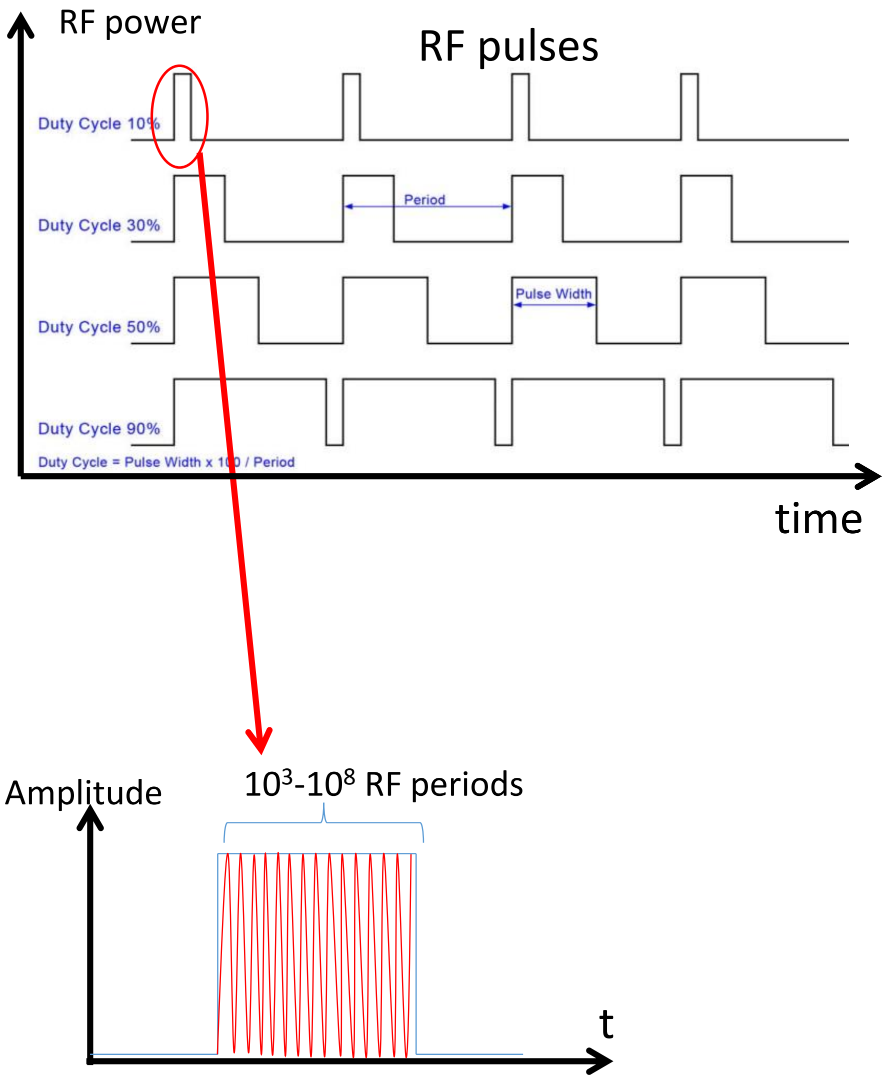
A multi cell SW cavity, operating on the **π-mode at 1 GHz**, accelerates protons at **β=0.5**. The cavity is a **9 cell structure**. Assuming a negligible variation of the particle velocity through the cavity itself calculate:

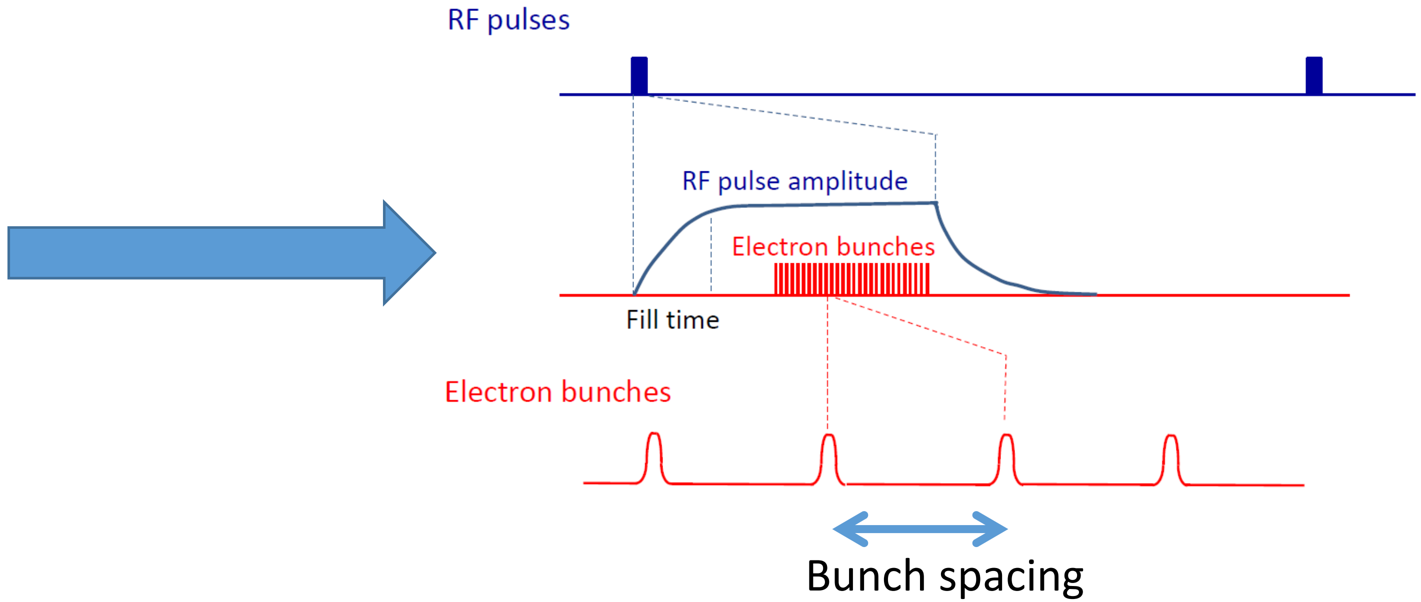
1) the **distance between the centers of the accelerating cells**;
2) assuming a shunt impedance of the single cell (R) of 1 MΩ, calculate the dissipated power to have a peak accelerating voltage on the overall structure of **V_{acc}=10 MV**;
3) Calculate the maximum average accelerating field;
4) If the cavity is fed by 4 μs rf pulses with a repetition rate of 100 Hz, calculate the **Duty Cycle**.

RF STRUCTURE AND BEAM STRUCTURE: NC vs SC

The “**beam structure**” in a LINAC is directly related to the “**RF structure**”. There are two possible type of operations:

- **CW** (Continuous Wave) operation ⇒ allow, in principle, to operate with a continuous (bunched) beam
- **PULSED** operation ⇒ there are RF pulses at a certain repetition rate (**Duty Cycle (DC)=pulsed width/period**)





⇒**SC structures allow operation at very high Duty Cycle (>1%) up to a CW operation (DC=100%)** (because of the extremely low dissipated power) **with relatively high gradient (>20 MV/m)**. This means that a continuous (bunched) beam can be accelerated.

⇒**NC structures can operate in pulsed mode** at **very low DC** (10^{-2}-10^{-1} %) (because of the higher dissipated power) with, in principle, **larger peak accelerating gradient**(>30 MV/m). This means that one or few tens of bunches can be, in general, accelerated. NB: NC structures can also operate in CW but at very low gradient because of the dissipated power.

EXERCISE 9: SOLUTION

1) $$d = \frac{\beta c}{2 f_{RF}} = \frac{\beta \lambda_{RF}}{2} = 7.5 cm$$

2)

R_{TOT}=N*R=9 MΩ

P_{diss}=V_{acc}^2/R_{TOT}=11.1 MW

3)

L_{TOT}=d*N=67.4 cm

E_{acc}=V_{acc}/L_{TOT}=14.8 MV/m

4)

DC=T_{imp}/T_{period}=4e-6/0.01=4e-4 (0.04%)

EXERCISE 10: ENERGY ACCEPTANCE

A RF accelerating structure operating at f_{RF}=400 MHz, is used to accelerate protons at an input nominal kinetic energy W_{in}=10 MeV. Assuming that the **nominal synchronous phase $\phi_s=-\pi/6$** and that the average accelerating field is E_{acc}=2 MV/m, calculate the **maximum kinetic energy of the protons that is possible to capture in the RF bucket** (assuming that it is injected at a phase corresponding to the synchronous one).

APPENDIX: LARGE OSCILLATIONS AND SEPARATRIX

To study the longitudinal dynamics **at large oscillations**, we have to consider the **non linear system of differential equations** without approximations. In the **adiabatic acceleration** case it is possible to easily obtain the following relation between w and φ that is the **Hamiltonian of the system** related to the total particle energy:

$$\frac{1}{2}\left(\frac{\omega_{RF}}{cE_0\beta_s^3\gamma_s^3}\right)^2 w^2 + \frac{\omega_{RF}\, q\hat{E}_{acc}}{cE_0\beta_s^3\gamma_s^3}\left[\sin(\phi_s+\varphi)-\varphi\cos\phi_s-\sin(\phi_s)\right] = \text{const} = \text{H}$$

⇒**For each H we have different trajectories** in the longitudinal phase space

⇒the oscillations are **stable** within a region bounded by a special curve called **separatrix**: its equation is:

$$\frac{1}{2}\frac{\omega_{RF}}{cE_0\beta_s^3\gamma_s^3} w^2 + q\hat{E}_{acc}\left[\sin(\phi_s+\varphi)-(2\varphi_s+\varphi)\cos\phi_s+\sin(\phi_s)\right] = 0$$

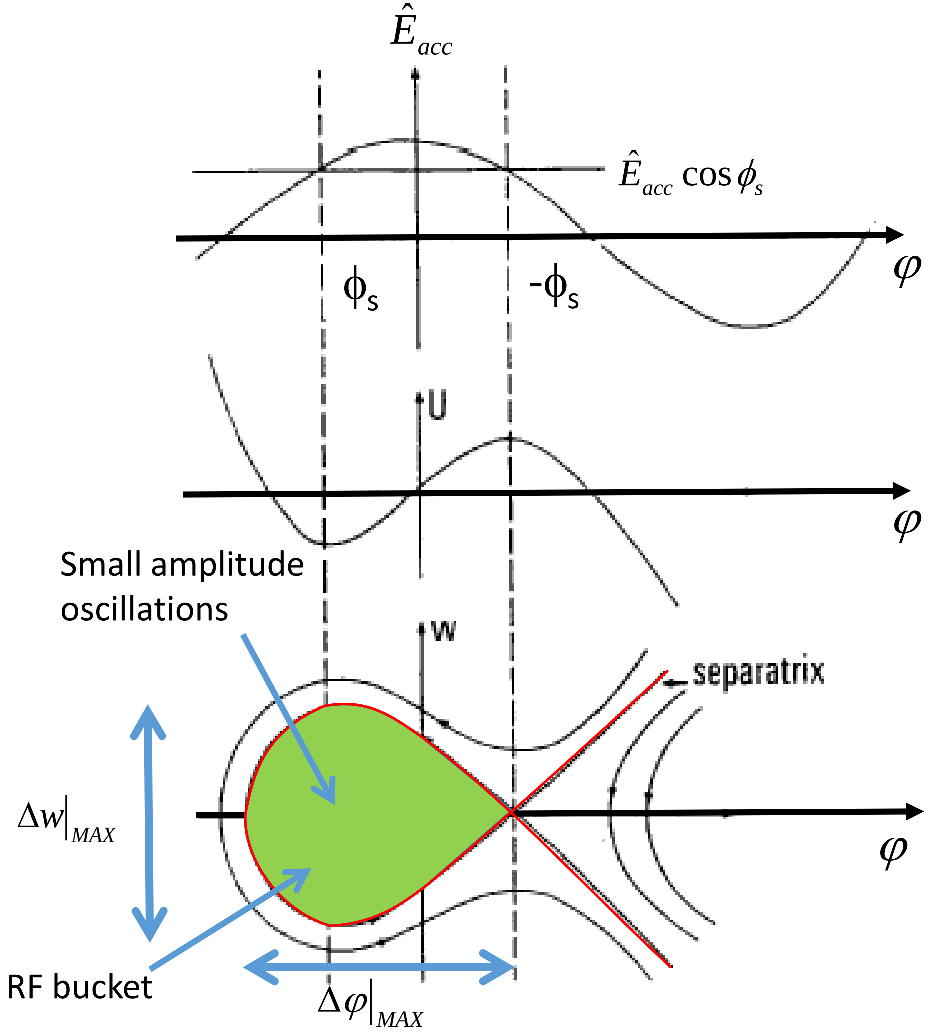
⇒the region inside the separatrix is called **RF bucket**. The dimensions of the bucket shrinks to zero if ϕ_s=0.

⇒trajectories outside the RF buckets are **unstable**.

⇒we can define the **RF acceptance** as the maximum extension in phase and energy that we can accept in an accelerator:

$$\Delta\varphi\big|_{MAX} \cong 3\phi_s$$

$$\Delta w\big|_{MAX} = \pm 2\left[\frac{qcE_o\beta_s^3\gamma_s^3\hat{E}_{acc}(\phi_s\cos\phi_s-\sin\phi_s)}{\omega_{RF}}\right]^{\frac{1}{2}}$$



EXERCISE 10: SOLUTION

γ_{in}=(W_{in}+E_{0_P})/E_{0_P}=1.0107

β_{in}=sqrt(1-1/γ_{in}^2)= 0.1449

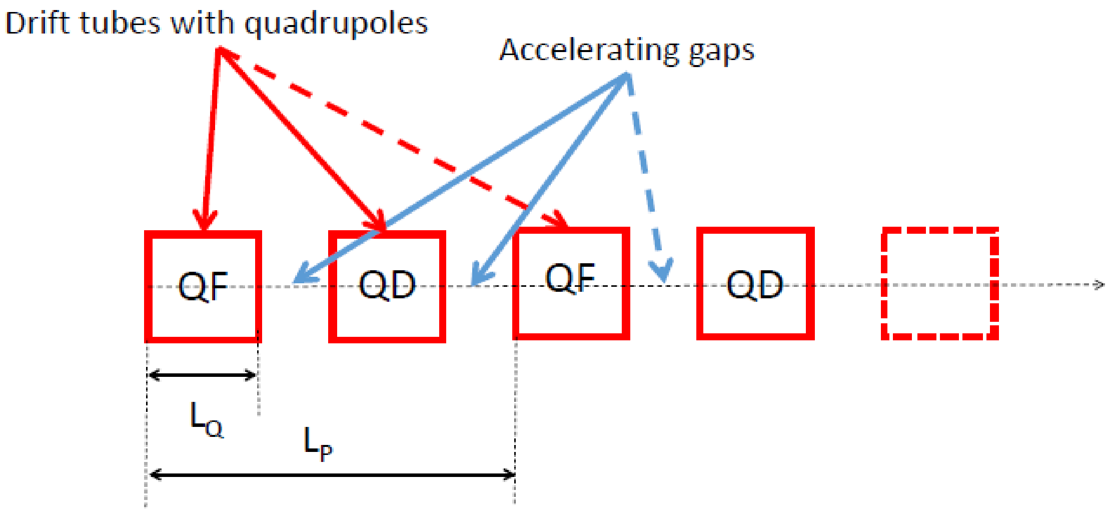
$$\Delta w\Big|_{MAX} = 2\left[\frac{qcE_o\beta_s^3\gamma_s^3\hat{E}_{acc}(\phi_s\cos\phi_s - \sin\phi_s)}{\omega_{RF}}\right]^{\frac{1}{2}} = 0.362\text{MeV}$$

$W_{MAX=}W_{in}+\Delta w$=10.362 MeV

EXERCISE 11: TRANSVERSE BEAM DYNAMICS

A DTL (Alvarez structure) working at f_{RF}=300 MHz accelerate protons with an injection energy W_{in}=4 MeV, permanent magnet quadrupoles are inside the drift tubes and the focusing system is equivalent to a **FODO lattice**, as sketched below. The quadrupoles inside the drift tubes have a length **L_Q=5 cm**.

If the average accelerating field per cell is E_{acc}=2 MV/m and the nominal synchronous phase $\phi_S=-\pi/6$, calculate, using the "**smooth approximation**" approach, the quadrupole gradient (G) that is necessary to have, in the first cells, of the structure in order to achieve a **transverse phase advance per period (σ) equal to π/3**, supposing that the period of the FODO (L_P) is exactly twice the distance between two accelerating gaps.

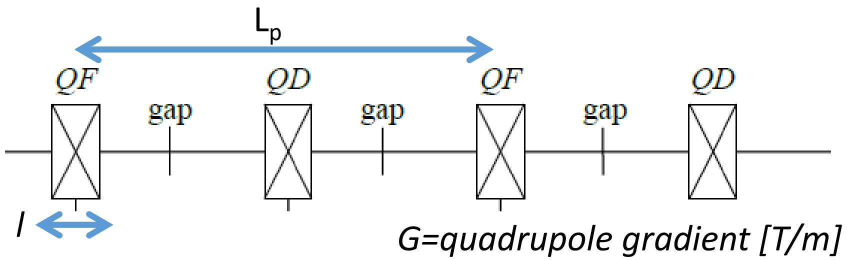


proton rest energy $m_0c^2=E_0$=938 MeV

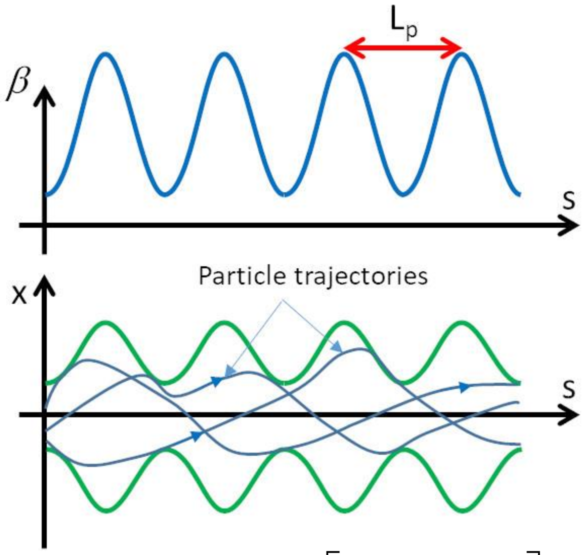
velocity of light c=2.998e8

SMOOTH APPROXIMATION OF TRANSVERSE OSCILLATIONS

⇒In case of "**smooth approximation**" of the LINAC (we consider an average effect of the quadrupoles and RF) we obtain a simple harmonic motion along s of the type (β is constant):

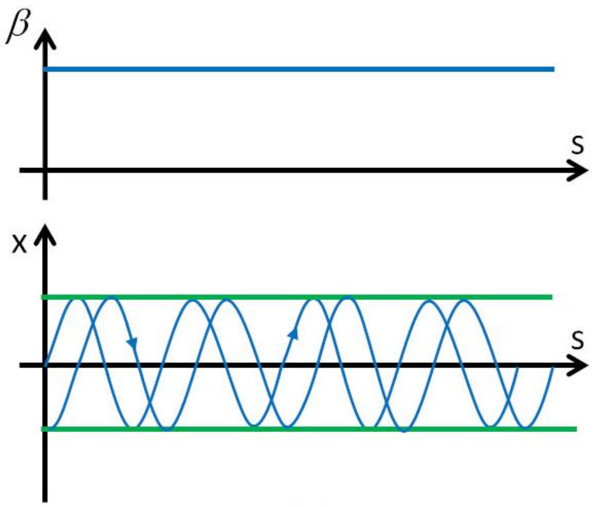


G=quadrupole gradient [T/m]
l=quadrupole length



$$x(s)=\sqrt{\varepsilon_o\beta(s)}\cos\left[\int_{s_0}^{s}\frac{ds}{\beta(s)}+\phi_0\right]$$

$$\sigma=\int_{L_p}\frac{ds}{\beta(s)}\approx\frac{L_p}{\langle\beta\rangle}$$



$$x(s)=\sqrt{\varepsilon_o}\sqrt{1/K_0}\cos(K_0 s+\phi_0)$$

$$\sigma=\int_{L_p}\frac{ds}{\beta(s)}=K_0L_p$$

Phase advance per unit length (σ/L_p)

$$K_0=\sqrt{\left(\frac{qGl}{2m_0c\gamma_s\beta_s}\right)^2-\frac{\pi q\hat{E}_{acc}\sin(-\phi_s)}{m_0c^2\lambda_{RF}(\gamma_s\beta_s)^3}}$$

Magnetic focusing elements (for a FODO)

RF defocusing term

NB: the RF defocusing term ∝f sets a higher limit to the working frequency

If we consider also the **Space Charge contribution** in the simple case of an **ellipsoidal beam** (linear space charges) we obtain:

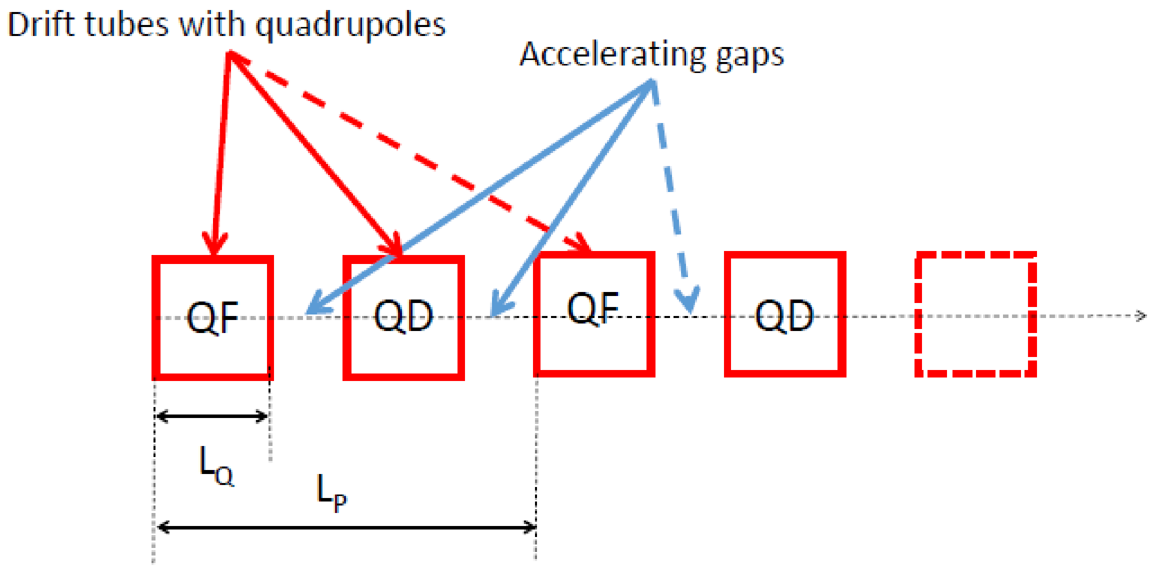
$$K_0=\sqrt{\left(\frac{qGl}{2m_0c\gamma_s\beta_s}\right)^2-\frac{\pi q\hat{E}_{acc}\sin(-\phi_s)}{m_0c^2\lambda_{RF}(\gamma_s\beta_s)^3}-\frac{3Z_0qI\lambda_{RF}(1-f)}{8\pi m_0c^2\beta_s^2\gamma_s^3r_xr_yr_z}}$$

Space charge term

I= average beam current (Q/T_{RF})
$r_{x,y,z}$=ellipsoid semi-axis
f= form factor (0<f<1)
Z_0=free space impedance (377 Ω)

For ultrarelativistic **electrons RF defocusing and space charge disappear** and the external focusing is required to control the emittance and to stabilize the beam against instabilities.

EXERCISE 11: TRANSVERSE BEAM DYNAMICS



λ_{RF}=c/f_{RF}=0.9993m

γ_{in}=(W_{in}+E_{0_P})/E_{0_P}=1.0043

β_{in}=sqrt(1-1/γ_{in}^2)= 0.0921

$L_{\_gaps}$=$\beta_{in}\lambda_{RF}$=92 mm

$$K_0 = \sqrt{\left(\frac{qGL_Q}{2m_0c\gamma\beta}\right)^2 - \frac{\pi q\hat{E}_{acc}\sin(-\phi)}{m_0c^2\lambda_{RF}(\gamma\beta)^3}} \Rightarrow G = 2\frac{m_0c^2\gamma\beta}{qcL_Q}\sqrt{K_0^2 + \frac{\pi q\hat{E}_{acc}\sin(-\phi)}{m_0c^2\lambda_{RF}(\gamma\beta)^3}} = 70\frac{T}{m}$$

L_P=2*L_{gaps}=184 mm

K_0=(π/3)/L_P=5.69 m^{-1}