Non-linear effects

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Bibliography

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  - E. Forest, Beam Dynamics - A New Attitude and Framework, Harwood Academic Publishers, 1998.
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  - J. Bengtsson, Nonlinear Transverse Dynamics in Storage Rings, CERN 88-05

Disclaimer

- This course is based on material from the JUAS course on non-linear dynamics by Y. Papaphilippou from the past.

- Parts of the slides were taken / inspired from the course on nonlinear dynamics of R. Bartolini (John Adams Institute, 2017) and the one of A. Wolski (Cockroft Institute, 2015).

Outline

- Introduction – nonlinear effects from a single sextupole
- Hamiltonian of the nonlinear betatron motion
- Resonance topology and onset of chaos
- Resonances are everywhere – can we do something?
- Non-linear map representation
- Lattice optimization by tracking
- Applications – making use of resonances
- Summary

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Example of a simple storage ring

- **To correct or control chromaticity in a storage ring we need to install sextupole magnets**

- **Nonlinear elements such as sextupole magnets can have significant impact on the particle motion (as we will see)**

- **To illustrate this, we start with a very simple example**
  - Assume a circular machine built of **identical cells**
  - There is **one sextupole per cell**, which is located at a point where the horizontal beta function is 1m, and the alpha function is zero (to control chromaticity in both planes we would need at least 2 sextupoles)
  - The **phase advance per cell can be tuned**
  - We **consider** for the moment **only horizontal motion** (i.e. y=0)
  - We build a small simulation code to study the particle behavior in phase space turn-by-turn

Example of a simple storage ring

- The map from the sextupole in one cell to the sextupole in the next cell is just a rotation in phase space (periodic linear transfer matrix with beta=1 and alpha=0)

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

- The change in the horizontal momentum of a particle moving through the sextupole is found by integrating the Lorentz force

$$\Delta p_x = -\int_0^L \frac{B_y}{B\rho} ds \quad \text{with} \quad \frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2 \quad \text{(assuming y = 0)}$$

- If the sextupole is short we can neglect the small change in the coordinate x as the particle moves through the sextupole, in which case we obtain (**thin lens approximation**)

$$\boxed{\Delta p_x = -\int_0^L \frac{1}{2} k_2 x^2 ds \approx -\frac{1}{2} k_2 L x^2}$$

Example of a simple storage ring

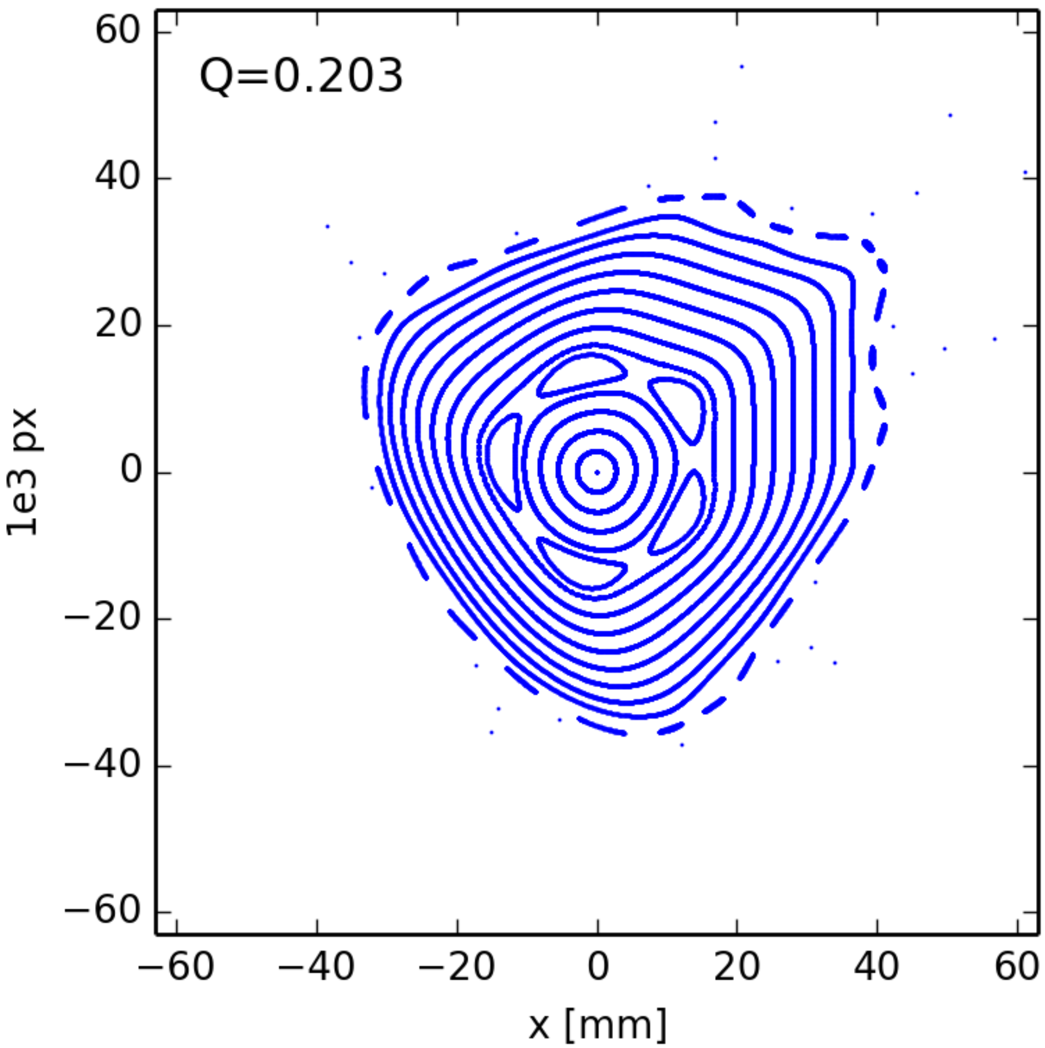
- The **map for a particle moving through a short sextupole** can be represented by a “**kick**” in the horizontal momentum

$$x \mapsto x$$

$$p_x \mapsto p_x - \frac{1}{2} k_2 L x^2$$

- For the moment we consider a **machine with a single cell**, for which the map consists of the linear transfer map and one sextupole kick
- We choose a **fixed value of k_2L** and look at the effects of the maps for **different tunes** (i.e. phase advances) of the machine
- For each case **we construct a phase space portrait** by plotting x, p_x turn after turn for **a range of initial conditions**

Example of a simple storage ring



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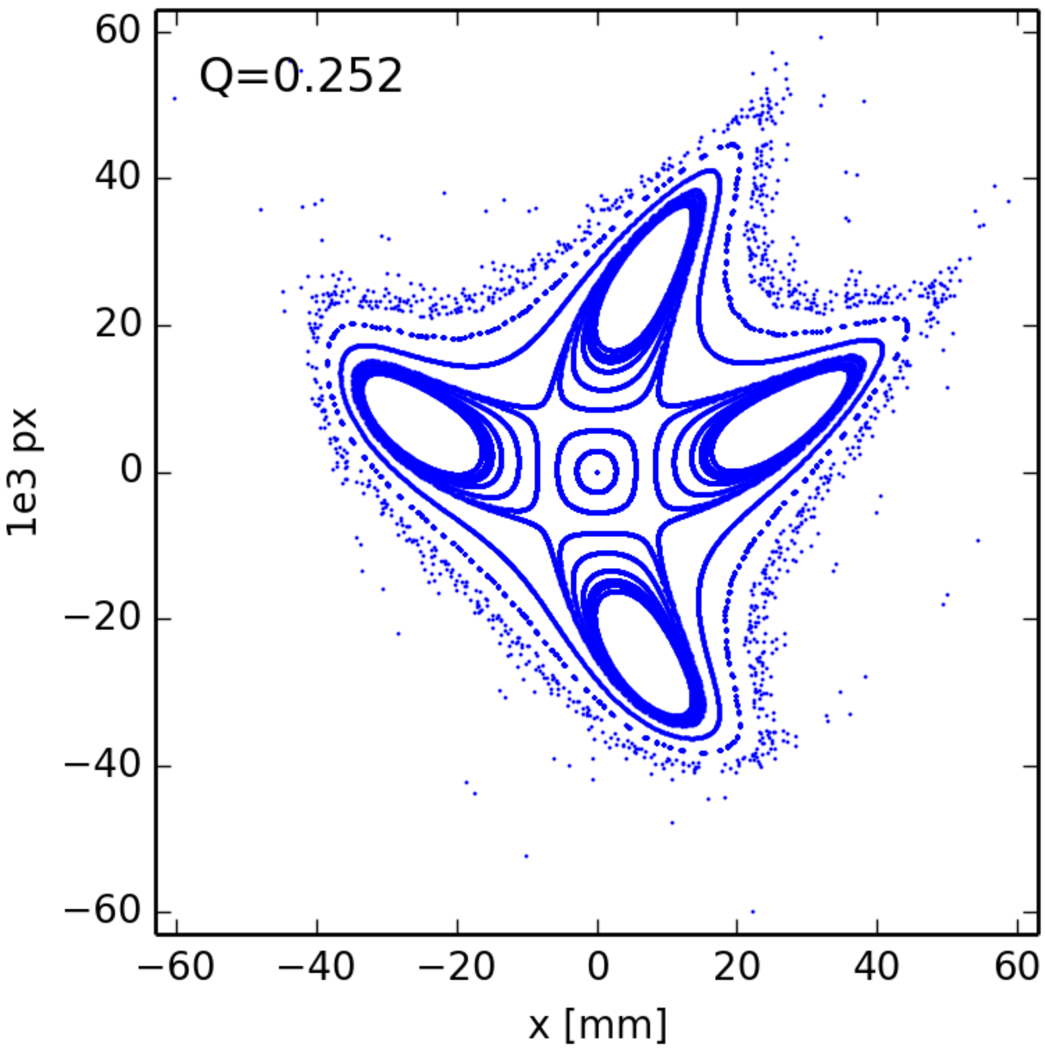
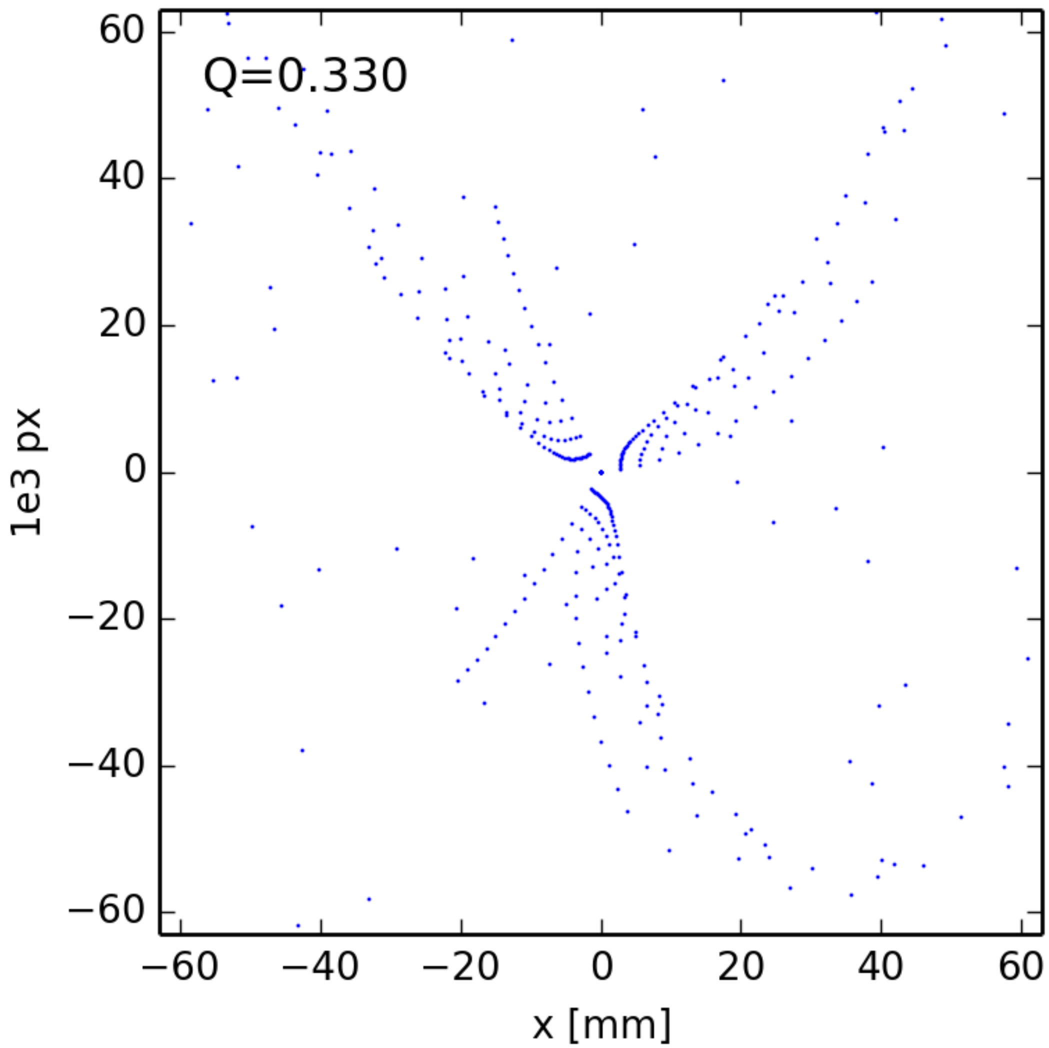


Example of a simple storage ring



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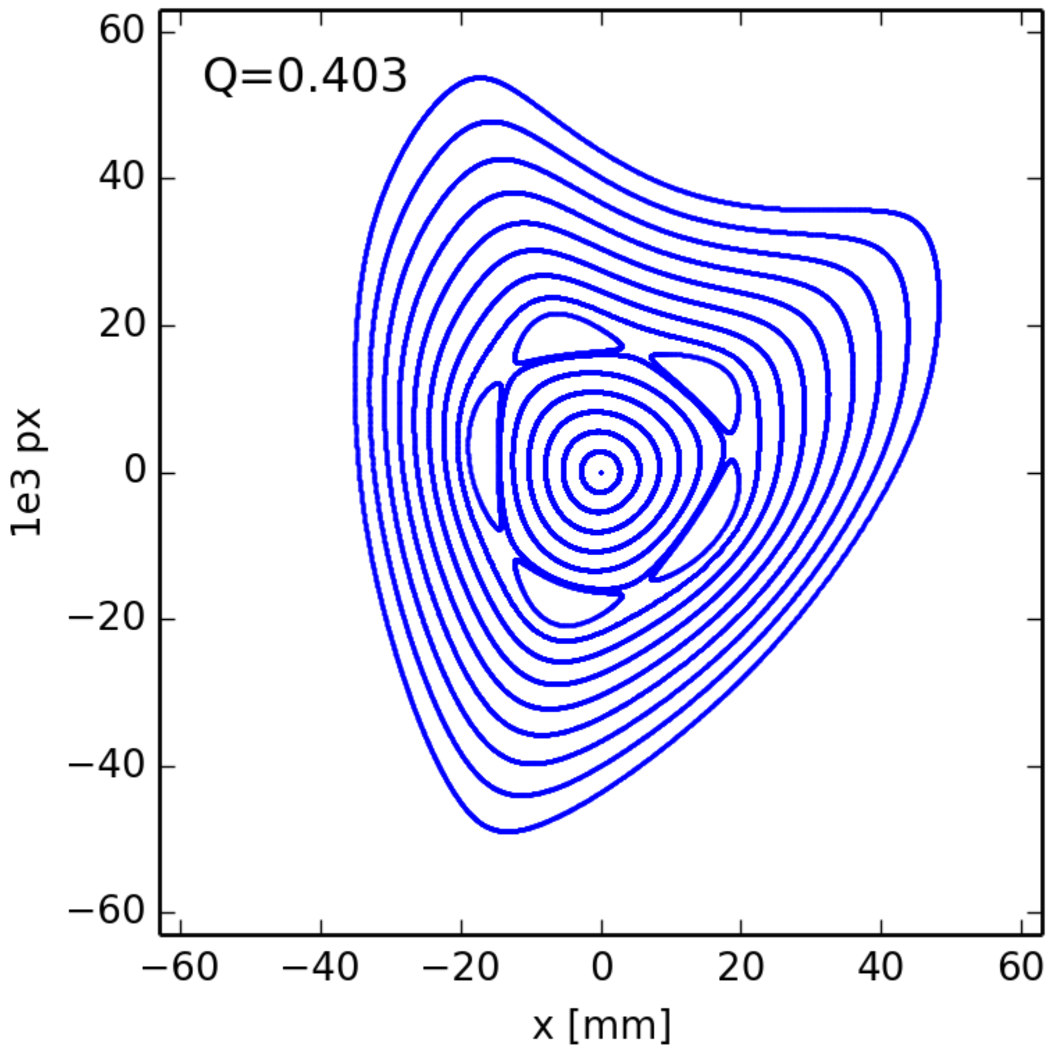

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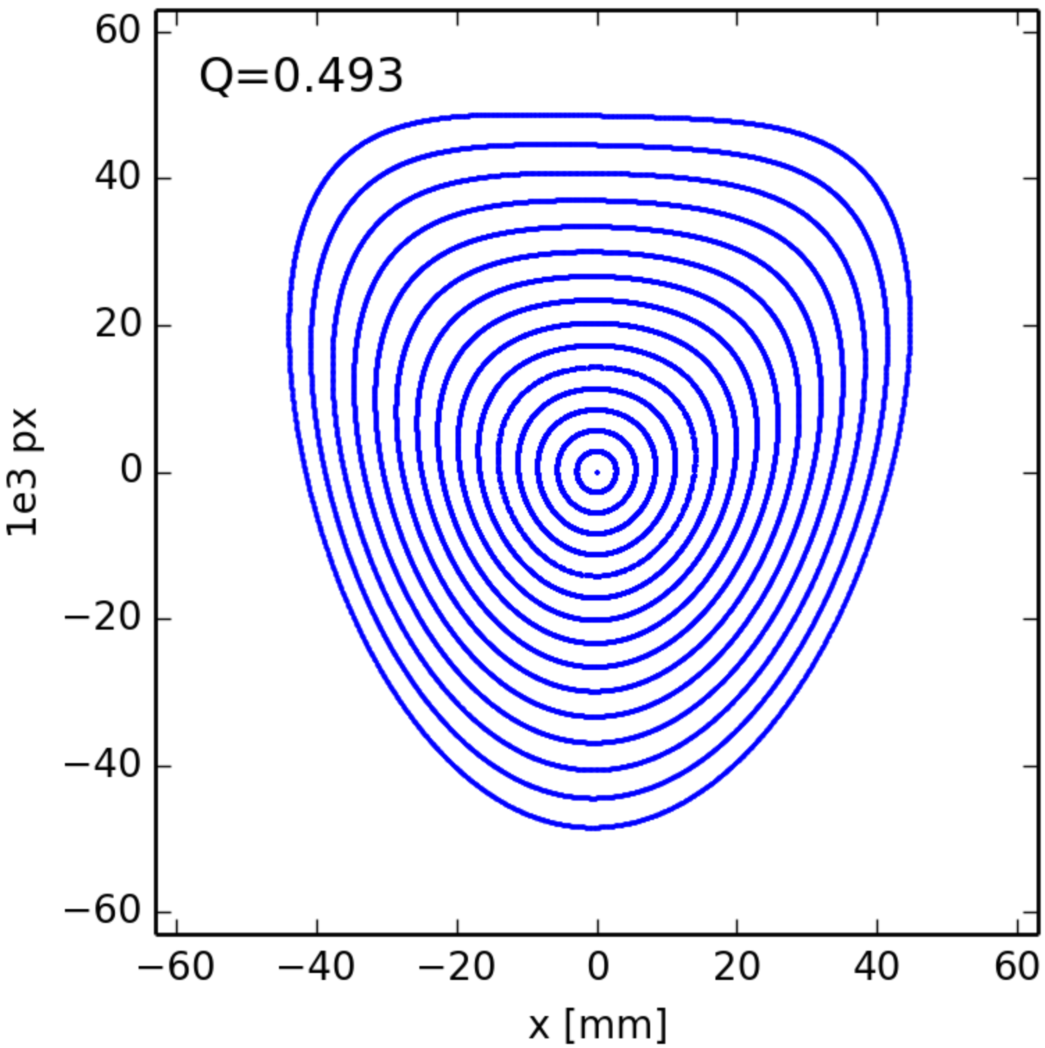
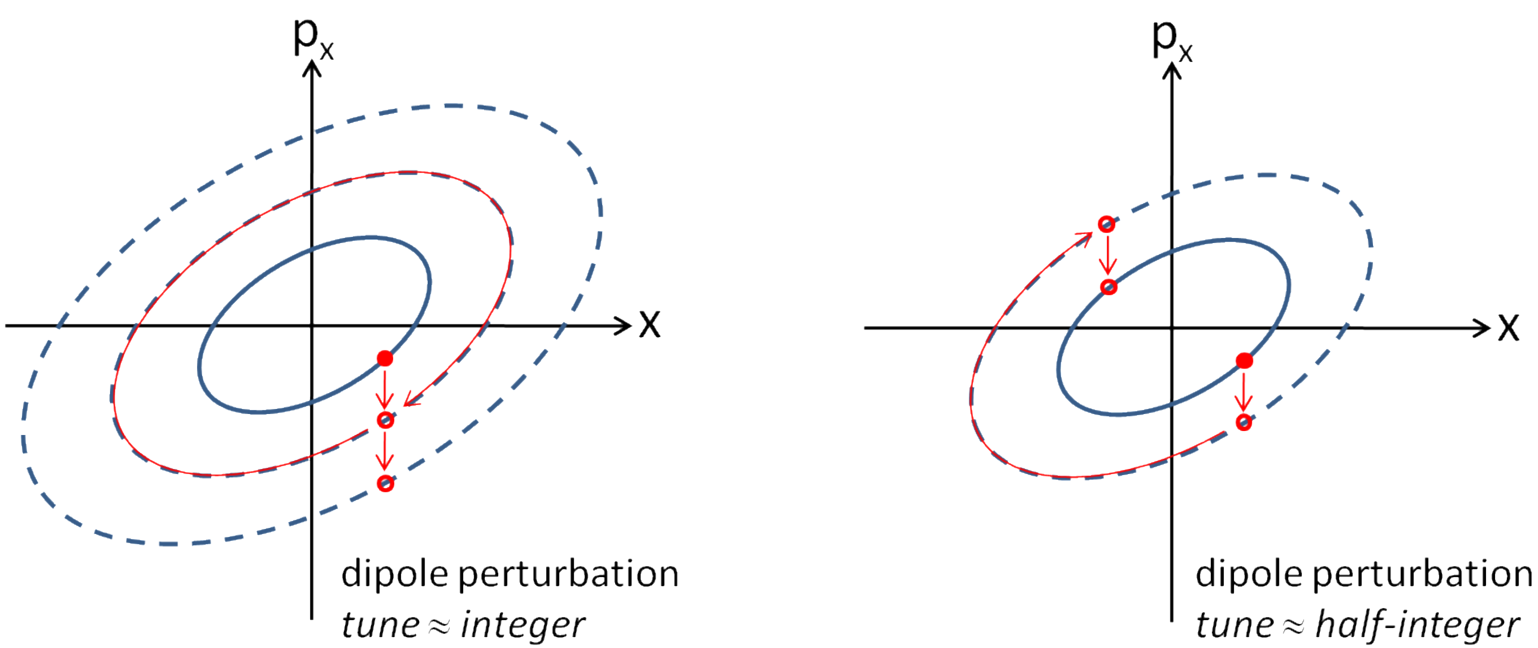


Some observations

- **There are some interesting features in these phase space portraits to which it is worth drawing attention:**
  - For small amplitudes (small x and p_x), **particles trace out closed loops around the origin**: this is what we expect for a purely linear map.
  - As the **amplitude is increased**, "**islands**" appear in phase space: the phase advance (for the linear map) is often close to m/p where m is an integer and p is the number of islands.
  - Sometimes, a larger number of islands appears at larger amplitude.
  - Usually, there is a **closed curve that divides a region of stable motion from a region of unstable motion**. Outside that curve, the amplitude of particles increases without limit as the map is repeatedly applied.
  - **The area of the stable region depends strongly on the phase advance**: for a phase advance close to $2\pi/3$, it appears that the stable region almost vanishes altogether.
  - It appears that **as the phase advance is increased towards π, the stable area becomes large**, and distortions from the linear ellipse become less evident.

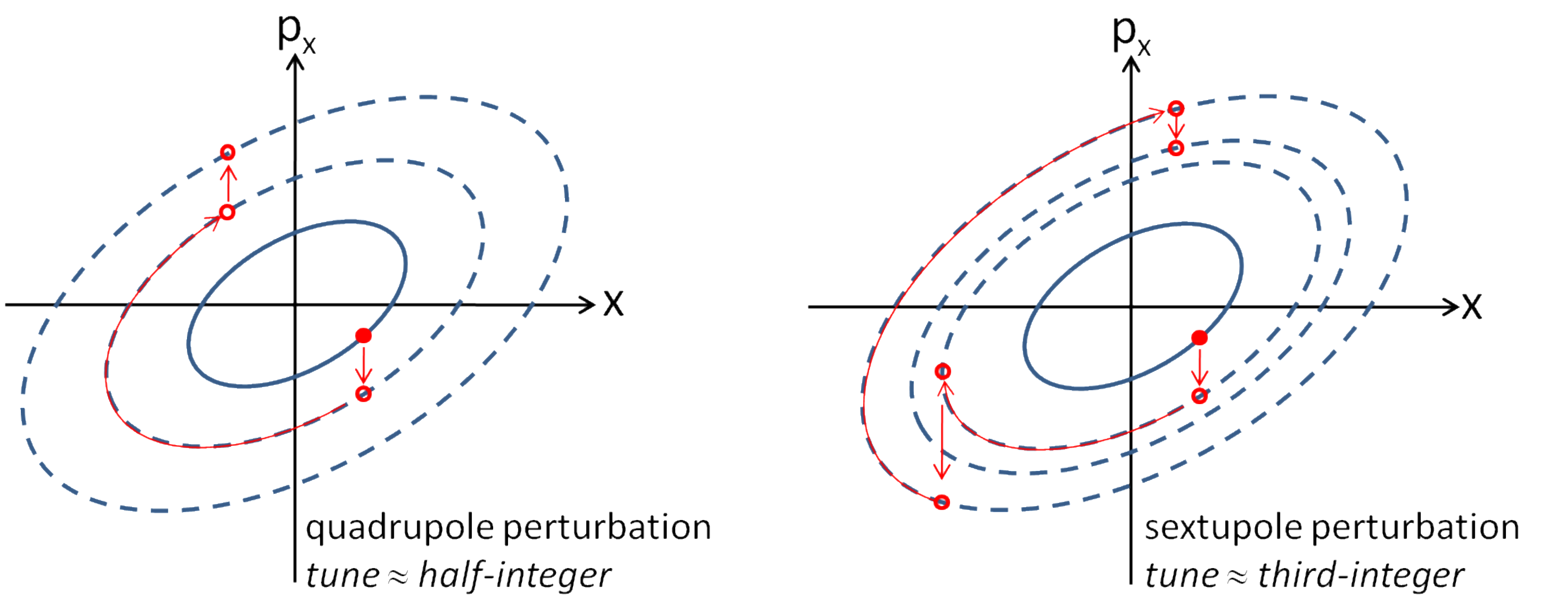
Impact of phase advance

- The effect of the phase advance on the sextupole “kicks” is similar to the effect on **perturbations** arising from **dipole** and **quadrupole errors** in a storage ring
- In the case of **dipole errors**, the **kicks add up** if the phase advance is an **integer**, and **cancel** if the **phase advance** is a **half integer**



Impact of phase advance

- In the case of **quadrupole errors**, the **kicks add up** if the phase advance is a **half integer** times 2π
- **Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram



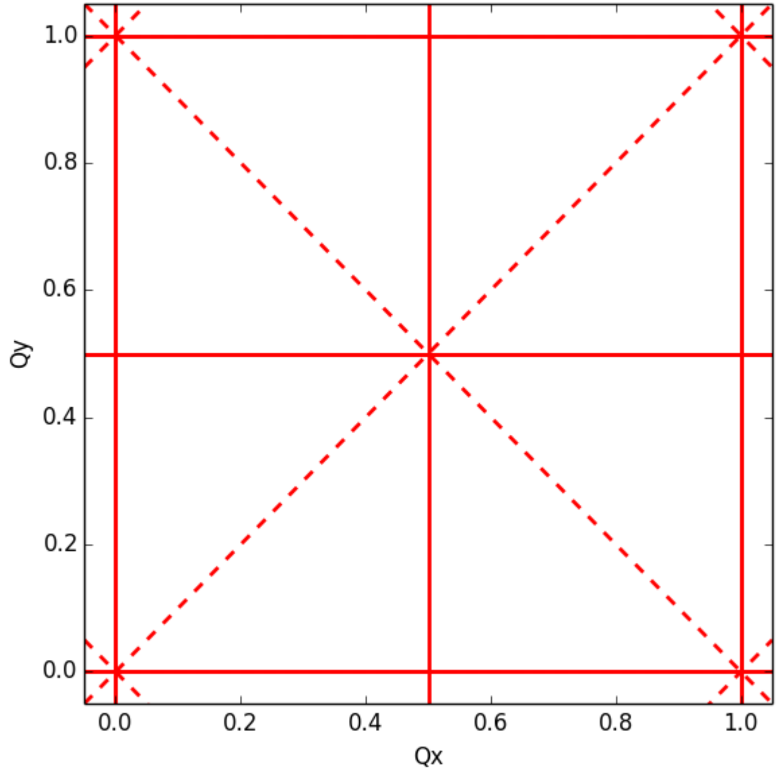
Resonances

- If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

$$\boxed{n_x Q_x + n_y Q_y = r}$$

where n_x, n_y and r are integers. The resonance is of **order $|n_x| + |n_y|$**

Resonances up to order 2



normal resonances (= even n_y)

skew resonances (= odd n_y)

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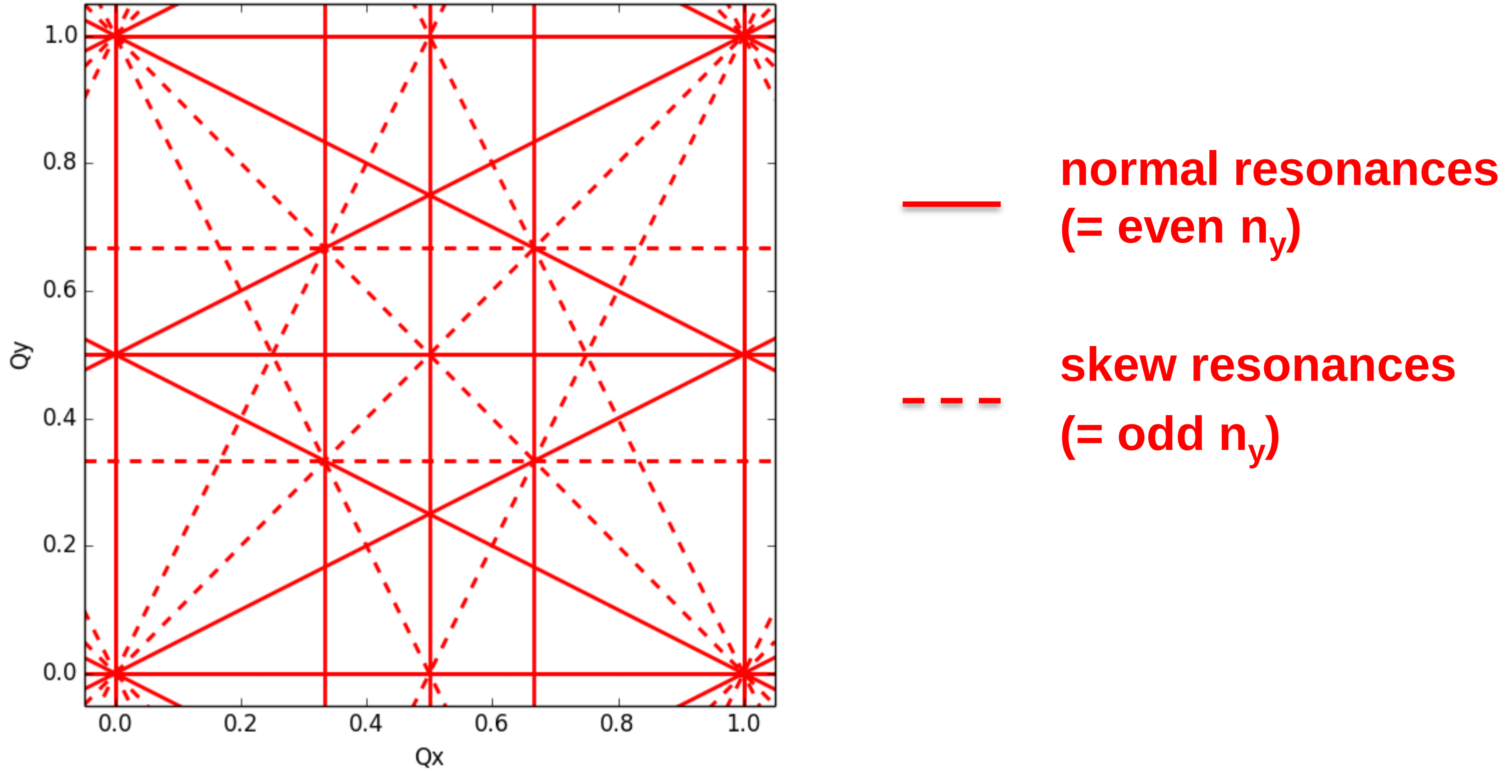
Resonances up to order 3



Resonances

- If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

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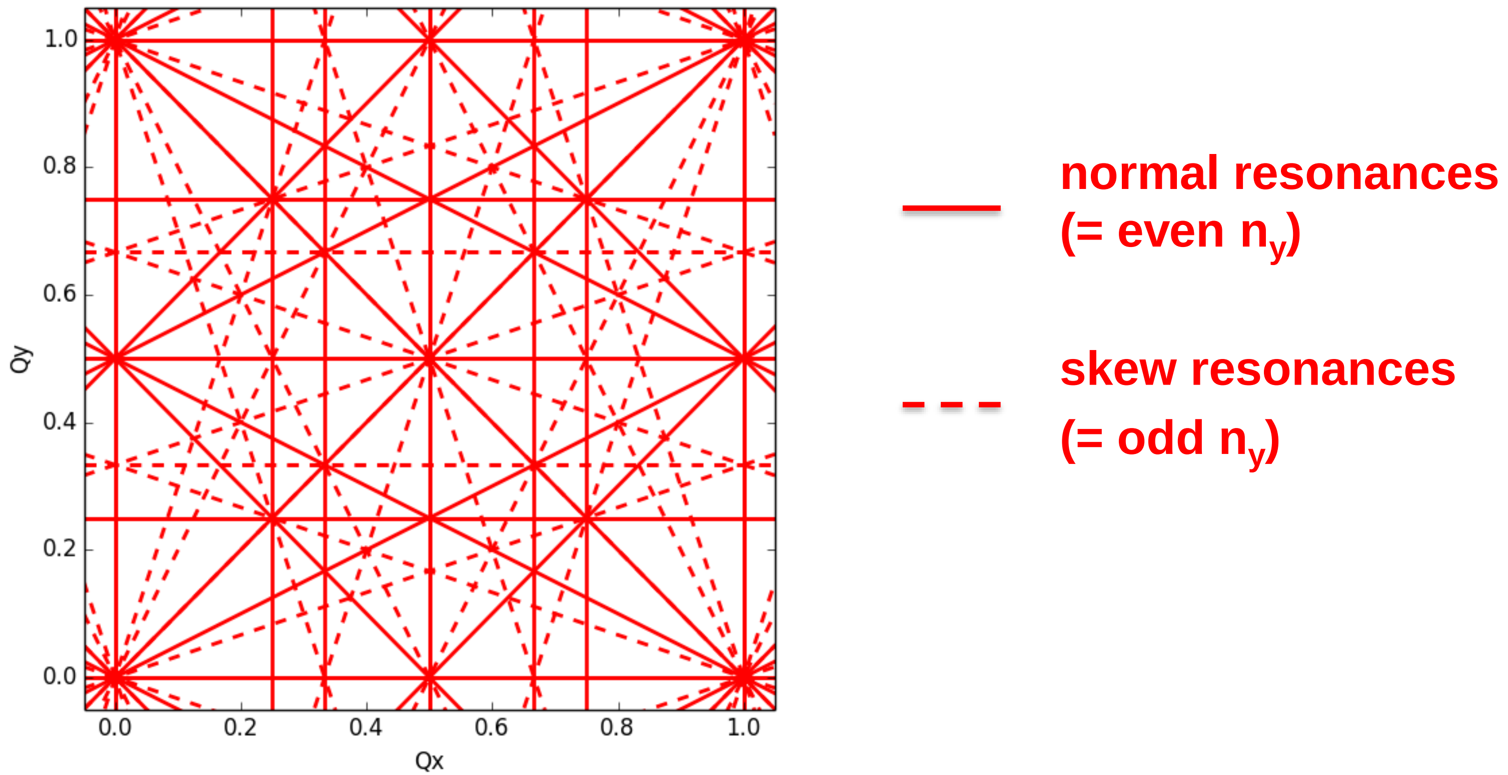
Resonances up to order 4



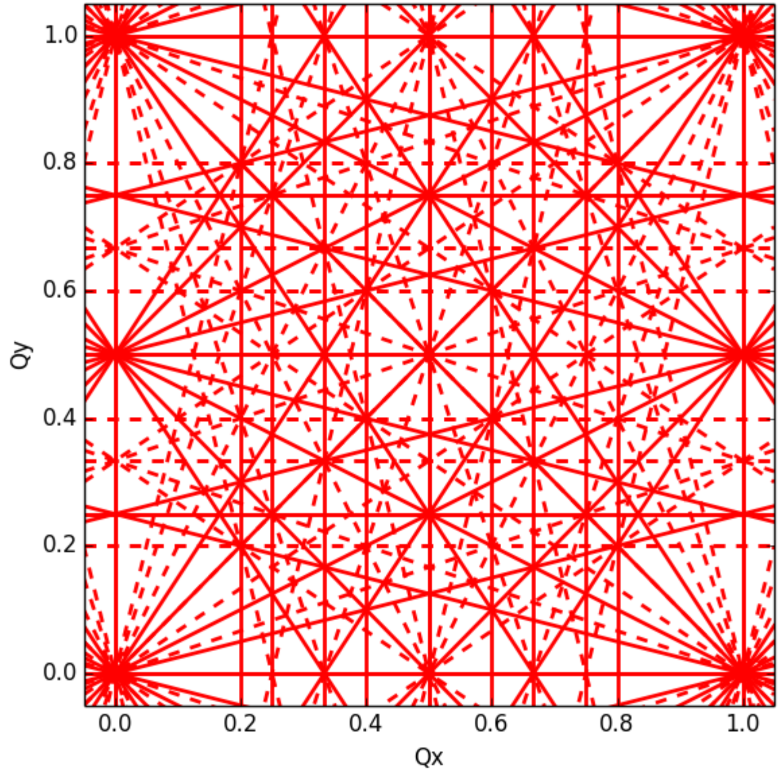
Resonances

- If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

$$\boxed{n_x Q_x + n_y Q_y = r}$$

where n_x, n_y and r are integers. The resonance is of **order $|n_x| + |n_y|$**

Resonances up to order 5



normal resonances (= even n_y)

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Hamiltonian mechanics

- Given a function H(x, p; t) (called the **Hamiltonian**), the equations of motion for a **dynamical system** are given by Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$$

- In Hamiltonian mechanics, the "state" of a system at any time is defined by specifying values for the coordinates x (or more generally q) and the conjugate momentum p
- "Physics" consists of writing down a Hamiltonian
- **All Hamiltonian systems are "symplectic"**: areas in phase space are conserved as the system evolves even when the dynamics are nonlinear. This important result is known as **Liouville's theorem**.

Hamiltonian mechanics

- It follows from Hamilton's equations that the Hamiltonian itself is conserved if the independent ("time-like") variable does not appear explicitly in the Hamiltonian:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p_x}\frac{dp_x}{dt} + \frac{\partial H}{\partial t}$$

- Using Hamilton's equations, we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

- **If the Hamiltonian does not depend explicitly on t, then the Hamiltonian is conserved**

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

Hamiltonian for linear betatron motion

- In a lattice made from dipoles and quadrupoles the Hamiltonian reads (in our usual coordinate system)

$$H(s) = \underbrace{\frac{\bar{p}_x^2 + \bar{p}_y^2}{2}}_{\text{kinetic term}} - \underbrace{\frac{x^2}{2\rho(s)^2}}_{\text{dipole term}} + \underbrace{\frac{k_1(s)}{2}(x^2 - y^2)}_{\text{quadrupole term}}$$

(quadratic in x and y!)

where we have used the normalized momenta $\bar{p}_x = \frac{p_x}{p_0}$ and $\bar{p}_y = \frac{p_y}{p_0}$

- The Hamiltonian consists of a **kinetic term** and a term for the **vector potential** accounting for the magnetic fields

- Using Hamilton's equations $\frac{dq}{dt} = \frac{\partial H}{\partial p}$ and $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$ (for x and y) we find back **Hill's equations**

$$x'' - \left(k_1(s) - \frac{1}{\rho(s)^2}\right) x = 0$$

$$y'' + k_1(s) y = 0$$

Hamiltonian with nonlinear fields (I)

- The more general form of the Hamiltonian describing the motion of a charged particle in the accelerator coordinate system with any order of multipoles looks like this

$$H(s) = \frac{\bar{p}_x^2 + \bar{p}_y^2}{2} - \frac{eA_s}{p_0}$$

- We have only a longitudinal component of the magnetic potential, i.e. A_s, since we restrict ourselves to **pure transverse magnetic fields** (hard edge approximation), with the following **multipole expansion**:

$$\frac{eA_s}{p_0} = \frac{x^2}{2\rho^2} - \mathbf{Re}\sum_{n=1}^{M}(k_n + ij_n)\frac{(x+iy)^{n+1}}{(n+1)!}$$

$$\left.\begin{aligned} B_x &= \frac{\partial A_s}{\partial y} \\ B_y &= -\frac{\partial A_s}{\partial x} \end{aligned}\right\}$$

$$B_y + iB_x = B_0\rho_0 \sum_{n=0}^{M}(k_n + ij_n)\frac{(x+iy)^n}{n!}$$

$$k_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_y}{\partial x^n}\right|_{(0,0)} \qquad j_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_x}{\partial y^n}\right|_{(0,0)}$$

normal multipoles **skew multipoles**

Hamiltonian with nonlinear fields (II)

- The Hamiltonian for the nonlinear betatron motion is then written like this

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho^2} + \mathbf{Re} \sum_{n=1}^{M} \frac{k_n + i j_n}{(n+1)!} (x+iy)^{n+1}$$

- We define H_0 the **linear part** (dependent only on dipoles and normal quadrupoles)

$$H_0(\bar{p}, \bar{q}; s) = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho^2} + \frac{k_1 x^2 - k_1 y^2}{2}$$

and V the **nonlinear part** dependent on the nonlinear multipoles

$$V(\bar{p}, \bar{q}; s) = \mathbf{Re} \sum_{n \geq 2} [k_n(s) + i j_n(s)] \frac{(x+iy)^{n+1}}{(n+1)!} = \sum_{n \geq 3} V_{mn} x^m y^n$$

short hand notation collecting terms according to powers of x and y

Normalizing linear part of Hamiltonian

- We define a **canonical transformation** that reduces the linear part of the Hamiltonian to a rotation

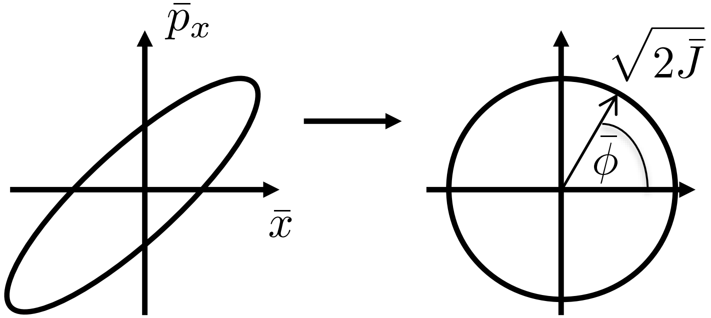
$$(\bar{x}, \bar{p}) \rightarrow (\bar{J}, \bar{\phi})$$

- In detail

$$J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$ linear Courant-Snyder invariant

$$\phi_x = -\mathbf{arctan}\left(\beta_x \frac{p_x}{x} + \alpha_x\right) - \int \frac{d\tau}{\beta_x}$$

- This transformation reduces ellipses in phase space to circles and the motion to a rotation along these circles



Resonance driving terms (I)

- The new **Hamiltonian in action angle variables** reads

$$H(\bar{J},\bar{\phi};s) = \frac{Q_x J_x + Q_y J_y}{R} + V(\bar{J},\bar{\phi};s)$$

$$V(\bar{J},\bar{\phi};s) = \frac{\epsilon}{R} \sum_{\substack{j=0 \\ j+k=m_x}}^{m_x} \sum_{\substack{l=0 \\ l+m=m_y}}^{m_y} J_x^{\frac{j+k}{2}} J_y^{\frac{l+m}{2}} h_{jklm} e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

- The **complex** coefficients h_{jklm} are called resonance driving terms – they **generate angle dependent terms in the Hamiltonian that are responsible for the resonant motion of the particles** (i.e. motion on a chain of islands or on a separatrix)
- The **resonant driving terms are integrals over the circumference** of the accelerator **of functions which depend on the s-location** of the **multipolar magnetic elements**

$$h_{jklm} = \frac{1}{2^{\frac{j+k+l+m}{2}}} \binom{j+k}{j} \binom{l+m}{l} \int_{s_0}^{s_0+2\pi R} V_{j+k,l+m}(s)\, \beta_x^{\frac{j+k}{2}}(s)\, \beta_y^{\frac{l+m}{2}}(s)\, e^{i[(j-k)\phi_x(s)+(l-m)\phi_y(s)]} ds$$

Resonance driving terms (II)

- The solution for the stable betatron motion can be written as a **quasi periodic signal** (to first order in the multipole strengths)

$$x(n) - ip_x(n) = \sqrt{2J_x}e^{i(2\pi Q_x n+\phi_{x0})}$$
$$-2i\sum_{jklm} s_{jklm}(2J_x)^{\frac{j+k-1}{2}}(2J_y)^{\frac{l+m}{2}}e^{i[(1-j+k)(2\pi Q_x n+\phi_{x0})+(m-l)(2\pi Q_y n+\phi_{y0})]}$$

… to first order in the multipole strengths

with $$s_{jklm} = \frac{1}{1-e^{-2\pi i[(j-k)Q_x+(l-m)Q_y]}}h_{jklm}$$

→ **solutions for stable betatron motion contain the driving terms**

- **On the islands the betatron tunes satisfy a resonant condition** of type

$$\boxed{n_x Q_x + n_y Q_y = r}$$

(n_x, n_y) resonance

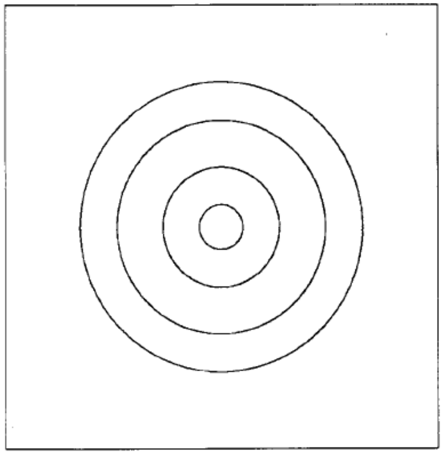
$n_x = j-k$ $\quad$ $n_y = l-m$

Resonance driving terms (III)

- Terms of **type h_{jjll}** are **independ of the angle**. They produce detuning with amplitude to the lowest order in the multipolar gradient (resulting in a tune spread for a beam!)

$$V(\bar{J},\bar{\phi};s) = \frac{\epsilon}{R} \sum_{\substack{j=0 \\ 2j=m_x}}^{m_x} \sum_{\substack{l=0 \\ 2l=m_y}}^{m_y} J_x^j J_y^l h_{jjll}$$

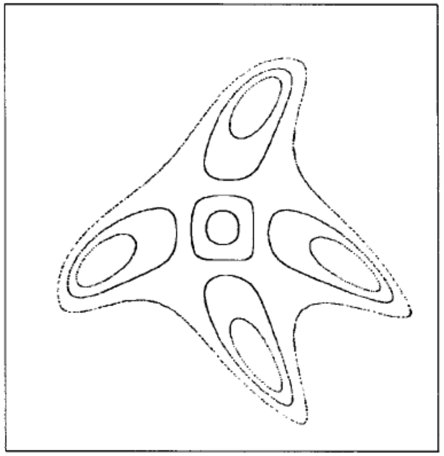
- The dynamics with only detuning terms (amplitude dependent phase advance)



- **Angle dependent terms** excite resonances creating fixed points and island structures in phase space. E.g. for the fourth order resonance (4,0)

$$V(\bar{J},\bar{\phi};s) = \frac{\epsilon}{R} J_x^2 h_{4000} e^{i[4\phi_x]}$$

- The phase space for the (4,0) resonance looks like this



Driving terms from sextupoles

$$h_{jklm} = \frac{1}{2^{\frac{j+k+l+m}{2}}} \binom{j+k}{j} \binom{l+m}{l} \int_{s_0}^{s_0+2\pi R} V_{j+k,l+m}(s)\, \beta_x^{\frac{j+k}{2}}(s)\, \beta_y^{\frac{l+m}{2}}(s)\, e^{i[(j-k)\phi_x(s)+(l-m)\phi_y(s)]} ds$$

- Starting from the general definition of driving terms we substitute the function that give the azimuthal distribution of the normal sextupoles

$$V(\bar{x};s) = b_2(s)(x^3 - 3xy^2) = V_{30}(s)x^3 + V_{12}(s)xy^2$$

- Sextupoles generate the following resonant driving terms (see Guignard, Bengtsson)

from V_{30} $\longrightarrow$ h_{3000} h_{2100}

resonances: (3,0) (1,0)

from V_{12} $\longrightarrow$ h_{1020} h_{1011} h_{1002}

resonances: (1,2) (1,0) (1,-2)

Note:

- **No detuning terms** (in first order of the sextupole strength) – they are generated only in second order
- **Pure horizontal but no pure vertical resonance terms** (since no skew sextupoles)

Driving terms from octupoles

$$V(\bar{x};s) = b_3(s)(x^4 - 6x^2y^2 + y^4) = V_{40}(s)x^4 + V_{22}(s)x^2y^2 + V_{04}(s)y^4$$

- In an analogous way we can see that the **normal octupoles** in the circular ring generate the following resonant driving terms (see Guignard, Bengtsson)

from V_{40} ⟶ h_{4000} h_{3100} h_{2200}

resonances: (4,0) (2,0) (det.)

from V_{22} ⟶ h_{2020} h_{1120} h_{2011} h_{1111}

resonances: (2,2) (0,2) (2,0) (det.)

from V_{04} ⟶ h_{0040} h_{0031} h_{0022}

resonances: (0,4) (0,2) (det.)

Note:

- **Detuning terms** (in first order of octupole strength)
- **Also pure vertical resonances excited**

Sextupole excites 4-th order resonance

- Let us consider the **nonlinear Hill's equation** for the case of a linear lattice where a single sextupole kick is added

$$\frac{d^2x}{ds^2} + K(s)x = \frac{k_2}{2}x^2 \qquad K(s) = \frac{1}{\rho^2(s)} - k_1(s)$$

- Use **perturbative procedure** and solve this equation by successive approximations. The perturbation parameter ε is proportional to the sextupole strength k_2. We look for a solution of the type:

$$x(s) = x_0 + \epsilon x_1(s) + \epsilon^2 x_2(s) + O(\epsilon^3)$$

- Substituting, ordering the contributions with the same perturbative order we have

$$\frac{d^2x_0}{ds^2} + K(s)x_0 = 0 \qquad \frac{d^2x_1}{ds^2} + K(s)x_1 = k_2(s)x_0^2(s) \qquad \frac{d^2x_2}{ds^2} + K(s)x_2 = 2k_2(s)x_0(s)x_1(s)$$

order zero: ε^0 first order: ε^1 second order: ε^2

Sextupole excites 4-th order resonance

- At each step we are using functions already calculated at the previous steps

$$x_0(s) = \sqrt{\epsilon_x \beta_x(s)} \cos[\phi_x(s) + \phi_{x0}]$$

Linear solution

$$x_1(s) \propto A \cos[2\phi_x(s) + \phi_{x0}]$$

Term generated by the 3rd order resonance; linear with k_2 (first order)

$$x_2(s) \propto C \cos[3\phi_x(s) + \phi_{x0}] + D \cos[\phi_x(s) + \phi_{x0}]$$

Terms generated by the 4th order and 2nd order resonance; quadratic with k_2 (second order)

- The series obtained from the successive approximation are in general divergent. However, the canonical perturbation method shows that **sextupoles can excite 4th order resonances in second order with the sextupole strength k_2**

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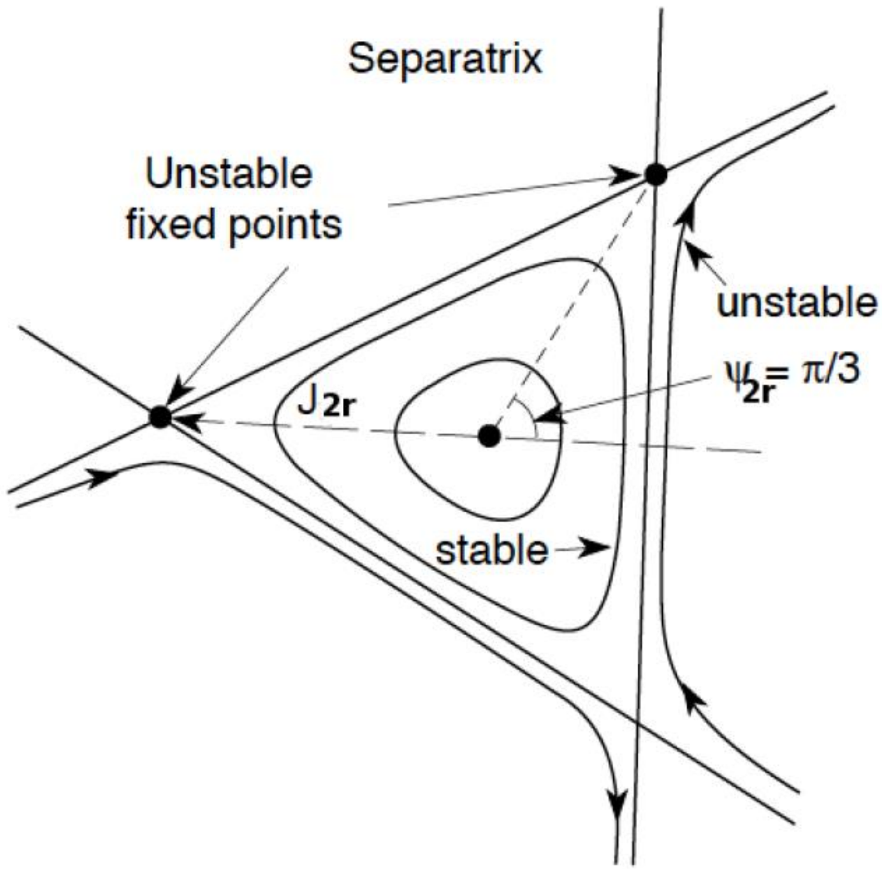
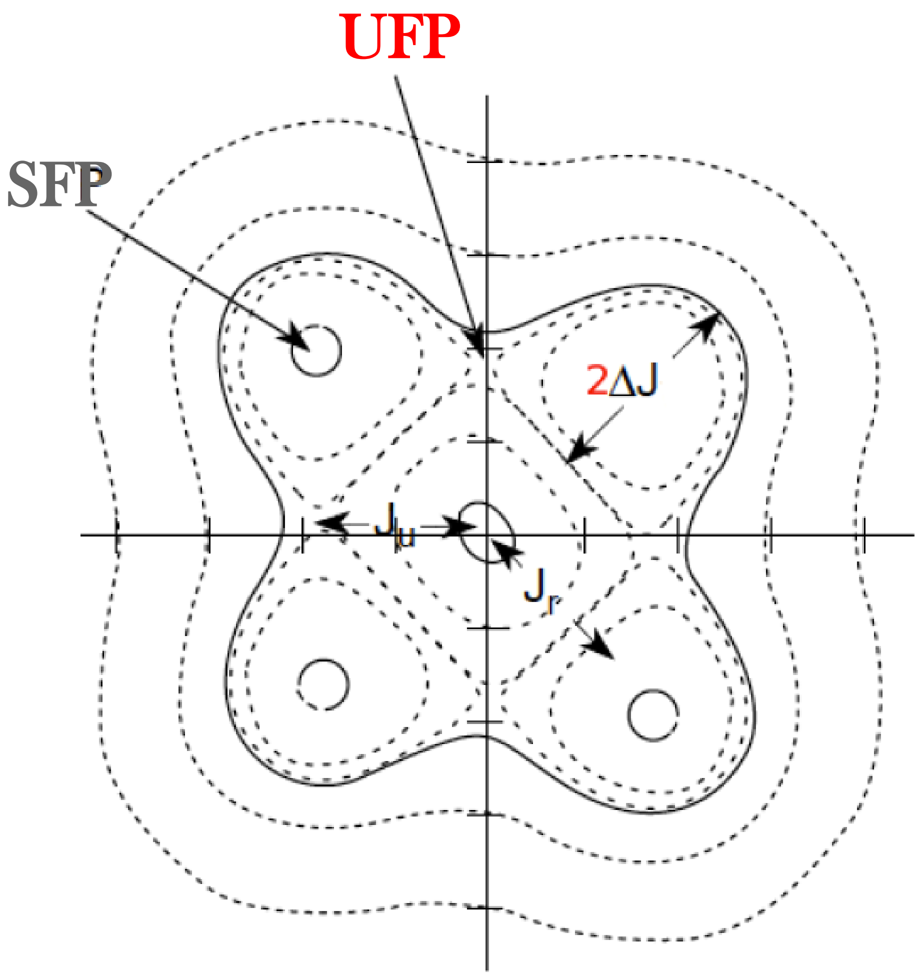
Topology of 3rd order resonance

- **Regular motion** near the center
- For increasing amplitudes the circles get deformed towards a **triangular shape** until the resonance condition is met
- The **separatrix** (barrier between stable and unstable motion) passes through 3 unstable fixed points



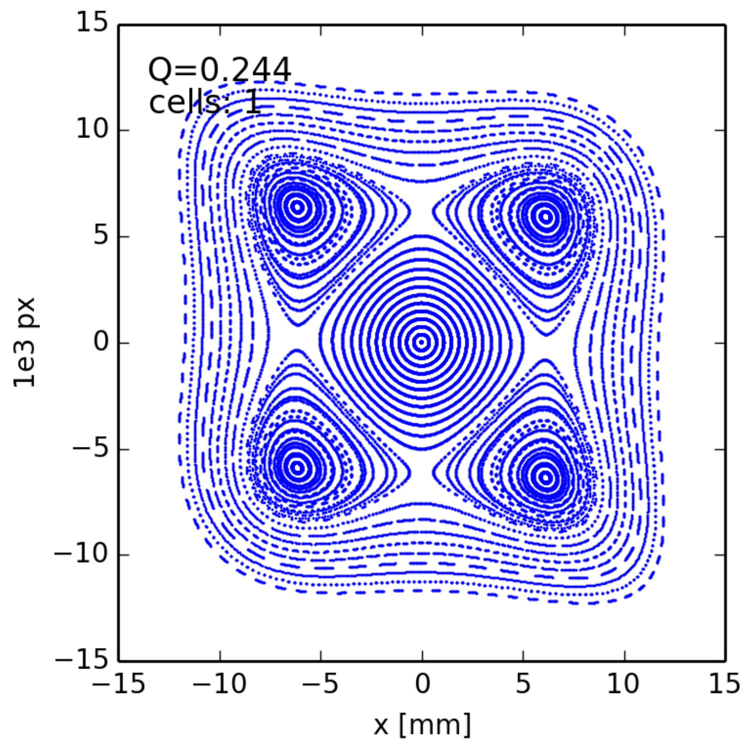
Topology of 4^{th} order resonance

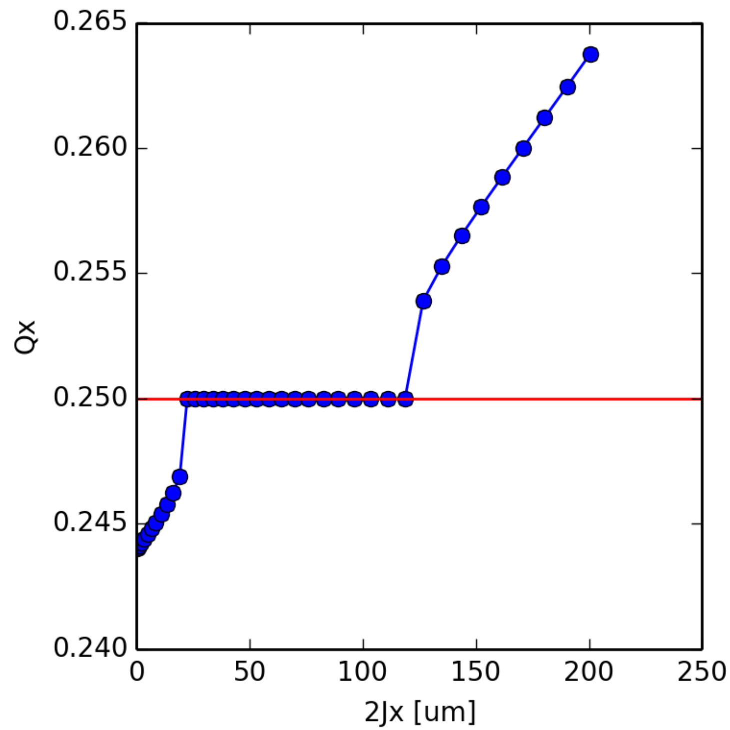
- **Regular motion** near the center, with curves getting more deformed towards a **rectangular shape**
- The **separatrix** passes through 4 unstable fixed points, but motion seems well contained
- **Four stable fixed points** exist and they are surrounded by stable motion **(islands of stability)**



Particle trapped in 4th order resonance

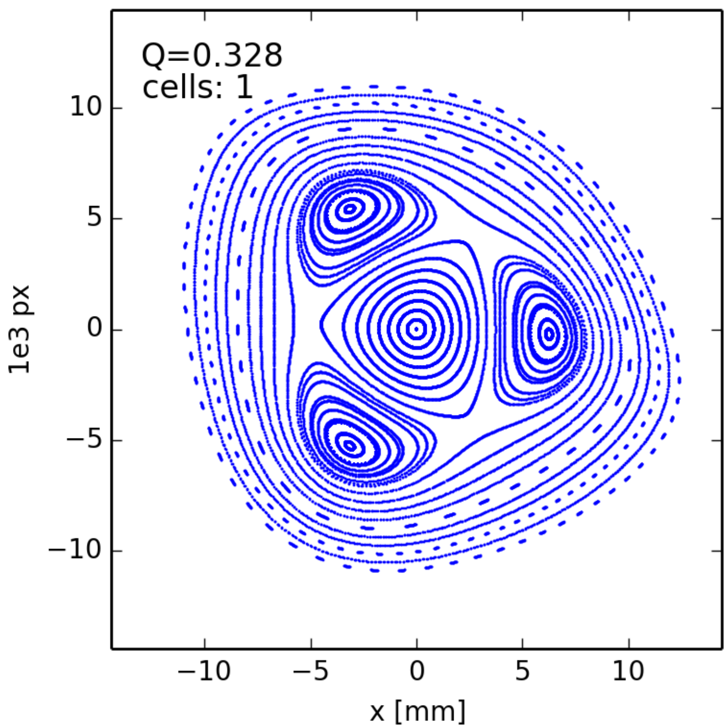
- Simulation of simple storage ring with a **single octupole close to 4th order resonance**
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have tune locked to resonance

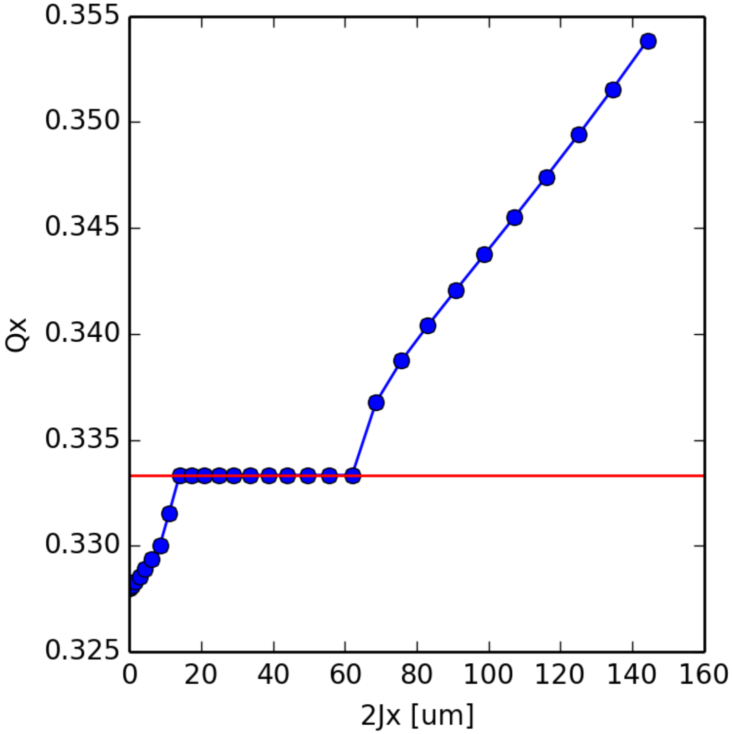




Particle trapped in 3rd order resonance

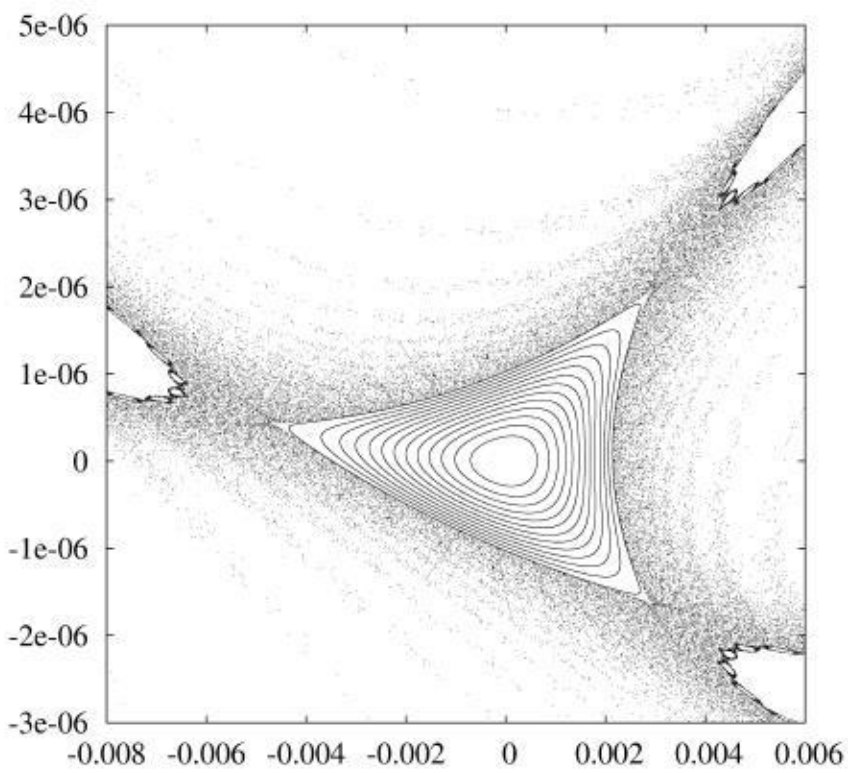
- Simulation of simple storage ring with **a sextupole and an octupole close to 3rd order resonance**
- **The amplitude detuning induced by the octupole can create stable islands even for the 3rd order resonance** (if the resonance is weak enough) – the tune of particles in islands is locked to the resonance while particles at higher amplitudes do not meet the resonance condition any longer → "stabilizing" effect





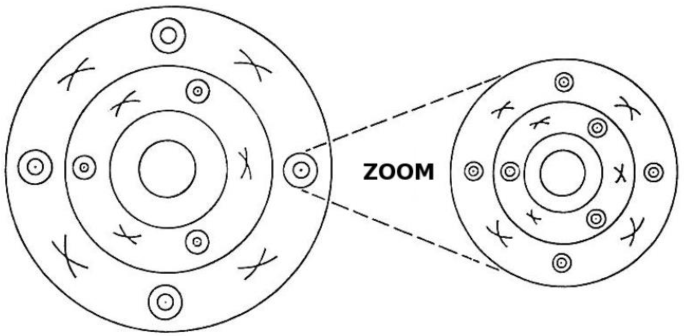
Path to chaos

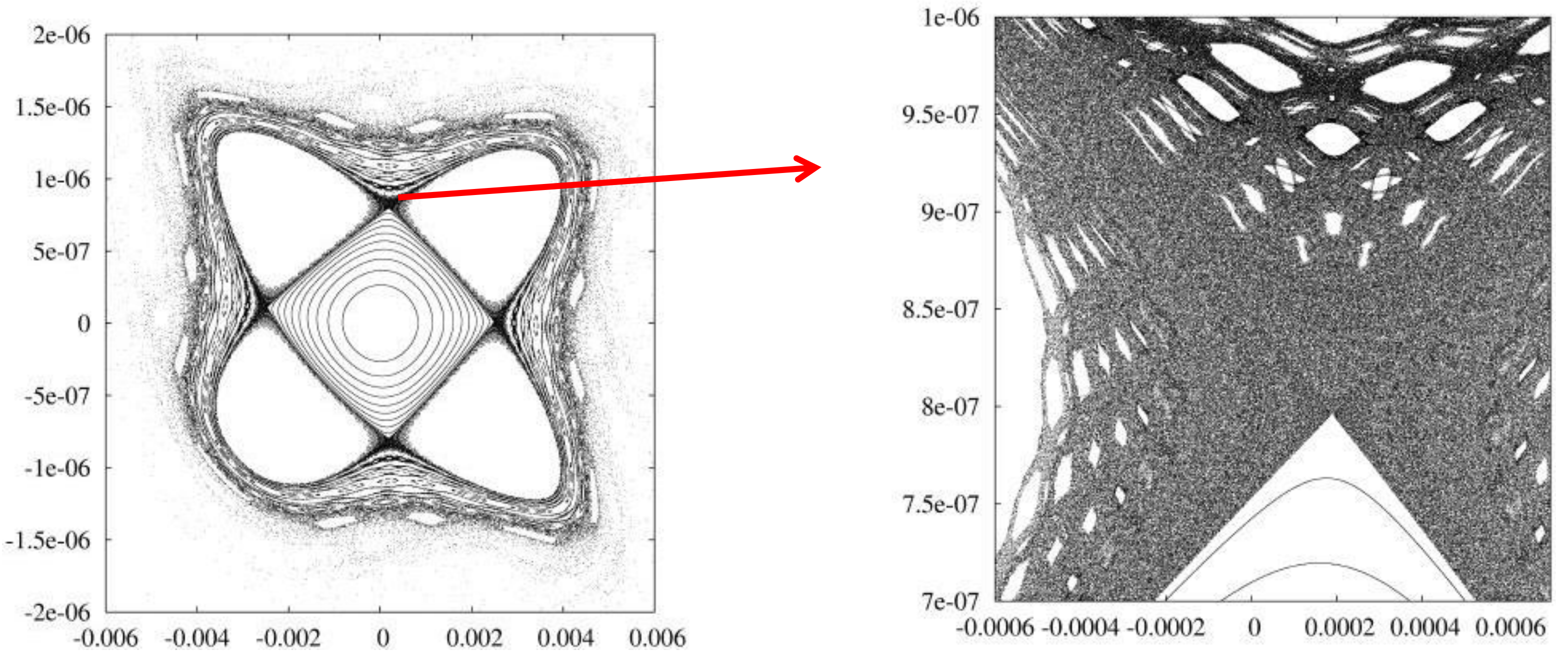
- When **perturbation** becomes **higher**, **motion around the separatrix becomes chaotic** (producing tongues or **splitting of the separatrix**)
- **Unstable fixed points** are indeed the **source of chaos** when a perturbation is added



Chaotic motion

- **Poincare-Birkhoff** theorem states that under perturbation of a resonance only an even number of fixed points survives (half stable and the other half unstable)
- Get destroyed when perturbation gets higher, etc. (self-similar fixed points)
- Resonance islands grow and resonances can overlap allowing diffusion of particles



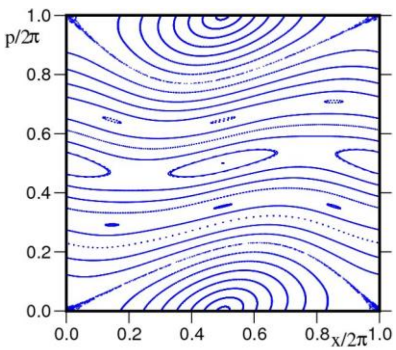


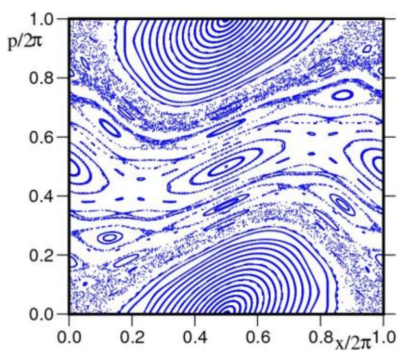
Resonance overlap criterion

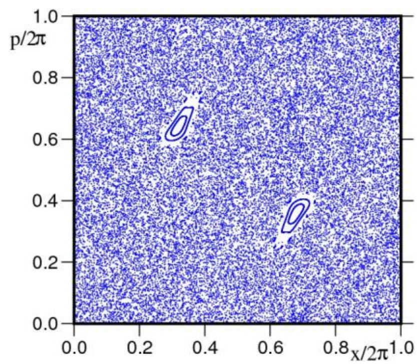
- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two **neighboring resonances** and the onset of orbit diffusion
- The **distance** between two resonances is $\delta \hat{J}_{1\ n,n'} = \frac{2\left(\frac{1}{n_1+n_2} - \frac{1}{n'_1+n'_2}\right)}{\left| \left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^2}\right|_{\hat{J}_1=\hat{J}_{10}} \right|}$
- The **simple overlap criterion** is

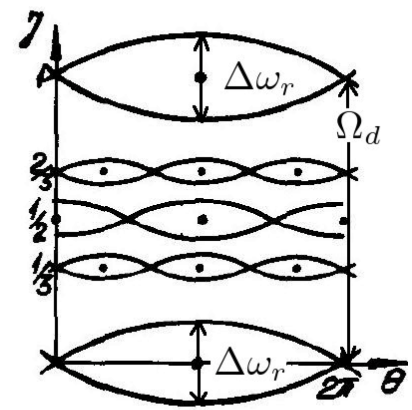
$$\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \delta \hat{J}_{n,n'}$$

- Considering the **width** of **chaotic layer** and **secondary islands**, the **"two thirds" rule** applies $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \frac{2}{3} \delta \hat{J}_{n,n'}$
- The main limitation is the **geometrical nature** of the criterion (**difficulty** to be **extended** for > **2 degrees of freedom**)









Outline

- Introduction – nonlinear effects from a single sextupole
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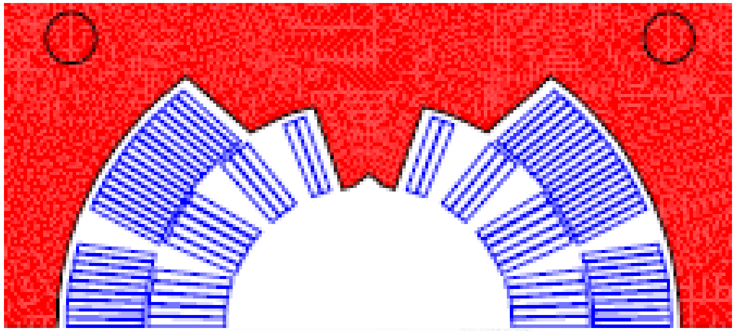
Sources of nonlinear magnetic fields

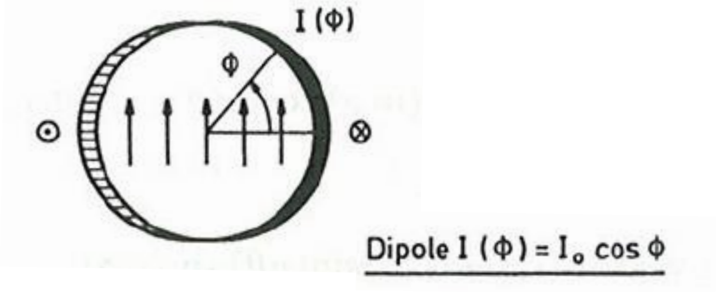
- Resonances can be excited by nonlinear elements installed intentionally (e.g. sextupoles for chromaticity correction) and / or by unavoidable multipolar errors from magnet imperfections
- Especially superconducting magnets can have strong multipolar errors up to very high orders due to the finite size of the coils reproducing only partially the cos-θ dependence of the current distribution necessary to achieve pure dipole fields

TABLE I

MEASURED MULTIPOLES IN THE MBP2N1 PROTOTYPE: AVERAGE OF 18 MEASUREMENTS ALONG THE MAGNET AXIS. UNITS OF 10^{-4} AT R_{REF} = 17 mm.

	Collared		Assembled		After cryo	
	Ap. 1	Ap. 2	Ap. 1	Ap. 2	Ap. 1	Ap.2
a2	0.94	0.43	0.98	0.75	0.89	0.81
b2	-0.96	1.25	-5.48	5.73	-4.99	5.13
a3	-0.11	0.29	-0.38	-0.01	-0.46	0.00
b3	2.08	2.71	8.09	8.68	8.17	8.71
a4	0.06	0.05	0.05	0.10	0.07	0.11
b4	-0.07	0.20	-0.66	0.75	-0.67	0.77
a5	-0.06	-0.05	-0.07	-0.02	-0.08	-0.02
b5	-0.63	-0.60	-0.69	-0.64	-0.76	-0.71
a6	0.03	0.03	0.02	0.03	0.02	0.03
b6	0.00	-0.01	-0.02	0.03	-0.03	0.03
a7	0.03	0.03	0.02	0.00	0.02	0.01
b7	0.65	0.70	0.57	0.61	0.58	0.61
b9	0.25	0.26	0.26	0.26	0.21	0.20
b11	0.73	0.73	0.63	0.62	0.63	0.62





What can we do about resonances?

- The **number of resonance lines in tune space is infinite**: any point in tune space will be close to a resonance of some order
- Remember that the driving terms creating resonances are complex numbers that are obtained by integrating contributions from individual multipoles around the machine taking into account the phase advance. By properly arranging these nonlinear elements around the machine circumference, some resonance driving terms can be cancelled
- **Cancellation of resonance driving terms can be achieved by**

1. Lattice periodicity or designing machine sections with symmetry (e.g. arranging sextupoles in families with certain phase advances, …)
2. Add sufficient multipole correctors to control driving terms

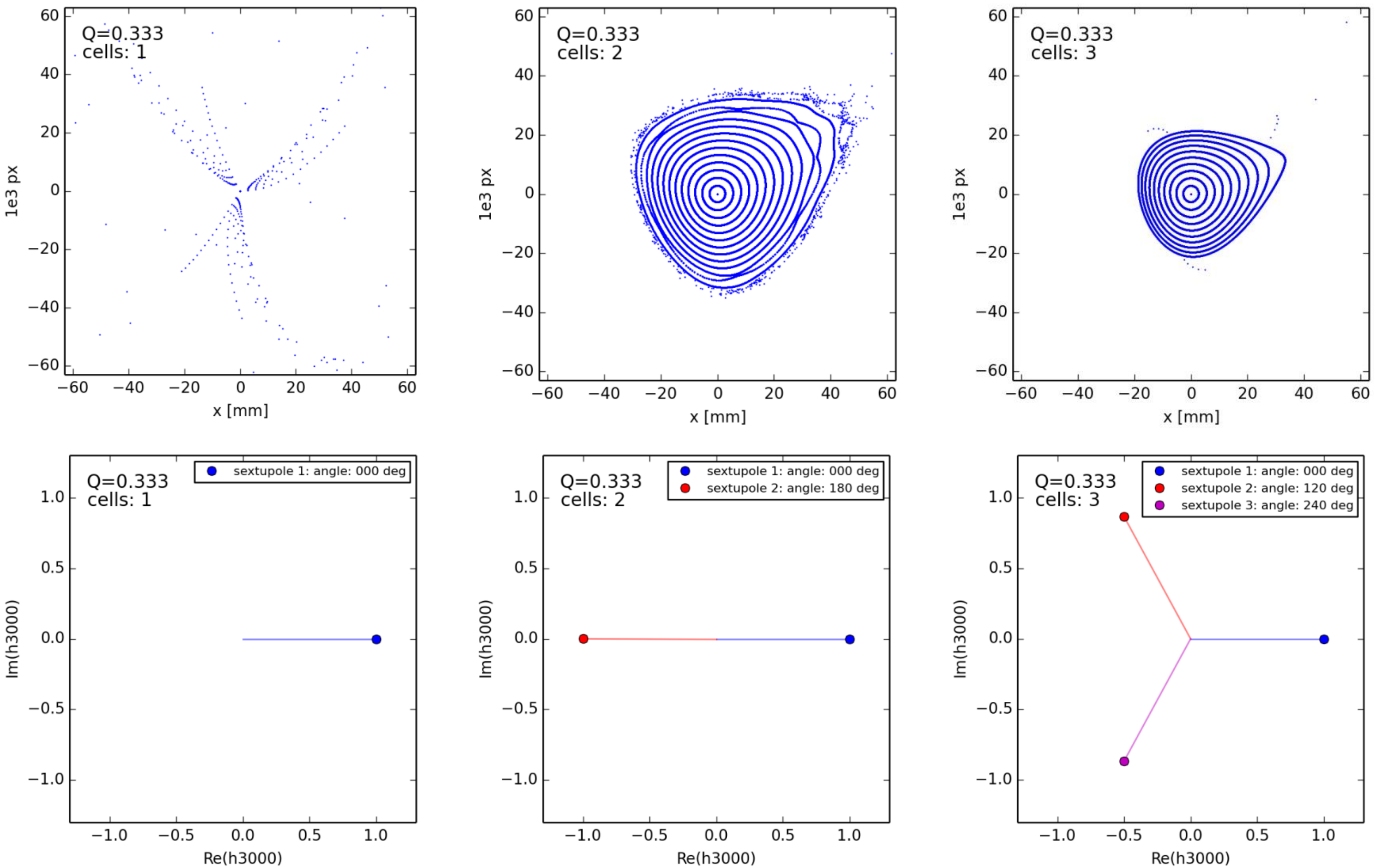
Lattice periodicity

- Consider a machine built of a number of identical cells. If a particular resonance is excited or suppressed depends on the resonance harmonic and the periodicity. In fact, the dynamics of a machine with P identical cells and tune of Q is the same as the one of a single sector with tune Q/P.
- Let's have a look what happens in our simple storage ring when we **increase the number of cells** but **adjusting the phase advance per cell** such that the **overall tune remains unchanged**. At the same time we compute the resonance driving term contribution for each sextupole of the machine and plot it together with the phase space obtained from tracking

Simple storage ring with periodicity

3 Qx = 1 resonance

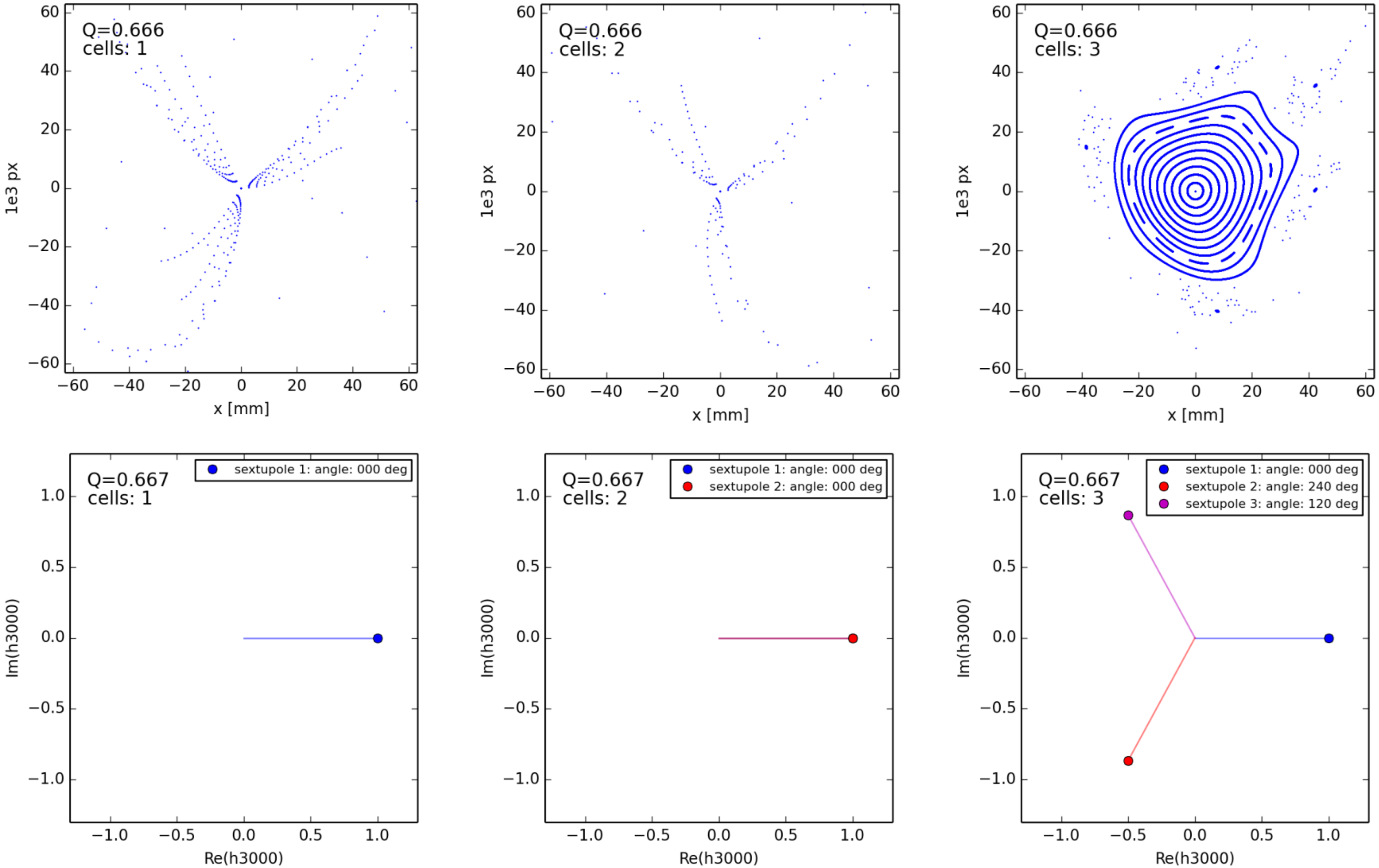
$$h_{3000} = \frac{1}{2^{\frac{3}{2}}} \int_{s_0}^{s_0+2\pi R} V_{30}(s) \cdot \beta_x^{\frac{3}{2}}(s) e^{i[3\phi_x(s)]} ds$$



Simple storage ring with periodicity

3 Qx = 2 resonance

$$h_{3000} = \frac{1}{2^{\frac{3}{2}}} \int_{s_0}^{s_0+2\pi R} V_{30}(s) \cdot \beta_x^{\frac{3}{2}}(s) e^{i[3\phi_x(s)]} ds$$



Resonance cancellation by periodicity

- By imposing a **periodicity P** in the lattice (i.e. building a machine from P identical cells) the resonance condition becomes

$$\boxed{n_x \frac{Q_x}{P} + n_y \frac{Q_y}{P} = r} \quad \rightarrow \quad \boxed{n_x Q_x + n_y Q_y = Pr}$$

… the resonance condition needs to be satisfied by each cell, as conceptually there is no difference between passing one cell P turns or passing a lattice consisting of P identical cells only once

Resonance cancellation by periodicity

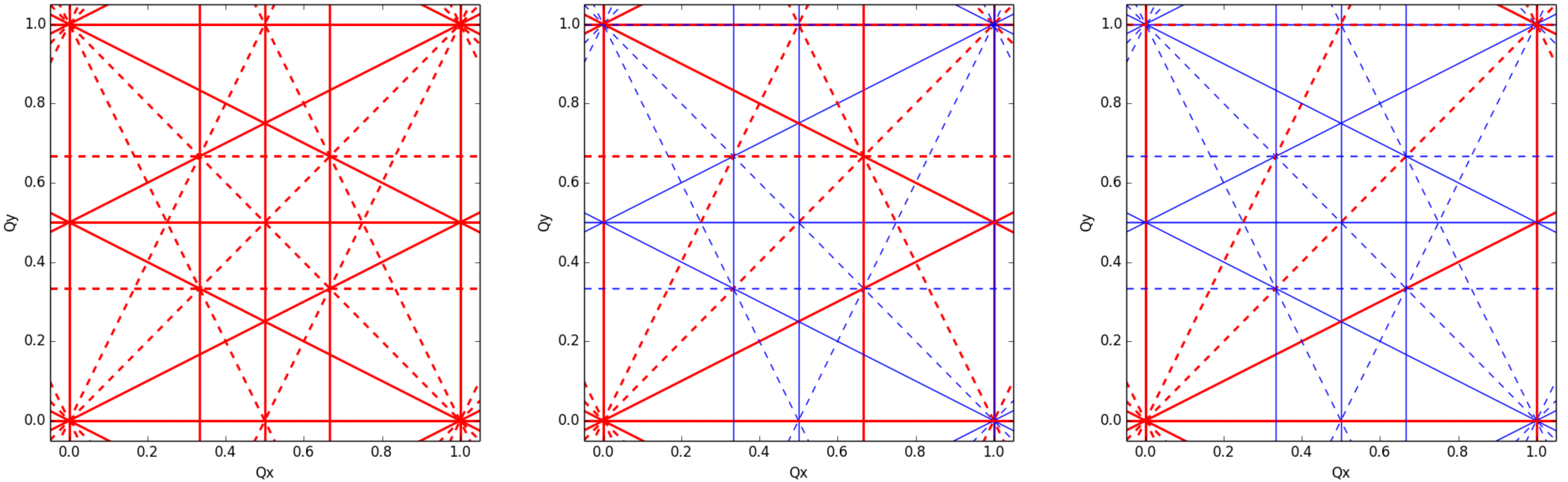
- By imposing a **periodicity P** in the lattice (i.e. building a machine from P identical cells) the resonance condition becomes

$$\boxed{n_x \frac{Q_x}{P} + n_y \frac{Q_y}{P} = r} \rightarrow \boxed{n_x Q_x + n_y Q_y = Pr}$$

- Resonances for which *r* is integer → **systematic** (resonance condition satisfied in each cell, driving terms add up constructively)

- If *r* is **NOT** integer, the driving term cancels → **non-systematic**

periodicity P=1 **periodicity P=2** **periodicity P=3**

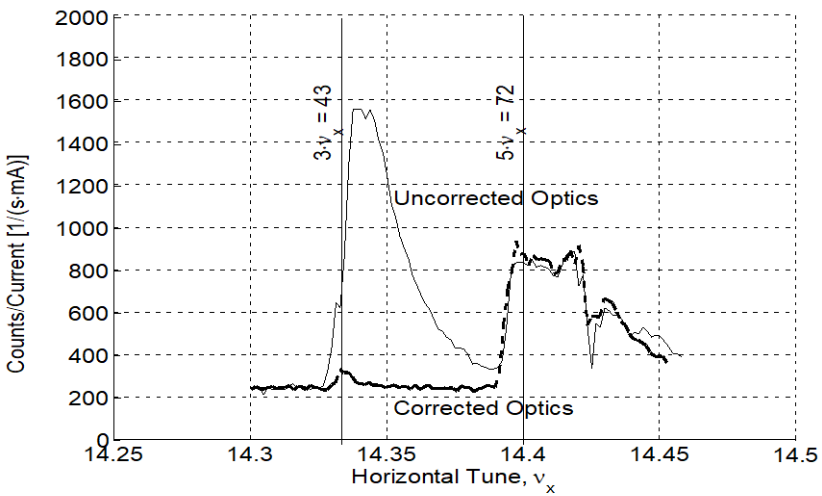


solid lines: normal resonances dashed lines: skew resonances

Real life example for periodicity: ALS

Advanced Light Source design lattice periodicity: 12

Measurement of beam loss as function of tune



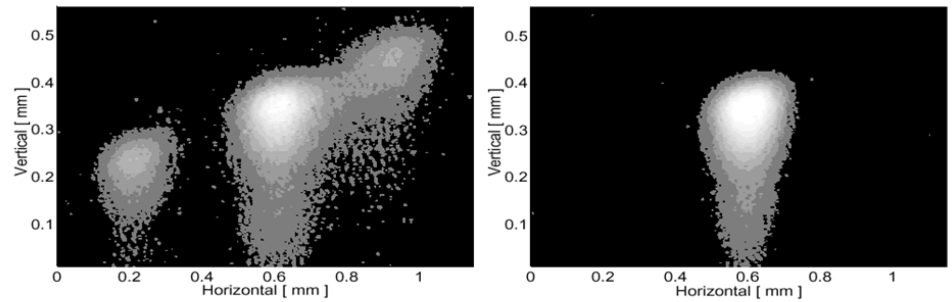
Beta beating

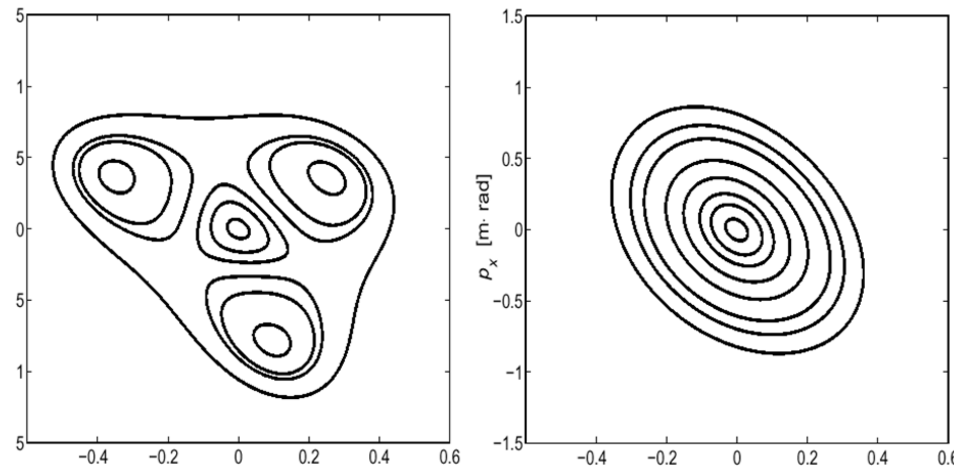
Before optics correction: ~30%

After optics correction: <1%

Synchrotron light beam spot

Uncorrected optics **Corrected optics**





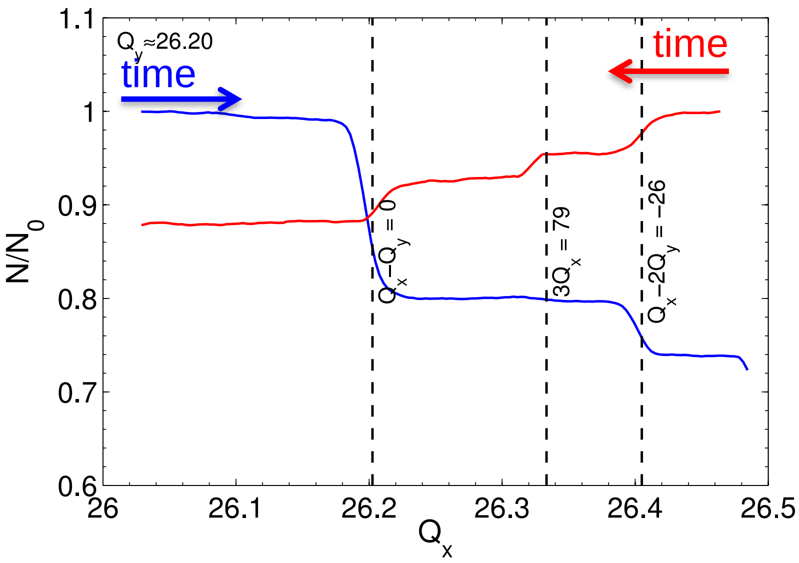
Simulated phase space

D. Robin, C. Steier, J. Safranek, W. Decking, “Enhanced performance of the ALS through periodicity restoration of the lattice,” proc. EPAC 2000.

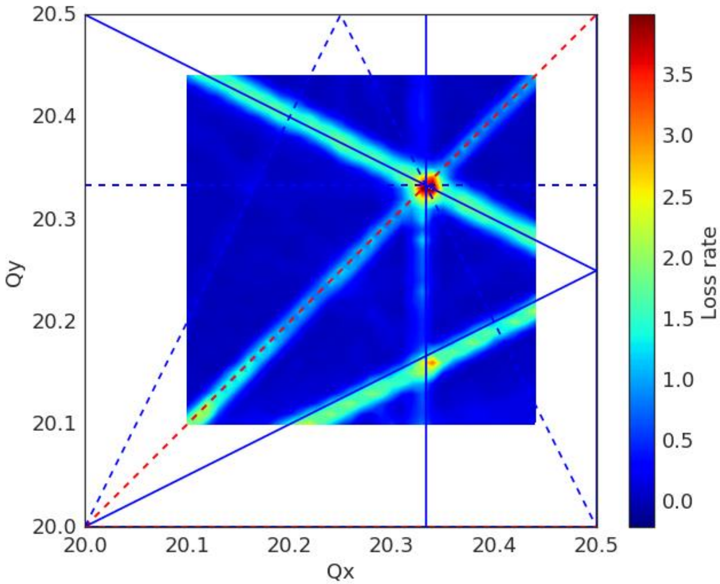
Real life example for periodicity: SPS

- SPS (hadron machine) has design lattice **periodicity of 6**
- Some indication for the strength of individual resonance lines can be inferred from the beam loss rate during dynamic tune scans, i.e. the derivative of the beam intensity at the moment of resonance crossing
- **Sextupole resonances can be clearly identified** although they should be suppressed by lattice periodicity … but SPS has no individual quadrupoles to restore optics functions distortions

Measured losses during tune scan

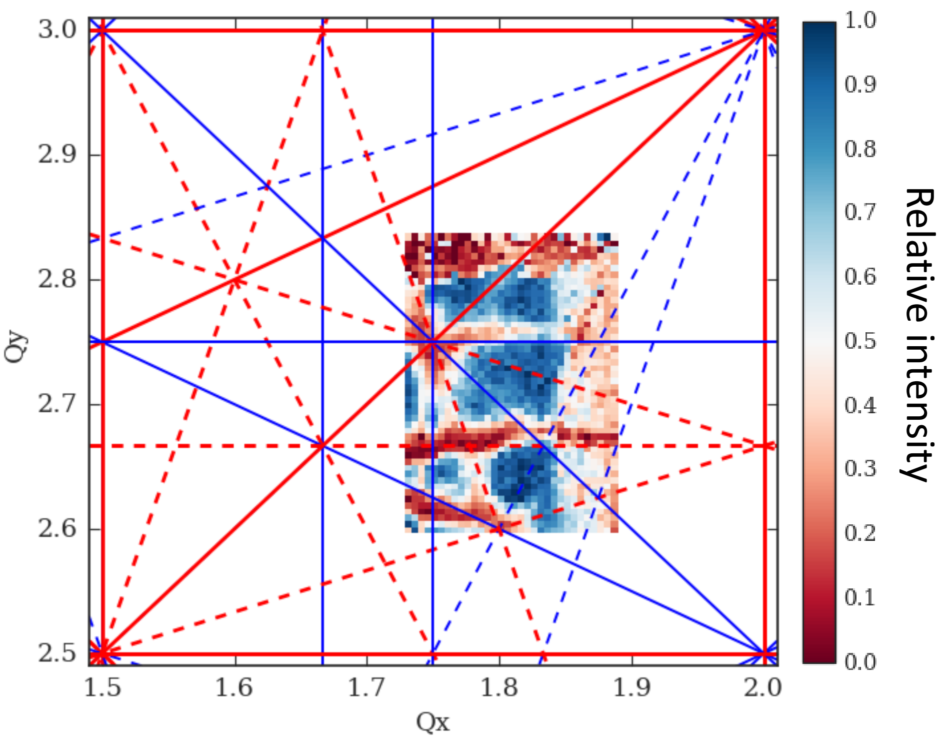


Measured loss rate in 2D scan



Real life example for periodicity: LEIR

- The Low Energy Ion Ring (LEIR) at CERN is a small ion accumulator with lattice **periodicity ≤ 2** (optics perturbations due to e-cooler distort 2 fold symmetry)
- Many resonances observed in measurements
- Sources for some resonances not clear and presently under study (e.g. Qy = 2.66)

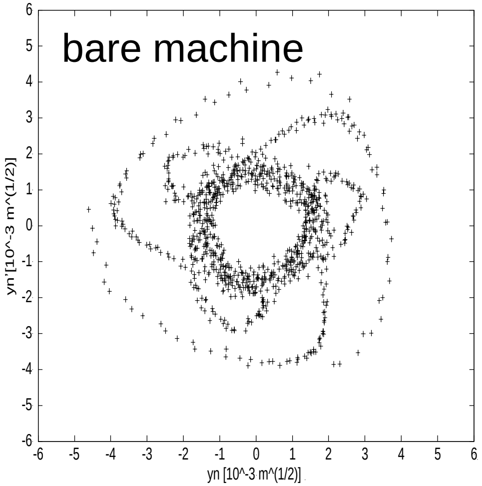


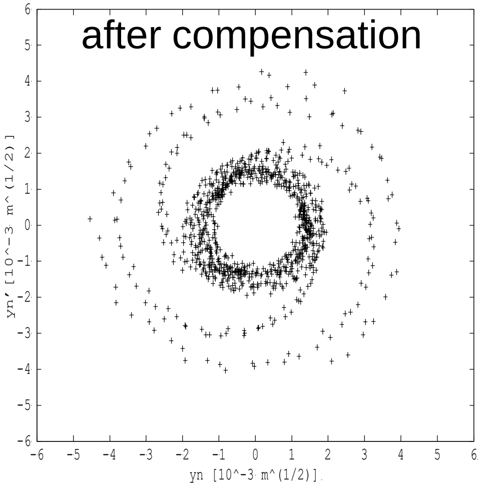
A. Saa Hernandez, D. Moreno, et al.

Compensation of individual resonance

- If a resonance is sufficiently weak, one can try to globally minimize the corresponding resonance driving term
- A pair of multipole correctors that are ~orthogonal in the corresponding resonance driving term is needed to cover all phases. Ideally these multipole correctors are installed in regions with zero / low dispersion in order not to change the (non-linear) chromaticity
- Note: A setting of multipole correctors that compensate a given resonance might unfortunately excite other resonances

Example: phase space reconstructed from measured turn-by-turn data in PSB ring 1 close to 3 Qy = 16 (systematic skew resonance)

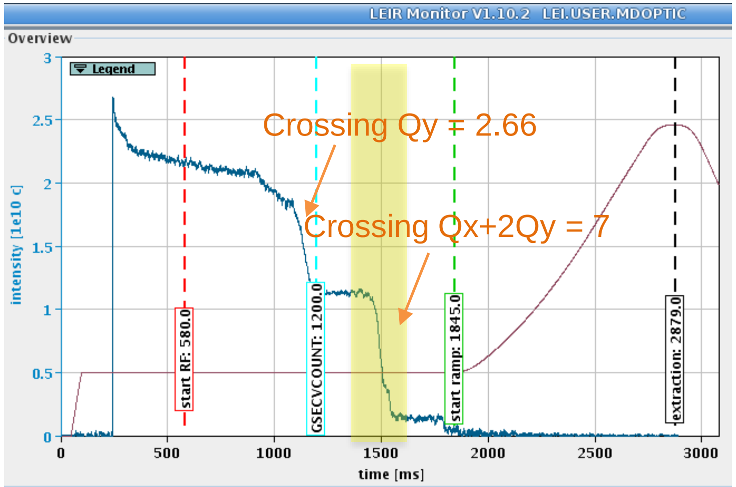


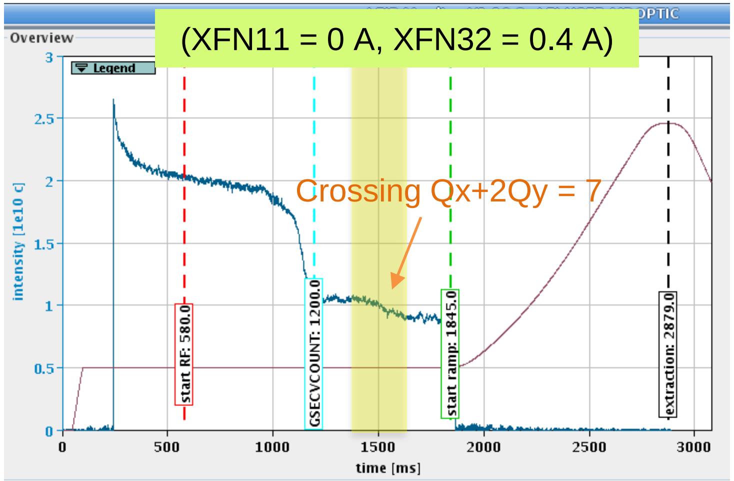


P. Urschütz et al.

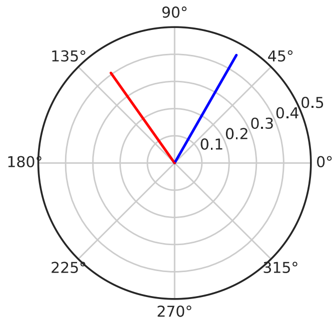
Resonance compensation at LEIR

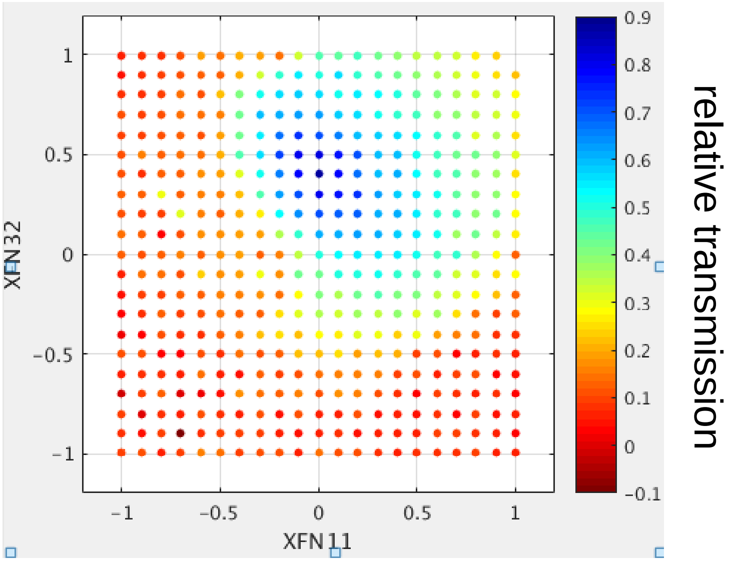
- Brute force technique: sweep tune through resonance and observe beam loss for different settings of pair of multipole correctors





2 sextupole corrector acting on h1020 resonance driving term

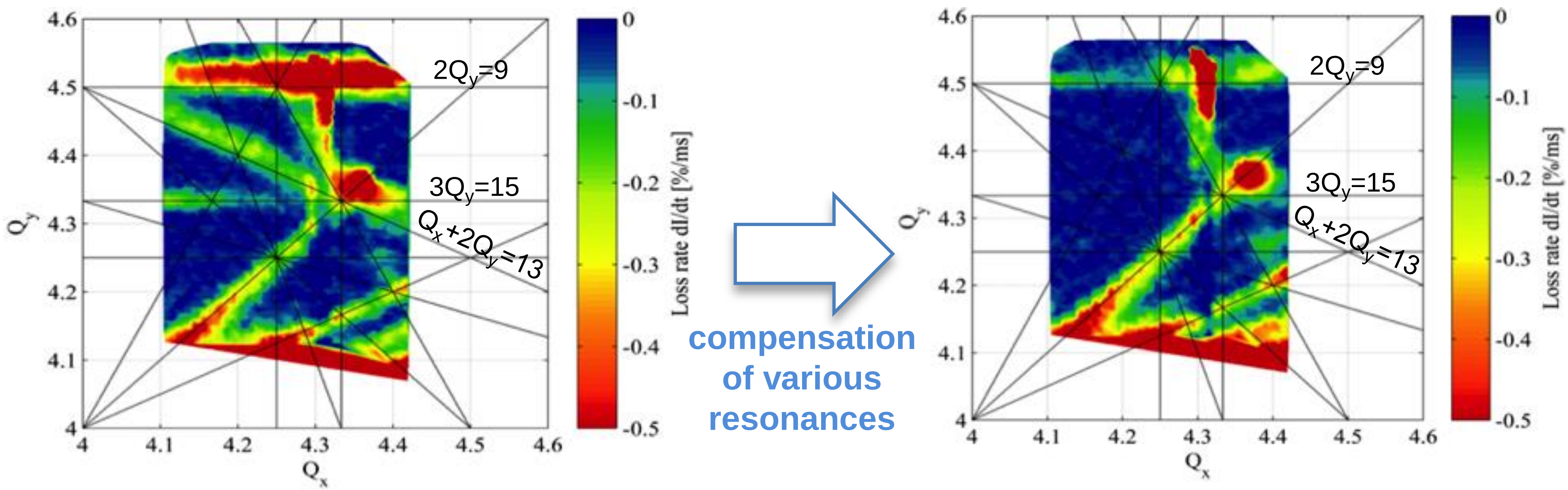




A. Saa Hernandez et al.

Resonance compensation at PSB

- PSB is a machine with 4 rings and **periodicity 16**
- Each ring has a stack of multipole correctors (quadrupoles, sextupoles and octupoles, all normal and skew!) with appropriate phase advances
- Allows to compensate various resonances around the working point (actually needed because tune spread is large due to space charge)



V. Forte et al.

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Taylor maps

- For any dynamical variable x_j the **Taylor map** up to **3rd order** can be written as

$$x_j^{\text{new}} = \sum_{k=1}^{6} R_{jk} x_k + \sum_{k=1}^{6} \sum_{l=1}^{6} T_{jkl} x_k x_l + \sum_{k=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} U_{jklm} x_k x_l x_m$$

- **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline
- The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number** of **terms**
- In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map
- In particular, a **truncated map** will be usually **non-symplectic**

Symplectic maps

- Consider two sets of canonical variables $\vec{x}_i,\ \vec{x}_f$, which represent the evolution of the system between two points in phase space
- A **map** $\mathcal{M} : \vec{x}_i \mapsto \vec{x}_f$ describes the transformation from one set to the other
- This map is **symplectic,** i.e. it **conserves phase space volumes**, if

$$\boxed{J^T S J = S}$$

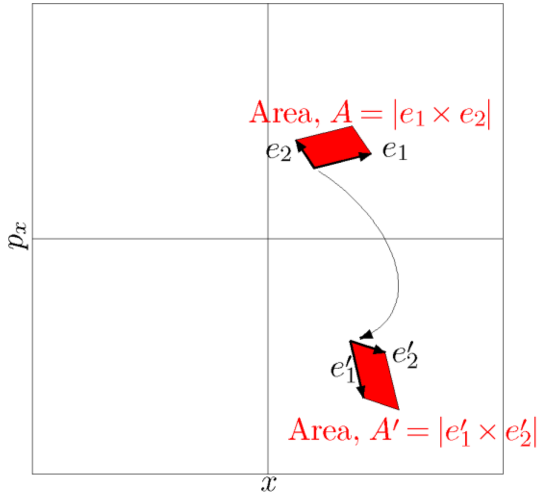
symplecticity condition

$$J_{mn} \equiv \frac{\partial x_{m,f}}{\partial x_{n,i}}$$

Jacobian matrix of the map

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

antisymmetric matrix with block diagonals

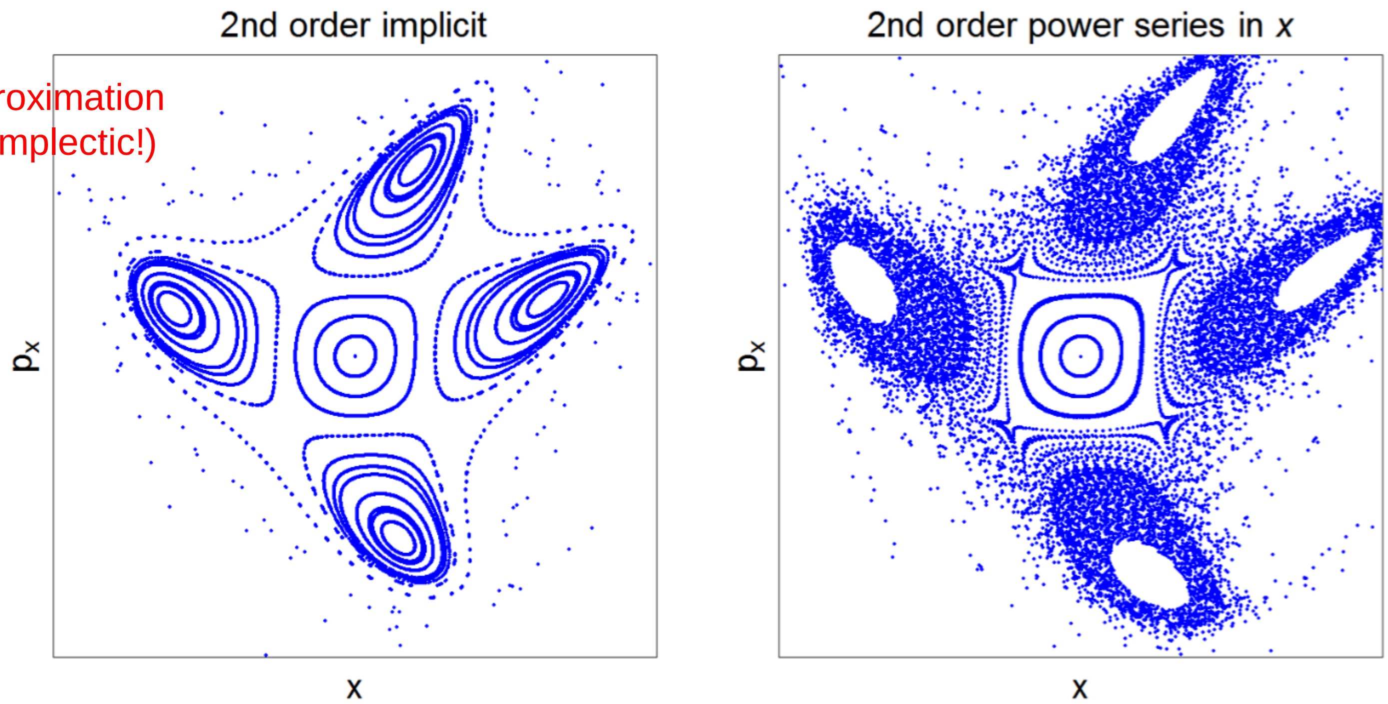


… this is Liouville's theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation

Symplectic maps

- The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.

Thin lens approximation
(is always symplectic!)



- Modelling a storage ring using non-symplectic maps can lead to an inaccurate estimate of the dynamic aperture and the beam lifetime

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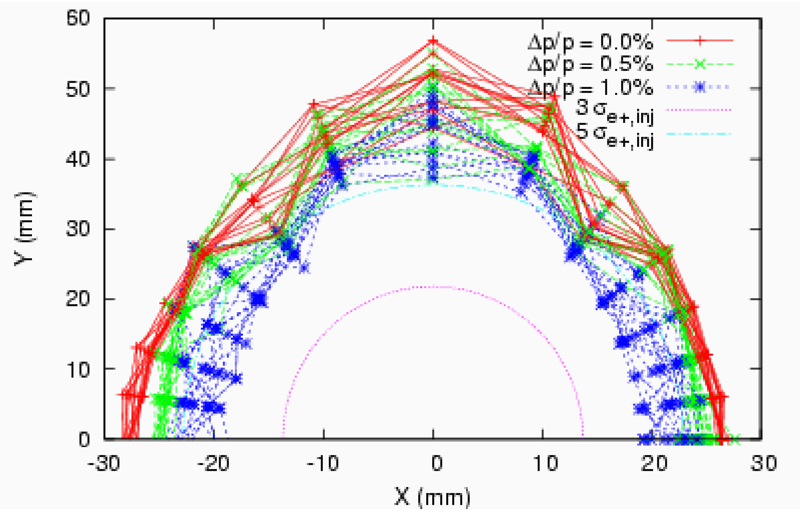
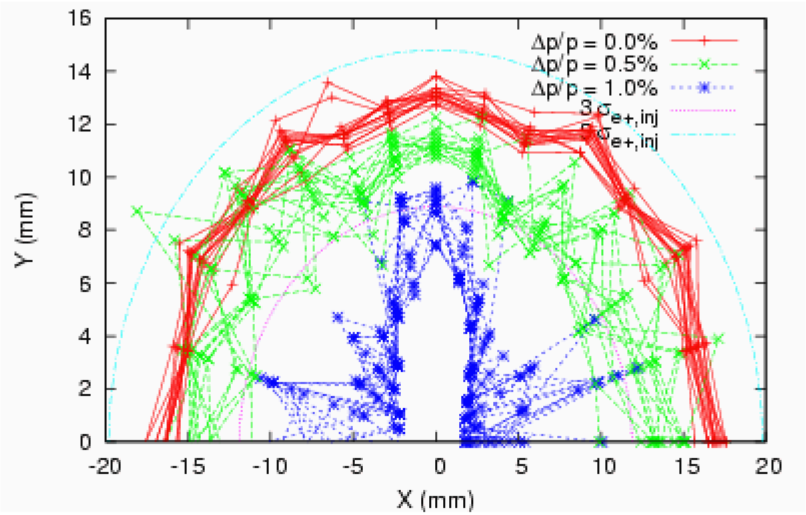
Dynamic aperture

- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture (short: DA)**
- Particle motion due to multi-pole errors is generally non-bounded, so **chaotic particles can escape to infinity**
- This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic** tracking code to follow particle trajectories (a lot of initial conditions) for a **number of turns** (depending on the given problem) until the particles start getting lost. This **boundary** defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through **several machine models** built by **random distribution** of these errors
- One could start with 4D (only transverse) tracking but certainly needs to simulate 5D (constant energy deviation) and finally 6D (synchrotron motion included)

Dynamic Aperture plots

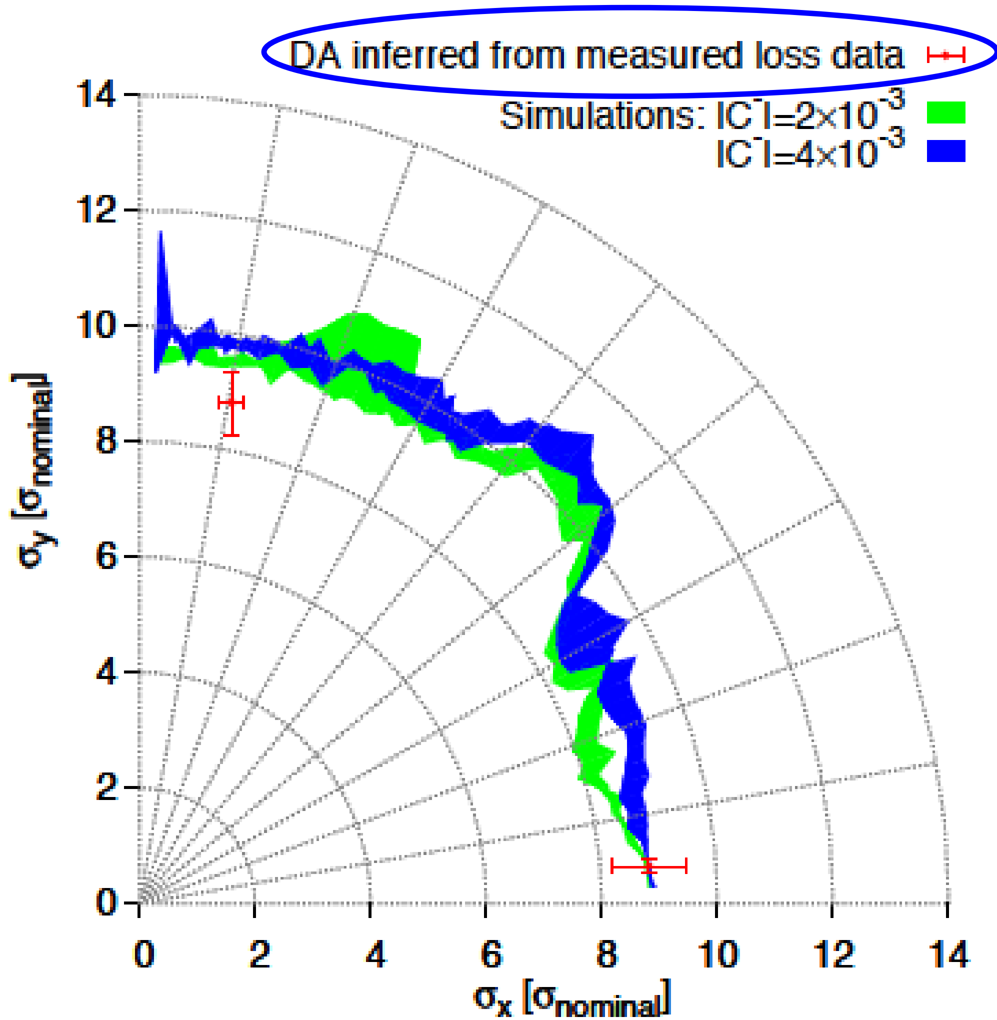
- **Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space** at a particular point in the lattice, for a range of energy errors
- The beam size can be shown on the same plot
- Generally, the goal is to allow some significant margin in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture





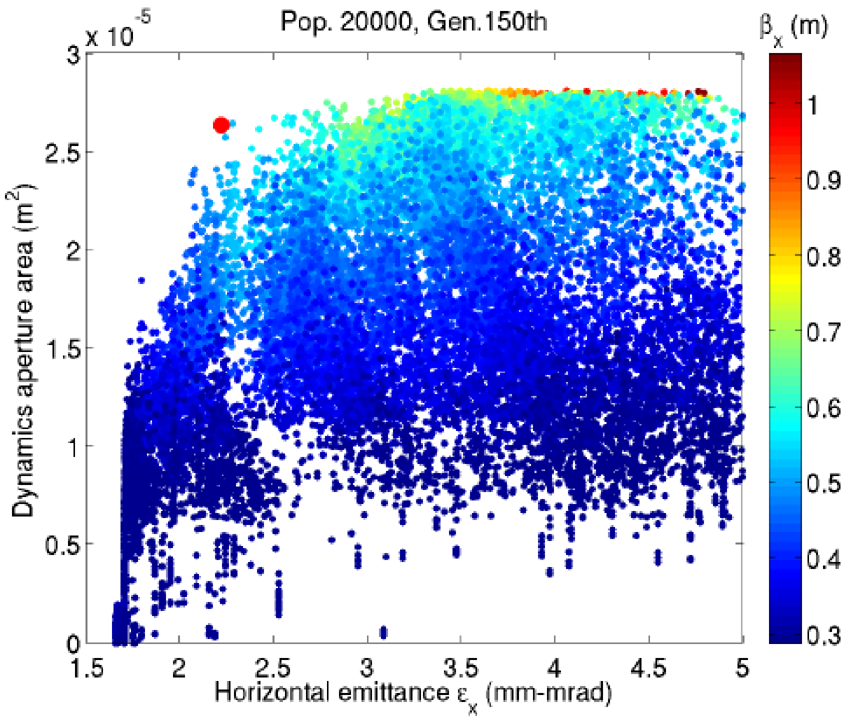
Dynamic Aperture of LHC

- During LHC design phase, DA target was 2x higher than collimator position, due to statistical fluctuation, finite mesh, linear imperfections, short tracking time, multi-pole time dependence, ripple and a 20% safety margin
- Good knowledge of the model led to good agreement between measurements and simulations for actual LHC



E.Mclean, PhD thesis, 2014

Genetic Algorithms



- MOGA – Multi Objective Genetic Algorithms are being used to optimise linear but also non-linear dynamics of electron storage rings
- Use knobs quadrupole strengths, chromaticity sextupoles and correctors with some constraints
- Target ultra-low horizontal emittance, increased lifetime and high dynamic aperture

Frequency Map Analysis

- **Frequency Map Analysis (FMA) is a numerical method which springs from the studies of J. Laskar (Paris Observatory) putting in evidence the chaotic motion in the Solar Systems**
- **FMA was successively applied to several dynamical systems**
  - Stability of Earth Obliquity and climate stabilization (Laskar, Robutel, 1993)
  - 4D maps (Laskar 1993)
  - Galactic Dynamics (Y.P and Laskar, 1996 and 1998)
  - Accelerator beam dynamics: lepton and hadron rings (Dumas, Laskar, 1993, Laskar, Robin, 1996, Y.P, 1999, Nadolski and Laskar 2001)

NAFF algorithm

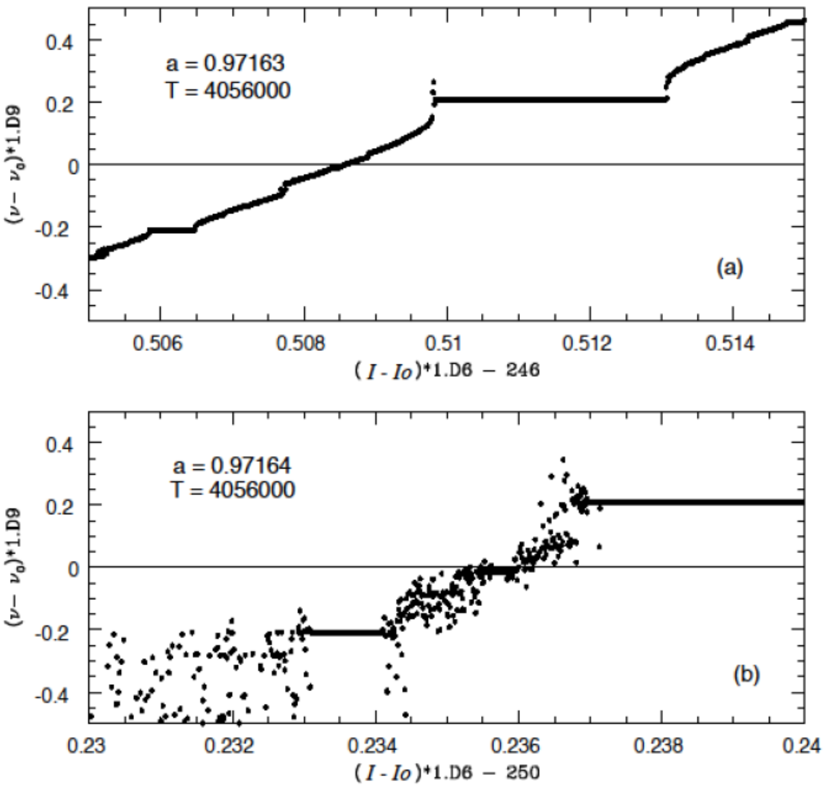
- When a quasi-periodic function $f(t) = q(t) + ip(t)$ in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation

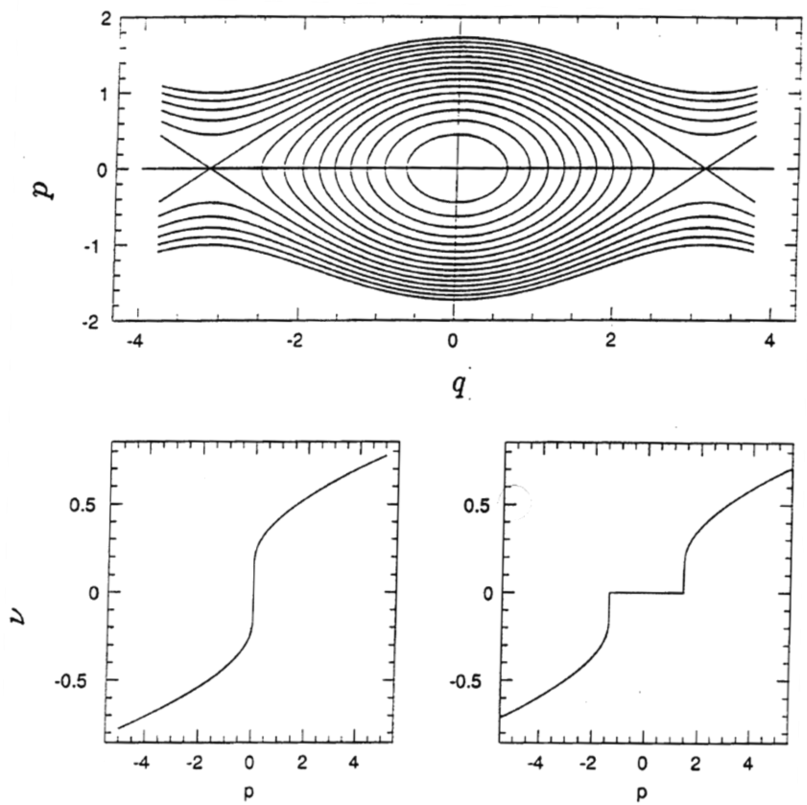
$$f'(t) = \sum_{k=1}^{N} a'_k e^{i\omega'_k t}$$

- in a very precise way over a finite time span $[-T, T]$ several orders of magnitude more precisely than simple Fourier techniques
- This approximation is provided by the Numerical Analysis of Fundamental Frequencies – **NAFF algorithm**
- The frequencies ω'_k and complex amplitudes a'_k are computed through an iterative scheme

Aspects of the frequency map

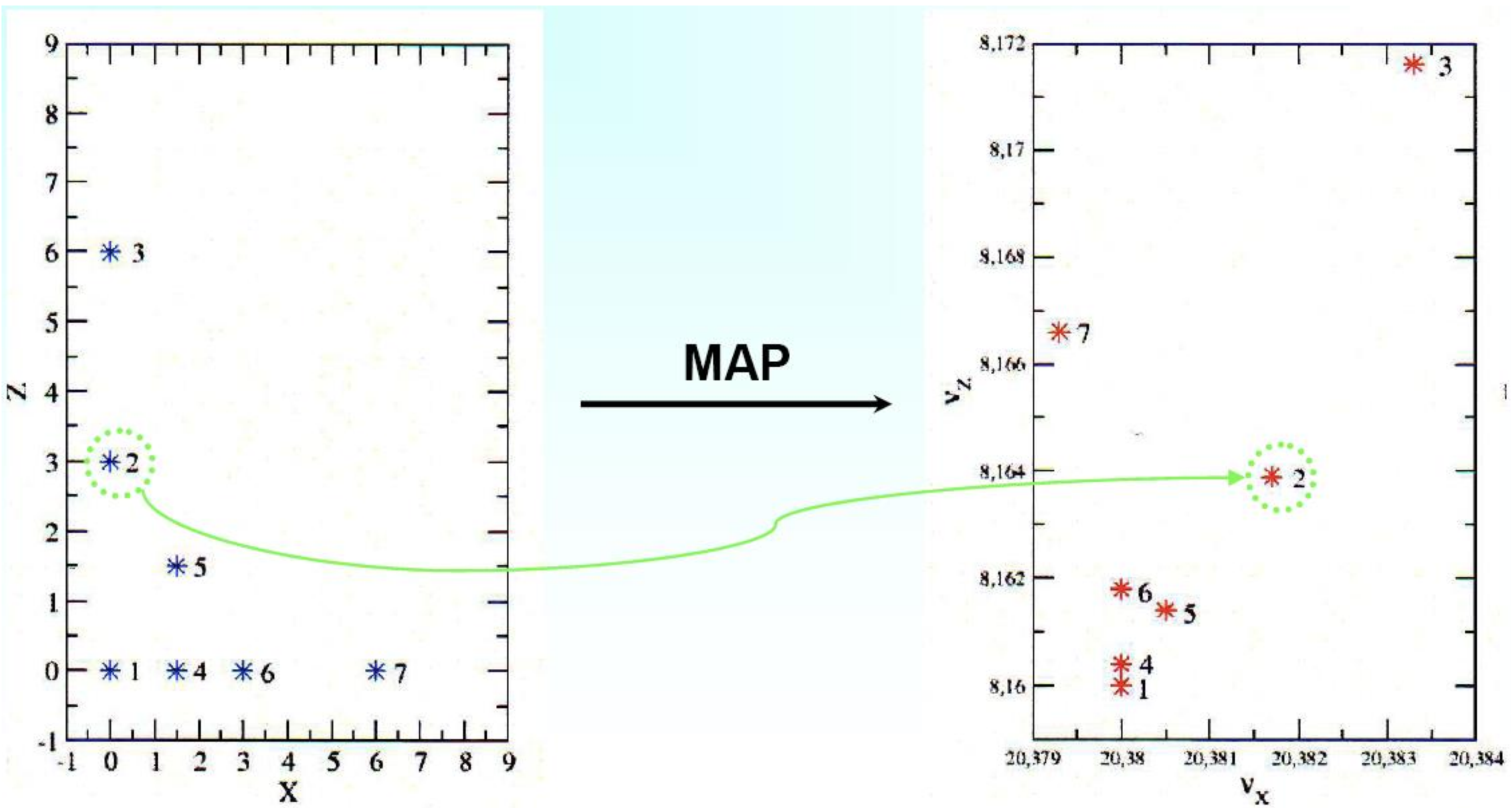
- **In the vicinity of a resonance the system behaves like a pendulum**
- Passing through the elliptic point for a fixed angle, a fixed frequency (or rotation number) is observed
- Passing through the hyperbolic point, a frequency jump is observed





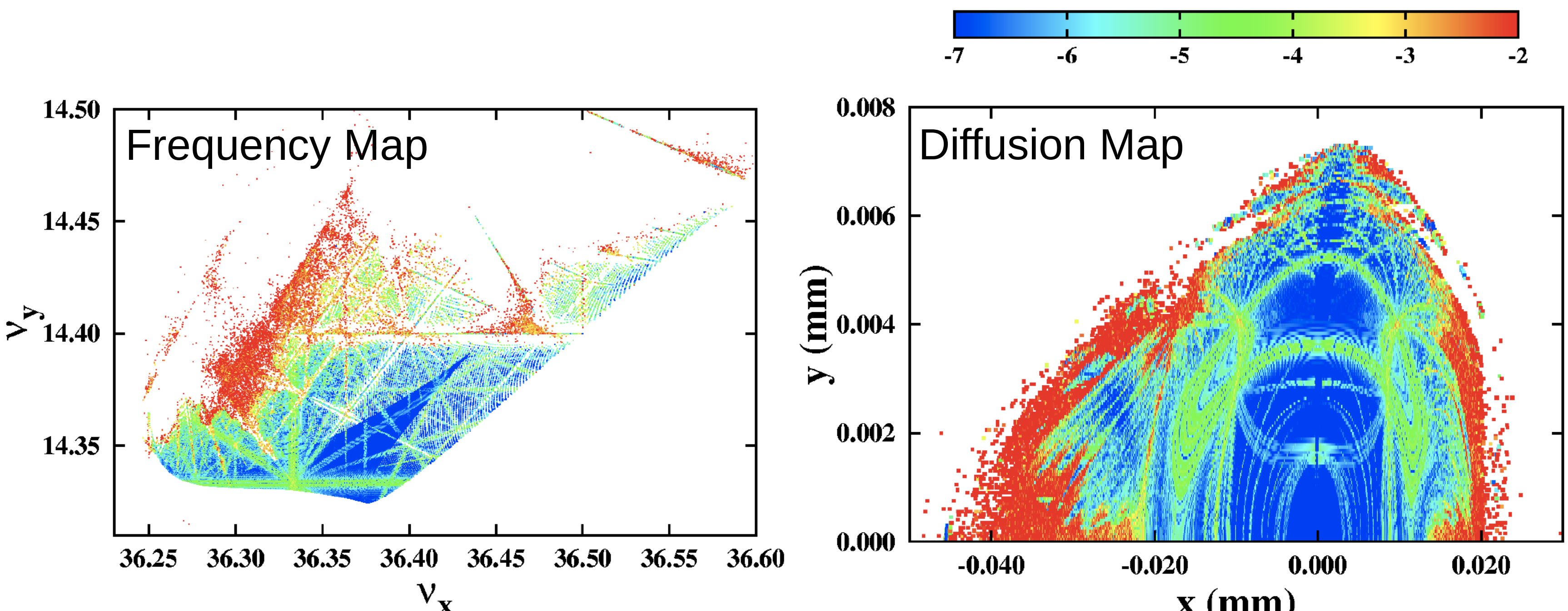
Building the frequency map

- Choose coordinates (x_i, y_i) with $p_x=p_y=0$
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram



Frequency Map for the ESRF

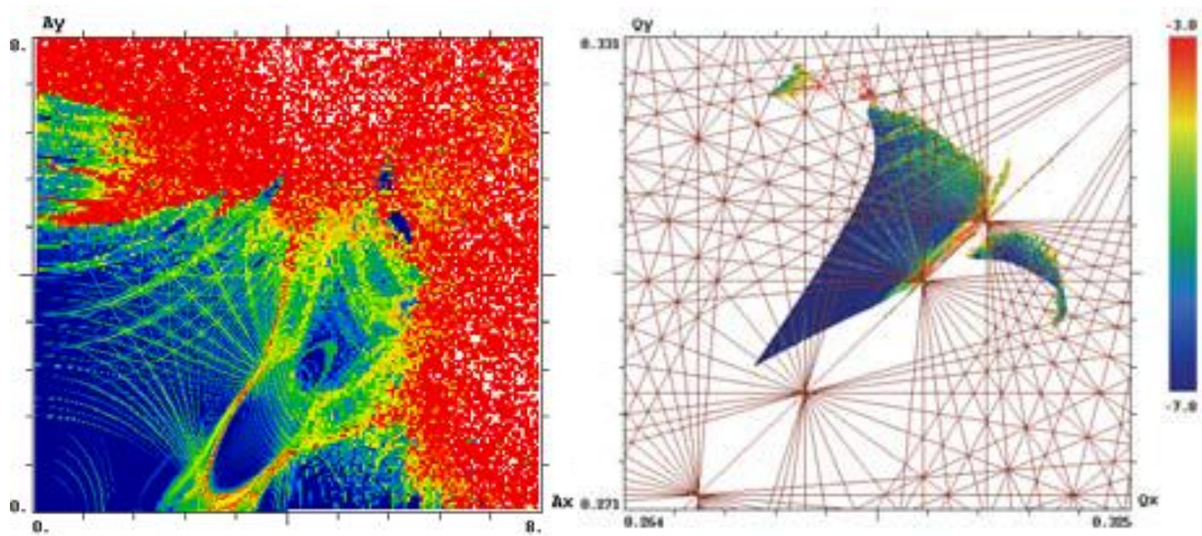
- All dynamics represented in two plots (Frequency Map / Diffusion Map)
  - Color indicates “Tune diffusion” (tune variation between two intervals)
  - Regular motion represented by blue colors
  - **Resonances appear as distorted lines in frequency space** (or curves in initial condition space)
  - Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine
- FMA shows nicely the detuning with amplitude



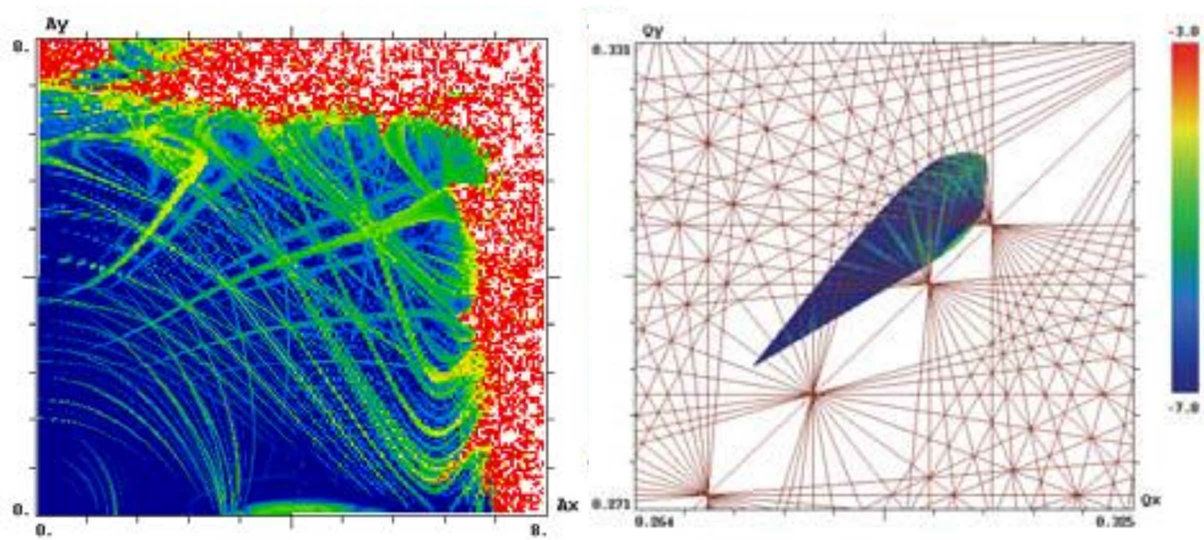
Frequency Map for LHC in collision

Joint Universities Accelerator School

- Frequency map analysis for LHC in collision



Large tune footprint and DA reduction due to "long range beam-beam" forces (electromagnetic field of other beam in interaction region)



DA clearly improved when compensating long range beam-beam with a wire

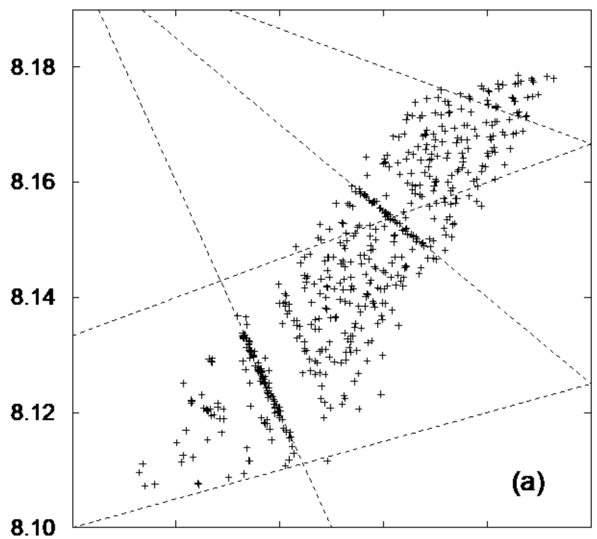
S. Fartoukh et al., PRSTAB, 2015

Experimental frequency maps

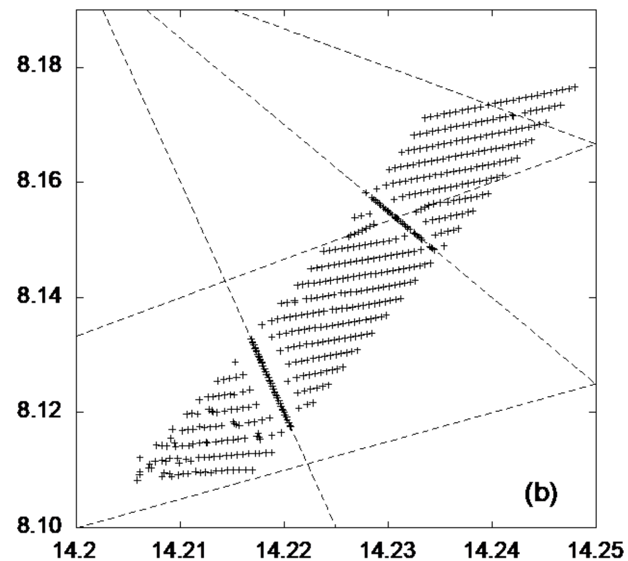
- Frequency analysis of turn-by-turn data of beam oscillations produced by a fast kicker magnet and recorded on a Beam Position Monitor
- Reproduction of the non-linear model of the Advanced Light Source storage ring and working point optimization for increasing beam lifetime

D. Robin, C. Steier, J. Laskar, and L. Nadolski, PRL 2000

experiment



simulation



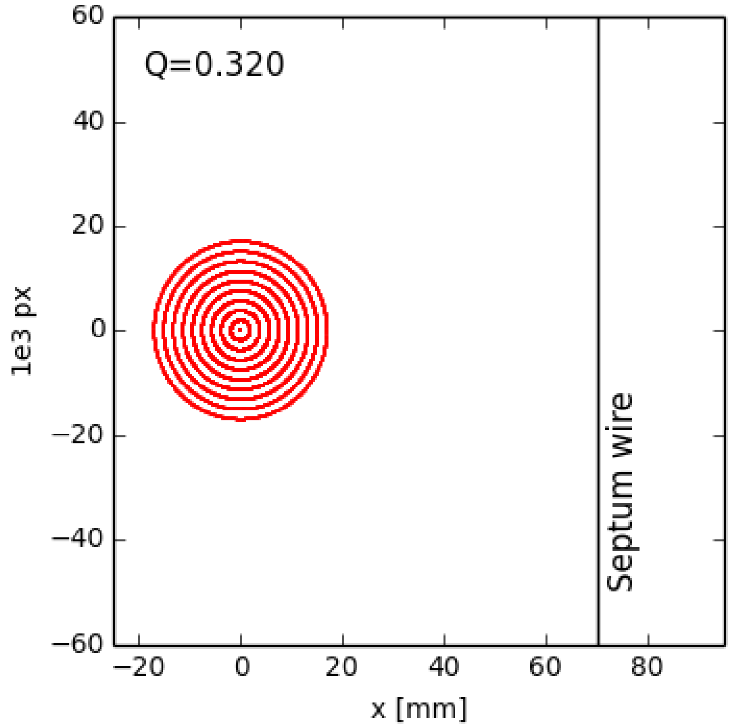
Outline

- Introduction – nonlinear effects from a single sextupole
- Hamiltonian of the nonlinear betatron motion
- Resonance topology and onset of chaos
- Resonances are everywhere – can we do something?
- Non-linear map representation
- Lattice optimization by tracking
- **Applications – making use of resonances**
- Summary

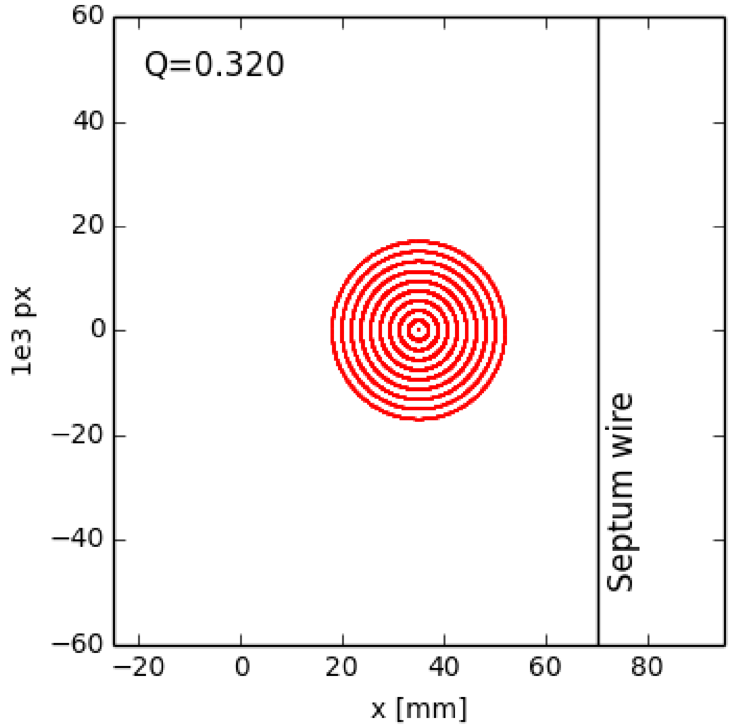
Where resonances can be of use

- **Resonances can be exploited to extract the beam in a controlled way**
  - Many physicists would like to have a continuous flux of particles to perform experiments with high energy (~low intensity) particles. **Resonant slow extraction** using 3rd order resonance is widely used to create a “spill” of the order of seconds, i.e. the beam is extracted over many thousands of turns.
  - **Resonant multi-turn extraction (MTE)** was invented to transfer the beam over 5 turns from the PS to the SPS at CERN with minimal losses based on exciting a 4th order resonance.
  - **Resonant fast extraction** is based on excitation of the half integer resonance by octupoles and a fast discharge of a quadrupole that pushes the particle tune onto the resonance so that they are extracted on a few ms.

Principle of resonant slow extraction

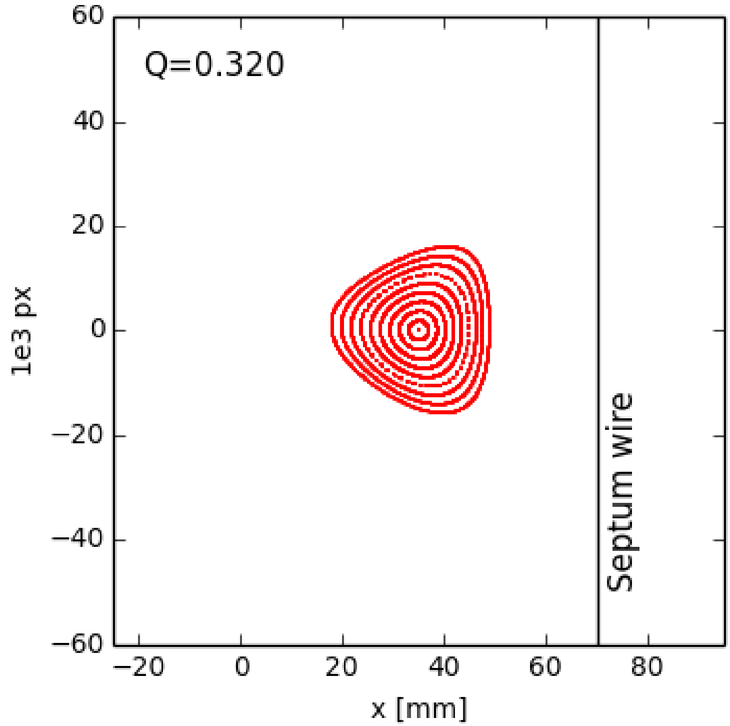


Principle of resonant slow extraction



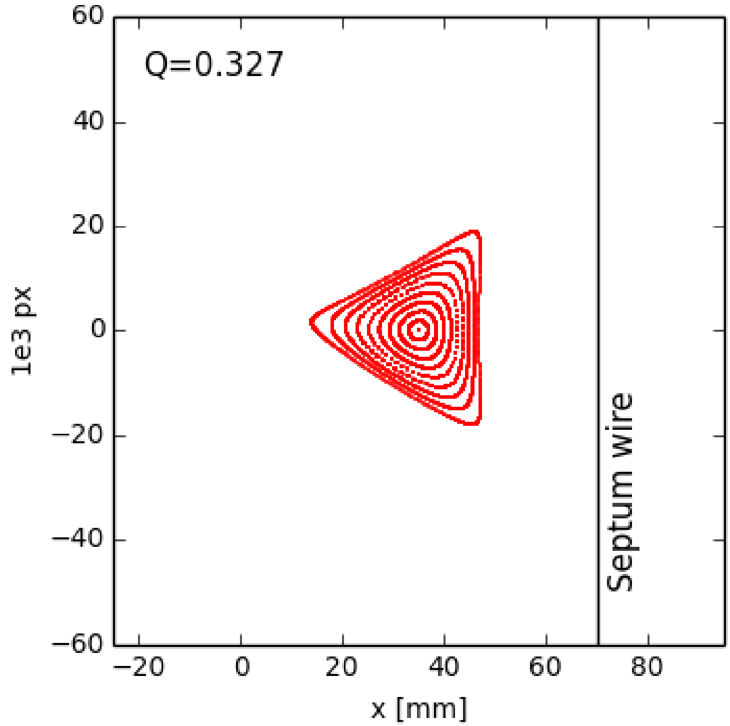
- Closed orbit bump to bring beam close to septum

Principle of resonant slow extraction



- Closed orbit bump to bring beam close to septum
- Sextupole magnets excite 3rd order resonance. Large tune spread (e.g. from chromaticity and not octupoles since we do not want to stabilize the particles)

Principle of resonant slow extraction



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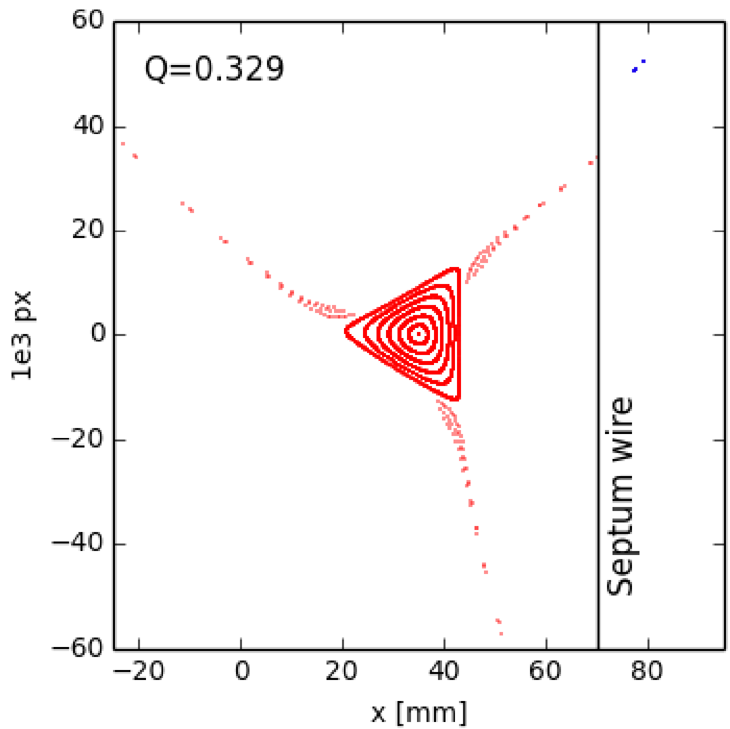
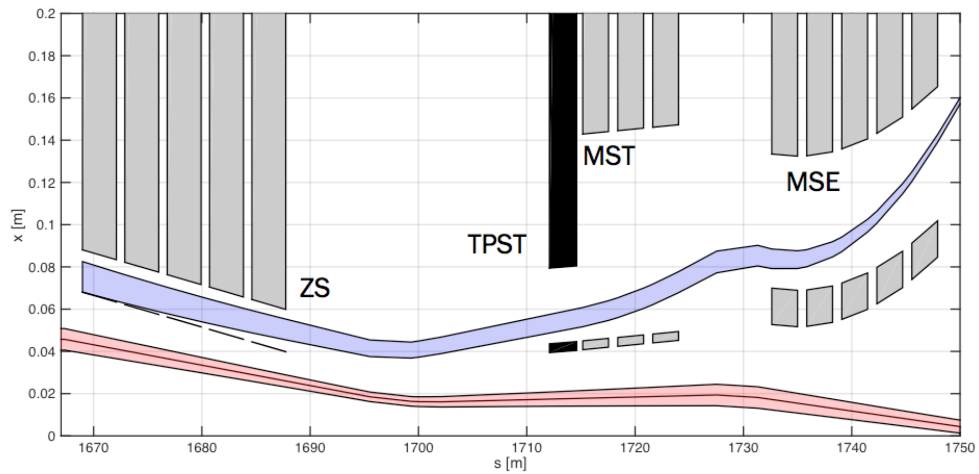
- ΔQ (distance to resonance) small – large amplitude particles close to separatrix

Principle of resonant slow extraction



Circulating (red, ±3σ) and extracted (blue) horizontal beam envelopes and apertures in the LSS2 extraction region.



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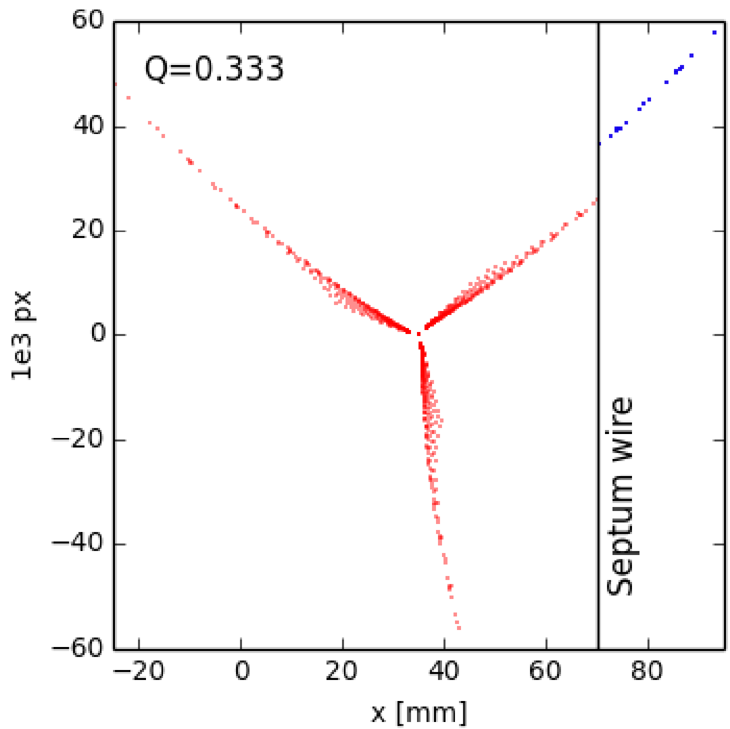
- ΔQ (distance to resonance) small – large amplitude particles close to separatrix
- ΔQ small enough that largest amplitude particles are unstable and follow separatrix with increasing amplitude - particles jump the septum and are extracted

Principle of resonant slow extraction



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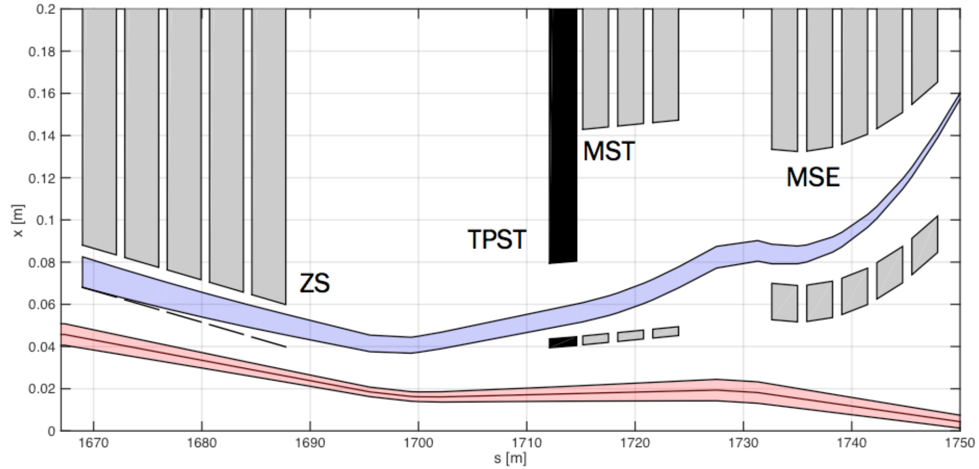

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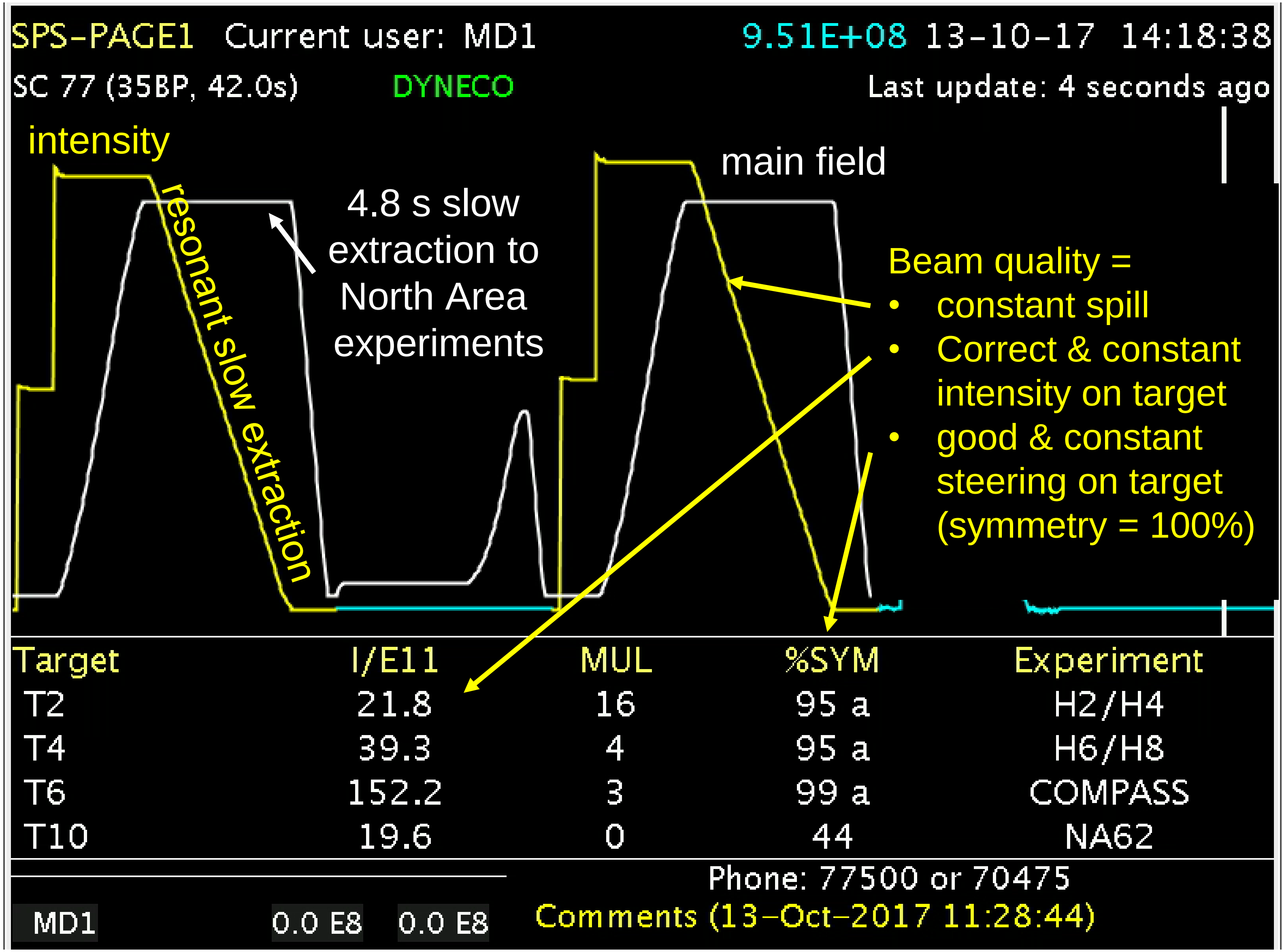
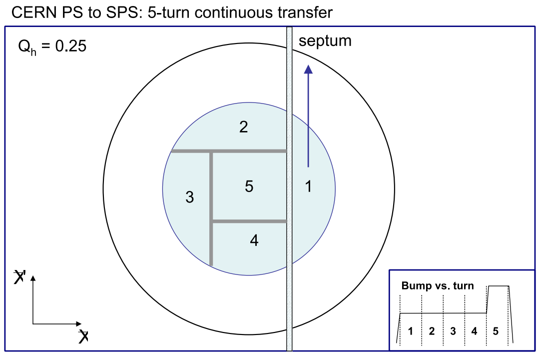
- As ΔQ approaches zero, finally particles with very small amplitude are extracted

Slow extraction at SPS



Multi-turn extraction – Introduction

- **Continuous Transfer (=non resonant multi-turn extraction) of high-intensity beams from CERN PS to SPS in use since 1973**
  - Drawback of high beam loss during the process due to physical slicing of the beam on septum
  - Issues with machine activation (radiation) due to losses

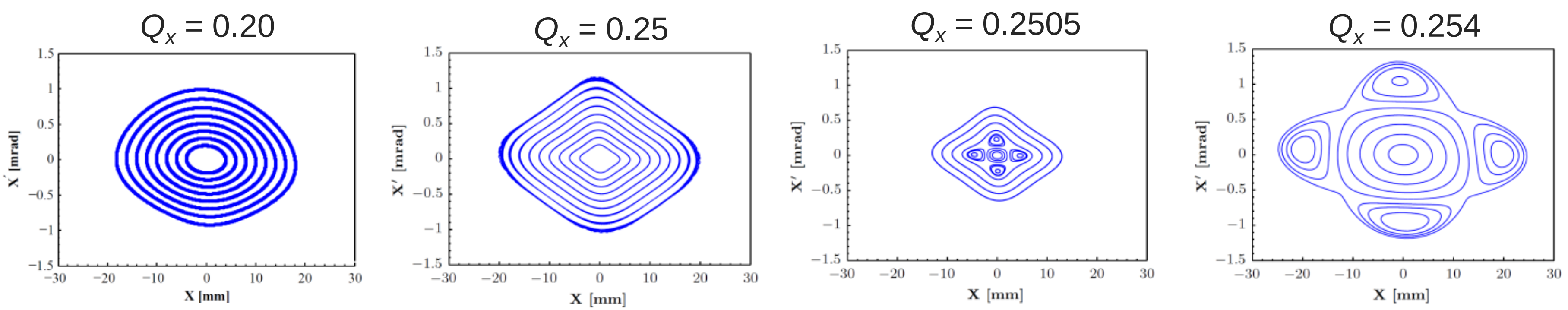


- **Resonant Multi-Turn Extraction (MTE) proposed in 2001 to reduce losses at PS-to-SPS transfer**

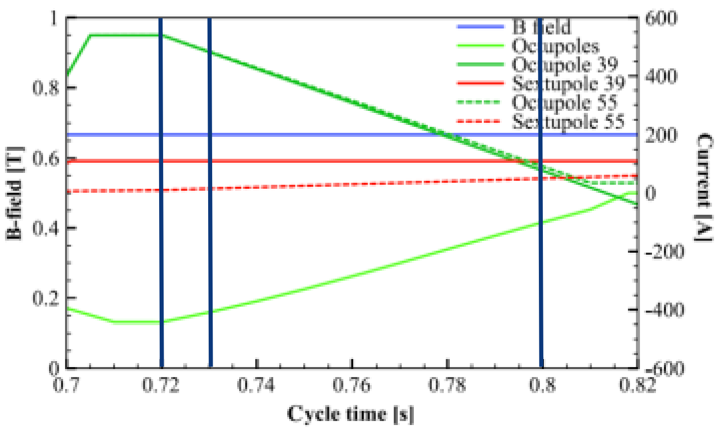
M. Giovannozzi et. al

  - MTE based on concepts of non-linear beam dynamics (Crossing of a stable 4th order resonance and particle trapping inside islands) to perform a "magnetic splitting" of the beam to avoid losses on septum
  - Unique extraction process, has never been done elsewhere
  - Used in routine operation for transfer of fixed target beam since 2015

Resonant multi-turn extraction

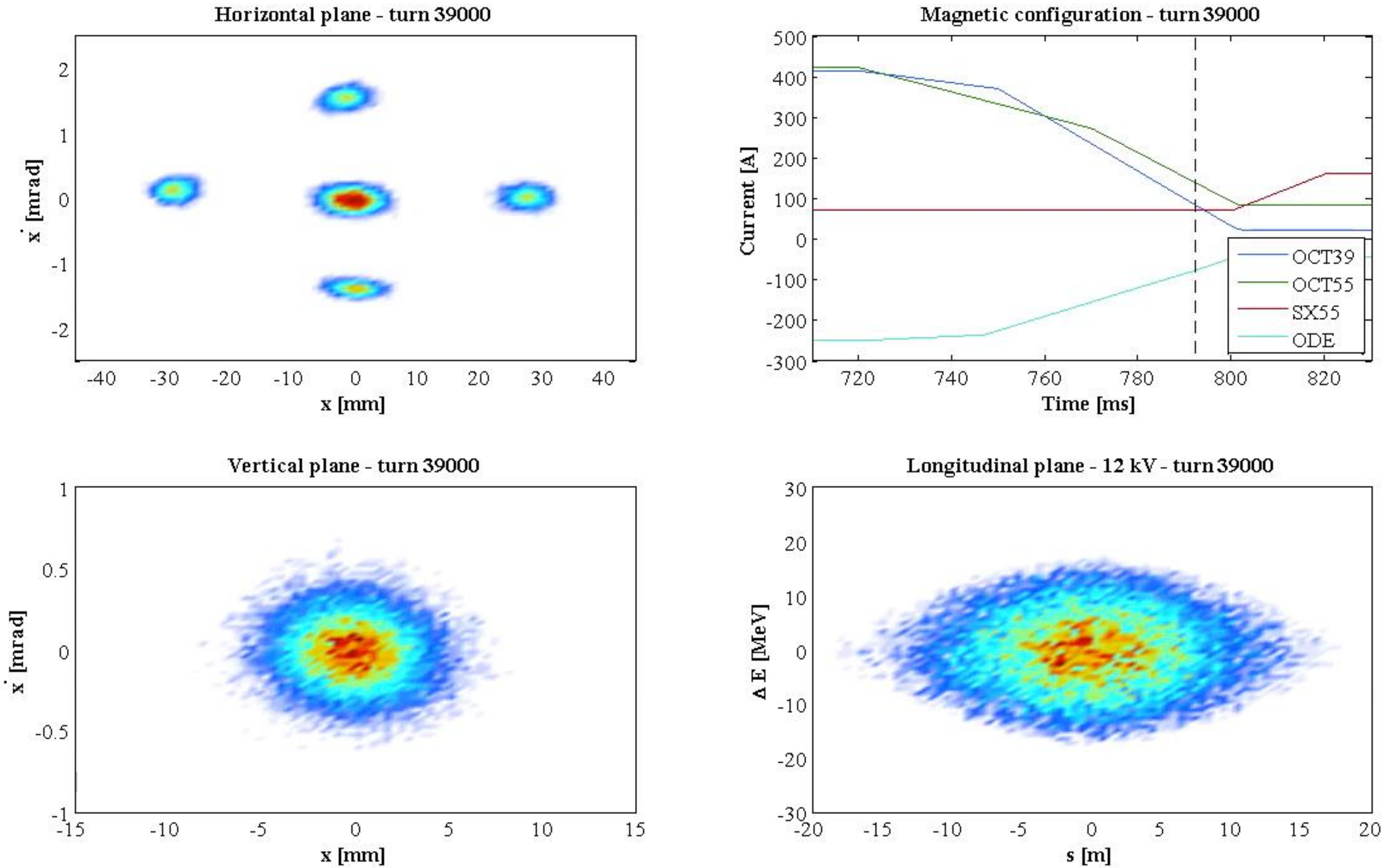


1) program non-linear elements to appropriate values to excite resonance (sextupole + octupole)
2) ramp horizontal tune across the resonant value
3) decrease current in the elements while increasing the tune
4) extract the beam once islands are sufficiently separated: 4 machine turns for the islands + 1 turn for the core



M. Giovannozi,
A. Huschauer et al.

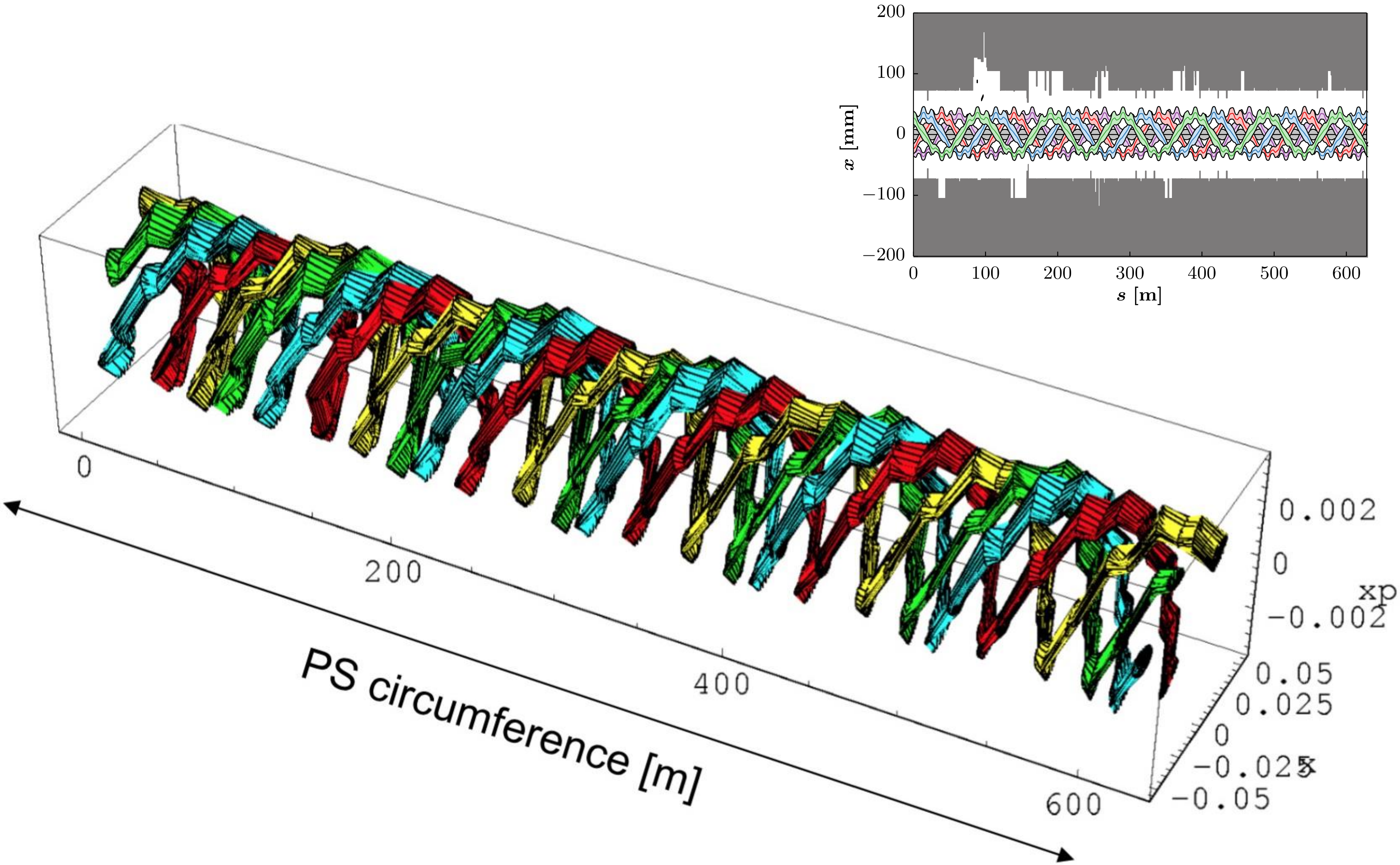
Resonant multi-turn extraction



A. Huschauer

Resonant multi-turn extraction

- Calculation of the 4 islands in phase space around the PS machine



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Summary

- Non-linear elements create detuning with amplitude and excite resonances
- Appearance of fixed points (periodic orbits) determine the topology of the phase space
- Perturbation of unstable (hyperbolic) points opens the path to chaotic motion
- Resonances can overlap enabling the rapid diffusion of orbits
- Individual resonances can be compensated (to some extent)
- Need numerical integration (tracking) for understanding impact of non-linear effects on particle motion (dynamic aperture)
- Frequency map analysis is a powerful technique for analyzing particle motion in simulations but also in real accelerator experiments
- Resonances can be used for beam extraction