Beam dynamics for cyclotrons

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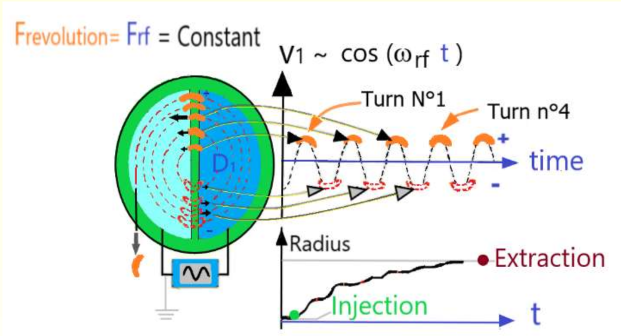


compact cyclotrons

Separated sectors (ring cyclotrons)

Synchrocyclotrons

Fixed energy

Variable energy

Superconducting

Normal conducting

Cyclotrons JUAS 2023

Chapter 1 : theory
- History & Principle
- isochronism
- Transverse dynamics (stability)

•Chapter 2 : different cyclotrons
- AVF cyclotron
- Synchro-cyclotron
- FFAG

Chapter 3 :specific problems
- Acceleration , RF
- Injection
- Extraction

Chapter 4 : design
- Design strategy
- Tracking

Chapter 5 :

-Theory vs reality (cost, tunes, Isochronism,...)

-Examples

Medical cyclotron

Research facility

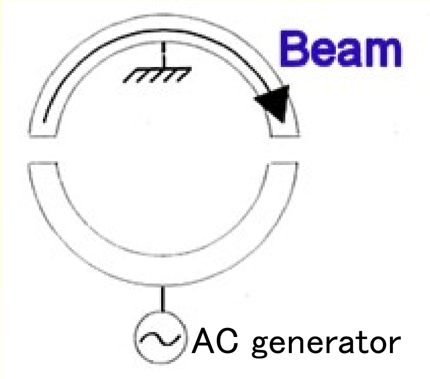
Cyclotron history

The Inventor, **E. Lawrence**, get the **Nobel** in Physics (1939)
(**first nuclear reactions Without alpha source**)

- Brilliant idea (E. Lawrence, Berkeley, 1929) : RF accelerating field is technically complex and expensive.

So Let 's use **only 1 RF cavity**, but many times

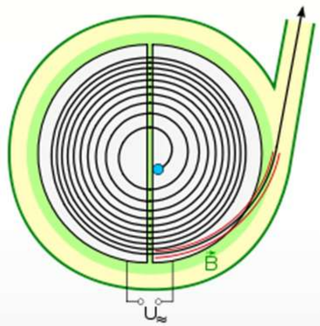
A device is put into a magnetic field, curving the ion trajectories and only one electrode is used several times.



From 1935-1980 : unique tool for Reseach facility (nuclear physics)

1980-today : Radio Isotope Production for the hospitals

In 2021 : 1300 cyclotrons in Operation !

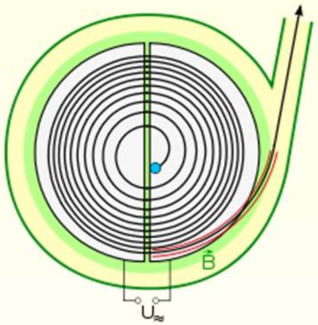


What is a cyclotron ?

- RF accelerator for the ions :

  from proton A=1 to Uranium A=238

- Energy range for proton 1MeV -1GeV (γ~1-2)
- Circular machine with 100% duty cycle
- Weak focusing
- Size Radius=30cm to R=6m
- RF Frequency : 10 MHz -60 MHz



APPLICATIONS : Nuclear physics :

from fundamental to applied research

Medical applications :

Radio Isotopes production (for PET scan,....)

Cancer treatment

Quality: Compact and Cost effective

Useful concepts for the cyclotrons

$$B\rho = \frac{P}{q} = \frac{\gamma\, m.v}{q}$$

$$E_K = (\gamma - 1).mc^2$$

Maxwell equations

$$\nabla \times \mathbf{B} = 0$$

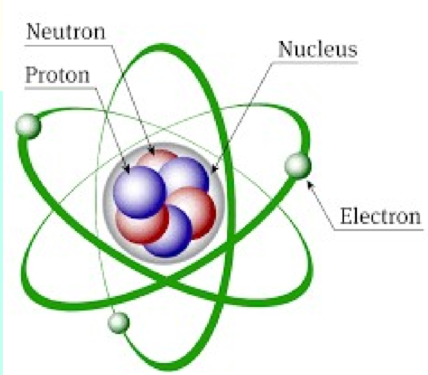
Cyclotron coordinates

r   Radial = horizontal

z   Axial = vertical

θ 《 Azimuth 》 = cylindrical angle

MeV/A= kinetic energy unit in MeV per nucleon



Ions :

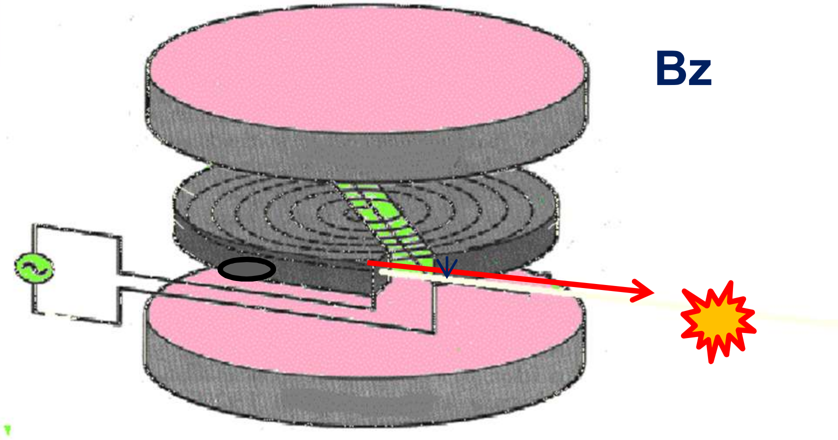
$${}^{A}_{Z}X^{Q}$$

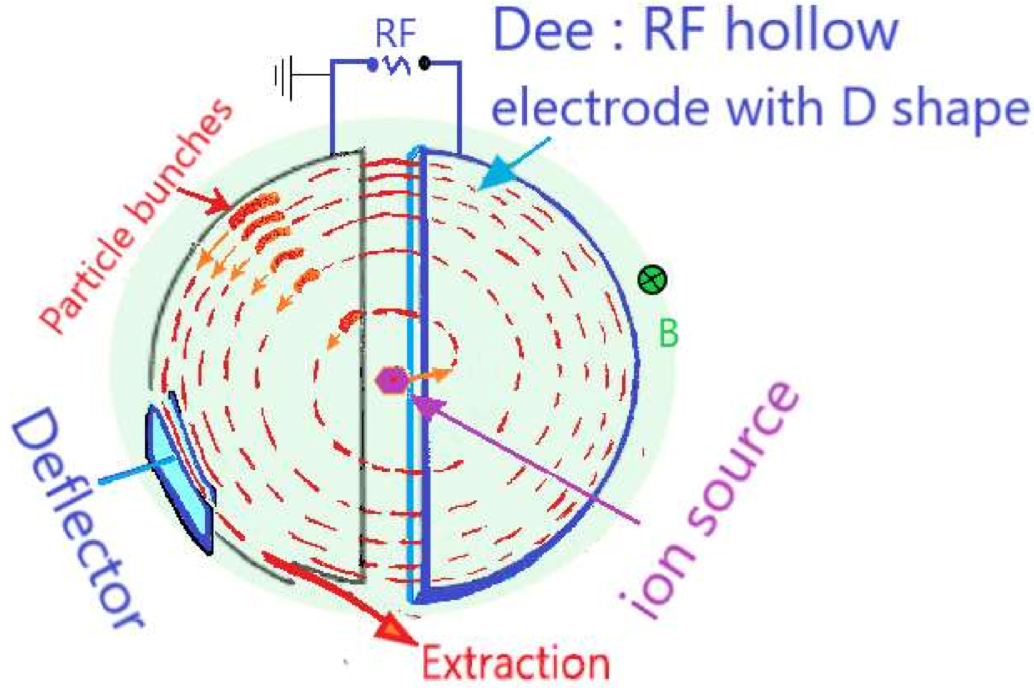
A : nucleons number

Z: protons number

Q : charge state : 0+,1+,2+,…..

Principle: spiraling trajectories





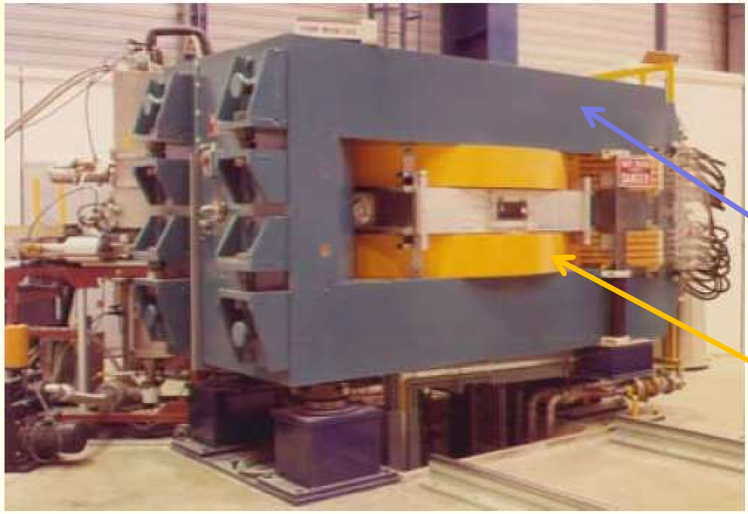
RF acceleration

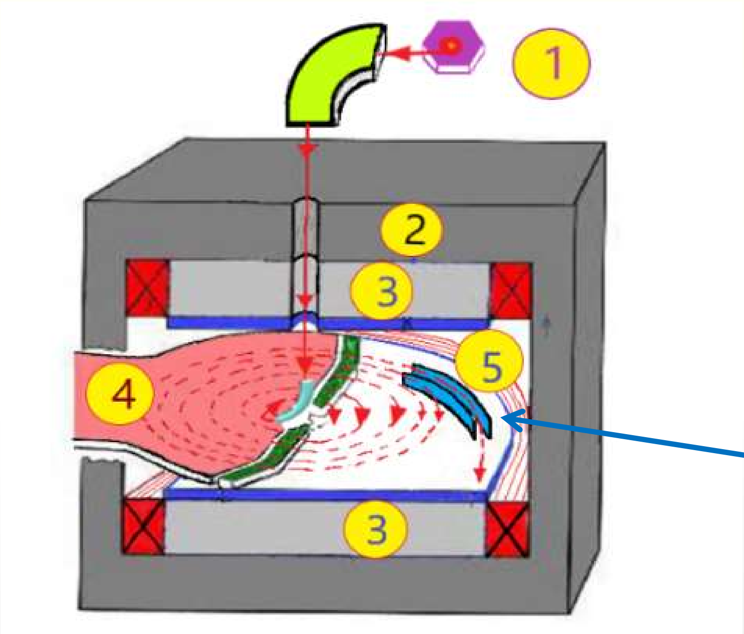
$$\vec{E}_{gap} = [V_1(t) - V_0]/d$$
$$= V\cos(\omega t)$$

Accelerating Dee's

2 Copper boxes $\neq$ potential
one is at the ground potential

A compact cyclotron in reality





1) Ion source

Magnet (Bz):

2) Yoke

2) Poles

3) Coils

RF cavities :

4) Dee

5) Electrostatic Deflector

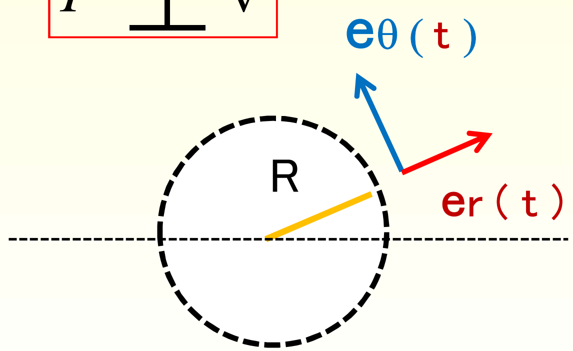
Trajectory in uniform B field

$$\frac{d(\gamma\, m\vec{\mathrm{v}})}{dt} = \vec{F}$$

A ion with a charge ***q*** and a mass ***m*** circulating at a speed $\boldsymbol{v}_\theta$
in a uniform induction field ***B***.=*(0,Bz,0)*
The motion equation can be derived from the **Newton's law** ***in a cylindrical coordinate system*** (er,,ez, eθ):

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{\mathrm{v}} \times \vec{B}) = -q\mathrm{v}_{\theta} B_z \vec{e_r}$$

$$\boxed{\vec{F} \perp \vec{\mathrm{v}}}$$



$$\text{since} \quad \boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e_r} & \boldsymbol{e_z} & \boldsymbol{e_\theta} \\ 0 & 0 & v_\theta \\ 0_r & B_z & 0 \end{vmatrix}$$

$$\frac{d\boldsymbol{p}}{dx} = m\gamma \frac{d\boldsymbol{v}}{dt} = q\ (v_\theta\ .B_z)\,.\,\boldsymbol{e_r}$$

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{\|\boldsymbol{v}\|^2}{R}\right).\,\boldsymbol{e_r}$$

$$m\gamma\ \frac{\boldsymbol{v}^2}{R}\boldsymbol{e_r} = \ q\ (v_\theta\ .B_z)\,.\,\boldsymbol{e_r}$$

$$m\gamma\ \frac{\boldsymbol{v}^2}{R} = \ q.vB_z \quad \Longrightarrow \quad \boxed{R = \frac{\gamma\, mv}{qB_z}} \quad \Longrightarrow$$

Trajectory in uniform Bz field

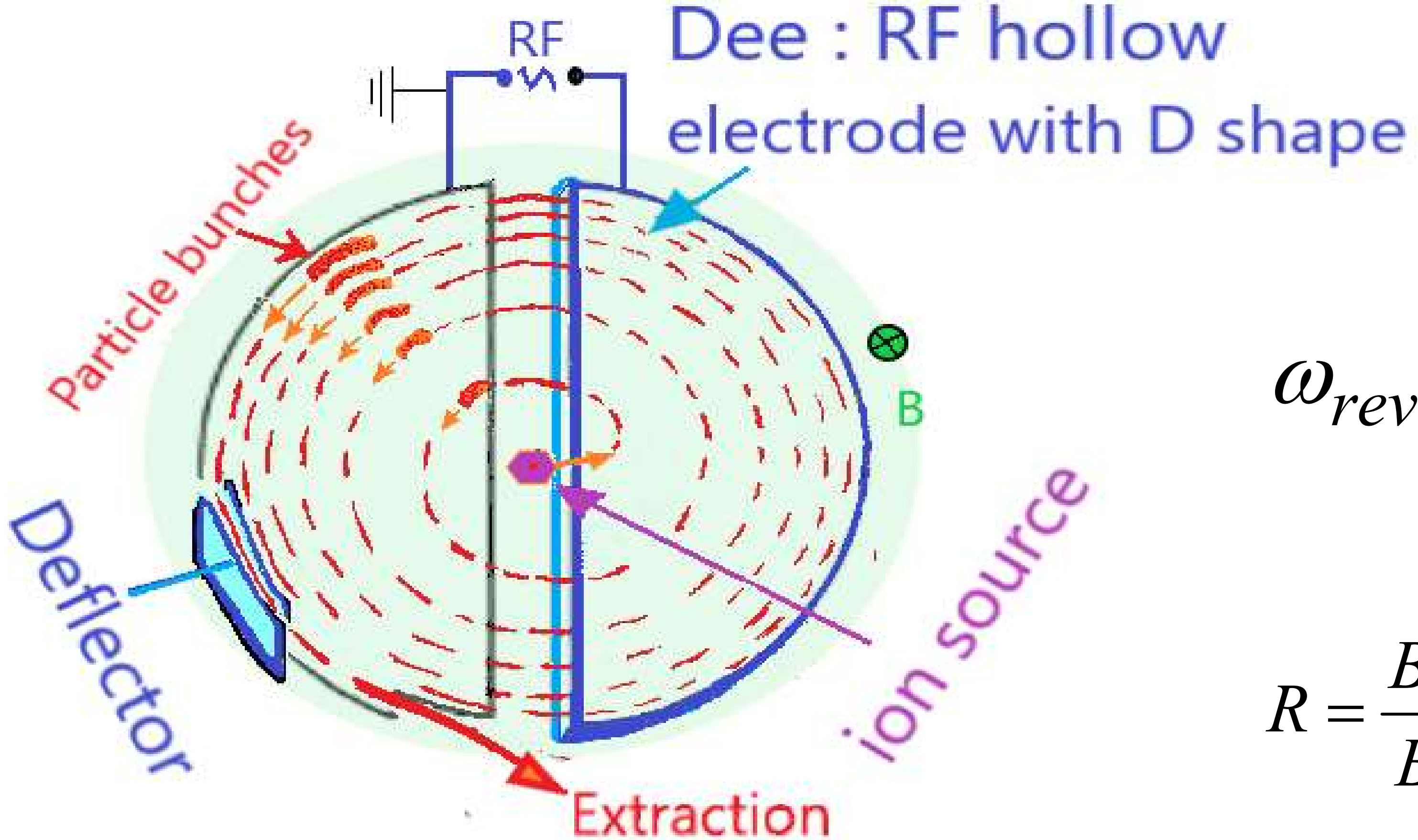
$$R = \frac{\gamma\, mv}{qB_z}$$

$$Frevolution = \frac{\mathrm{v}}{2\pi\, R} = \frac{1}{2\pi}\frac{qB}{\gamma\, m}$$

$$\omega_{rev} = 2\pi\, F_{rev} = \dot{\theta} = \frac{d\theta}{dt} = \frac{\mathrm{v}_\theta}{R} = \frac{qB}{\gamma\, m}$$

$$\boxed{\omega_{rev} = \frac{qB}{\gamma\, m}}$$

The acceleration in cyclotron



$$\omega_{rev} = \frac{qB}{\gamma\, m}$$

$$R = \frac{B\rho}{B_z} = \frac{\gamma\, m\mathrm{v}}{qB_z}$$

Cyclotrons Tutorials 0

- **Demonstrate that the revolution frequency** $\left(F_{rev} = \omega_{rev}/2\pi\right)$
of an ion in a perpendicular uniform field B_z is

$$\omega_{rev} = q\, B_z / m\gamma$$

1) **With the Newton-Lorentz equation (demo 1)**
2) **With magnetic rigidity (demo 2)**

Nota: Electric field is supposed to be Zero

so $||V||$ =constant and

hence $\gamma = [1-v^2/c^2]^{-1/2}$ = constant

Cyclotrons Tutorials 0

Demonstrate that the revolution frequency ***Frev*** **=** ω**rev /2**π

of an ion in a perpendicular uniform field ***Bz*** **is** ω**rev =q** ***Bz*** **/ m**γ

Demo N° 1 : Newton equation

d ***p*** /dt = q (**v x B**)

$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_z & \mathbf{e}_\theta \\ 0 & 0 & V\theta \\ 0 & B_z & 0 \end{vmatrix}$$

m γ (d v/ dt) = − q vθ Bz er

m γ (− v^2 /R) er = − q vθ Bz er

R = q v Bz / m γ

$$\boxed{Frevolution = \frac{v}{2\pi\ R} = \frac{1}{2\pi}\frac{qB}{\gamma\ m}}$$

Demo N° 2 : Bρ formula

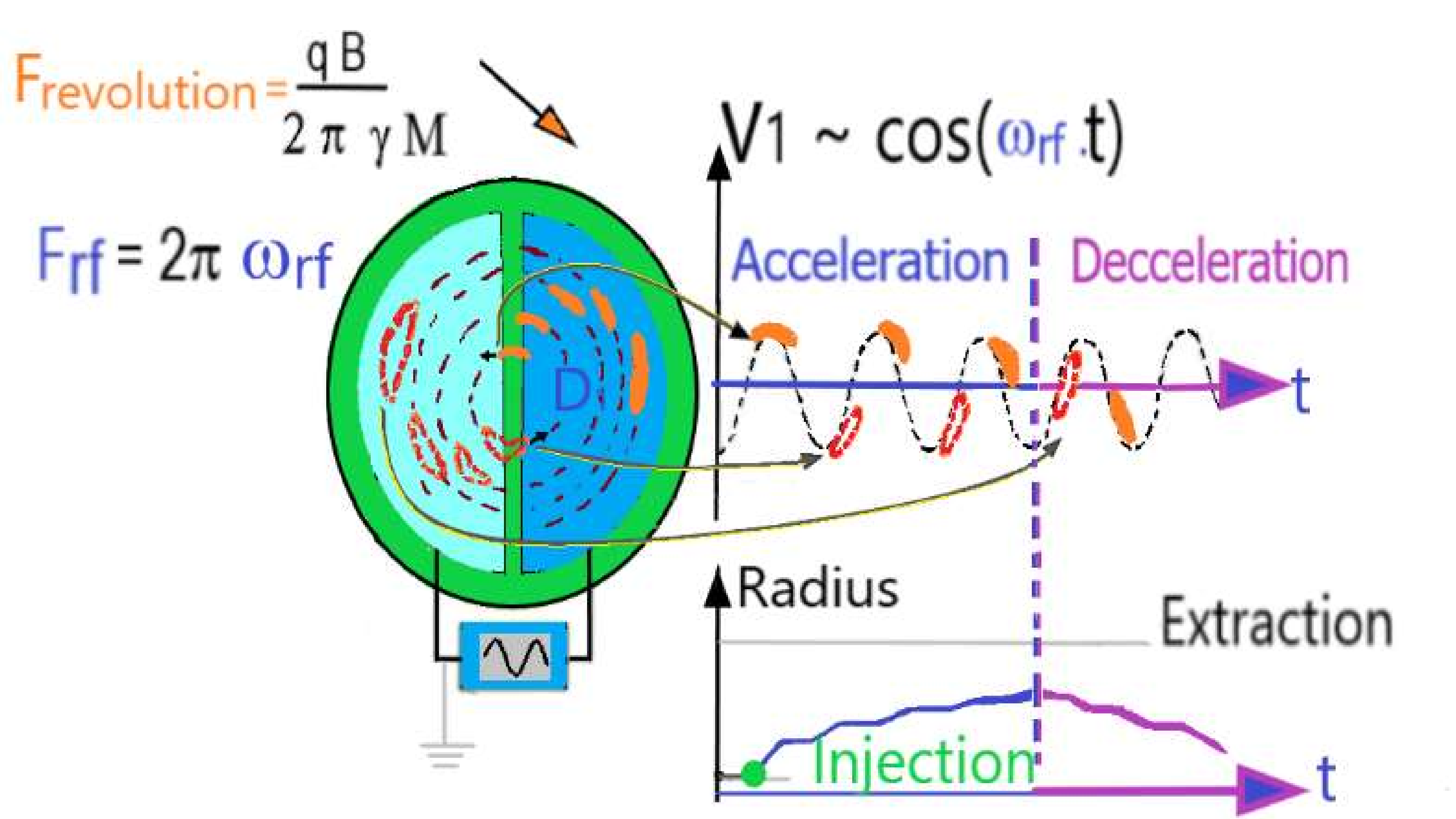
$$\boxed{R = \frac{B\rho}{B_z} = \frac{\gamma\ mv}{qB_z}}$$

$$\boxed{Frevolution = \frac{v}{2\pi\ R} = \frac{1}{2\pi}\frac{qB}{\gamma\ m}}$$

ω_{rev} = 2π *Frev* =q ***Bz*** / m γ

Desynchronization of bunches in uniform field cyclotron $B_z = B_0$

$$\omega_{rev} = \frac{qB_0}{m\gamma} = \frac{qB_0}{m} \cdot \sqrt{1 - {v^2}/{c^2}} = \frac{qB_0}{m} \cdot \sqrt{1 - {R^2\omega^2}/{c^2}} \quad .$$



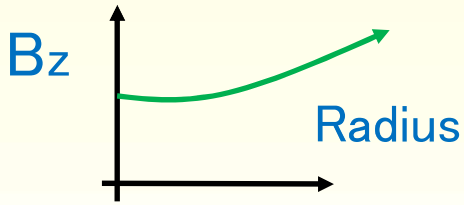
How to get constant revolution frequency

$$\omega_{rev} = \frac{qB_z(R)}{\gamma(R)\,m} = const$$

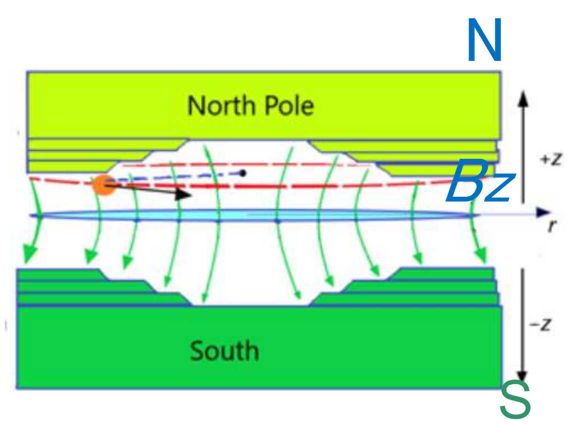
$$\gamma(R) = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (R\omega_{rev})^2/c^2}}$$

Isochronism condition

Bz =Bz(Radius) $\sim \gamma$



Pole gap
= F(Radius)



Bz(Radius) = obtained by pole gap shaping

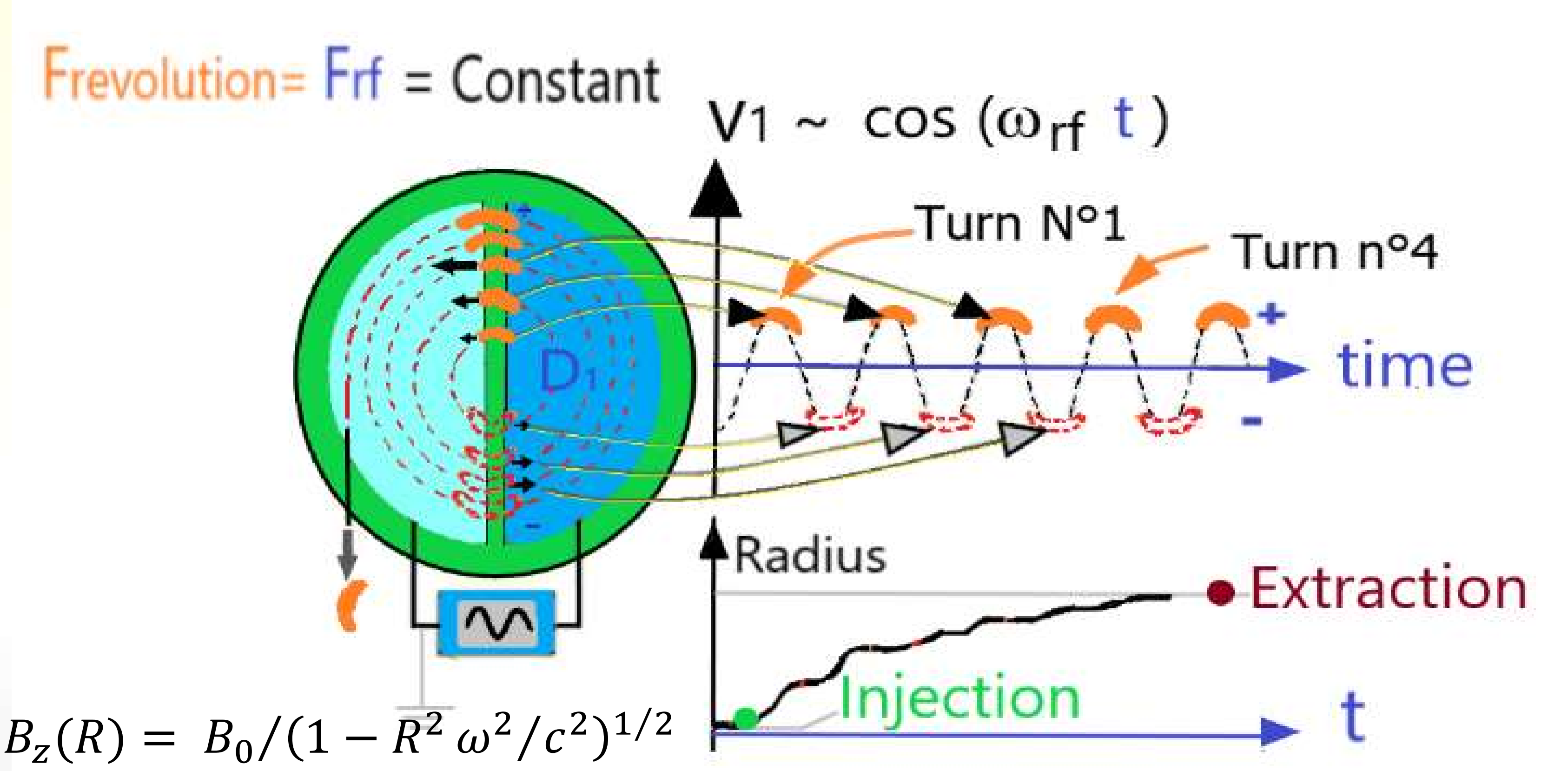
The distance between magnet pole (gap) evolves with Radius : *Bz* ~ 1/gap

Isochronism ω_{rev} = constant $B_z(R) = B_0/(1 - R^2\,\omega^2/c^2)^{1/2}$

Isochronism condition: ω_{rev}=constant

with $\omega_{rf} = H\omega_{rev}$, the particle is synchronous with the RF wave.

In other words, the particle arrives always at the same RF phase in the middle of the accelerating gap.



$$B_z(R) = B_0/(1 - R^2\,\omega^2/c^2)^{1/2}$$

Cyclotrons Tutorial 1

•<u>An isochronous cyclotron</u> uses a **RF cavity** at **60 MHz** at the RF harmonic **H=3**

a. Compute **the time needed to perform one turn T_{rev}** for the accelerated ions.

b. Compute **the average field B_z** needed to accelerate a proton beam (in a non relativistic approximation)

Cyclotrons Tutorial 1

•**An isochronous cyclotron uses a RF cavity at 60 MHz at the RF harmonic h=3**

a. Compute the time needed to perform one turn for the accelerated ions.
b. Compute the average field B needed to accelerate proton

in a non relativistic approximation

Answer ***a*** Revolution freq=60/3 =20 MHz $\Delta T = 1/F_{rev} = \mathbf{50ns}$

.

b $\boldsymbol{\omega = qB/\gamma m} = \omega_{rf} / \boldsymbol{H}$ *and we have* γ close to 1

proton mass~ 1.6 10^{-27} kg // proton charge~ 1.6 10^{-19} C

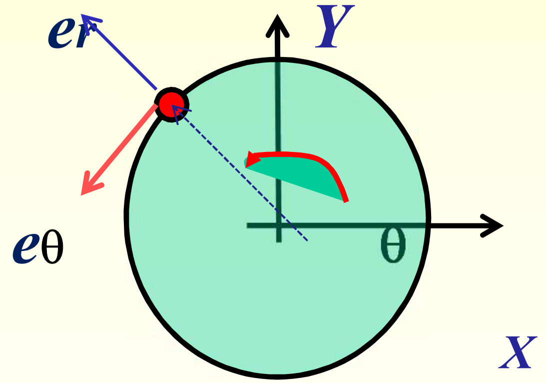
F_{rf}=60 MHz= $\omega_{rf}/2\pi$

$$\boldsymbol{B_z = m_p/q} \;.\; 2\pi F_{rf}/\boldsymbol{H} \;=\; 10^{-8}.\,10^{6}\;20\;.2\pi\;.= 1.26\ \text{Tesla}$$

Cyclotrons Tutorial 2

•Demonstrate than in a **uniform circular motion** , the radial acceleration is

$$a_r = |V^2 / R| \ .$$



Nota : *You can use parametric equations* :

$$X(t) = R\cos(\omega t)$$
$$Y(t) = R\sin(\omega t)$$

Then compute **the velocity** and the **acceleration.**
Demonstrate that the acceleration is radial

Nota : $\omega t = \theta$ $\omega = d\theta/dt$

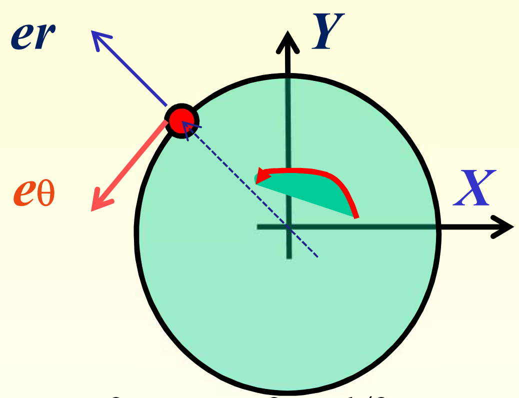
Cyclotron Tutorial 2

uniform circular motion

$X(t) = R\cos(\omega t)$

$Y(t) = R\sin(\omega\ t)$

*compute **the velocity** and the **acceleration.***



Answer

$$V_x = -\omega R \sin(\omega t)$$

$$V_y = +\omega R \cos(\omega t)$$

$$v_\theta = |\mathbf{v}| = (V_x{}^2 + V_y{}^2)^{1/2} = \omega R$$

$$|\mathbf{a}| = (a_x{}^2 + a_y{}^2)^{1/2} = \omega^2 R = v^2 / R$$

v perpendicular to **a** (since $\mathbf{v.a} = \mathbf{0}$)

$$\boldsymbol{a} = -v^2/R$$

Radial vector e_r and longitudinal vector e_θ

$$e_r = \left| \begin{array}{l} \cos(\omega t) \\ \sin(\omega t) \end{array} \right.$$

$$de_r/dt = \omega e_\theta$$

$$d^2 e_r/dt^2 = -\omega^2 e_r = v^2/R^2 e_r$$

Transverse dynamics in a cyclotron without acceleration

We will use cylindrical coordinates (e_r, e_θ, e_z)

The reference trajectory is $\mathbf{r} = R_0\, \mathbf{e_r}$ $R_0 = B\rho/B_z$

What happen to a particle with $\mathbf{r} = (R_0 + x_0)\, \mathbf{e_r} + z_0\, \mathbf{e_z}$

<u>In Radial plane (horizontal) :</u>

$$r(t) = R_0 + x_0 \cos(Q_r\, \omega_{rev}\, t)$$

<u>Radial tune Q_r</u>

<u>In the Vertical (axial) plane :</u>

$$z(t) = z_0 \cos(Q_z\, \omega_{rev}\, t)$$

<u>Axial tune Q_z</u>

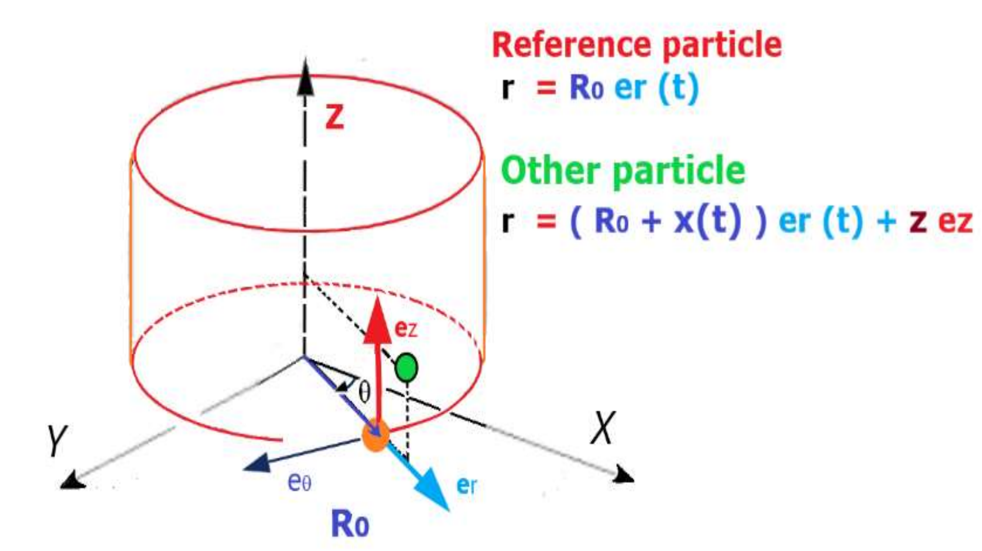
Transverse dynamics with Bz (R)

$$B_z(R) = B_0/(1 - R^2\,\omega^2/c^2)^{1/2}$$

Cylindrical coordinates ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$)

and

define **x & z** a small orbit deviation with **B***z*=Bz(r) (not constant)



$$B_z(R) = \gamma(R)\ B_0$$
$$= R^{-n}\ B_0$$

Isochron field

Uniform Circular motion x=0

Motion Equat. With x≠ 0 & z ≠ 0 ?

$$\boxed{m\frac{d(\vec{\mathbf{v}})}{dt} = m\frac{d^2(\mathbf{r})}{dt^2} = ?}$$

Vertical dynamics with Bz(R) (No RF)

- Taylor expansion of the field B_z around the median plane:

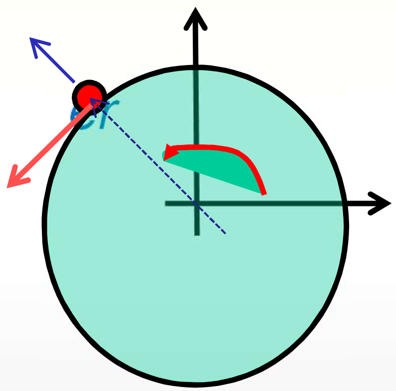
definition of $n(r)$ $B_z \sim B_0\,(r/R_0)^{-n}$ n=field index Bz is not uniform)

$$\text{with} \quad n = -\frac{R_0}{B_0}\frac{\partial B_z}{\partial r} = -\frac{R_0}{B_0}\frac{\partial B_z}{\partial x}$$

so $B_z\,(R_0+x) = B_0(R_0) + (dB/dx)\,x + \ldots = B_0\,(1-nx/R_0)$

How evolves an ion, in this non uniform Bz :

$\mathbf{r}(t) = (\,R + x(t)\,)\,\mathbf{e_r} + z(t)\,\mathbf{e_z}$



$m\,\gamma\,d^2\,(z\,\mathbf{e_z})/dt^2 = \mathbf{F_z}\,\mathbf{e_z}$ Vertical motion e_θ

Fz vertical plan : $\mathbf{e_z}$=constant

$$m\gamma\frac{d^2 z}{dt^2} = F_z = q\,(\mathrm{v}\times B)_z = -q(\dot{r}\,B_\theta - r\,\dot{\theta}\,B_r)$$

Vertical dynamics with B (r)

$$m\gamma \frac{d^2 z}{dt^2} = F_z = q\,(\mathrm{v}\times B)_z = -q(\dot{r}\cancel{B_\theta} - r\,\dot{\theta}\,B_r)$$

$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_z & \mathbf{e}_\theta \\ \dot{r} & \dot{z} & r\dot{\theta} \\ B_r & B_z & 0 \end{vmatrix}$$

$Br=?$ $\quad \nabla\times\mathrm{B}=0 \quad \dfrac{\partial B_r}{\partial z} - \dfrac{\partial B_z}{\partial r} = 0$ $\quad$ + $\quad B_z = B_0\, r^{-n}$

$$\Rightarrow \quad B_r = -n\frac{B_{oz}}{r}z$$

Motion equation $\quad \dfrac{d^2 z}{dt^2} = -\omega^2 . Q_z^2 . z$

Harmonic oscillator ?

$$z(t) = z_0 \cos(Q_z\, \omega_{rev}\, t)$$

For isochronism, we have n < 0 (therefore Q_z is imaginary)

$$Q_z^{\,2} = n < 0 \qquad Q_z = i\,|n| \ \ldots$$

Watch the vertical oscillations !!

Isochronism condition : **n < 0** **: Bz(r)~ r^{-n}** **: increase with Radius**

Vertical tune $Q_z^{\,2} = n < 0$

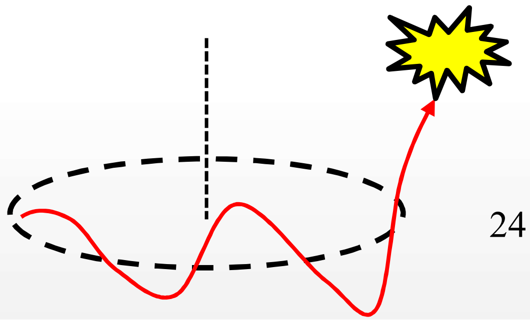
Isochronism condition will induce unstable oscillations

$$z(t) = z_0 \cos(Q_z\, \omega_{rev}\, t)$$

$$z(t) \sim z_0 \exp(\pm\, |n|^{1/2}\, \omega_{rev}\, t)$$

Unstable oscillations in Z

= exponential growth = beam losses



Radial (horizontal) dynamics with Bz(R) *(No RF)*

- Taylor expansion of the field B_z around the median plane:
- *definition of n(R)* $B_z(R) \sim B_0 (R/R_0)^{-n}$ n= called field index

so $B_z(R_0+x) = B_0(R_0) + (dB/dx)\, x + \ldots = B_0(1 - nx/R_0)$

•How evolves an ion, in this non uniform Bz : $r(t) = R_0 + x(t)$

$$m\gamma \frac{d^2 \vec{r}}{dt^2} = -q\ \mathbf{v} \times \mathbf{B}$$

$$\frac{d^2 \mathbf{er}}{dt^2} = -\omega^2\, \mathbf{er} = \mathrm{v}^2/r^2\, \mathbf{er}$$

$$\frac{d\mathbf{er}}{dt} = \omega\, \mathbf{e}\theta$$

$$\frac{d^2(r \cdot \vec{er})}{dt^2} = (\ddot{x} - \frac{\mathrm{v}_\theta^2}{r})\vec{er} + 2\dot{x}\dot{e}_r$$

$$r = R(1 + x/R)$$

$$= \left[\ddot{x} - \frac{\mathrm{v}_\theta^2}{R}(1 - \frac{x}{R})\right]\vec{\mathbf{er}} + 2\dot{x}.\vec{\mathbf{e_\theta}}$$

radial motion (projection on **er***)*

$$\frac{1}{r} = \frac{1}{R\,(1 + \frac{x}{R})} \approx \frac{1}{R}(1 - \frac{x}{R})$$

$$m\gamma\left(\ddot{x} - \frac{\mathrm{v}_\theta^2}{R}(1 - \frac{x}{R})\right) = -q\,\mathbf{v}_\theta B_{0z}(1 - n\frac{x}{R})$$

Radial dynamics with Bz(R)

$$m\gamma\left(\ddot{x} - \frac{v_\theta^2}{R}\left(1 - \frac{x}{R}\right)\right) = -q\,B_{0z}\left(1 - n\frac{x}{R}\right).v_\theta$$

After simplification :

$$\text{and} \qquad \omega_{rev} = \frac{qB_{0z}}{\gamma\, m} = \omega_0 \approx \frac{v_\theta}{R}$$

$$\frac{d^2x}{dt^2} = -\omega^2 {Q_r}^2\, x \qquad\qquad Q_r = (1-n)^{1/2}$$

Horizontal stability condition (Qr real) : $\boxed{n < 1}$

We have n < 0 for isochronism : therefore *$Q_r^2 > 0$*

Horizontal stability is guarantee in isochronous cyclotron

Radial dynamics (Q_r real) : Stable oscillations

Harmonic oscillator with the frequency ($Q_r\,\omega_{rev}$)

$$\frac{d^2x}{dt^2} = -\omega^2(1 - n)x$$

$$x(t) = x_0 \cos(Q_r\,\omega_{rev}\,t)$$

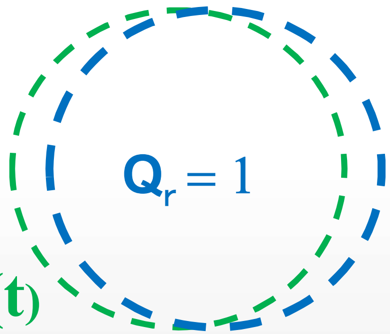
Horizontal stability if $n < 1$: always satisfied

$$Q_r^{\,2} = 1 - n \quad > 0$$

n< 0 : isochronism condition Bz **should increase**

Stability condition ($Q_r^{\,2} > 0$) OK

$$r = R_0 + x_0 \cos(Q_r\,\omega_{rev}\,t)$$

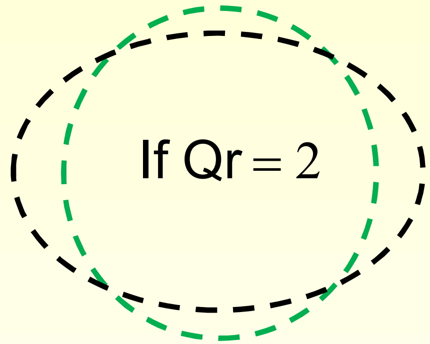


$r(t) = R_0(t)$

Tunes : Q_r & Q_z

oscillations around reference trajectory

$$r(t) = R_0(t) + x_0 \cos(Q_r\, \omega_{rev}\, t)$$



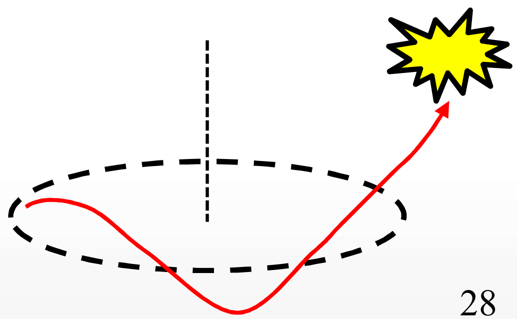
Q_r : Number of radial oscillations per cyclotron turn
in horizontal (radial) plan

$r = R_0(t) + x_0 \cos(2\,\omega_0\, t)$

$Q_r^2 = 1 - n$ stable oscillations

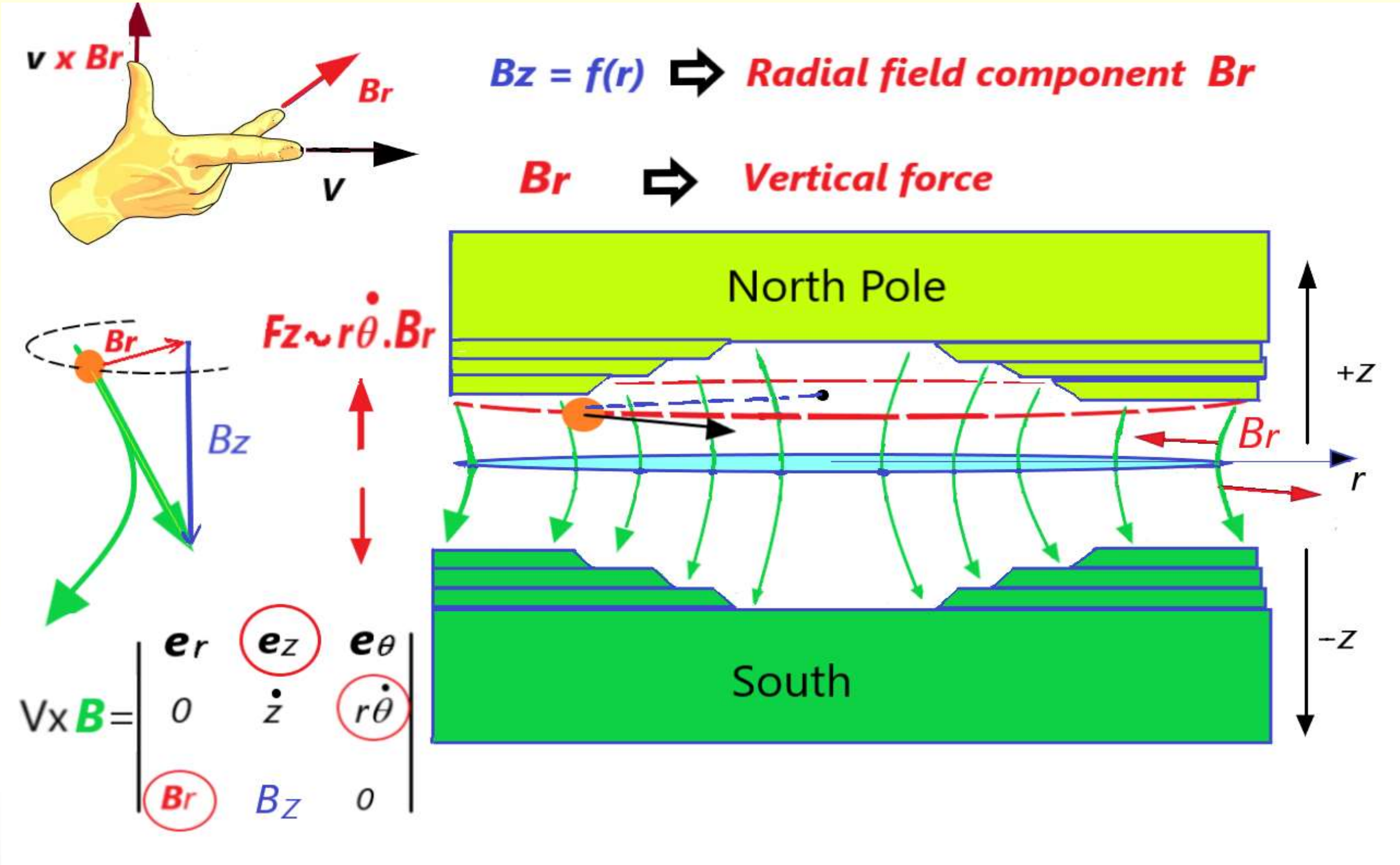
$$z(t) = z_0 \cos(Q_z\, \omega_{rev}\, t) = z_0 \cos(Q_z\, \theta)$$

$Q_z^2 = n < 0$ unstable oscillations
$(Q_z = i\,|Q_z|)$



$$z(t) \sim z_0 \exp(\pm |Q_z|\ \theta)$$

The vertical instability generated by Bz(R)



Vertical stability ≠ Isochronism

Isochronism condition
(longitudinal)

B=Bz(R)

$$\omega_{rev} = \frac{qB_z(R)}{\gamma(R)\, m}$$

Bz should increase with radius （$B_z \sim B_0\, r^{-n}$　n<0）

Unstable Vertical oscillations（ *Br* defocus in z plane)

Additive Vertical focusing is needed

$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_z & \mathbf{e}_\theta \\ \dot{r} & \dot{z} & r\,\dot{\theta} \\ B_r & B_z & B_\theta \end{vmatrix}$$

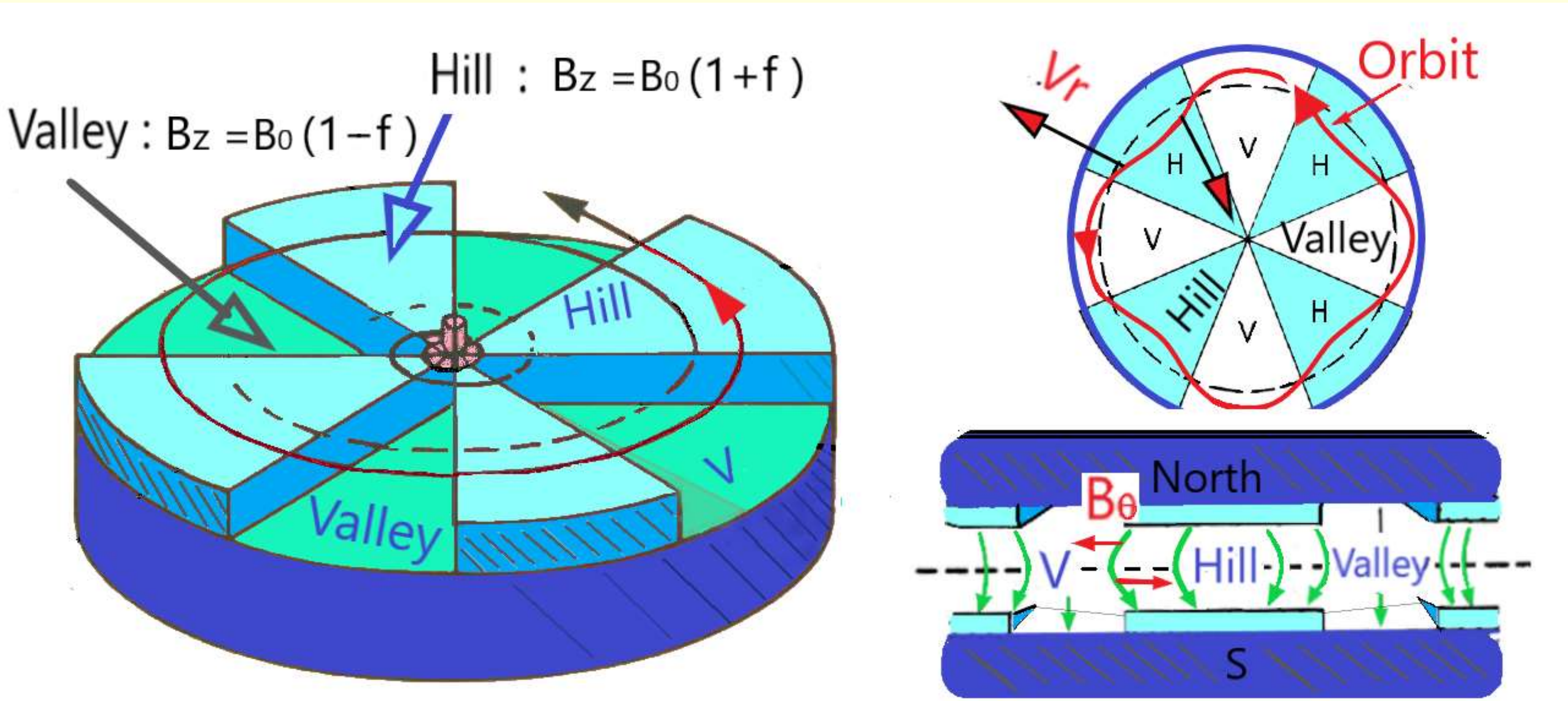
Bθ component needed（$F_z = v_\theta . B_r - q\, v_r B_\theta$）:

《 Azimutally Varying Field 》 Cyclotron　B= B(r,θ)

Azimuthally Varying Field cyclotron

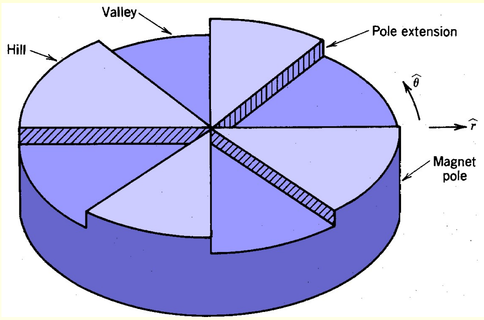
Restore vertical stability even with Bz=B0 (R)

$$\mathbf{v}\times\mathbf{B}=\begin{vmatrix}\mathbf{e}_r & \mathbf{e}_z & \mathbf{e}_\theta\\ \dot{r} & \dot{z} & r\dot{\theta}\\ B_r & B_z & B_\theta\end{vmatrix}$$



$\langle F_z\rangle = q\langle v_r \cdot B_\theta\rangle$: additional Vertical focusing force

What is the azimuthal field B_θ



Maxwell Equation Curl B=0

$$\nabla \times \boldsymbol{B} = \left(\frac{dB_\theta}{dz} - \frac{dB_z}{Rd\theta}\right).\boldsymbol{e_r} + \left(\frac{d(RB_\theta)}{RdR} - \frac{dB_\theta}{dz}\right).\boldsymbol{e_z} + \left(\frac{dB_z}{dR} - \frac{dB_r}{dz}\right).\boldsymbol{e_\theta} = \mathbf{0}$$

$$\frac{dB_r}{dz} = \frac{dB_z}{dR} = -\ n\,K\,R^{n-1} = -n\,.\frac{B_0}{R}$$

$$\boxed{B_\theta = z.\frac{dB_z}{Rd\theta} + \ldots}$$

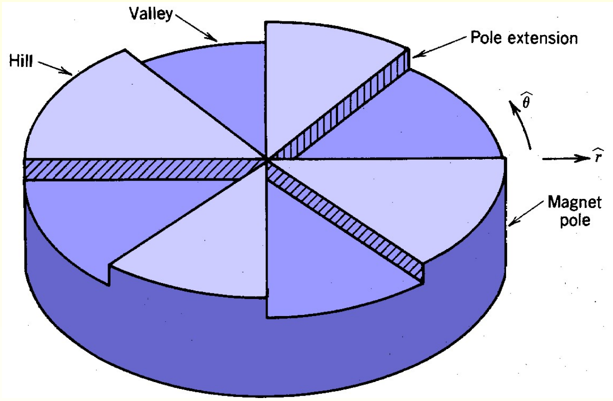
$$\boxed{B_r = -n\frac{B_{oz}}{r}z}$$

$\langle F_z \rangle = q\langle v_r\,.\,B_\theta\rangle \sim -z$: Vertical focusing force

Azimuthally Varying Field ("A.V.F.")

Vertical weak focusing : $B_z = F(R, \theta)$

• $F_z \sim \langle q\, v_r\, B_\theta \rangle$: Vertical focusing



$B_z = f(R, \theta)$ $\quad$ $\nabla \times B = 0$

$B_\theta = g(R, \theta)$

Isochronism n <0 : Bz(R) increases with Radius R

Vertical stability : Bz(R) Defocus in z + Bθ *Focus in z*

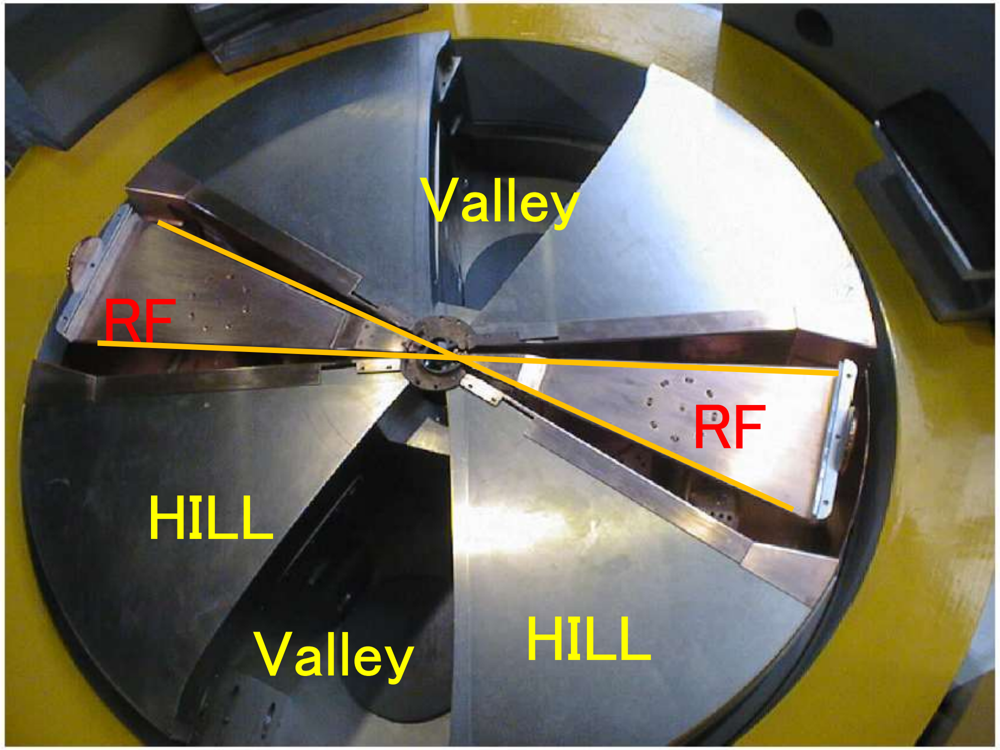
Bz should oscillate with θ to compensate the instability

Azimuthally varying Field (AVF)

Exemple : 30 MeV compact proton cyclotron

4 straight sectors

C30 poles and valleys



-2 RF cavities
Inserted in the valleys
= 4 accelerating gaps

4 Hills + 4 Valleys

$B = B(R,\theta)$

Bz Field varies
with azimuth θ