Cyclotrons, synchrocyclotrons and FFAGs: Draft Version

B. Jacquot
GANIL, Caen, France

Abstract
The cyclotrons are the most used hadron accelerators: it represents a compact and efficient solution with 100% duty cycle, very well adapted for the medical applications and for the nuclear physics research. The maximal energy of a cyclotron is typically 1 GeV for a proton beam, since relativistic effects and transverse focusing limit the cyclotrons. In this introductory lecture, we present the underlying concepts. The longitudinal and transverse beam dynamics in these accelerators are covered. Some specific cyclotrons are presented (compact cyclotron at low energy, synchrocyclotron, FFAG, superconducting cyclotron, separated sector cyclotrons). The concepts used in cyclotron are numerous, and the topic is an ideal application of many ideas introduced in a basic course in accelerator physics. We provide many exercises for a better understanding.

Keywords
Isochronous cyclotrons; synchrocyclotrons; FFAG; beam dynamics; accelerator physics.

1 Introduction: from physics research to medical application

The cyclotron concept [1- 4] has been developped in the years 1929-1931 by Ernest Orlando Lawrence. The proposed accelerator is a generic and powerfull idea : a unique acceleration radiofrequency gap crossed many times by a beam having a spiraling trajectory : Such general idea has been adapted to most of cyclic accelerators (cyclotrons, synchrocyclotron, microtron, FFAG, recirculation linac).

Hence the first cyclotrons permitted to the scientists to overcome the technical difficulties of the high voltage DC accelerators (Van de Graaff). The numerous potential discoveries induced by the cyclotron idea have been recognized very soon, and a Nobel prize in Physics has been awarded to E.O Lawrence in 1939 for his pionnering work and for the large quantity of new results obtained with cyclotrons (especially with regard to artificial radioactive elements). Then the research in nuclear and particle physics made considerable progress during the period 1930-1970 thanks to proton and ion cyclotrons.

Since the 1980’s, developpement of cyclotron facilities for the production of radionucleides used in the hospitals has opened up a new era for the cyclotrons: Nowadays more 1300 cyclotons are in operation in the word (AIEA report in 2021). A cyclotron can be bought on a catalalog at several manufacters (IBA, BEST, VARIAN, SIEMIENS, SUMITOMO, GE...). For such applications, the R&D in cyclotron is led mainly by these industrial manufacturers, which aim to reduce the cost and ease the operation in a medical context.

In the near future, a larger number of cancer treatment facilities using 250 MeV proton isochronous cyclotron or synchrocyclotron could complement the standard radiotherapy technics. The X-rays radiotherapy uses low cost 5-15 MeV electron linac producing photons by bremstrahlung mechanism.

The advantage of hadron irradiation over photon is related to the very precise tumour irradiation in a narrow range of depth, the so-called Bragg peak, minimizing radiation to the healthy nearby tissues.

On the research context, the cyclotrons are very limited in their maximal energy compared to synchrotron. However many research facilities still used some very specific cyclotrons, let us cite :

- PSI (in Switzerland, providing the most powerfull proton beam of 1.4 MW at 590 MeV)
- RIBF(in Japan, operating the largest superconducting cyclotron in the world)
- TRIUMF(in Canada, housing the world's largest cyclotron, which delivers H^- at 520 MeV)
- GANIL (in France, having 5 cyclotrons in operation)

Besides, large projects could emerge using advanced cyclotron concept for muon acceleration or accelerator driven nuclear reactor.

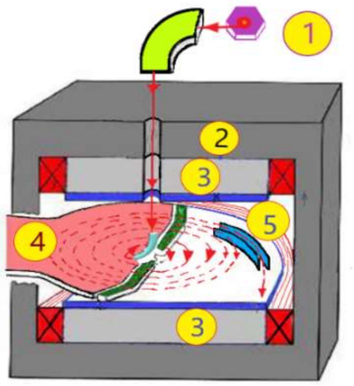

Fig. 1: The "compact cyclotron" hardware : 1) External ion source : A beam line direct the beam inside the cyclotron through an axial hole, then the beam is deviated in the cyclotron plane with an electrostatic inflector. 2) The magnet yoke: One compact magnet provide the bending force to the beam. 3) The magnet poles excited by copper coils define a complex magnetic field with modulations. 4) The 180° RF Dee for acceleration: Half of the cyclotron is at the ground potential while an hollow electrode is at sinusoidal voltage. 5) An electrostatic deflector is used for beam extraction.

2 The cyclotron principle and the longitudinal dynamics

2.1 Generalities

A cyclotron consist in a very large dipolar magnet operating with a vertical magnetic field B_z generated by an electro-magnet. Inside the magnet, two semi-circular hollow electrodes with a D shape (the so-called « Dees ») are excited by a radiofrequency generator. A sinusoidal electric field is generated in the gap between the two cavities.

The operating mode of cyclotron is quite different from the synchrotron. Synchrotrons are pulsed accelerators: the beam is injected, then accelerated, then extracted. During the acceleration, no beam is injected into the ring, the magnet field are ramped up and the RF synchronized with beam revolution frequency. The beam is delivered to the users in pulses of a given length ΔT (typically few microseconds with a fast extraction) at a given repetition frequency F_{pulse} (usually between 1 Hz and 10 Hz). The duty cycle, i.e. the product of pulse length and repetition frequency, is very low (< 0.01%), but synchrotron can deliver beams (ions or electrons) at ultra-relativistic energy.

A cyclotron will take the continuous particle beam coming out of an ion source. The beam is bunched at a given RF frequency with a pre-buncher and then accelerated continuously. The cyclotron delivers a continuous stream of particle bunches at RF frequency (we call it a continuous wave (CW) accelerator).

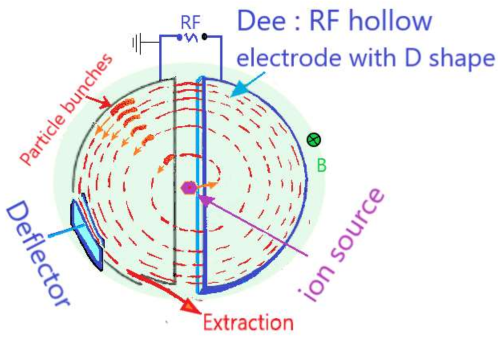

Fig. 2 : The cyclotron principle .The beam is injected in the centre of the cyclotron magnet from a source in the gap between the two semi-circular cavity (the "Dee") .The RF voltage of the Dees produces a sinusoidal electric field in the gap. The particle bunches are accelerated after each half turn. Nota: one of the Dee is at the ground potential.

Table 1: A comparison between cyclotron and synchrotron.

Comparaison	Isochronous cyclotron	Synchrotron
Revolution frequency	Constant	Variable (~ v)
Rf frequency	Constant cw acceleration 100% RF duty cycle	Variable : RF ramped pulsed accelerator very Low RF duty cycle
Orbit Radius r	Variable	Constant
Magnetic field Bz	Constant	Variable : Ramped B_{dipole}= f(time)=Bρ/ R_{dipole}
Transverse focusing	Weak focusing	Strong focusing with quadrupoles
Limits	Beam energy (γ<2)	No limit in energy (except synchrotron radiation and cost)
Particles	Protons, ions	Electrons, electons, ions

Therefore, the main advantages of the cyclotron over the synchrotron are its 100% duty cycle and its compactness (and the associated relatively low cost). However, most of cyclotron accelerators are restricted to low energy hadron beams (E<1GeV for protons) as we will see.

2.2 Revolution frequency: $\omega=\omega_{rev} = qB/ m\gamma$

Using a cylindrical coordinate system ($\mathbf{e_r}$, $\mathbf{e_z}$, $\mathbf{e_\theta}$) , we compute to the beam revolution frequency during the acceleration in the cyclotron. A particle is injected in horizontal plane of cyclotron and the vertical magnetic field B=(0,B_z ,0) produces a radial force which bend the trajectories in a circular motion between each acceleration.

$$\boldsymbol{F} = q\,(\boldsymbol{v} \times \boldsymbol{B}) = q\,(v_\theta\,.\,B_z)\,.\,\boldsymbol{e_r} \;\; \text{since} \;\; \boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e_r} & \boldsymbol{e_z} & \boldsymbol{e_\theta} \\ 0 & 0 & v_\theta \\ 0_r & B_z & 0 \end{vmatrix}$$

Between two successive accelerations, the relativistic Newton equation reads:

$$\frac{d\boldsymbol{p}}{dx} = m\gamma \frac{d\boldsymbol{v}}{dt} = q\ (v_\theta\ .\ B_z)\ .\ \boldsymbol{e_r}$$

The magnetic force, being always perpendicular to the motion produces an uniform circular motion. This circular motion corresponds to a radial acceleration (related to the centrifugal force, see exercise n°1*)*

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{\|\boldsymbol{v}\|^2}{R}\right).\ \boldsymbol{e_r}$$

$$m\gamma \frac{d\boldsymbol{v}}{dt} = m\gamma\ \frac{\boldsymbol{v^2}}{R}\boldsymbol{e_r} =\ q\ (v_\theta\ .\ B_z)\ .\ \boldsymbol{e_r}$$

Rearranging the last equation*,* we express the radius R of the trajectory

$$R =\ \mathrm{m}\gamma v_\theta/\mathrm{q}\ B_z\ \ =\ (\mathrm{P}/\mathrm{q})\ /B$$

The particle revolution frequency reads*:*

$$F_{revolution} = \frac{v}{2\pi\mathrm{R}}\ =\ \frac{\mathrm{q}\ B_z}{2\pi\ \ m\ \gamma}$$

Hence, in the non-relativistic approximation *(γ ~1),* the revolution frequency is independent of the energy and the beam radius in the cyclotron. We express generally the angular velocity ω_{rev} .

$$\omega = \omega_{rev} = \frac{d\theta}{dt} = \frac{v_\theta}{R} = 2\pi\ F_{rev} = \frac{qB}{m\gamma}$$

Exercise 1

a. Demonstrate that in a uniform circular motion, the radial acceleration is $\boldsymbol{a} = \left(\frac{v^2}{R}\right).\boldsymbol{e_r}$ **.**

You can use parametric equations for a circular motion $X(t) = Rcos(\omega t)\quad and\ Y(t) = R\ sin(\omega t)$

b. Computing the velocity and the acceleration, demonstrate that the acceleration is radial .

Answer : $v_x = \frac{dX(t)}{dt} = -\omega R\ sin(\omega t)\quad and\ v_y = \frac{dY(t)}{dt} = +\omega R\ cos(\omega t)$

So the velocity modulus is $v = \|\boldsymbol{v}\| = \sqrt{v_x^2 + v_y^2} = \omega R$

Then the acceleration is

$$a_x = \frac{dv_x}{dt} = -R\omega^2\ cos(\omega t)\quad and\ a_y = \frac{dv_x}{dt} = -R\omega^2\ sin(\omega t)\qquad \|a\| = R\omega^2 = v^2/R$$

The "longitudinal velocity **v** is perpendicular to **a** (check that $\boldsymbol{v}.\boldsymbol{a} = v_x.a_x + v_y.a_y = 0$) **,** so the acceleration is radial. We can conclude that, in an uniform circular motion, the acceleration vector is $\boldsymbol{a} = \left(\frac{\|v\|^2}{R}\right).\boldsymbol{e_r}$

2.3 Beam synchronisation with RF

In fact, there is two alternative solutions to guarantee a proper synchronization with the RF field during the acceleration:

- Making the revolution frequency ω_{rev} constant to match the fixed RF frequency (isochronous cyclotron)
- Matching the RF frequency to the variable revolution frequency (this is the synchrocyclotron and FFAG options, see chapter V)

We will concentrate on the first technics (isochronous) since it provides 100% duty cycle accelerators, while the second, less used nowadays, delivers pulsed beam.
If we choose a RF generator with a constant frequency and with a voltage $V=U_0\ cost(\omega_{rf}\ t\ )$, the synchronization between the beam and accelerating RF cavity requires first a careful tuning of the field B_0 at injection:

$$\omega_{rf} = H.(2\pi\ F_{rev}) = H\ qB_0/m \quad .$$

where H is an integer called Harmonic.
The early cyclotron was designed with an uniform axial field $B_z=B_0.\mathbf{e_z}$. If the magnetic field is uniform, the beam revolution frequency will decreases progressively with the energy and radius due to the special relativity. Since the velocity in a circular motion is $|v|= R\ \omega_{rev}$, we have

$$\omega_{rev} = \frac{qB_0}{m\gamma} = \frac{qB_0}{m}.\sqrt{1-{v^2}/{c^2}} = \frac{qB_0}{m}.\sqrt{1-{R^2\omega^2}/{c^2}} \quad .$$

The early cyclotrons, having a uniform B_z field, was not able to provide high energy beams: at large energy, because of non-constant revolution frequency, the bunches arrive out of phase at the gap, as explained in the following fig. 3:

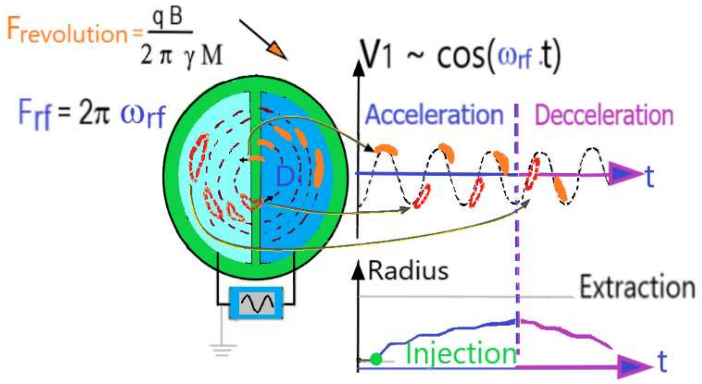

Fig. 3 Desynchronization of bunches in uniform field cyclotron**.** With an uniform field*,* the bunches arrive out of phase after few turns when the γ factor increases. The bunches start then to be decelerated and no beam reaches the extraction. The B field should evolve with the radius as the Lorentz factor in order to maintain an acceleration : $B_z= f(radius) \sim \gamma$

2.4 Isochronous cyclotron

In order to keep the synchronisation with the RF acceleration, the vertical field B_z should follow the evolution of the gamma factor

$$B_z = B_0.\gamma(Radius) = {B_0}\Big/{\sqrt{1-R^2\omega^2/c^2}} \quad \rightarrow \quad \omega = \frac{qB_z}{m\gamma} = \frac{qB_0}{m} \quad .$$

With this field $B_z(R)$, the time to perform one turn is constant whatever the energy and radius. The cyclotron is said to be ISOCHRONOUS. In the fig. 4, the bunches are represented with respect to the RF accelerating wave at each gap crossing between the two Dees. With the isochronous conditions (F_{rev}=Constant), the beam arrives always in the gap at the same optimum accelerating RF phase during the acceleration.

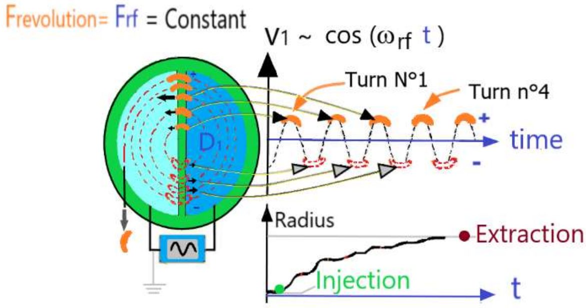

Fig. 4: Bunch synchronization with RF in an isochronous cyclotron**.** For a perfect acceleration, the two following conditions are required:

- Synchronism with RF at injection $\omega_{rf} = H\omega_{rev} = H.qB_0/m$.
- Isochronism ω_{rev} = constant during the acceleration, with $B_z(R) = B_0/(1 - R^2\omega^2/c^2)^{1/2}$

2.4.1 Choice of RF harmonic $H=\omega_{rf} / \omega$

In the gap, it is required that the electric field to be directed always in the right direction to obtain the acceleration. The AC generator must alternate the polarity of the 180° Dees in order to give to the ion an accelerating electric field every half period. With this cavity geometry, the harmonic H should an odd value: ω_{rf} =H . qB_0/m. Besides, the harmonics H will define the number of bunches per turn in the cyclotron.

2.4.2 Variable energy cyclotron

Most of industrial cyclotrons for the medical application have a fixed energy. In the research labs, the flexibility is more important, whatever the technical complexity. The ion energy variation is associated with the beam velocity at the extraction radius extraction: which impose to modify the magnetic field and its radial dependence (to be adapted to the new γ factor), and the RF frequency.

The variable energy cyclotrons having a variable final velocity $(v_{final} = R_{extraction}\, \omega)$, requires :

- A RF cavity with variable frequency ($\omega_{rf} = H\, \omega$) adapted for each required energy
- A magnet with many correction coils* for the modification of radial field evolution

$$B_z(R) = B_0/(1 - R^2\omega^2/c^2)^{1/2} \text{ with } \quad B_0 = \omega m/q$$

***Nota :** Correction coils (called trimming coils) are a set of adjustable concentric coils located on the pole pieces inside the magnet gap. With a variable frequency cavity and adjustable trimming coils (excited with independent power supplies), a cyclotron can accelerate a wide range of ion species (q,m) at diverse energies (γ).

Exercise n°2 : The isochronous cyclotron, numerical application

An isochronous cyclotron uses a RF cavity at 60 MHz at the RF harmonic H=3

a. Compute the time needed to perform one turn for the accelerated ions.

b. Compute the average field B needed to accelerate proton in a non-relativistic approximation

Answer :

a. Revolution frequency $F_{rev}=F_{rf}/H=20$ *Mhz* $\Delta T=1/(20.\ 10^{-6}\ s) =50ns$

b. $\omega = qB/\gamma m = \omega_{RF}/H$ *:* at relatively low energy the Lorentz factor γ is close to 1.

So using the proton mass m_p ~ 1.6 10^{-27}kg and proton charge~ 1.6 10^{-19} C, F_{rf}= 60 MHz= $\omega_{RF}/2\pi$

$B_0 \sim m_p /q\ .\ (2\pi F_{RF}/ H)= 10^{-8}.\ 10^6\ 20\ .\ 2\pi .=1.26$ Tesla

2.5 Acceleration with RF dees

2.5.1 The classical α_{cav}=180° RF dee

The energy gain, in the accelerating gaps of the RF cavity (called "dee"), depend upon the phase at the gaps. In a isochronous cyclotron, with constant revolution frequency, the particle's azimuth θ is connected revolution frequency:

$\theta = \omega_{rev}\ t + constant$. While the RF phase evolves like $\phi_{rf} = H\theta + constant$.

With a Dee angular width α_{cav} =180° (fig.1,2), there are two accelerations per turn. Using as a reference the phase φ_{mid} in the middle of the Dee, the phase φ_{gap1} is , at the entrance of the dee :

$$\varphi_{gap1} = \varphi_{mid} - {}^{H\alpha_{cav}}/_{2}$$

Taking account that the voltage should be negative at the entrance of the Dee to accelerate the positive ions, the total energy gain per turn is:

$$\delta E_{turn} = \delta E_{tap1} + \delta E_{tgap2} = -qV(\varphi_{gap1}) + qV(\varphi_{gap2})$$

$$= -q\,U_0 \sin\left(\varphi_{mid} - {}^{H\alpha_{cav}}/_{2}\right) + U_0 \sin\left(\varphi_{mid} + {}^{H\alpha_{cav}}/_{2}\right)$$

$$= +\,2q\,U_0 \cos(\varphi_{mid}).\sin\left({}^{H\alpha_{cav}}/_{2}\right) = 2q\,U_0 \cos(\varphi_{mid}).\sin(H.90°)$$

We demonstrate here that the even harmonics H=2,4... produce no energy gain (sin 180°=0), since the energy gain in the first acceleration is compensated by a deceleration in the second.

In principle, all particles with a phase satisfying$-\pi/2 < \varphi_{mid} < \pi/2$, are accelerated ($\delta E_{turn}$ >0). But the particles with low energy gain are lost either at injection, extraction, or during the acceleration. The longitudinal acceptance Δϕ do not exceed generally 40°(out of 360°), therefore, the optimization of the transmission requires a RF buncher upstream of the cyclotron.

2.5.2 RF dee with smaller angle α_{cav}

Some cyclotrons possess RF-cavities with an angular extent α_{cav} much smaller than 180°, which permit to increase the number of acceleration per turn. We present in fig.5, a cyclotron with 2 independent cavities producing four accelerations per turn.

$$\delta E_{turn} = N_{gap}\, q\, U_0.\sin\left({}^{H\alpha_{cav}}/_{2}\right).\cos(\varphi_{mid})$$

The kinetic energy of any particle after N_{turn} is connected the phase in the cavity:

$$E(N_{turn}) = E_N = E_{injection} + N_{turn}\, N_{gap}\, q\, U_0.\sin\left({}^{H\alpha_{cav}}/_{2}\right).\cos(\varphi_{mid})$$

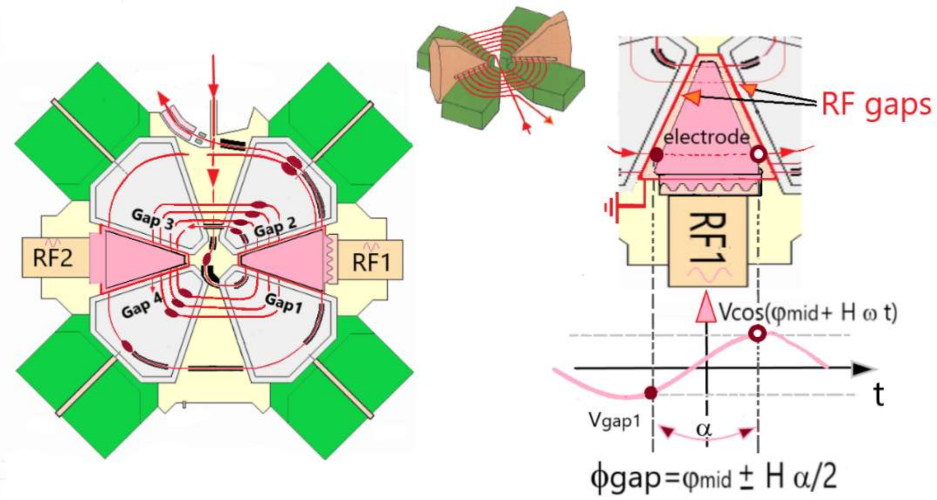

Fig. 5 The Isochronous Cyclotron "CSS2" (Ganil, France) has two RF cavities with an angular width of α_{cav}=34°, the RF is operated with H=2 at a frequency between 7 and 14MHz, depending on the chosen heavy ion beam. U_0 reach 230 kVolts. The incoming beam has been already pre-accelerated with 2 consecutive cyclotrons. The positive ions ($^{12}C^{6+}$, $^{40}Ca^{20+}$, …,$^{238}U^{58+}$) are accelerated by a negative voltage at the cavity entrance, and by a positive voltage at the exit.

2.5.3 RF acceleration, radial size ΔR and bunch separation δR

The acceleration of a bunch having a finite length $\Delta\varphi = \omega_{rf}\Delta t = H\,\omega\,\Delta t$, increases the bunch radial size: Two particles arriving at different time in accelerating gap will get a different energy :

$$E_1 = E_0 + q\,U_0 . \cos(0) \quad \text{and} \quad E_2 = E_0 + q\,U_0 . \cos(\omega_{rf} . \delta t)$$

So the bunch length Δt induces an energy dispersion which ends up with a radial dispersion ΔR of the bunch . The reference particle after N turns is located at $R_0 = B\rho_0 / B_z$, and the horizontal size of the bunch ΔR satisfies the relation :.

$$\Delta R / R = \Delta B\rho / B\rho = \Delta p / p = \gamma / (\gamma + 1) \,.\, \Delta E / E$$

Nota : We have used the relativistic formula $dE/dp = \gamma / (\gamma + 1) \,.\, (E / p)$

For low energy ions γ ~1-1.5, we have

$$\Delta R/R \approx 1/2 \,.\Delta E/E \approx 1/2 \;\; \Delta\cos(\varphi) \approx 1/4\, \Delta\varphi^2$$

The radial beam size ΔR is very sensitive to the bunch length Δt and RF Harmonic H: ($\Delta\varphi = H\,\omega\,\Delta t$)

Besides, the radial separation between two bunches δR, (i.e. the distance between two successive turns) is decreasing with the radius R in the cyclotron:

$$E_N = E_{injection} + N_{turn} . \delta E_{turn} \approx 1/2 \;.m\,v^2 \approx 1/2 \;.m\,(R\omega)^2$$

$$\delta R/R \approx 1/2 \,.\delta E_{turn}/E_N \sim 1/R^2$$

As a consequence, the bunch spacing δR ~ 1/R is often very small at the cyclotron extraction at large radius, : The bunches overlap each other and the extraction of the bunches independently is often rather difficult (see chapter 7).

Exercise n°3 : the K_b value of a cyclotron and its maximal capability

a. A cyclotron is designed to accelerate ions with A nucleon and a charge state Q. Demonstrate than the maximal kinetic energy (E_K/A) of a cyclotron can be written as $E_K/A = K_b.(Q/A)^2$

Nota : Give the K_b factor in a non-relativistic approximation using the extraction radius R_{extr}, the magnetic field B. Assuming the mass of the ions is m~ Am_u and the charge Q of the ions is q= Qe_0 . The quantity m_u is the atomic mass unit (m_u= 1.66 10^{-27}kg)

b. A cyclotron with K_b= 30 MeV can accelerate proton up to 30 MeV. What would be the maximal energy of a carbon ion beam $^{12}C^{4+}$ in this cyclotron.

c. What will be the revolution frequency with a field B_0= 1.26 Tesla for proton beam (H^{1+}) and for or the $^{12}C^{4+}$ beam .

Answer :

a. The kinetic energy is $E_K = (\gamma - 1)mc^2 \sim \frac{1}{2}mv^2 = \frac{1}{2}m.(R_{extr}.\omega)^2 = \frac{1}{2}m.(R_{extr}.qB/m)^2$

$$E_K/A = \frac{1}{2}m/A.\left(R_{extr}.\frac{qB}{m}\right)^2 = \frac{1}{2}m_u.\left(R_{extr}.\frac{e_0B}{m_u}\right)^2.\left(\frac{Q}{A}\right)^2 = 1/2\ (R_{extr}.e_0B)^2/m_u.(Q/A)^2$$

Therefore $E_K/A = \frac{1}{2}\frac{(R_{extr}.e_0B)^2}{m_u}.\left(\frac{Q}{A}\right)^2 = K_b.\left(\frac{Q}{A}\right)^2$

b. For a proton (1 nucleon), we have A=1 and q=+1 e_0 *$E_K/A = 30 . (Q/A)^2 = 30$* MeV/A
For $^{12}C^{4+}$: 12 nucleons (6 protons,6 neutrons), q=+4 e_0 .So the maximal energy for $^{12}C^{4+}$ is E_K/A= 30. $(4/12)^2$= 3.33 MeV/A (« MeV per nucleon »)
$F_{rev}=\omega_{rev}/2\pi$= (qB/m) / 2π = 60 MHz with protons and F_{rev}=20 MHz with $^{12}C^{4+}$.

Let us note that the revolution frequency for $^{12}C^{4+}$, $\omega=qB_0/m \sim q\,B/(A\,m_u)$, will be 3 times smaller than the one of the proton beam. We could imagine to operate the cyclotron at F_{rf}=60 Mhz for the two beams, corresponding to $H=\omega_{RF}/\omega=1$ for protons, and H=3 for Carbons (a slight B adjustment would be needed, since the carbon ion mass is not exactly twelve times the proton mass).

3 Transverse dynamics and orbit stability

In the cyclotron magnet, the particles travel a long way before the extraction corresponding to many turns in the magnetic field. Therefore, it is important to study if a particle starting with a slight deviation to the reference orbit, is transported correctly.

To study this aspect, we will introduce the following concepts: cylindrical coordinates, field index n, transverse stability and tunes.

3.1 Equation of motion in cylindrical coordinates

We will provide rigorous formulation of the particle trajectories in the vicinity of an ideal circular trajectory over one turn: we use a cylindrical **coordinate system ($e_r(t)$, e_z, $e_\theta(t)$) .**

How evolves an arbitrary particle, in the field B(**r**):

$$\boldsymbol{r}\,(t) = \mathrm{R}\,.\boldsymbol{e_r} + \mathrm{z}.\boldsymbol{e_z} = (\mathrm{R}_0 + x(t)).\boldsymbol{e_r} + \mathrm{z}.\boldsymbol{e_z}$$

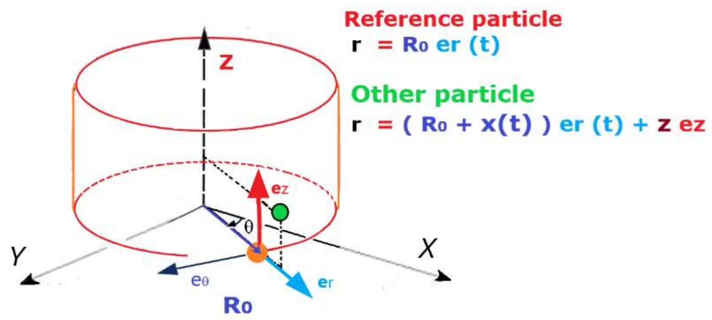

Fig. 6 In the cylindrical system, the basis vector $\boldsymbol{e_r}$ and $\boldsymbol{e_\theta}$ (azimuthal vector) follow the reference particle and evolve as function of time. Let us note that the frame ($\mathbf{e}_r$, $\mathbf{e}_z$, $\mathbf{e}_\theta$) is direct : $\mathbf{e}_r \times \mathbf{e}_z$, = + $\mathbf{e}_\theta$. In the equations of motion, computing the derivative of the position vector **r** is required. Using a Cartesian frame (X,Y), basic vectors are : $\mathbf{e}_r$=(cos(ωt), sin(ωt)) and $\mathbf{e}_\theta$=(sin(ωt), cos(ω t)). Then, the vector derivatives required in the equations of motion are: $d\boldsymbol{e_r}/dt = \boldsymbol{\omega}\,.\boldsymbol{e_\theta} = d\theta/dt.\boldsymbol{e_\theta}$ and $d\boldsymbol{e_\theta}/dt = -\boldsymbol{\omega}\,.\boldsymbol{e_r}.$ Finally, we have $\boldsymbol{d^2e_r}/dt^2 = -\omega^2\,.\boldsymbol{e_r} = v^2/R^2.\boldsymbol{e_r}$, since ω=v/R.

The equations of motion will be determined by : $\frac{d\boldsymbol{p}}{dt} = m\gamma\frac{dv}{dt} = m\gamma\frac{d^2\boldsymbol{r}}{dt^2}$

In the horizontal plane, the time derivatives will act on R(t), $\mathbf{e}_r$(t), and $\mathbf{e}_\theta$(t) .

Since we have $d\boldsymbol{e_r}/dt = \boldsymbol{\omega}\,.\boldsymbol{e_\theta} = d\theta/dt.\boldsymbol{e_\theta}$: (see Fig. 6 for a demonstration)

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dR}{dt}.\boldsymbol{e_r} + \frac{dz}{dt}\ \boldsymbol{e_z} + R\frac{d\theta}{dt}.\boldsymbol{e_\theta}$$

While the acceleration **a** is determined (by using $d\boldsymbol{e_\theta}/dt = -\boldsymbol{\omega}\,.\boldsymbol{e_r}$):

$$\boldsymbol{a} = \frac{d^2r}{dt^2} = \frac{d\boldsymbol{v}}{dt} = [\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2].\boldsymbol{e_r} + \frac{d^2z}{dt^2}\ \boldsymbol{e_z} + [R.\frac{d^2\theta}{dt^2} + 2r.\frac{dR}{dt}.\frac{d\theta}{dt}].\boldsymbol{e_\theta}$$

Outside the gap, there is no electric field, the velocity |**v**| is constant, and the Lorentz factor γ is constant: we can neglect dω/dt=d²θ/dt² ~ *0* (we have no longitudinal acceleration). The relativistic Newton-Lorentz equation is:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{q}{m\gamma}\ (\boldsymbol{v}\times\boldsymbol{B})$$

Expressing the cross product in the cylindrical frame:

$$\boldsymbol{v}\times\boldsymbol{B} = \begin{vmatrix}\boldsymbol{e_r} & \boldsymbol{e_z} & \boldsymbol{e_\theta}\\ v_r & v_z & v_\theta\\ B_r & B_z & B_\theta\end{vmatrix}$$

This leads to the 3 equations used to study the stability of motion in an arbitrary field **B**=(B_r,B_z,B_θ)

$$\frac{d^2\boldsymbol{r}}{dt^2}.\boldsymbol{e_r} = [\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2] = \frac{q}{m\gamma}\ (v_z.B_\theta\ - v_\theta.B_\theta)$$

$$\frac{d^2\boldsymbol{r}}{dt^2}.\boldsymbol{e_z} = \frac{d^2z}{dt^2} = \frac{q}{m\gamma}\ (v_\theta.B_r\ - v_r.B_\theta)$$

$$\frac{d^2\boldsymbol{r}}{dt^2}.\boldsymbol{e_\theta} = R.\frac{d^2\theta}{dt^2} + 2\frac{dR}{dt}.\frac{d\theta}{dt} = \frac{q}{m\gamma}\ (v_\theta.B_r\ - v_r.B_\theta)$$

3.2 The definition of local field index n

In an isochronous cyclotron, the magnetic field increases with the radius to fullfill the ischronism condition. As we will see later, the magnetic will generate defocusing force. Let's suppose that the field evolution can be described locally as a power law :

$$B_z(r) = K\,R^{-n}$$

The *n* factor is called "field index". This definition of "*n*" eases the analytical calculation of the orbit stability but does'nt restrict the generality. The sign is a convention.

Nota : If the field is B_z=B_0 at r=R_0, the field can be written as $B_z(r) = B_0.(r/R_0)^{-n}$

Around a given orbit with a radius R_0, the vertical fied can be expanded a radius $R = R_0 + x$ as

$$B_z(R = R_0 + x) = B_z(R_0) + x.\left(\frac{dB_z}{dR}\right) + \cdots = B_0 - x.n\left(\frac{B_0}{R_0}\right) + \cdots$$

$$B_z(R = R_0 + x) = B_0(1 - x.\frac{n}{R_0}) + \cdots$$

The field index n (*R*) can be seen as the fractional change in field assiociated with a fractional change in radius.

$$n(R) = ((dB_z)/B_z)/(dR/R) = \left({}^{R}\!/_{B_z}\right).\left({}^{dB_z}\!/_{dR}\right)$$

Historical note : In the early synchrotron, the vertical focusing was not ensured with quadrupoles,but by dipoles having a slight a positive field index (a decreasing field with radius : n >0) :This technics is called weak focusing. In the isochronous cyclotrons, the field index is ajusted for isochronism and so the field should increase with increasing energy and radius (negatif field index n <0).

Between the poles, in the vaccum chamber, the static magnetic field satisfies **Curl B=0.** The dependence of the vertical field B_z, produces a radial field component B_r, as it can be seen from the maxwell equation.

$$\nabla \times \boldsymbol{B} = \left(\frac{dB_\theta}{dz} - \frac{dB_z}{Rd\theta}\right).\boldsymbol{e_r} + \left(\frac{d(RB_\theta)}{RdR} - \frac{dB_\theta}{dz}\right).\boldsymbol{e_z} + \left(\frac{dB_z}{dR} - \frac{dB_r}{dz}\right).\boldsymbol{e_\theta}$$

-Using $B_z(R) = K\,R^{-n}$ and $(\nabla \times \boldsymbol{B}).\boldsymbol{e_\theta} = 0$, we have $\frac{dB_r}{dz} = \frac{dB_z}{dR} = -\,n\,K\,R^{n-1} = -n\,.\frac{B_0}{R}$

so $B_r = -z.n.B_0\left(\frac{z}{R_0}\right) + Constant$ (the constant is zero)

-Since the radial component of **curl B** is zero *(so* $dB_\theta/dz = dB_z/Rd\theta$ *)*, therefore to the first order in z, the azimutal component B_θ is :

$$B_\theta = z.\frac{dB_z}{Rd\theta} + \ldots$$

Finally,we expect a general magnetic field $\boldsymbol{B} = (B_r,\, B_z, B_\theta\,)$ in the cyclotron with the following appearence :

$$B_r = -z.n.B_0.\frac{z}{R} + \cdots \qquad B_z = \frac{B_0}{\left(1 - \frac{R^2\omega^2}{c^2}\right)^{\frac{1}{2}}} = K\,R^{-n} \qquad B_\theta = z.\frac{dB_z}{Rd\theta} + \cdots$$

Exercice 4 : Demonstrate that the field index *n*, is related to the Lorentz factor in an isochronous cyclotron : $n(R) = 1 - \gamma^2$. Remember that $n = \left({R}/{B_z}\right).\left({dB_z}/{dR}\right)$

Answer : Since in isochronous cyclo. $B_z = B_0.\gamma(R)$, let's computes $\frac{d\,B_z}{dR} = B_{0.}\frac{d\gamma}{dR}$

$$\frac{dB_z}{dR} = B_{0.}\frac{d\gamma}{dR} = B_0\frac{d\left(1-\frac{R^2\omega^2}{c^2}\right)^{-1/2}}{dR} = B_0 \times \left(\frac{2R\,\omega^2}{c^2}\right) \times -\frac{1}{2} \times \left(1-\frac{R^2\omega^2}{c^2}\right)^{-\frac{3}{2}}$$

$$\frac{dB_z}{dR} = -B_0.\left(\frac{\beta^2}{R}\right).\left(1-\frac{R^2\omega^2}{c^2}\right)^{-3/2} = -\frac{B_0}{R}.\beta^2.\,\gamma^3 = -\left(\frac{B_0.\gamma}{R}\right).\beta^2.\,\gamma^2 = -\left(\frac{B_z}{R}\right).\beta^2.\,\gamma^2$$

Remember $\gamma^2 = 1/(1-\beta^2)$ so it implies that $\beta^2\,\gamma^2 = 1 - \gamma^2$

Finally, we have $\frac{dB_z}{dR} = -\left(\frac{B_z}{R}\right).\beta^2.\,\gamma^2 = -\left(\frac{B_z}{R}\right).(1-\gamma^2)$

Besides $\frac{dB_z}{dR} = -n\,K.R^{-n-1} = -n\left(\frac{B_z}{R}\right)$

So, by identification, we have in isochronous cyclotron : $n(R) = 1 - \gamma^2$

3.3 Horizontal stability and the radial tune Q_r

A reference particle (m, q, **v**=v_0 e_θ) is injected in a cyclotron having a field $B_z(r)=B_0\,(r/R_0)^{-n}$ at a radius $r=R_0$. The field B_0 is adjusted to $B_0=(P_0/q)/R_0= B\rho/R_0$.

This particle will describe a perfect circle of radius $R=B\rho/B_z=R_0$ which correspond to the radius of injection.

If a particle with the same velocity $v_0 = R_0\omega$ is injected a radius R= $R_0 + x$, it will oscillates around the ideal trajectory.

$$\boldsymbol{r}\,(t) = \mathrm{R}\,.\boldsymbol{e_r} + \mathrm{z(t)}.\boldsymbol{e_z} = (R_0 + x(t)).\boldsymbol{e_r} + \mathrm{z(t)}.\boldsymbol{e_z}$$

We project the Newton equation on horizontal plane (the radial plane), as obtained in *3.1 :*

$$\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] = \frac{d^2R}{dt^2} - R(\omega)^2 = \left[\frac{d^2R}{dt^2} - R\left(\frac{v_0}{R}\right)^2\right] = \frac{q}{m\gamma}\,(v_z.B_\theta - v_\theta.B_\theta)$$

For this particle, we have $\frac{d\theta}{dt} = \omega = \frac{v}{R} = \frac{v_0}{R}$

So, with $R = R_0 + x$

$$\left[\frac{d^2x}{dt^2} - \frac{{v_0}^2}{R_0 + x}\right] = \frac{q}{m\gamma}\,(v_z.B_\theta - v_\theta.B_z)$$

to the first order in x : $(R_0 + x)^{-1} = {R_0}^{-1}.(1 + x/R_0)^{-1} = {R_0}^{-1}.(1 - \frac{x}{R_0} + 0(x^2)\ )$

so we get

$$\left[\frac{d^2x}{dt^2} - \omega^2 R_0.(1 - \frac{x}{R_0} + \cdots)\right] = \frac{q}{m\gamma}\,(v_z.B_\theta\ - v_\theta.B_\theta)$$

-The quantity v_z. B_θ is zero since B_θ =0 in our case (no azimuthal component)

- The velocity of our particle is $v_\theta = v_0 = R_0\omega$ and $qB_0/m\gamma = \omega$ (see 2.2)

- The field around R_0 is $B_z = B_0\left(1 - x.\frac{n}{R_0}\right) + 0(x^2)$

$$\left[\frac{d^2x}{dt^2} - \omega^2 R_0. + -\omega^2 x + \cdots)\right] = -\frac{q}{m\gamma}\,[v_\theta\,].[B_z] = \frac{qB_0}{m\gamma}\,[R_0\omega].[(1 - x.n/R_0\,) + \cdots\,]$$

Rearranging the equation, we get

$$\left[\frac{d^2x}{dt^2} - \omega^2 R_0. -\omega^2 x + \cdots)\right] = -\omega[R_0\omega].[(1 - x.n/R_0\,) + \cdots\,]$$

$$\frac{d^2x}{dt^2} = -\omega^2(1 -\ n)x + \cdots$$

The field index *n*, being negative in an isochronous cyclotron since $dB_z/dR > 0$, the quantity $(1-n)$ is positive : indicating a sinus like solution. Hence, a particular solution can be found using initial condition $x(t{=}0) = x_0$. At first order,

$$x(t) = x_0\cos(\omega\,Q_r\ t + \varphi)\quad \text{with}\quad Q_r = (1-n)^{1/2}$$

The quantity $Q_r = (1-n)^{1/2}$ is called the "radial tune", it correspond to the number of oscillations per turn in the radial plane around the reference orbit. The motion correspond to a stable oscillation if Q_r is real, i.e. $(1-n) > 0$.

3.4 Vertical stability

Let us follow a particle starting at radius $R=R_0$ at the altitude $z=z_0$ between the pole. The study of the motion is obtained by projecting the newton equation on the $\mathbf{e_z}$ axis (3.1)

$$\frac{d^2\boldsymbol{r}}{dt^2}.\boldsymbol{e_z} = \frac{d^2z}{dt^2} = \frac{q}{m\gamma}\,(v_\theta.B_r\ - v_r.B_\theta)$$

Using $\nabla \times \boldsymbol{B} = \boldsymbol{0}$, the field index produces the radial component $B_r = -n\,B_0\,z/R$, While the angular component could come from an angular modulation of B_z: $B_\theta = z.\frac{dB_z}{R\,d\theta} + \cdots$

The revolution frequency being $\omega = qB_0/m\gamma$, we have to the first order in z :

$$\frac{d^2z}{dt^2} = \frac{\boldsymbol{q}}{m\gamma}\left(-R\omega.n.\frac{B_0\,z}{R} - v_r.z.\frac{dB_z}{Rd\theta}\right) = -\omega^2.\left(n + \frac{v_r}{\omega B_0}.\frac{dB_z}{Rd\theta}\right).z$$

We define the so called vertical tunes Q : $\quad Q_z^2 = \left(n + \frac{v_r}{\omega B_0}.\frac{dB_z}{Rd\theta}\right)$

and we get $\frac{d^2z}{dt^2} = -\omega^2.Q_z^2\,.z$

The nature of the dynamics in the vertical plane will depend of the sign of Q_z^2 .

- Case [Q_z^2] <0 : vertical instability

if the azimuthal component is zero $B_\theta=0$, $Q_z^2 = n$ since $dB_z/d\theta = 0$. A particular solution of the equation is an exponential: the field index is negative in isochronous cyclotron (see ex. n°4, $n=1-\gamma^2$):

$$z(t) = z_0 \exp(\omega\, |-n|^{1/2}\, t) = z_0 \exp(\, |-n|^{1/2}\, \theta)$$

All the trajectories with $z_0 \neq 0$ will diverge exponentially (the transmission of such cyclotron would be very low).

- Case [Qz^2] > 0 : vertical stability

If some angular modulations are done on the magnet poles, providing non zero B_θ components, the quantity Q_z^2 can be positive: $Q_z^{\,2} = \left(n + \left(\frac{v_r}{\omega B_0}\right) . \frac{dB_z}{Rd\theta}\right) > 0$

In that case, a particular solution of the equation is a cosinus :

$$- \qquad z(t) = z_0 \cos(\omega\, Q_z\, t + \varphi) \quad \text{with} \quad Q_z = \sqrt{\left(n + \left(\frac{v_r}{\omega B_0}\right) . \frac{dB_z}{Rd\theta}\right)}$$

The quantity Q_z correspond to the number of oscillation per turn of any particle around the reference orbit in the vertical plane. An angular field modulation $B_z = B_0.F(r,\theta)$ associated with the B_θ component are used to provide an additional vertical focusing . The cyclotron with such field dependence is called "Azimuthally Varying Field" (see "AVF cyclotron" chapter). The motion in the vertical plane should be stabilized with angular field modulations $B_z = f(\theta)$.

3.5 Qualitative understanding of the vertical instability

In the isochronous cyclotrons, the vertical field B_z should increase with the radius in such way to compensate the increase of the γ factor $B = B_0\, \gamma(r)$ to keep ω constant. This can be obtained by using correction coils or, by reducing the gap a large radius (since the local field B_z is inversely proportional to aperture between the two poles: $B_z(r) \sim 1/\text{gap}(r)$.

The non-uniformity of the magnetic field B_z generates a radial field component B_r. The Lorentz force, coupled to the circular motion ($v=v_\theta = R.\omega$), generates a vertical defocusing force. The particles that are not injected exactly on the median plane will hit the cyclotron pole. The solution to overcome the defocusing force will be to add an azimuthal field component B_θ, as we will see in the chapter 4.

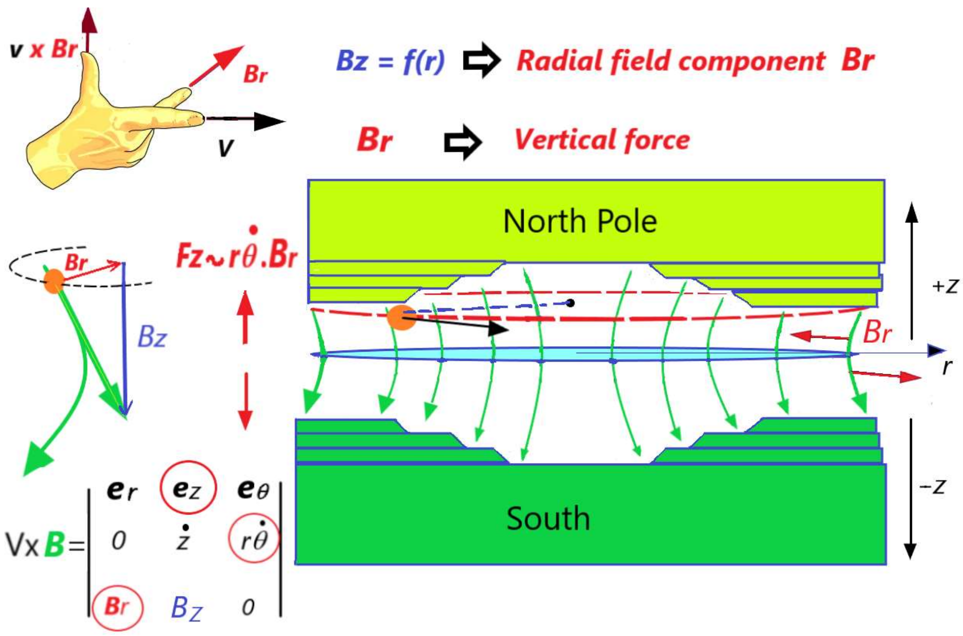

Fig. 7 : The magnetic field lines are perpendicular to the pole surfaces, due to Maxwell equations (**Curl B**=0). The curvature of the magnetic field lines depicted correspond to a radial component B_r. As we have seen in the last paragraph $B_r = -n B_0 z / R$

-In the upper plane, B_r is directed the centre of the cyclotron, this generates the vertical force F_z toward the top, As it can be seen with the right hand rule : $F_z = v_\theta . B_r \sim -z$

-In the lower plan, B_r is directed outwards, generating a vertical force F_z directed downward

-In the median plane, B_r=0 because of the symmetry, there is no force.

This force F_z proportional to z correspond to a vertical defocusing induced by the increasing function B_z (R).

4 Azimuthally Varying Field (AVF) cyclotron

4.1 Hill and valley

As we have seen, in the isochronous cyclotrons, we have a radial field B_r that gives a vertical defocusing force : $F_z = q\,(v_\theta . B_r) = q\, v_\theta . (n. B_0\, z/R) < 0$

This defocusing force is linear in z (like a defocusing quadrupole). With a field with B_θ component, we could theoretically add a new force which can compensate the defocusing force:

$$F_z = q\,(v_\theta . B_r - v_r . B_\theta)$$

A modulation of the magnet gap, function of the azimuth θ produces such B_θ component. If we realize a magnet gap modulation adding N angular sectors, i.e. a succession of hills and valleys. We get a vertical field : $B_\theta(R,\theta) = B_0 . [1 + f \, \sin N\theta]$

- It produces an azimuthal field (**Curl B**=0) $B_\theta = z. dB_z / R d\theta = +B_0 . f . N \cos N\theta / R$

- Besides, the local orbit curvature radius ρ(θ), in the high field sectors (hill) is reduced since locally $R_{Hill} = B\rho / B_z = B\rho / B_0 (1+f) \approx R_0 (1-f)$. While ρ(θ) increases in the valley (low field region) $R_{Valley} \approx R_0 (1+f)$

The evolution of the trajectory radius corresponds to a radial velocity $v_r = dR/dt$. Therefore the combination of the two effects generate a new vertical force $F_z = -q(v_r B_\theta)$. This effect has been discover by L.H. Thomas in 1938, which has improved a lot the transmission of the early cyclotrons.

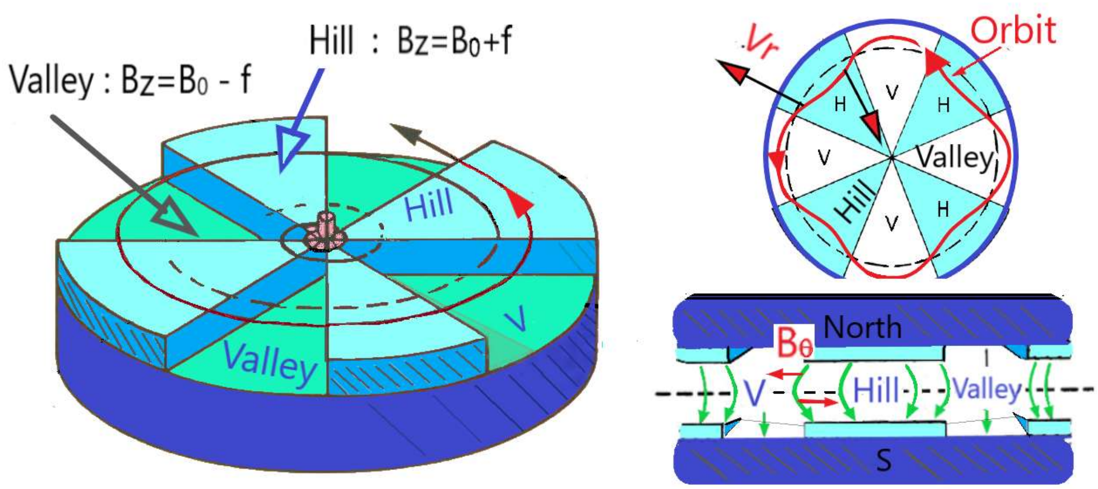

Fig. 8 : Principle of pole modulation with a 4 "straight sectors" cyclotron. The pole modulation generates:
-An oscillation of the trajectories in the horizontal plane corresponding to non-circular orbit $v_r = dR/dt \neq 0$ ($v_r < 0$ at hill entrance and $v_r > 0$ at the exit)
-An angular field component proportional to z : $B_\theta = z.\frac{dB_z}{Rd\theta} + \cdots$

The created force Fz is always inverse to z (this is a vertical focusing force as you can realized applying the right hand rule) : $F_z^{AVF} = q\,(-v_r.B_\theta) \sim -z$

4.2 Flutter F and averaged field index k

The effect of the vertical focusing is maximal when the particle crosses the edge, and zero in the middle of a sector. The evaluation of the average focusing effect over one turn, is related to the flutter function $F(R)$. F(R) is defined as the mean-squared relative azimuthal fluctuation of the magnetic field B_z along a circle of radius R :

$$F(R) = \frac{\langle\ (\ B_z\,(R,\theta) - \langle B_z\,(R,\theta)\ \rangle\ )^2\ \rangle}{\langle B_z\,(R,\theta)\ \rangle^2} = \frac{{\sigma_{B_z}}^2}{\langle B_z\ \rangle^2}$$

Where $\langle B_z\,(R,\theta)\,\rangle = \frac{1}{2\pi}\int_0^{2\pi} B_z\,(R,\theta)\,d\theta = B_0$ is the average field over one turn.

The flutter F is a useful quantity, because the tune Q_z is connected to the betatron oscillation frequencies, and it can be expressed quite precisely in terms of F(r).
The local curvature radius $\rho(\theta)$ in a complex field $B_z(\theta)$ does not coincide anymore with the radius R (the coordinate R) . Let us define an equivalent radius $\langle R \rangle$, being connected to the path length C over one turn: $\langle R\,\rangle = C/2\pi$

The magnetic rigidity is the product of the average field and radius :

$$B\rho = P/q = \langle B(\theta) \rangle . \langle R \rangle = C/2\pi$$

Demonstration: Since locally $B(\theta) = B\rho/\rho(\theta)$, we have

$$\langle B \rangle = \frac{1}{C}\int B.ds = \frac{B\rho}{C}\int \frac{1}{\rho(\theta)}.ds = \frac{B\rho}{C}\int_0^{2\pi} \frac{1}{\rho(\theta)}.\rho d\theta = \frac{B\rho}{C}2\pi = B\rho/\langle R \rangle$$

Exercise 5: Considering a four sectors cyclotron having a field : $B_\theta(R,\theta) = B_0.[1 + f \cos N\theta]$, Compute the "Flutter" F(r) .

Answer : We have the average field $\langle B_z(R,\theta) \rangle = B_0(R)$

$$\langle (B_z - \langle B_z \rangle)^2 \rangle = \frac{1}{2\pi}\int_0^{2\pi} {B_0}^2.[1 + f \cos N\theta]^2 - {B_0}^2 \, d\theta \quad \text{with N=4}$$

since $\cos N\theta = \left(e^{iN\theta} + e^{-iN\theta}\right)/2 \quad \rightarrow \cos^2 N\theta = (1 + \cos 2N\theta)/2$

The flutter is $F = \frac{\langle (B_z - \langle B_z \rangle)^2 \rangle}{\langle B_z \rangle^2} = \frac{1}{2\pi}\int_0^{2\pi} [f \cos N\theta]^2 \, d\theta = f^2/2$ *(* whatever sector number N)

Though the trajectories are complex in an AVF cyclotron, a simple formula holds: $B\rho = \langle B \rangle \langle R \rangle$
The field index should be defined of as an average and the field evolution over one turn is:

$$\langle B_z \rangle = \langle B_0 \rangle . \left(\frac{\langle R \rangle}{\langle R_0 \rangle} \right)^k$$

So, locally in magnet, we can used the local field index n :

$$n(\rho,\theta) = -(dB/d\rho).(\rho/B)$$

where $\rho(\theta) = B\rho/B(\theta)$ is the curvature radius in the magnet.

While an AVF machine the average field index k over one turn is

$$k(\langle R \rangle) = +(dB/d\langle R \rangle).(\langle R \rangle/B)$$

Where $\langle R \rangle = \frac{1}{2\pi}\int ds$ is the average radius

Nota : the sign + of k corresponds to the convention of synchrotron community and is not coherent with *n* (sorry...).

Exercise 6: Compute the average field index k, in a separated sector cyclotron.

Answer : In a cyclotron without azimuthal field modulation, the trajectories are circular, and we have $Bz(\theta) = \langle B \rangle$ and $\langle R \rangle = B\rho/\langle B \rangle = \rho$, so $k = -n$.

A Separated Sector Cyclotron with N radial sectors, alternates straight lines of Length L_0 and circular sections with $\Delta\theta = 2\pi/N$. The ion path for a reference ion is $2\pi\rho + N L_0 = 2\pi <R>$.

The constant factor $\lambda = <R>/\rho$ drives the field in the sectors: for a given trajectory, we have $B_{sector} = B\rho/\rho = <B><R>/\rho = \lambda <B>$.
We have demonstrated that $B\rho = \langle B \rangle \langle R \rangle$ therefore $\langle R \rangle = B\rho / \langle B \rangle = \gamma\, m\, v / q \langle B \rangle$
Using the average radius <R>, the particle revolution is given by $\omega = 2\pi F_{rev} = 2\pi \frac{v}{2\pi\langle R \rangle} = \frac{q \langle B \rangle}{m\gamma}$

For isochronism (ω=Constant), we should have $<B> = B_0\gamma(<R>)$. Finally, we recover the usual formula for the average field index (except the sign convention)

$$k(\langle R \rangle) = +\left(\frac{dB}{d\langle R \rangle}\right) \cdot \left(\frac{\langle R \rangle}{B}\right) = \gamma^2 - 1 \qquad \text{(see exercise n° 3)}$$

4.3 Edge focusing and AVF cyclotron

The mechanism of vertical focusing in a cyclotron with straight radial sectors is very similar to the edge focusing that occurs at the entrance and exit of the rectangular bending magnets in synchrotron.
A dipole magnet, which has an entrance edge non-perpendicular to the reference trajectory, generates focus or defocus as a function of the sign of the edge angle.

In the deviation plane (horizontal), a positive edge corresponds to less deviation of external trajectories (x>0), which ends up as an equivalent defocusing lens (see fig. 9).

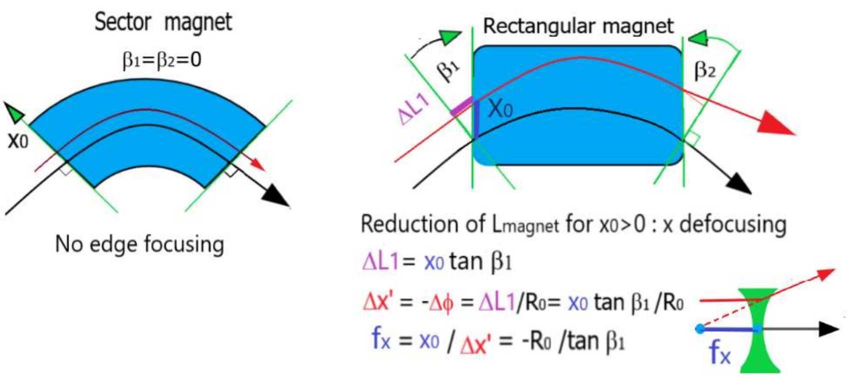

Fig. 9: In a sector magnet, the edge crossing has no effect. In a rectangular magnet a positive edge angle at entrance β_1 and exit β_2 defocus in horizontal plane : fx<0.

With a positive entrance edge angle (β_1>0), like the rectangular magnet, the focal length associated with an horizontal plane is

$$f_x = - R_0/ \tan \beta_1$$

Like a quadrupole, the effect in the vertical plane is inverse to the one in horizontal plane and $f_y = - f_x = R_0/ \tan \beta_1$, the edge is vertically focusing (see a basic course in beam optics for a full demonstration).

Exercice 7: Write down the transfer matrix R of the entrance edge with positive angle β_1 in vertical plane (y,y')= R. (y,y') with a thin lens approximation.

Answer : in the edge the vertical angle change $\Delta y' = y_0/f_y = y_0/(R_0/\tan\beta_1)$

After the dipole edge, we have $y = y_0$ and $y' = y_0 \tan\beta_1/R_0 + y'_0$

This can be written on the a matrix form: $\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/f_y & 1 \end{pmatrix} . \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ tan\beta_1/R_0 & 1 \end{pmatrix} . \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$

In the AVF cyclotrons, the crossing of the hill edge is such that the equivalent edge angle is positive: though the reference frame (R,z,θ) is different than the usual (x,y,s) optical frame of a synchrotron, the mechanism is the same.

We can demonstrate (ex. 8) the vertical tune is :

$$Q_z^2 = -k + \frac{1}{2} f^2 . \frac{N^2}{N^2 - 1} = -k + F \frac{N^2}{N^2 - 1}$$

Exercise 8 : How to compute the vertical tune in an AVF field : $B_\theta(R,\theta) = B_0.[1 + f \sin N\theta]$

- First compute a periodic solution for the radial motion R =r (θ)
- Then compute the particle dynamic in the vertical plane.

Answer : The field perturbation in AVF generates $\delta F_r = -q\, v_\theta . \delta B_z = -q\, v_\theta . \delta B_z . f \sin N\theta$
So, we expect an evolution as $R = R_0 + x(t) = R_0$ A. $\sin(N\omega t)$, let's compute A.The effect of field index k is neglected radially, because the smooth variation of B(R) is smaller than the variation of B(θ) for the equilibrium orbit.

$$\frac{d^2x}{dt^2} = -\omega^2 x - q\, v_\theta . B_0 \frac{f \sin(N\theta)}{m\gamma} = \omega^2 x - \omega^2 R_0 . f \sin(N\theta)$$

(since $v_\theta = R_0\, \omega$ and $\omega = qB_0/m\gamma$)

$$-\omega^2 N^2 A \sin(N\theta) = -\omega^2 A \sin(N\theta) - \omega^2 R_0 f \sin(N\theta)$$

Hence the amplitude is $A = f\ R_0\ /\ (N^2 - 1)$

The radial velocity $v_r = dR/dt = (d\theta/dt).(dR/d\theta) = \omega\ N\ A\ \cos(N\theta)$

while the azimuthal field is $B_\theta = z.dB_z/Rd\theta = +z \langle B \rangle N . f \cos(N\theta) / R_0$

The vertical force is $F_z^{AVF} = -q\, v_r . B_\theta = -\omega z . q\langle B \rangle A f N^2 \cos^2(N\theta)$

$\omega = q\langle B \rangle / m\gamma$, we compute the vertical motion *z(t)* as in 3.

$$\frac{d^2z}{dt^2} = -\omega^2 \langle n \rangle z + \frac{F^{AVF}}{m\gamma} = -\omega^2 \langle n \rangle z + \omega^2 z f^2 cos^2(N\theta) . \frac{N^2}{N^2-1}$$

The average effect over one turn , $\langle \frac{F^{AVF}}{m\gamma} \rangle = \langle \omega^2 f^2 N^2 \cos^2(N\theta) \rangle = \omega^2 \frac{f^2}{2} \frac{N^2}{N^2-1} = \omega^2 . F . \frac{N^2}{N^2-1}$

Finally we get : $Q_z^2 = \frac{d^2z}{dt^2} / \omega^2 z = < n > + F \frac{N^2}{N^2-1}$

4.4 Spiralled sectors

We can use edges with spiral shape in AVF machines to enhance the vertical focusing effect over one turn. Shifting progressively the hill boundary with a function *g(R) gives a* field

$$B_z(R,\theta) = B_0.[\,1 + f\ \sin(\,N(\theta - g(R)\,)\,)\,]$$

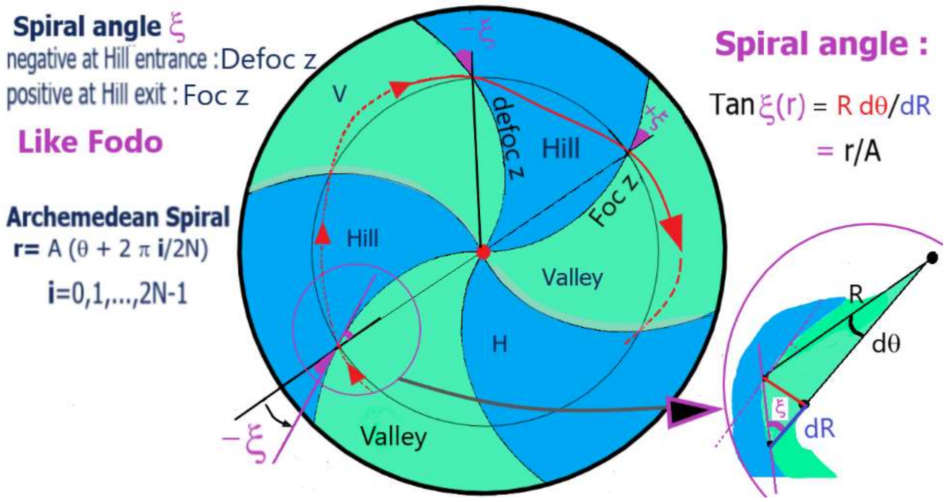

Fig. 10: Spiralled sectors with spiral inclination angle ξ and evolution of the tangent of the hill edges. The inclination of the sector edge ξ is related to the evolution of the edge g(R). For an archemedean spiral g(R)=R/A.

On the fig.10, we understand the effect:

-At the hill entrance, we get a vertical defocusing due to inclination of the field boundary (the inclination angle is negative $\xi<0$)

- At the hill exit, we get a vertical focusing ($\xi>0$)

The overall effect is focusing, like in a FoDo channel: alternating the gradient of the quadrupoles provides a net focusing effect. For the spiral geometry, we can use an archimedean spiral *g(R) = R/A,* in that case the inclination of the field boundary become simple

$$\tan(\xi(r)) = R.\ d(\theta_{edge})/dR = rR.\ d(R/A)/dR = R/A$$

The tangent of the spiral angle ξ, increases linearly with radius in archimedean spiral : So the associated z-focusing effect increases at large radius, compensating the increase of the relativistic vertical defocusing effect.

4.5 Separated Sector Cyclotron (S.S.C.)

When the Lorentz factor becomes large ($\gamma>1.4$) one has to increase the flutter term F to keep an efficient vertical focusing. This is possible by lowering the field in the valley down to zero : separating the sectors lead to the Separated Sector Cyclotron. The SSCs provide optimal beam quality, are adapted to high-energy beams. They require generally a pre accelerator for the beam injection.

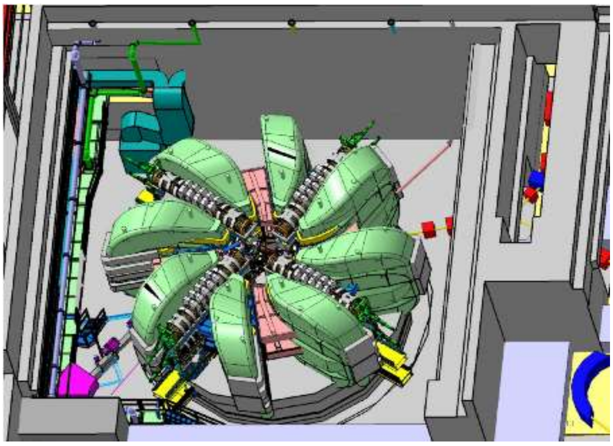

Fig. 11: PSI Ring cyclotron .The largest cyclotron of PSI (Villigen, Switzerland) provides a very intense proton beam (up to 2.5 mA) at 590 MeV. The total size is about 15 m diameter ($R_{extraction}$= 4.5m) .There are 8 independent magnets with a spiral shape, to increase as much as possible the z-focusing. Beside four RF cavities in the valleys are required for a separated turn extraction (chapter 7) to reduce the beam losses. The large beam power permit to produce high intensity secondary particles (neutrons, muons, pions) for different research fields.

4.6 Tunes in AVF cyclotron with spiralled sectors

With a complex calculation, the tunes in AVF cyclotrons and synchrocyclotron with N sectors appears as

$$Q_z^2 = -k + \frac{N^2}{N^2-1} F(1 + 2\tan^2(\xi)) + \ldots$$

$$Q_R^2 = 1 + k + \frac{3N^2}{(N^2-1)(N^2-4)} F(1 + 2\tan^2(\xi)) + \ldots$$

-The term $1/(N^2-4)$ in the radial tune means that a cyclotron geometry with two sectors ($N=2$) is unstable in horizontal plane. Therefore $N=3$ sectors is the minimal geometry.

- In isochronous cyclotron, the average field index k is constrained by isochronous requierement(exercice 4 and 6) $k= \gamma^2-1$

4.6.1 Tunes and resonances

The tune calculation is important to establish whether the motion is stable ($Q^2>0$), but it is not sufficient. At a certain energy, if the tunes reach an integer value (Q_z =N), it means than any particle will go back nearly at the same position after one turn: So if a magnet produces a field defect (no magnet is perfect), the particles will see the same defect at each turn, producing large amplitude oscillations and finally beam losses. Moreover, If the tune corresponds to Q_z =N/K the particles will go back to the same vertical position after K turns, it can excite a resonance except if K is very large. In a machine with large turn number, the oscillations in vertical plan can even excite the horizontal resonances (the non-linearities couple the vertical and horizontal motion).

Thus, in the synchrotrons and cyclotrons, we try to adjust the tunes to avoid a resonance, by requiring $K.Q_r + L.Q_z \neq N$ during the acceleration, where K,L N are integers. In synchrotron tune is often constant, while in cyclotron the tunes evolves progressively. Only large cyclotron with a large turn number (N_{turn}>500) can excite a resonance, while in synchrotron is the worry is important since particles can perform more than 10^7 turns.

4.6.2 Limits of the tunes formulas

The previous approximations for Q^2 are not very precise, since these formulas correspond to a simplified field : $B_z(R,\theta) = B_0.[\,1 + f\ \sin(\,N(\theta - g(R)\,)\,)\,]$

In reality the magnetic field of real magnets is often more complex and better approximations has been computed for more general field with several harmonics N [7]

For a more accurate determination of the tune $Q_{z,r}$, we can determine numerically with a multiparticle code using a realistic field map to extract the first order transport matrix R over one turn. We use a reference particle, which describe a closed orbit (the trajectory comes back at the same position after one turn without acceleration). The dynamic is periodic without acceleration and the matrix over one turn has the following form:

$$R(0,2\pi) = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ \beta \sin(\mu) & \cos(\mu) + \alpha \sin(\mu) \end{pmatrix}$$

Where (α β,γ) are the Twiss parameters

The phase advance per turn $\mu=2\pi\,Q$ is drawn from the half trace of the transport matrix :

$$Cos\ \mu = ½\ Trace\ (R)$$

So, we can extract μ_z and μ_r. Finally, we can obtain the tunes numerically

$$Q_{z,r} = {}^{\mu_{z,r}}/_{2\pi} = {}^{\mathrm{acos}(\,{}^{Trace\ (R)}/_{2}\,)}/_{2\pi}$$

Another limit of tunes formulas can be underlined: the tune concept does not describe fully the particle motion, since it's a first order approach. The complexity of the cyclotron magnet and the complexity of the trajectories with the acceleration produces many non-linear effects. Therefore, a start-to-end multi-particle simulation is always required.

5 Frequency modulated cyclotrons (Not isochronous)

Two machines resolve the problem of the detuning between particle revolutions and RF-field by cycling the RF frequency: Synchrocyclotron and FFAG [10,11].

5.1 Synchrocyclotron

The RF modulated cyclotron called synchrocyclotron are less used than the isochronous cyclotrons. In the synchrocyclotron, the *B field* is not shaped for isochronism, and the RF ω_{rf} is varied to synchronize few injected bunches during their acceleration:

$$\frac{qB_0}{m\,\gamma(R)} = f(time) = \omega_{rf}(t)/H$$

During one cycle, the RF frequency is first adapted to the injection energy (γ =1). Progressively, the frequency is decreased to follow the evolution of the gamma factor of some particle. The frequency then reaches the value corresponding to the extraction energy. Then, the frequency restart at the highest value to be re-synchronized the acceleration of few other bunches.

Any particle injected at RF phase φ close to an ideal phase φ_s will oscillate during the acceleration. This is the synchrotron oscillations. The phase φ_s is called synchronous phase and corresponds to a particle phase permitting to cross the gaps at the same RF phase during the whole acceleration process. The particle revolution time T_{rev} is not constant during the acceleration.

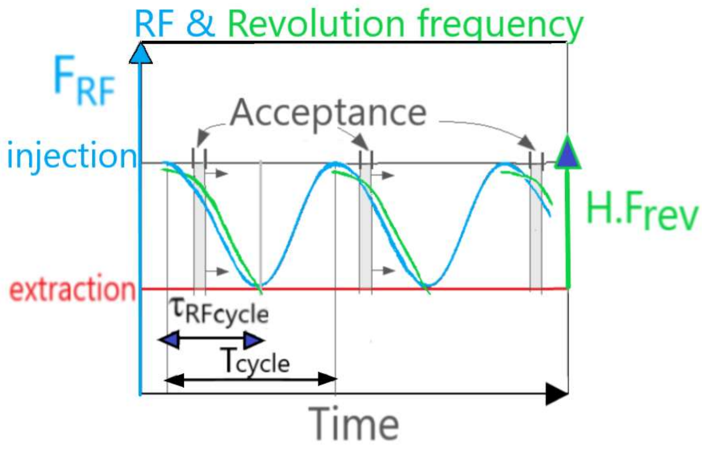

Fig 12: The cycle of the RF frequency. The revolution frequency decreases with $\gamma(R)$ from the injection to extraction. The total acceleration ΔE should take place during the decreasing part of the RF cycle $\tau_{RFcycle}$. The acceleration per turn must be on average:

$$\langle \delta E_{turn} \rangle = \Delta E / N_{turn} \quad \text{with} \quad N_{turn} = \tau_{RFcycle} / \langle T_{rev} \rangle$$

The energy gain per turn is related to the gap Number N_{gap} ,the cavity aperture α_{cav}, and the RF voltage U_0:

$$\langle \delta E_{turn} \rangle = q\, N_{gap} U_0 \sin(H\, \alpha_{cav}/2) . \cos(\langle \varphi \rangle) = q\, V_0 \ \cos(\varphi_s)$$

Technically, the cycle of RF can hardly be faster than 1kHZ and the acceleration take a rather long time $\tau_{RFcycle}$~ 1ms, while the rf cycle is short $\langle T_{rev} \rangle = H / \langle F_{RF} \rangle$ ~10-20ns, finally we have a large turn number in the synchrocyclotron: $N_{turn} = \tau_{RFcycle} / \langle T_{rev} \rangle \sim 10^4 - 10^5$

The total acceleration voltage V_0 per turn and the synchronous phase φ_s are chosen to match the cycle of the RF (see fig. 12), therefore the energy gain δE_{turn} must be small:

$$\langle \delta E_{turn} \rangle = q\, V_0 \ \cos(\varphi_s) = \frac{\Delta E}{N_{turn}} = \left(E_{extraction} - E_{injection} \right) . \langle T_{rev} \rangle / \tau_{RFcycle}$$

The couple (V_0, φ_s) have to fullfil the equation : $q\, V_0 \ \cos(\varphi_s) \ = \Delta E . \langle T_{rev} \rangle / \tau_{RFcycle}$

The synchronous phase φ_s should be chosen to maximise the phase acceptance of the synchrocyclotron: a typical value like φ_s =60° provide a large acceptance in phase such that many particles having different phases at the source output will be captured and describe stable oscillations in the plane (phase, energy) [2]. The frequency $\Omega_{synchro}$ of the synchrotron oscillations is generally much lower than the revolution frequency ω: $\Omega_{synchro} \sim 10^{-3}\, \omega$

For the proton therapy facility, the IBA company proposes an ultra-compact machine which is a superconducting synchrocyclotron called S2C2 [12] and operated at 5Tesla (fig. 13). The machine aims to replace the less compact isochronous superconducting cyclotron "C235" (fig. 14).

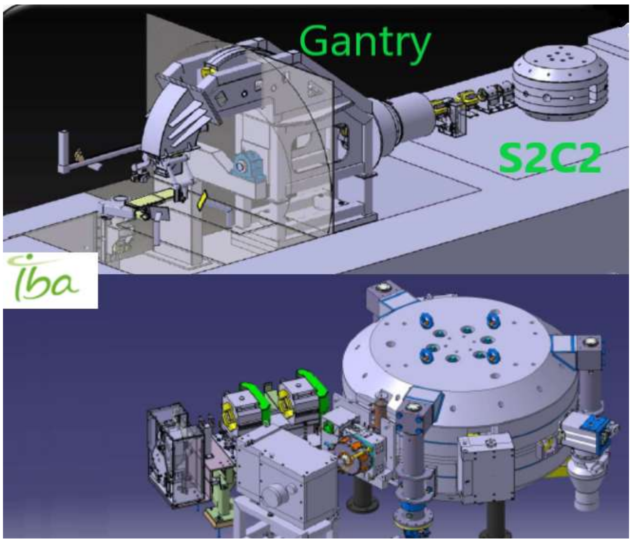

Fig. 13: The S2C2 IBA proton synchrocyclotron and the associated rotating gantry. The machine because of very high field is more compact than isochronous cyclotron, but it has a very low duty cycle.
Accelerator parameters: E= 230MeV ; Weight 50 tons.
Field at extraction <B> = 5.Tesla , k<0. Extraction radius R=49cm. Tpulse = 7 μs Repetion rate : 1kHz (duty cycle=0.7%) . Frf =[93 Mhz , 63 Mhz] (cycled) . Vrf= 10kV → turn number~ 40000.

Fig. 14 : C235_IBA isochronous cyclotron. This cyclotron have 100% duty cycle but has a greater cost than the very compact synchro-cyclotron S2C2.
Accelerator parameters : E= 235MeV; Weight 210 tons
The Field at extraction is <B> = 2.2Tesla , k>0 . Compact magnet with spiralled sectors is used;
Extraction radius R=108 cm (duty cycle=100%). The RF frequency is F_{RF} = 106 Mhz (fixed)
Vrf= 150kV → turn number~ 700

Historical Note: The synchrocyclotrons was the precursors of the synchrotrons, they provided the highest energy beam from 1946 to 1954. The first accelerator of the CERN facility was a 600MeV proton synchrocyclotron with a radius of 2.3m and has been operated in the years 1958-1990. Nowadays, for research applications, which require often a high beam intensity, the 100% duty cycle isochronous cyclotron are preferred. Regarding very high-energy applications, synchrotrons have still no rival.

The main drawback of a synchrocyclotron is its low duty cycle. Because of the frequency variation of the cavity, only a small fraction of the ions leaving the source are synchronized with RF acceleration.

Finally, the average beam current is rather low, compared to the one of an isochronous cyclotron. As an advantage, the magnet of a synchrocyclotron can be much simpler than the one of isochronous cyclotron. The vertical stability of the beam is guaranty without the trick of azimuthal field modulation.

Exercise 9 : Establish the relation of revolution time T_{rev} and the momentum in a synchrocyclotron, noting that the field <B_z> is slightly decreasing at large radius ($B \sim R^k$, so k<0). Use the concept of momentum compaction factor $\alpha_p = (p/C).(dC/ dp)$.

Answer : For a given particle orbit with length C, traversed in the time T_{rev}, we have $v.T_{rev} = C$. Using the logarithmic differentiation : $dT_{rev}/ T_{rev} = dC/ C - dv/ v = dC/ C - d\beta/ \beta$

$$\frac{dT_{rev}}{T_{rev}} = \frac{dC}{C} - \frac{d\beta}{\beta} = \left[\frac{p}{C}\frac{dC}{dp} - \frac{p}{\beta}.\frac{d\beta}{dp}\right].\frac{dp}{p}$$

$$dp/ d\beta = d(\gamma m\beta c)/ d\beta = \ldots = (p/\beta).\gamma^2$$

- We have $dp/ d\beta = d(\gamma m\beta c)/ d\beta = \ldots = (p/\beta).\gamma^2$ so $(p/\beta).(d\beta/dp) = 1/\gamma^2$

- The quantity $\alpha_p = {dC/ C}/{dp/ p}$ is called momentum compaction factor.

Since $p = q.B\rho = \langle B\rangle\langle R\rangle$ and <B>~<R> ^{k}, $\frac{dp}{p} = \frac{d\langle B\rangle}{\langle B\rangle} + \frac{d\langle R\rangle}{\langle R\rangle} = (k+1)\frac{d\langle R\rangle}{\langle R\rangle}$

so $\alpha_p = 1/{k+1}$ $dT_{rev}/ T_{rev} = \left(\alpha_p - 1/\gamma^2\right) .dp/p$

-In a synchrocyclotron $\left(\alpha_p - \frac{1}{\gamma^2}\right) = \left(\frac{1}{k+1} - \frac{1}{\gamma^2}\right) > 0$, the revolution time T_{rev} increases with momentum p, and the revolution frequency F_{rev} decreases.

-in an isochronous cyclotron $k = -n = \gamma^2 - 1$,(ex n°4), therefore $\left(\alpha_p - 1/\gamma^2\right) = 0$ the revolution time T_{rev} is independent from p (this is needed)

5.2 FFAG : Fixed Field Alternating Gradient accelerator

The idea of FFAG [10,11] (also abbreviated **FFA**) is to provide a very high energy like in synchrotron (γ>>1) but at higher duty cycle. The ramping of the magnets in a synchrotron limits the repetition rate to a range 1Hz -50Hz. While, in a synchrocyclotron the magnet has fixed field, and the cycling of the RF can be as fast as 1kHz. However, the vertical focusing in synchrocyclotron is not optimal, and a FFA could provide a better beam quality at even larger energy. The FFAG idea has been proposed in the 1950s, Few proof of principle machines have been constructed (KEK KURRI with 30 and150 MeV protons).

A FFAG is a synchrocyclotron with complex magnets ensuring a better focusing by alternating gradient (dB_z/dr).The FFAG field should oscillate azimuthally $B_z(R,\theta) = \pm B_0.(R/ R_0)^{-n}.\mathrm{F}(\theta - g(R))$

-F(θ- g(r)): is a periodic oscillating function with N symmetry
-g(r) : is the spiral edge angle, if needed
-*n(r, θ)* : is the local field index in each magnet

The average field scale as follows $\langle B_z(\theta)\rangle = \langle B_0\rangle.(\langle R\rangle/\langle R_0\rangle)^k$ .The beam dynamic in a FFAG is not isochronous ($k \neq \gamma^2$-1). Since we have demonstrated that $B\rho = \langle B(\theta)\rangle.\langle R\rangle = C/2\pi$, the particle momentun increases with the average radius $\langle R\rangle = C/2\pi$

$$P_{extraction} = q\,\langle B_z\rangle\langle R\rangle = P_{injection}.\left(\langle R_{extraction}\rangle / \langle R_{injection}\rangle\right)^{k+1}$$

The average field index k in a FFAG is generally rather large (k>>1) to reduce the horizontal size (and cost) : for a given extraction momentum $P_{extraction}$ you can reduce <$R_{extraction}$> if k is large. The vertical motion (Q_z^2=-k+…) is stabilized with the alternating gradient magnets. The use of a serie of dipole magnets (with a gradient) with alternate field $\pm B_0$ permit to provide an additive transverse focusing coming by an alternating-gradient focusing effect to get a stable motion ($Q_z^2=-k+\ldots>0$).

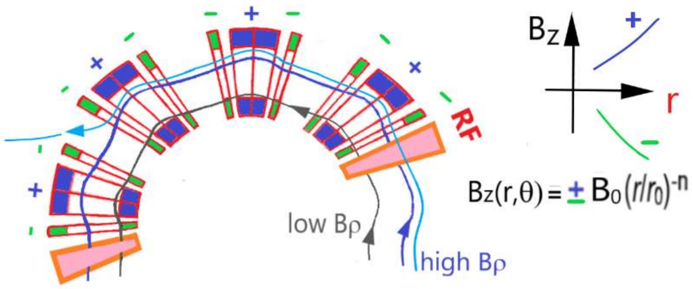

Fig. 15 : The scaling FFAG geometry. The scaling conditions constant focusing properties during the acceleration conversely to the cyclotrons. Such FFAG use many cell containing a triplet of bending magnets. In the main dipole (+ : horizontal focusing) $dB/dR= -n\,B_0\,(R/R_0)^{-n} > 0$. While In the 2 reverse field dipoles (- : means horizontal defocusing) $dB/dR = (-n)\,(-B_0)\,(R/R_0)^{-n} <0$

5.2.1 Scaling FFAG

The simplest idea was to find a geometry which provide homothetic trajectories, keeping the same tunes during the acceleration (one reason is to avoid to cross resonance during the acceleration in the tune diagram: Scaling FFAG means that the orbit shape (optics) is kept fixed, independent of energy, just as in synchrotrons. The 3 conditions to get a constant tune are the following

- The average field index k independent of r: We take local index n=constant in the two dipole kind $B_1=-B_2$
- The flutter F, should be independent of r
- A Spiral shape permitting to get a constant (tan ξ) : we have two possibilities
  - -With a straigth sector tan(ξ) =0 no spiral for the main magne
  - -With $r_{spiral}=R_0\exp(A\,\theta)$, we get $\tan(\xi(r)) = r\,d\theta/dr = A$ (the reverse field magnet)

5.2.2 New applications and non-Scaling FFAG

In FFAG, the reverse field region increases the average radius of the machine (and the cost) by an important factor compared to an equivalent synchrocyclotron or cyclotron. But the FFAG can be a solution when a rapid cycling synchrotron is not possible and when a cyclotron cannot reach the desired energy. For instance, FFAG accelerators were reconsidered for muons acceleration for a neutrino factories and for Muon Colliders : It was indeed required to accelerate muons to an energy of about 20 GeV, not reachable with cyclotrons :The muons have a short half-life and the acceleration in a rapid cycling synchrotron would be difficult.

The rapid acceleration if with a small turn number (<20 turns) would be important for unstable muons ($T_{1/2}$=2.2μs). A small number of turns allows to relax the constraint of scaling FFAG, since the betatron oscillations and resonances will have no time to develop and damage beam quality. So, the field index can evolve k=k(r). In non-scaling FFAGs, a linear variation of magnetic field can be employed: using constant-gradient "*linear"* magnets greatly increases dynamic aperture and simplifies construction, while employing the strongest possible gradients minimizes the real aperture. Besides, the linear field variation provides a large dynamic aperture, allowing the acceleration of large emittance beams.

EMMA (Electron Model for Many Applications), the first non-scaling FFAG, has been built at the Daresbury Laboratory (UK) in 2010. This is a proof of principle machine delivering 20MeV electrons, requires a ring of 16 m circumference: The ring uses a combined function magnet with a dipolar and a quadrupole component (a gradient): The magnets implemented seems in fact quadrupoles, and the dipole component is obtained by using them off-axis.

A serie of 42 cells composed with 2 magnets (focusing and defocusing) produces at the same time a deviation (dipolar component) and AG focusing (quadrupole components).

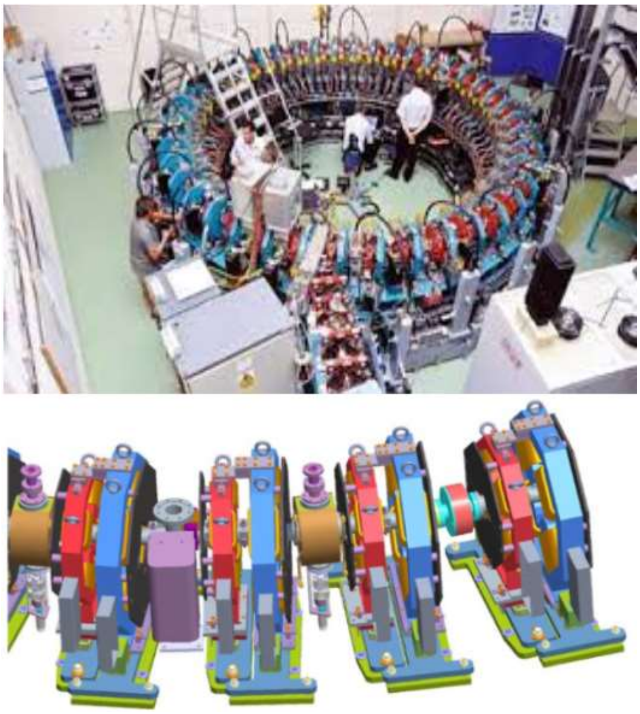

Fig. 16 : The EMMA FFAG [13] constructed at STFC Daresbury Laboratory**.** The EMMA ring accelerates electrons from 10 MeV to 20 MeV. The geometry is called "non-scaling" since the tunes Q_r,Q_z are evolving from the injection up to extraction. Since it crosses the resonances very quickly (3 Q_r=1, 2Q_r+2Q_z =1) within 20 turns EMMA has demonstrated a good stability. There are 42 cells of two magnets. The two magnets are quadrupoles used off axis to provide two field component : we have B_z (r) = B_0 ± G (r-r_0) .The alternate linear gradients ± G generates a small momentum compaction factor α_p=(dC/C)/ (dp/p) which provide a good dynamics aperture with small magnets. Four cells of the doublet are shown in detail.

6 Introduction to cyclotron injection

Designing the central region of a cyclotron is a very complex task [14, 15, 16,17, 18], and we gives only a first approach. Several injection systems are used, depending of the cyclotron model.

- For the low-energy cyclotrons, two technics are used to inject a beam in a cyclotron:
  - A very compact internal ion source inserted directly inside the cyclotron center
  - An external ion source connected to a beam line inject the beam axially (vertically) and bend the beam with a compact electrostatic inflector (hyperboloid or spiral inflector) on the first orbit
- The high-energy cyclotrons (like separated sector cyclotrons) are located downstream a pre-accelerator. In that case, the beam is more rigid and a compact axial injection system is not feasible. We use a radial injection beam line.

6.1 Internal ion sources

Most of industrial cyclotron (K_b=5-30 MeV) use a very compact internal source (PIG=Penning ion gauge), which can be inserted in the central region of the cyclotron. This system avoid a complex injection beam line (vacuum, quads, solenoids, inflector), this reduce a lot the cost of the machine. This technology can deliver positive and even negative hydrogen ions.

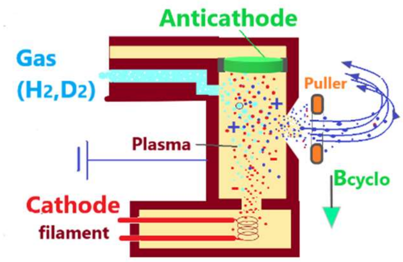

Fig. 17 : Penning Ion Gauge source with a hot cathode (filament).
Electrons are emitted by thermo-ionic effect and ionize the gas atom generating a plasma. The plasma is confined with the cyclotron magnetic field. Ions emerge from the plasma, with the electric field between the anode, and the puller.However, a very old technology**,** specific design aim to improve current or to increase cathode lifetime, thus reducing the cost and maintenance of medical cyclotron. I=100 μA is a typical performance for H+ beam.

6.2 Axial injection with external sources

An external ion source is often preferred for various reasons: The external source is less constrained geometrically and can be much bigger and performant than a very compact internal ion source.

– For heavy ion, it is important to get the higher charge state Q to increase the maximal energy of a given cyclotron: $(E/A)_{max}$= Kb $(Q/A)^2$. The ECR ion source, which produces heavy ion beams with high charge states, require a magnetic confinement with two large solenoids and a hexapolar magnet and can not fit inside a cyclotron.
- For negative hydrogen isotopes, Multicusp sources are very efficient but cumbersome.
- In general situation, a complex injection beam line associated with an external ion source permit 6D matching (radial, axial, and longitudinal) which is better to obtain a better transmission of any cyclotron.

In particular, using a pre-buncher, operating at the cyclotron frequency, permit to focus longitudinally the beam and increase the beam current and the cyclotron phase acceptance.

At the energy provided by the ion source extraction (typically 20kV), the beam magnetic rigidity is very low. The only possibility is to inject the beam vertically toward the centre of the cyclotron, by way of a hole in magnet yoke: in the axial direction, beam velocity is parallel to the cyclotron field B_z so no radial force perturbs the injection axially.

The beam is then bent onto the cyclotron median plane with an electrostatic device, called inflector. As soon as the beam is deflected into the horizontal plane, the beam experience a magnetic force F_r due to the cyclotron field. The full motion is in 3D. After the inflector, the beam starts a circular radial motion, meet rather quickly the first acceleration gap. The inflector is designed to inject the beam on the first cyclotron orbit R_m

Two inflector geometries are used nowadays:

- Hyperboloid inflector (Müller Inflector)
- Spiral inflector (called Pabot-Belmont inflector)

The injection geometry in a cyclotron is determined by the injection energy (the voltage of the ion source) and the mass over charge ratio of the desired beam: the two parameters define the magnetic rigidity at injection and the magnetic radius of the first orbit is hence $R_m = B\rho_0/B_0$.
The inflector should drive the beam on the first orbit taking of magnetic force generated in the horizontal plane and the electric force generated by the inflector electrodes.
The axial injection problem is generally treated in the backward direction:

i) we start from a trajectory rotating around cyclotron center Ω at large radius
ii) then we go back toward the first turn orbit
iii) then we search to bend the beam backward from the radial plane toward the vertical plane of the injection beam line with a vertical electric field $\mathbf{E}_z$.

6.2.1 Hyperboloid inflector ("Müller Inflector")

The principle of Hyperboloid inflector 18] [is to find a geometry whose electrodes are surfaces of revolution. An hyperbolic electric potential $V = Kz^2/2 + KR^2/4$, is the simplest potential which satisfies $\Delta V = 0$ which possesses a radial symmetry. The two electrodes follow an equation $R^2 - 2z^2 = \text{Constant}$. Assuming a constant field B_0, the parametric equations of the central trajectory in the inflector are:

$$X(t) = [\mathrm{a}\cos(kt) - b\cos(kt)].r_0/2$$
$$Y(t) = [\mathrm{a}\sin(kt) - b.\sin(kt)].r_0/2$$
$$Z(t) = A\,[\sin(kt)]$$

(kt) is included in $[0, \pi/2]$ and corresponds to the azimuth $kt = v_0 t/A$. The parameter k is related to the potential V : $k = qK/\mathrm{m}$. While A is the inflector height.

The other parameters a, b and r_0 are given below:

$$a = (3/2)^{1/2} - 1 \quad \text{and} \quad b = (3/2)^{1/2} + 1 \qquad r_0/2 = (6)^{1/2} R_m = A$$

The magnetic radius of the first orbit is given by injection energy and the field of the cyclotron.

$$R_m = B\rho/B_0 = r_0/(24)^{1/2} = A/(6)^{1/2}$$

Therefore, if R_m is fixed, there is no free parameter: the hyperboloid inflector height A is sometines too large to be adapted in small cyclotron, in that case the spiral inflector is preferred.

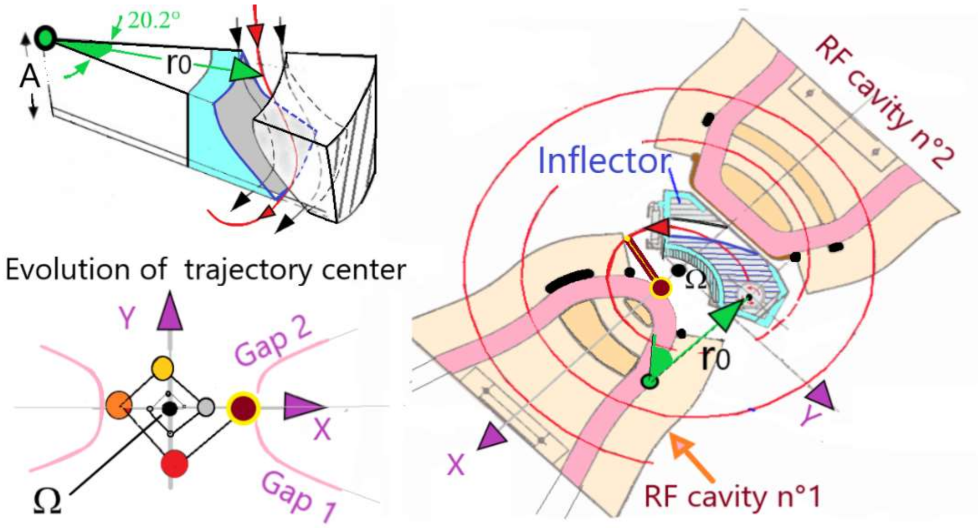

Fig 18 : hyperboloid inflector and beam centring. The inflector electrodes are represented in 3D with 3 trajectories. The use in a real cyclotron is represented (right, cyclotron CIME at Ganil). Two RF cavities produce four acceleration per turn. The positions of the centre Ω of the ion trajectory (curvature centre) during the acceleration evolve (left). The centre of curvature should converge toward the cyclotron centre after the after few accelerations. The position of the inflector should be chosen carefully.

The magnetic radius R_m is fixed by cyclotron geometry and injection energy, so we have only one adjustable parameter A (the height): Therefore, it is often used with an off-axis injection, since the axis of the cyclotron centre Ω does not coincide with the entrance point in the inflector.
For a given radius R_m, the required height A is bigger than the Spiral inflector, which is more compact. The advantage of a hyperboloid is that the two transverse sub-spaces are not correlated and can be matched independently.

6.2.2 Spiral inflector (called "Pabot-Belmont" inflector)

The main issue of the spiral inflector [19,20] is to find the electrode shape such that electric field vector ***E*** along the central trajectory is always perpendicular to the velocity vector of the ion. This assumption insures that the central ion trajectory will always lie on an equipotential surface, and this allows us to construct an inflector working at a low voltage. Several geometries can be found ensuring **v. E**=0 : The electrode can be tilted progressively around the central trajectory : varying this tilt k' permit to adapt the geometry to the cyclotron constraint

The electric field changes as a function of the parameter $\theta = v_0 t/A$, evolving from 0 to $\pi/2$:

$$E_x = E_0.[\cos(\theta).\cos(2K\theta) - k'.\sin(\theta).\sin(2K\theta)]$$
$$E_z = E_0.[\cos(\theta).\sin(2K\theta) - k'.\sin(\theta).\cos(2K\theta)]$$
$$E_z = E_0.[\sin(\theta)]$$

The parametric equation of the central trajectory can be derived in 3D:

The integration of the equation of motion for the reference trajectory

$$X(t) = [\cos(a\theta)/a - \cos(b\theta)/b].\frac{A}{2} - \lambda$$
$$Y(t) = [\sin(a\theta)/a - \sin(b\theta)/b].\frac{A}{2}$$
$$Z(t) = A\ [\sin(kt)]$$

The parameter are : a=2 K-1 b=2 K+1 λ= A/(4K^2-1) With K= A/2R_m + k' .
Physically, A is the inflector height, and R_m= Bρ/ B_o is the Magnetic radius and k' is the electrodes tilt.

The height A is constrained by the maximal vertical dimension available in the axial hole after the injection solenoid.
The tilt k' is related to the angle ϕ between the electric field with the vertical axis

$$k' = \tan(\phi)/\sin(\theta)$$

The tilt k' is chosen to adapt the entrance point of the inflector for a fixed arrival point in the cyclotron median plane.

Though the spiral inflector is the most compact inflector, it suffers from two defects:
- The beam is defocused vertically and require often a small electrostatic quad at the inflector exit, which increase the complexity of the injection.
- The electrode geometry requires a complex machining.

6.3 Radial injection for separated sector cyclotron

This technic is devoted to specific separated sector cyclotron. The separated sector cyclotrons (SSC) consist of magnet sectors separated by empty valleys, which can house an injection beam line (SSC2_ganil, picture 19). The injection beam line is comprised of several magnets and a high voltage electrostatic inflector, having two planar electrodes.

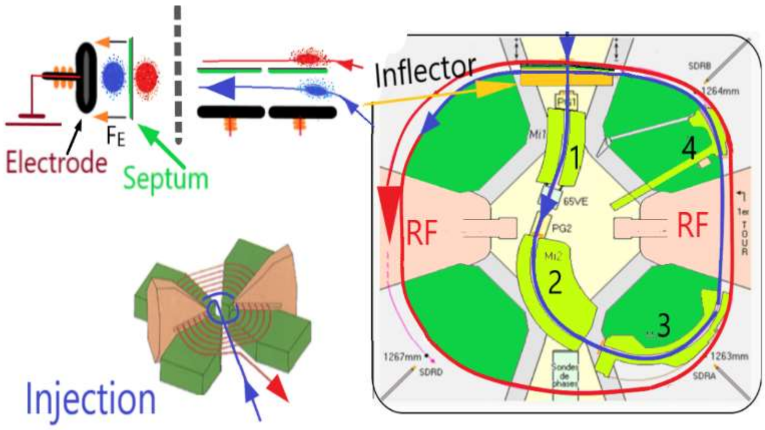

Fig 19: Radial injection of SSC2 (Ganil, France).The space available between separated sector magnets is used to insert injection magnet (1,2,3,4) and an electrostatic inflector in the radial plane. The inflector gives the few milliradians deviation to put the beam on the accelerated orbit. while the trajectory of the accelerated beam (red) is not perturbed. The position of the electrostatic inflector can be moved to generate a precession (see extraction chapter

7 Extraction

The most frequently used extraction methods are stripping extraction and extraction by electrostatic deflectors.

7.1 Stripping extraction

Hydrogen isotopes (^1H or deuterium ^2H) can be produced in negative ion state: i.e. a hydrogen atom can capture an extra-electron. Though the extra-electron on an H^- ion is very loosely bound, we can accelerate these anions. After the acceleration, when passing through a thin foil, the ion loses its electrons (stripping). By positioning a thin carbon foil in the cyclotron magnetic field close to the extraction radius, the accelerated H^- ions are transformed into H^+. The change of the ion's sign changes the direction of the bending force (from F_r=- e_0 vB to F_r=+e_0 vB), and the positive ions are directed outside the magnetics field.

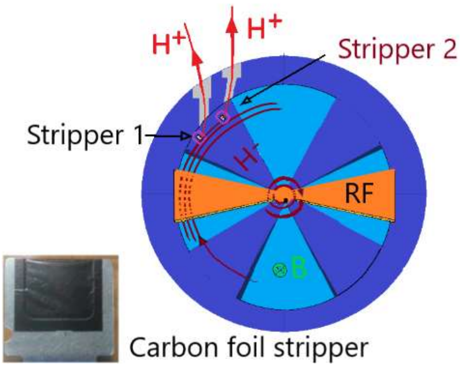

Fig. 20: Stripping extraction for H^- **.** At large radius, a thin carbon foil strips off the electrons. The residual ions are H^+. The positive ion trajectories are bent in the opposite direction. Several strippers can be inserted producing simultaneous beams. The radial location of the stripper define the beam energy. The simultaneous beam can be delivered for different applications at the same time, which is very cost effective.

The stripping extraction technics present many advantages and is very cost effective:

- The efficiency is closed to 100%, much better than an electrostatic deflector.
- The foil lifetime even with large current is rather long (~10^4 μA.h)
- Simultaneous extraction of two beams toward two external lines is possible with two foils corresponding to different energies and with different beam currents.
- A large energy range may be covered by changing the foil position in the magnetic field (correcting magnets are needed to ensure a constant angle in the exit beam line)
- Negative ion source can provide H^- or negative deuterium D^- at high current up to 0.5 mA (with an external multiscup source)

7.2 Deflector extraction: Single turn extraction vs multi-turn extraction

For positive ions such as H+, the stripping extraction cannot be used and several optical element are used to bend the beam out of the cyclotron.

The extraction hardware for a cyclotron usually consists of an electrostatic deflector followed by a magnetic channel. The curvature of the electrodes of the deflector must correspond to the shape of the orbits of extracted ions. The deflector provides an angle kick of typically 50mrd with an electric field around E=100kV/cm.

If the bunches are well separated radially (bunch size ΔR smaller than δR turn separation), it is possible to adjust on line the acceleration voltage to direct most of particle inside the electrostatic deflector without beam losses: We talk about single turn extraction.

If the bunches are not separated radially (size ΔR> turn separation δR), the bunch having performed N turn overlap the N-1 turn bunch, the deflector will cut the continuous stream of particle and generating important beam losses: and we talk about multiturn extraction, which result in activation and HV sparking of the deflector.

The reduction of beam losses in the deflector requires to minimize beam size while maximizing the turn separation.

- Minimization Bunch size Δr : The minimization of the beam size Δr is obtain through transverse matching an longitudinal bunching.

- Transverse matching: The cyclotron is a (quasi) periodic accelerator and the magnetic structure possesses an ideal periodic emittance in the transverse plane (the eigen ellipse).
A mismatch between the cyclotron periodic ellipse and the injected beam will result in an increase of the beam emittance. It is always important to design the injection beam line to match properly the beam ellipse to the ideal periodic ellipse and avoid emittance dilution, and getting the minimal bunch size at the extraction.
- Longitudinal bunching: The energy dispersion can be reduced by a minimization of the phase width $\Delta\varphi = H\ \omega_{rf}\Delta t$ using a buncher, which result in a reduction of the bunch size.

$$\Delta R/R = \Delta B\rho / B\rho = \Delta p/p = \gamma/\gamma+1 \cdot \Delta E/E \sim 1/2\,[\cos(0) - \cos(\Delta\varphi/2)] \approx 1/4\,\Delta\varphi^2$$

- Maximisation of the orbit separation δR : During acceleration the bunch spacing δR is reduced progressively, the energy gain per turn is constant while the energy is increasing. The conditions to obtain a single turn extraction rely on the maximisation of the turn separation δR. Three technics are used:
  1. Acceleration // 2. Precession // 3. Resonant extraction

7.2.1 Acceleration

In high-energy isochronous cyclotron, the energy gain per can be increased by using more accelerating gaps: For example, the 8 separated sector cyclotron of PSI (K_b=590 MeV, **fig. 11**), has 4 accelerating RF cavities located in the valleys. It provides 8 accelerating gaps. The number of valleys available restricts the number of the accelerating cavities. A large bunch separation δR that overpasses 1cm is rather difficult to obtain. Therefore, additive mechanisms are required.

Exercise 8 : Demonstrate that the turn separation δR is dominated but the energy gain per turn δE_{turn} in the cyclotron, but is influenced as well by the average field index k:

$$\frac{\delta R}{R} = \frac{\gamma}{\gamma+1} . \frac{\delta E_{turn}}{E_N} . \frac{1}{1+k}$$

Answer :

We can compute the radial gain δR per turn, in a non-homogenous field $\langle B\rangle \sim \langle R\rangle^k$:

Since $\frac{d\langle B\rangle}{dR} = \frac{d(R^k)}{dR} = k.R^{k-1} = k\frac{\langle B\rangle}{R}$, we have $\frac{\delta\langle B\rangle}{\langle B\rangle} = k\frac{\delta\langle R\rangle}{\langle R\rangle}$

Besides, we have $B\rho = \langle B\rangle\langle R\rangle$ so $\langle R\rangle = B\rho/\langle B\rangle$

$\frac{\delta\langle R\rangle}{\langle R\rangle} = \delta\ln(R) = \delta\ln\left(\frac{B\rho}{\langle B\rangle}\right) = \frac{\delta B\rho}{B\rho} - \frac{\delta\langle B\rangle}{\langle B\rangle} = \frac{\delta p}{p} + k\frac{\delta\langle R\rangle}{\langle R\rangle}$ = acceleration + field variation over one turn

So rearranging δR on two side : $\frac{\delta\langle R\rangle}{\langle R\rangle} = \frac{\delta p}{p} . \frac{1}{1+k} = \frac{\gamma}{\gamma+1} . \frac{\delta E_{turn}}{E_N} . \frac{1}{1+k}$

The energy after N_{turn} is $E_N = E_{injection} + N_{turn}.\delta E_{turn} \approx N_{turn}.\delta E_{turn}$

Therefore the radial separation of two consecutive turn δR is :

$$\delta R = \langle R\rangle\frac{\gamma}{\gamma+1} . \frac{\delta E_{turn}}{E_N} . \frac{1}{1+k} = \langle R\rangle\frac{\gamma}{\gamma+1} . \frac{1}{N_{turn}} . \frac{1}{1+k} = \langle R\rangle\frac{\gamma}{\gamma+1} . \frac{1}{N_{turn}.Q_r^2}$$

So for a desired cyclotron energy, if you have to increase the turn separation δR at extraction.
- Build cyclotrons with a large radius R
- Make the energy gain per turn δE_{turn} as high as possible (it reduces N_{turn})
- Accelerate the beam into the fringe field, where <B> and Q_r^2 drops

if δR is not sufficient to get a single turn extraction, we have to use precession and resonant extraction.

7.2.2 Precession

An orbital precession induced by an off-centring injection enhance The turn separation δR.. Detuning slightly the beam at the cyclotron injection permits to shift the radial position by few mm. Then, because of the betatron oscillation, the beam will oscillate with a frequency (Q_r . ω t)=. Q_r θ (see 3.3). The position of a bunch after N turns at azimuth θ_1=2πN will be :

$$R(\theta_1 = 2\pi N) = R_0(2\pi N) + x_N \cos(\,Q_r.(2\pi N) + \varphi\,) = R_{0N} + x_N \cos(\,[Q_r - 1].(2\pi N) + \varphi\,)$$

Where the ideal orbit after N turns is R_{0N} = R_0 (θ_1 =2πN), given by acceleration, and the amplitude of the precession is given the injection error x_0 : $x_N = R_{0N}. (x_0/R_0)$. Since Q_r ~1 in isochronous cyclotron, we can replace cos(Qr.(2π)) by cos([Qr-1](2π)), only the fractional part of the tunes matters.

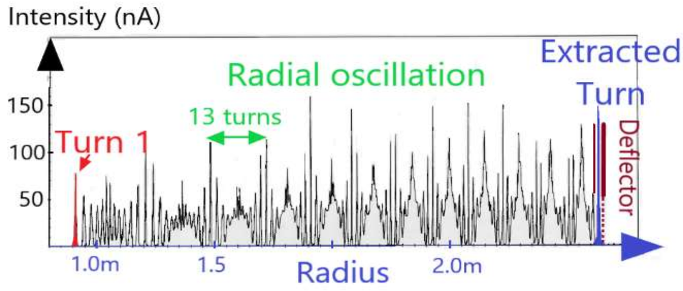

Fig. 21: Precession by off centering injection.. The beam intensity is measured as a function of the cyclotron radius on radial proble. The radial oscillation of the beam permit to increase the turn's separation at the deflector. In this cyclotron (Ganil CSS2), the precession is defined by the radial tune (Q_r-1)=1/13 generating a period of oscillation of 13 turns

7.2.3 Resonant extraction

The resonant extraction aims to generate an oscillation x(θ) close to the extraction with some magnetic perturbations (field bumps). The radius of the trajectory will oscillate around a non-perturbed trajectory R_0 : $R(\theta) = R_0(\theta) + x(\theta)$

Adding a magnetic perturbation at the extraction $\Delta B_z = B_P(R)\ cos(P\theta)$, using θ=ωt, we get as in 3.3:

$$\left[\frac{d^2x}{d\theta^2} + Q_r x\right] = A\cos(P\theta)$$

Nota : a full derivation gives $A = (\langle R\rangle/\langle B_z\rangle) . (dB_P(R)\ /dR)$

This equation correspond to a driven oscillator, and we can search a particular solution x(θ)~cos(P θ) which gives $\chi = \left[\frac{x(\theta)}{A}\right] = \frac{\cos(P\theta)}{(Q_r^2 - P^2)}$.

If the excitation frequency P corresponds to the natural frequency Q_r (P ~ Q_r), the response function $\chi = [\ x(\theta)/A]$ will diverge. This means, that a very small amplitude perturbation A, induces a large amplitude motion x(θ)=χ Δ. This is the definition of a resonance…

Nota on resonances: Most of time, in the synchrotrons and cyclotrons, we try to adjust the tune Qz, Qr to avoid the resonance, by requiring $K.Q_r + L.Q_z \neq N$ during the acceleration, where K,L N are integers. For a resonant extraction, we excite on purpose the radial resonance ($K.Q_r$ =N) , but locally close to the extraction radius to increase δR.

In isochronous cyclotron, the average field index is k~ γ^2-1, so $Q_r^2 = 1+k+\ldots = \gamma^2$ which defines the multi-polarities of the field perturbation to be used P~ γ. For low energy cyclotron (γ<1.2) a flat field bump localized at large radius with an aperture Δθ=20° is sufficient: it mimics a first harmonic perturbation Acos(θ) and corresponds to a "Q_r=1" resonance.

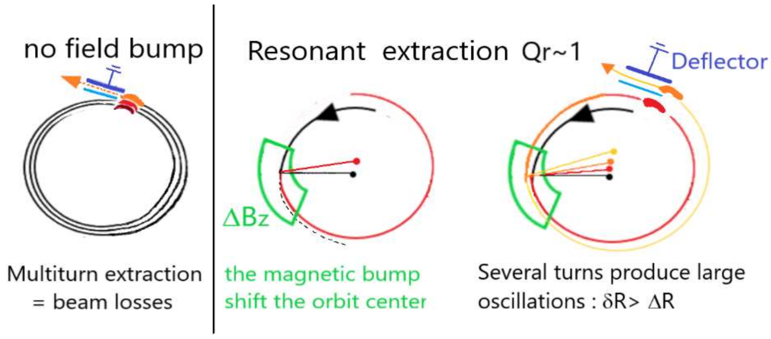

Fig. 22 : Principe of the excitation of the first order resonance using a magnetic bump. When Qr~ 1, a dipolar perturbation (ΔBz) shifts the orbit centre at each turn. If the orbit separation is sufficient (δR>ΔR), we can direct the full bunch through the extraction channel without beam losses : obtaining a single turn extraction.

Table 1: Overview of two resonance often used in cyclotron extraction

Resonance	Ideal perturbation	Field bump	Comments
(Q_r =1)	cos(θ)	Local dipolar bump $\Delta B(\theta) = C_1 f(\theta-\theta_0)$	Adapted to cyclotron with Qr ~ 1
($2Q_r$ =2)	cos(2 θ)	Local quadrupolar bump $\Delta B(\theta) = C_1(r) f(\theta-\theta_0)$ $- C_1(r) f(\theta-(\theta_0+90°))$ C_1 is a quadrupolear field.	Called regenerative extraction: The quadrupolar perturbation C_1 modify the field index k and the radial tune Q_r. $Q_r^2 = 1+k+\ldots$ If k~-1, δR increases exponentialy since $\delta R/R \sim 1/(Qr^2)$ Useful technic for low energy gain per turn and bad beam quality as in synchrocyclotrons.

8 Conclusion

The cyclotrons are cost effective hadron accelerators. The main application is the production of radioisotopes for medicine for diagnosis (imaging). The Cancer therapy centres using 250 MeV proton cyclotron represent as well one application with an increasing interest. Beside, large cyclotrons are still used in many laboratories for research, and can deliver various ions, even at very large beam power (1.4 MWatt proton beam has been obtained in PSI).

The cyclotrons are classified as a function of their bending power K_b (see exercise 3). The main formula which drives all the concept of isochronism is the particle revolution frequency in the cyclotron magnet:

$$\omega_{rev} = \frac{q \ \langle B_z(R,\theta)\rangle_\theta}{m\gamma}$$

We have seen that several variations of the initial concept leads to 3 cyclotron kinds:

Table 1: Summary of the cyclotron family

Isochronous cyclotrons (the most diffused)	**Synchrocyclotrons**	**FFAGs**
-Cw (ω_{rev} = Constant =H ω_{rf}) 100% duty cycle -Rather complex magnet (to guarantee $Qz^2>0$) $B_z=F(r,\theta)$ -Azimuthal field modulation are required for vertical stability. -limited in energy ($\gamma<2$) **Applications:** mainly Radio-isotope production, but also proton therapy, research in nuclear physics. 1300 cyclotrons was operated in the world in 2021 **Possible future Application :** accelerator-driven reactor (ADS)	-RF cycled ($\omega_{rev}\neq$ Constant) typically ~1% duty cycle -Simpler magnets than the isochronous (no azimuthal field oscillations are required) -Optimal average field (very compact for a given energy) **Applications :** low intensity applications, proton therapy (250 MeV protons)	-RF cycled (ω_{rev} $\neq$ Constant) typically ~1% duty cycle -Complex dipole magnets, certain have a reverse field. -Never competitive with the cyclotrons at low energy. -Faster cycling than a typical synchrotron (only RF is cycled) -Not limited in energy (except accelerator cost) and can accelerate light particles (electron or muon) **Possible future Application** : Muon acceleration project (20 GeV), ADS

Acknowledgement

I wish to thank specially F. Meot, J. Leduf. for enlightening comments on the present document. Many good lectures of other colleagues (F. Chautard, E. Baron, W. Kleeven and S. Zaremba ,M. Seidel) inspired the structure of this document.

References

General

[1] J. J. Livingood, D. Van Nostrand Comp. 1961, Principles of Cyclic Particle Accelerators.

[2] M.KCraddock, Physics & technology of cyclotrons & FFAGs, U.S Particle Accelerator School, 2010

[3] W. Kleeven and S. Zaremba, Cyclotrons: Magnetic Design and Beam Dynamics, CAS 2015

[4] F Chautard, Beam dynamics for cyclo, https://indico.psi.ch › Cyclotrons_ECPM_2012-B

[5] M. M. Gordon and F. Marti, Electric Focusing in cyclotrons…, *Particle Accelerators,* 1980 Vol. I I pp.161-172 0031-2460/81/1103-0161$06.50/0

[6] H.L. Hagedoorn and N.F.Verster, Nucl. Instrum. Methods 18–19 (1962) 201

[7] P. K. Sigg, RF for cyclotron, CAS 2005; https://cds.cern.ch/record/1005053/files/p231.pdf

[8] S. Zaremba, Magnets for cyclotron, CAS 2005, https://cds.cern.ch/record/1005055/files/p253.pdf

FFAG and Synchrocyclotrons

[9] Shinji Machida **FFAG,** CERN Accelerator School 2011: High Power Hadron Machines, Bilbao, 2011, https://doi.org/10.48550/arXiv.1302.2026

[10] M.K.Craddock, Proceedings of 2005 Particle Accelerator Conference, Knoxville, Tennessee, p261- 265

[11] R. Edgecock et AL . *11th European Particle Accelerator Conference EPAC'08*, Jun 2008, Genoa, Italy. ,pp.3380-3382

[12]] J. Van de Walle et Al., Proceedings of Cyclotrons 2016, Zurich, Switzerland, p285-289

Injection & extraction in Cyclotron

[13] P.Heikkinen, Injection and extraction for cyclotrons, CAS (1994), http://cds.cern.ch/record/398439/files/p819.pdf.

[14] W. Kleeven, Injection and extraction for cyclotrons, CAS (2004), https://cds.cern.ch/record/1005057/files/p271.pdf.

[15] J.I.M. Boltman and H.L. Hagedoorn, Extraction from cyclotron, CAS (1994), CERN 94-01, Vol. II, p169. https://cds.cern.ch/record/399427/files/p169.pdf

[16] D. Toprek, Centering of the ion trajectories…, Nuclear Instruments and Methods in Physics Research A 480 (2002) 379–386

[17] R.W. Müller, Novel inflectors for cyclic accelerators, NIM 54, 29 (1967).

[18] J.L. Belmont, Ph. D. Thesis, Grenoble, 1964.

[19] D. Toprek, Nuclear Instruments and Methods in Physics Research A 449 (2000) 435}445