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Space Charge Effects and Instabilities

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CONTENTS (SPACE CHARGE)

- Introduction, self induced fields
- $\vec{E}, \vec{B}$ of a point charge with uniform velocity
- Forces between two charges
- Collisional and space charge regimes, Debye length
- $\vec{E}, \vec{B}$ of a uniform infinite beam in free space
- Boundary conditions: image charges and currents
- Examples: beam in a circular pipe, in parallel plates
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- Simulation codes

EQUATION OF MOTION

Charged particles in a transport channel or in a circular/linear accelerator are accelerated, guided and confined by external electromagnetic fields. The motion of a single charge is governed by the Lorentz force through the equation:

$$\frac{d(m_0\gamma\vec{v})}{dt} = \vec{F}_{e.m.}^{ext} = e(\vec{E} + \vec{v}\times\vec{B})$$

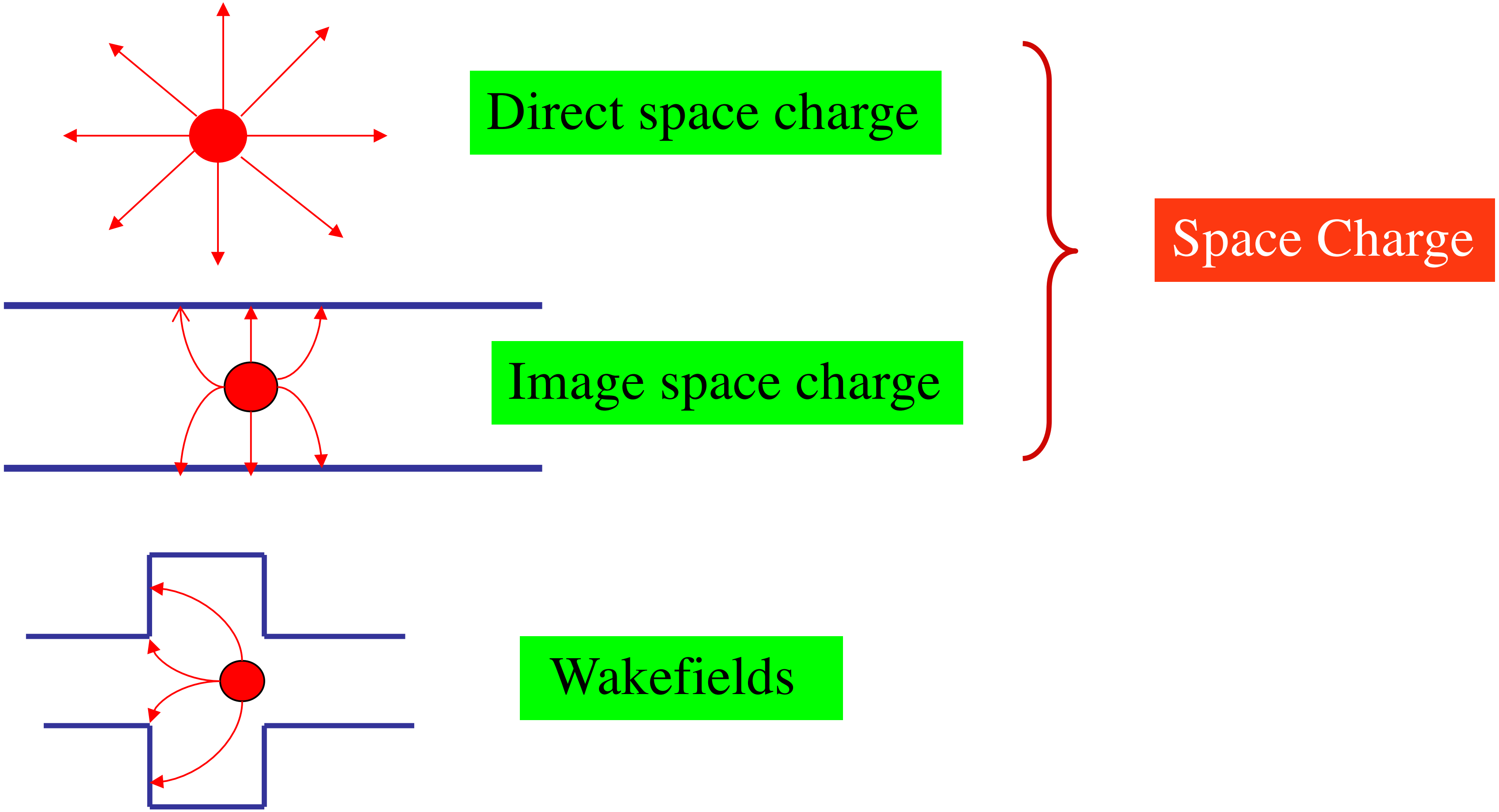
Where m_0 is the rest mass, γ is the relativistic factor and v is the particle velocity.

Acceleration is usually provided by the electric field inside of RF cavities. Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

However, there is another source of e.m. fields, the beam itself…

SPACE CHARGE AND WAKEFIELDS

In a real accelerator, in particular at high currents, there is an important source of e.m. fields to be considered, the beam itself, which circulating inside the vacuum chamber, produces additional e.m. fields:



Direct space charge

Image space charge

Space Charge

Wakefields

These self induced fields depend on:

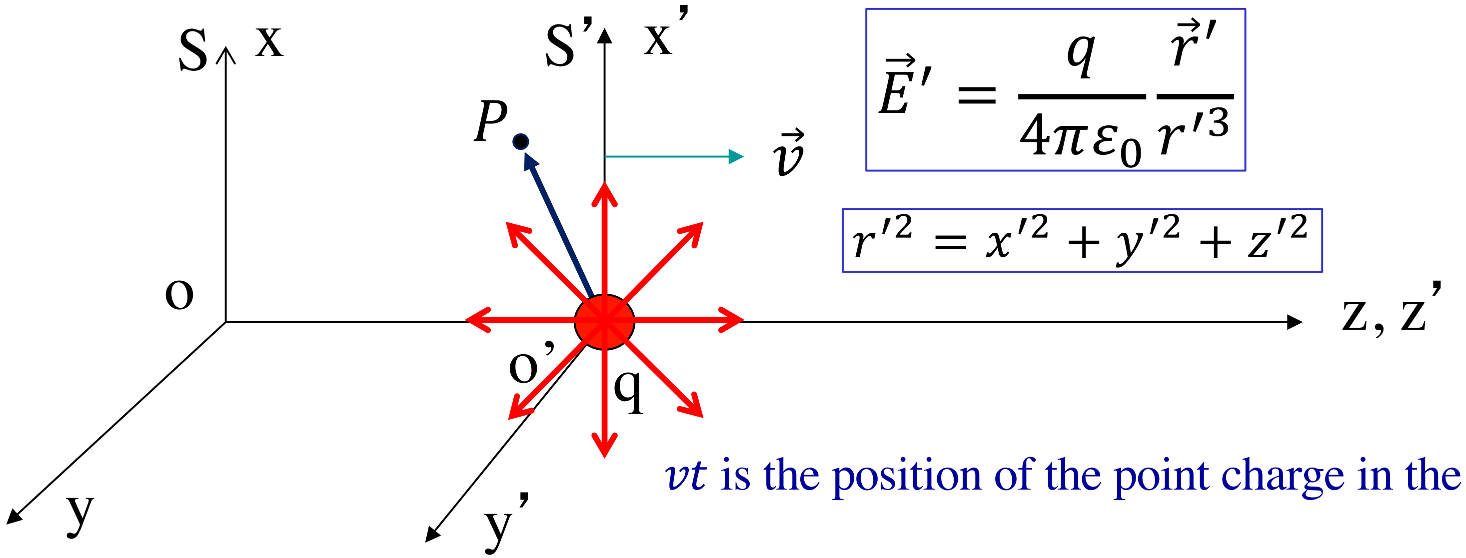
- the beam current and beam distribution
- the surrounding geometry and the beam pipe
- the surrounding material.

They are responsible of many phenomena of beam dynamics:

- betatron tune shift
- synchrotron tune shift
- energy loss
- energy spread and emittance degradation
- instabilities.

CONTENTS (SPACE CHARGE)

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}'}{r'^3}$$

$$r'^2 = x'^2 + y'^2 + z'^2$$

vt is the position of the point charge in the system O.

$$E'_x = \frac{q}{4\pi\varepsilon_0}\frac{x'}{r'^3}$$

$$E'_y = \frac{q}{4\pi\varepsilon_0}\frac{y'}{r'^3}$$

$$E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3}$$

- In O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero: $\vec{B}' = 0$

Relativistic transformations of the fields and coordinates from O' to O

$$\begin{cases} E_x = \gamma\left(E'_x + vB'_y\right) \\ E_y = \gamma\left(E'_y - vB'_x\right) \\ E_z = E'_z \end{cases}$$

$$\begin{cases} B_x = \gamma\left(B'_x - vE'_y/c^2\right) \\ B_y = \gamma\left(B'_y + vE'_x/c^2\right) \\ B_z = B'_z \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - vt) \end{cases}$$

$$r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$r' = [x^2 + y^2 + \gamma^2(z - vt)^2]^{1/2}$$

We now transform the electric and magnetic fields and the coordinates from *S'* to *S*

$$E_x = \gamma E_x' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}$$

$$E_y = \gamma E_y' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}$$

$$E_z = E_z' = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(z - vt)}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}$$

→ x-y symmetry

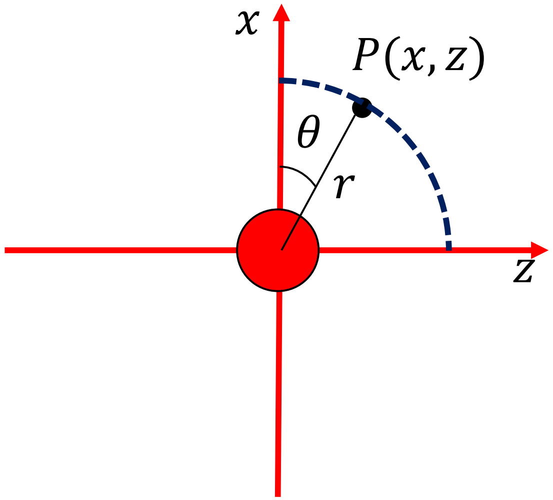
At time $t = 0$ $$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\vec{r}}{[x^2 + y^2 + \gamma^2 z^2]^{3/2}}$$

We can study the properties of the fields at time $t = 0$ because the fields move along z together with the charge. The fields at any time t are just the fields at $t = 0$ shifted by vt

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\gamma\vec{r}}{(x^2 + y^2 + \gamma^2 z^2)^{3/2}}$$

Due to the $x - y$ symmetry (cylindrical symmetry), we can consider $y = 0$ ($x - z$ plane) and use

$$x = r\cos\theta\,, \qquad z = r\sin\theta$$

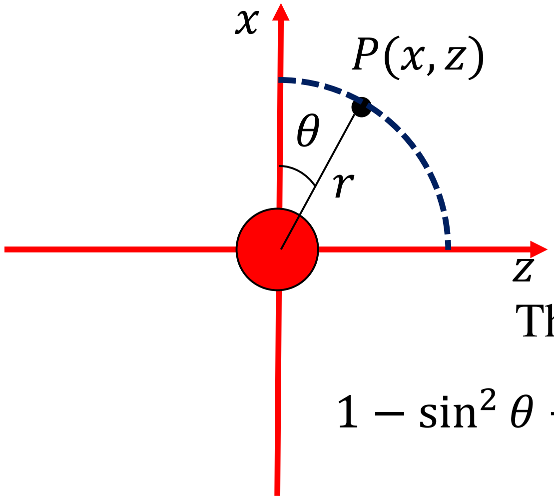


$$\left|\vec{E}\right| = \frac{q}{4\pi\varepsilon_0 r^2} \frac{\gamma}{(\cos^2\theta + \gamma^2\sin^2\theta)^{3/2}}$$

$\left|\vec{E}\right|_{max}$ when $\theta = 0$: $\quad \left|\vec{E}\right|_{max} = \dfrac{\gamma q}{4\pi\varepsilon_0 r^2}$

$\left|\vec{E}\right|_{min}$ when $\theta = \pi/2$: $\quad \left|\vec{E}\right|_{min} = \dfrac{q}{4\pi\varepsilon_0 r^2 \gamma^2}$

$$|\vec{E}| = \frac{q}{4\pi\varepsilon_0 r^2}\frac{\gamma}{(\cos^2\theta+\gamma^2\sin^2\theta)^{3/2}} = \frac{|\vec{E}|_{max}}{(\cos^2\theta+\gamma^2\sin^2\theta)^{3/2}}$$



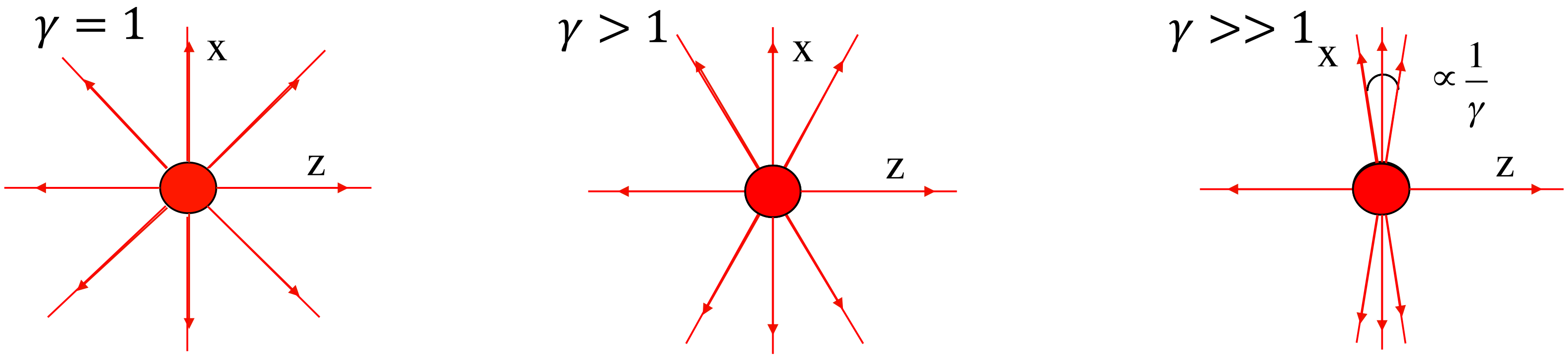
When $(\cos^2\theta+\gamma^2\sin^2\theta) = 2$, we have

$$|\vec{E}| = \frac{|\vec{E}|_{max}}{\sqrt{8}} \simeq \frac{|\vec{E}|_{max}}{3}$$

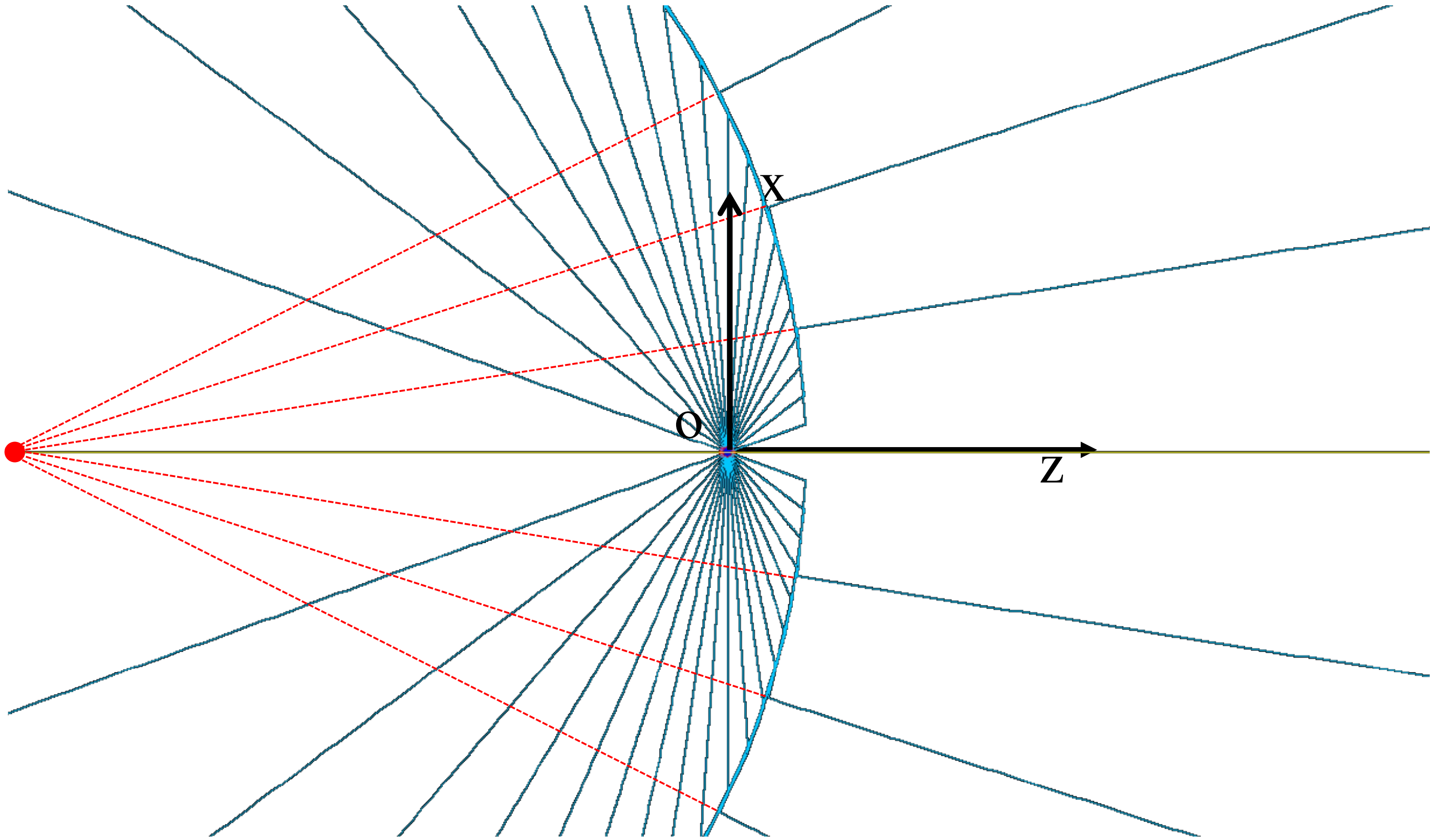
This happens when

$$1-\sin^2\theta+\gamma^2\sin^2\theta = 2 \;\rightarrow \sin^2\theta = \frac{1}{\gamma^2-1} \quad \rightarrow \;(\text{if}\,\gamma \gg 1) \;\rightarrow \theta \simeq \frac{1}{\gamma}$$

Electric field lines



Electric field lines of a charge moving with velocity βc



$$E_x = \gamma E_x' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$

$$E_y = \gamma E_y' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$

$$E_z = E_z' = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma (z-vt)}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$

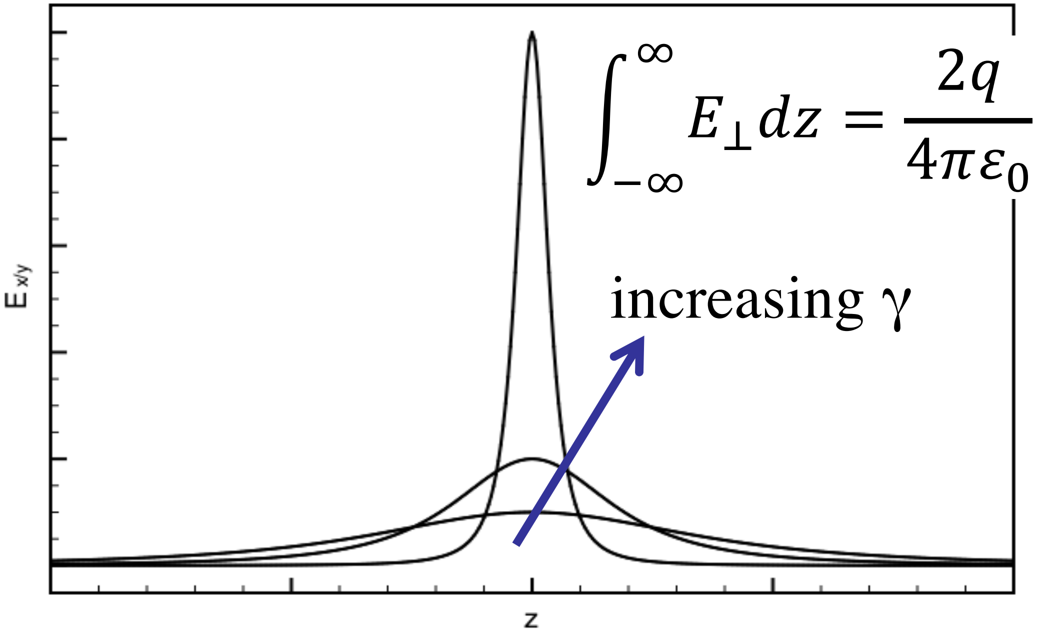
$\gamma \longrightarrow \infty$
$v \longrightarrow c$

$$\begin{cases} E_x = ? \\ E_y = ? \\ E_z \to 0 \end{cases}$$

$\boxed{z \neq ct \rightarrow \; E_\perp = 0}$ (1)

$\boxed{z = ct \rightarrow \; E_\perp = \infty}$ (2)

$$\int_{-\infty}^{\infty} E_\perp dz = \frac{2q}{4\pi\varepsilon_0}\frac{1}{r}$$ independent on γ (3)



$$\Rightarrow \boxed{E_\perp = \frac{2q}{4\pi\varepsilon_0}\frac{1}{r}\delta(z-ct)}$$

B is transverse to the motion direction

$$\begin{cases} B_x = \gamma\left(B_x' - vE_y'/c^2\right) \\ B_y = \gamma\left(B_y' + vE_x'/c^2\right) \\ B_z = B_z' \end{cases}$$

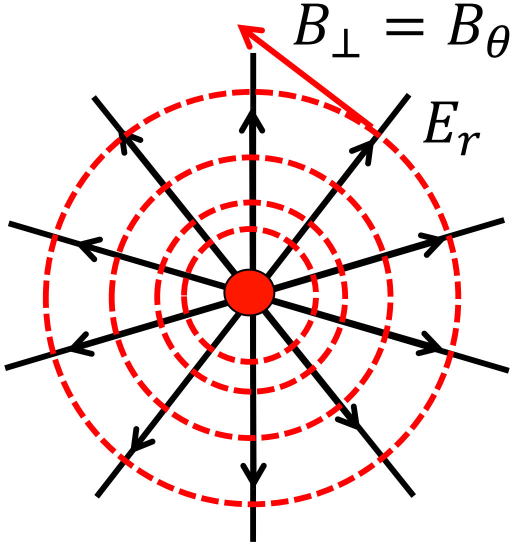
$$B_z = 0$$

$$B_x = -vE_y/c^2$$

$$B_y = vE_x/c^2$$

$$\vec{B}_\perp = \frac{\vec{v}\times\vec{E}}{c^2}$$

$$\boxed{B_\perp = B_\theta = \frac{vE_r}{c^2} = \frac{\beta E_r}{c}}$$



$\gamma >> 1$

Quiz # 1 - 2

What is the source of space charge and wakefields?

1) The conducting material of the beam pipe
2) The beam
3) The external electromagnetic fields of the many accelerator devices (as the electric field in RF cavities and magnetic field in magnets)

Is it possible that a pure magnetic field seen by one observer in one reference system is transformed into an electric and magnetic field in another reference system? (yes, no, you can never tell)

For reflection: are you able to repeat in few words the passages to obtain the electric and magnetic field of a point charge moving with constant velocity? Can we apply the same method to an accelerating charge?

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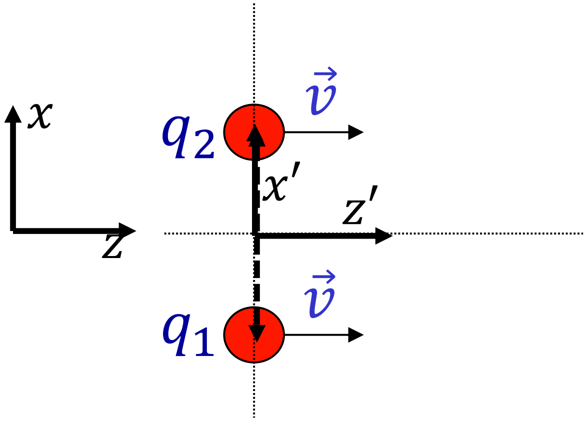
Two point charges with same velocity on parallel trajectories



In the frame S' in which particles are at rest

$$F_r' = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r'^2}$$

In the frame S in which particles are moving we use the relativistic transformations

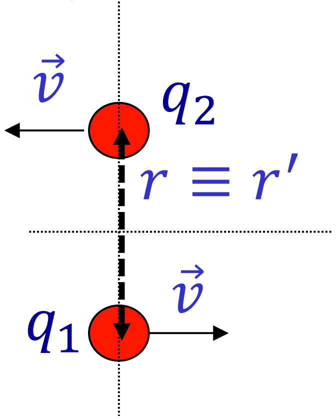
$$\Rightarrow \quad F_r = \frac{F_r'}{\gamma} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\boxed{\gamma} r^2}$$

Two point charges with same velocity on parallel trajectories

The same force can also be obtained considering the Lorentz force acting on a charged particle moving with velocity $\vec{v}$

$$F_r = q_2(E_r - vB_\perp) = q_2\left(E_r - \frac{v^2 E_r}{c^2}\right) = q_2(E_r - \beta^2 E_r) = \frac{q_2}{\gamma^2} E_r = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{\gamma r^2}$$

This force is important in low energy accelerators for space charge effects



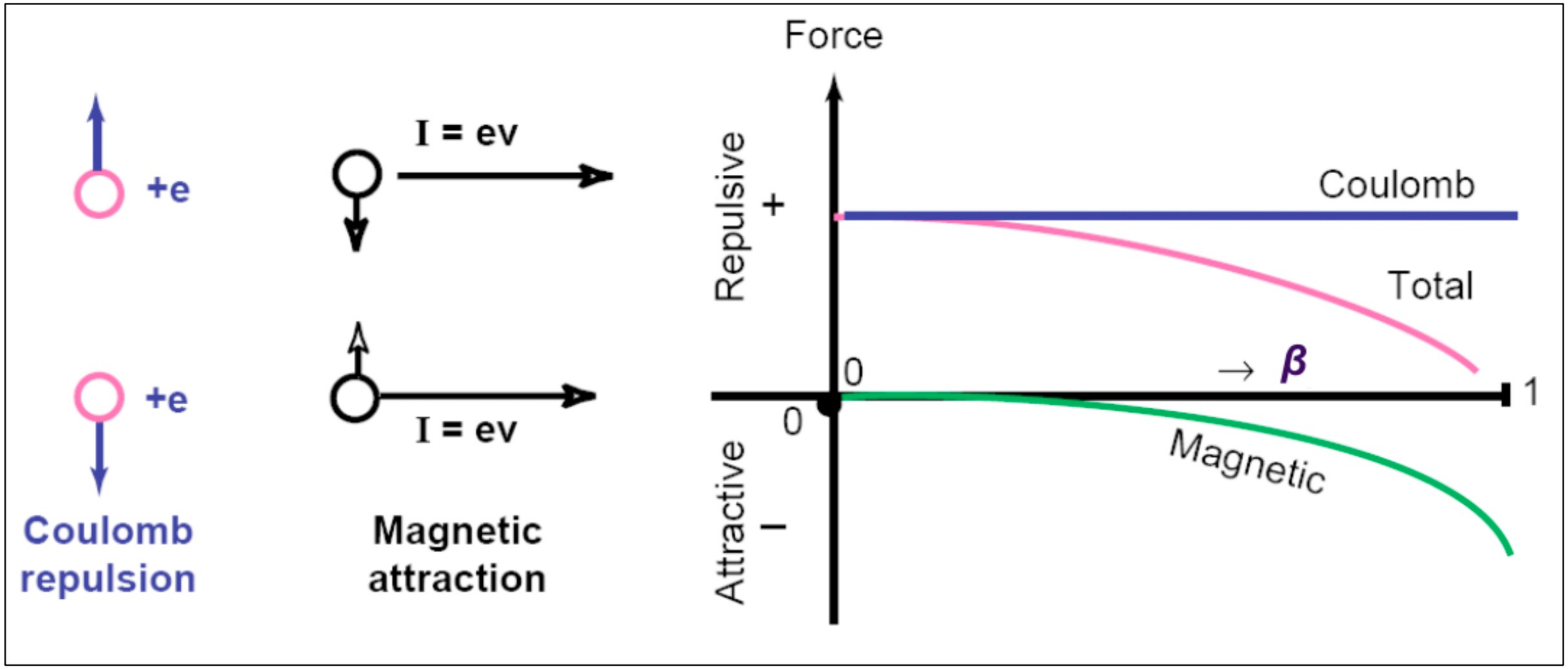
If the two charged particles are counter-rotating:

$$F_r = q_2(E_r + vB_\perp) = q_2\left(E_r + \frac{v^2 E_r}{c^2}\right) = q_2(1 + \beta^2)E_r$$

$$\rightarrow \quad F_r = (1 + \beta^2)\frac{\gamma}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

This force is important for beam beam effects in colliders

Courtesy of E. Métral



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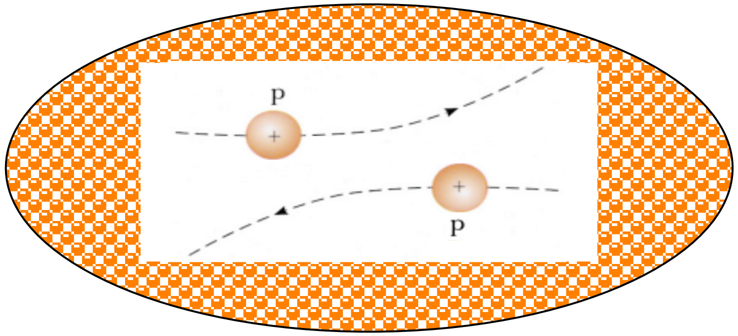
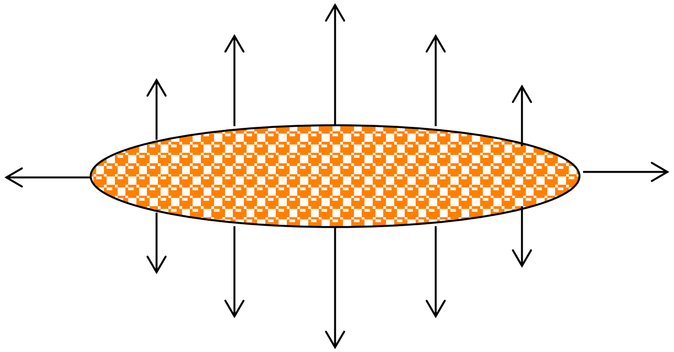
Space Charge

The effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** between particles ==> **Single Particle Effects (e.g. intra-beam scattering, but this is another story …)**



2) **Space Charge Regime** ==> dominated by the **self fields** produced by the entire distribution ==> **Collective Effects**



Collisional and Space Charge regimes

- The interaction of the charged particles in a beam can be represented by the sum of a “collisional” and a “smooth” force. The collisional part of the interaction force arises when a particle “sees” its immediate neighbours and is affected by their individual positions. This force will cause small random displacements of the particle’s trajectory and statistical fluctuations in the particle distribution as a whole. In most practical beams, however, this is a small effect, and the mutual interaction between particles is described largely by a smoothed force.

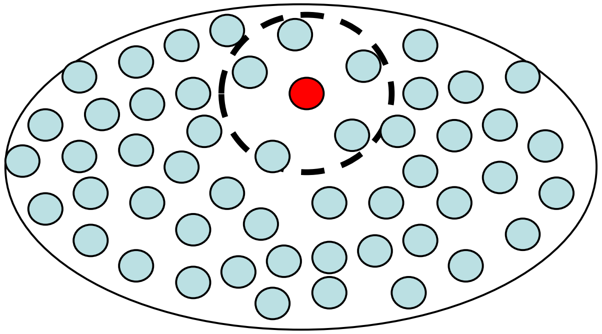
- A measure for the relative importance of collisional versus smoothed interaction, of single-particle versus collective effects, is the ***Debye length,*** λ_D: it is a distance over which a local perturbation in the equilibrium charge distribution of a beam with transverse temperature T and density n, confined by external focusing fields, is screened off.

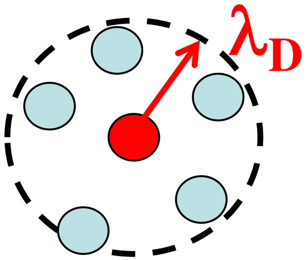
Collisional and Space Charge regimes

If the Debye length is large compared with the beam radius ($\lambda_D \gg a$), the screening will be ineffective and single-particle behaviour will dominate (motion of particles is influenced by local perturbations): collisional regime.

On the other hand, if the Debye length is small compared to the beam radius ($\lambda_D \ll a$), smooth functions for the charge and field distributions can be used, and collective effects due to the self fields of the entire beam will play an important role: space charge regime.

The charges surrounding a test particle have a screening effect at a distance λ_D





$$\boxed{\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{e^2 n}}}$$

k_B = Boltzman constant

T = Temperature

$k_B T$ = average kinetic energy of the particles = $\gamma m_0 \langle v_\perp^2 \rangle$

n = particle density (N/V)

Quiz # 3 - 7

Why have we introduced the Debye length?

1) It represents the maximum distance of action of the space charge force beyond which we can neglect it

2) It allows to understand if collective effects are important over collisions

3) It tells us what is the force between two particles with the same velocity

In simulation codes taking into account collective effects, do we must include the collisions between particles? (Yes always, no never, it depends)

The space charge force is more important:

1) At low energy

2) At high energy

3) It is independent on the beam energy

4) I prefer not to respond

Let’s consider the force acting on one of the two charges moving with the same velocity in the same direction. If two observers are measuring this force in the reference system S, where we see the charges moving with constant velocity, and in S’, where the charges are at rest, do they measure the same amplitude of the force?

(Yes, no, I don’t know)

A bunch of particles with the same type of charge does not ‘explode’ in a particle accelerator due to the Coulomb forces because:

1) The space charge (Coulomb) forces are always negligible
2) Focusing forces, such as those due to the quadrupoles, help to counteract the space charge defocusing effect
3) Counter rotating beam focuses the bunch
4) I’m too tired to answer this question

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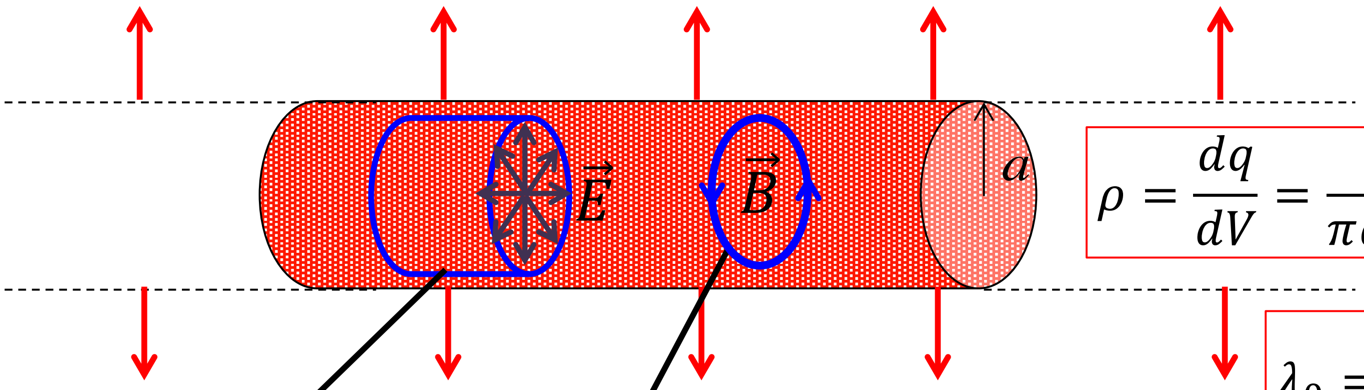
Example 1. Relativistic Uniform Cylindrical Beam

$$\boxed{J = \frac{I}{\pi a^2}}$$

$$\boxed{\rho = \frac{dq}{dV} = \frac{dq}{\pi a^2 dl} = \frac{dq/dt}{\pi a^2 dl/dt} = \frac{I}{\pi a^2 v}}$$

$$\boxed{\lambda_0 = \frac{dq}{dl} = \frac{dq/dt}{dl/dt} = \frac{I}{v} = \pi a^2 \rho}$$



Gauss's law

$$\varepsilon_0 \int \vec{E} \cdot d\vec{S} = \int \rho dV$$

Linear

$$2\pi r l \varepsilon_0 E_r = \rho \pi r^2 l$$

$$\boxed{E_r = \frac{\rho r}{2\varepsilon_0} = \frac{\lambda_0 r}{2\pi\varepsilon_0 a^2} \qquad \text{for } r \le a}$$

Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

$$\boxed{B_\theta = \frac{\beta}{c} E_r}$$

$$2\pi r B_\theta = \mu_0 J \pi r^2$$

$$\boxed{B_\theta = \mu_0 \frac{Jr}{2} = \frac{1}{\varepsilon_0 c^2} \frac{v \lambda_0 r}{2\pi a^2} = \frac{\beta}{c} \frac{\lambda_0 r}{2\pi\varepsilon_0 a^2} \qquad \text{for } r \le a}$$

$$\boxed{E_r = \frac{\rho r}{2\varepsilon_0} = \frac{\lambda_0 r}{2\pi\varepsilon_0 a^2}}$$

$$\boxed{B_\theta = \frac{\beta}{c} E_r}$$

The combination of these fields on a moving charge gives the

Lorentz Force

$$\boxed{F_r(r) = e(E_r - vB_\theta) = e(1-\beta^2)E_r = \frac{eE_r(r)}{\gamma^2}}$$

- has only **radial** component
- is a **linear** function of the transverse coordinate

The attractive magnetic force, which becomes significant at high velocities, tends to compensate the repulsive electric force. Therefore, space charge defocusing is primarily a non-relativistic effect.

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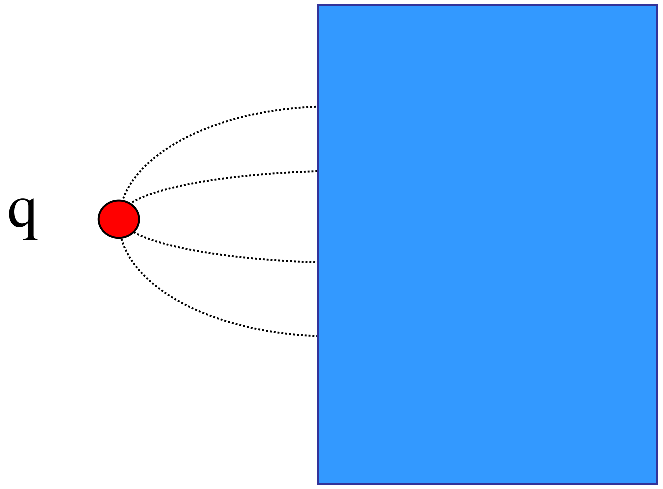
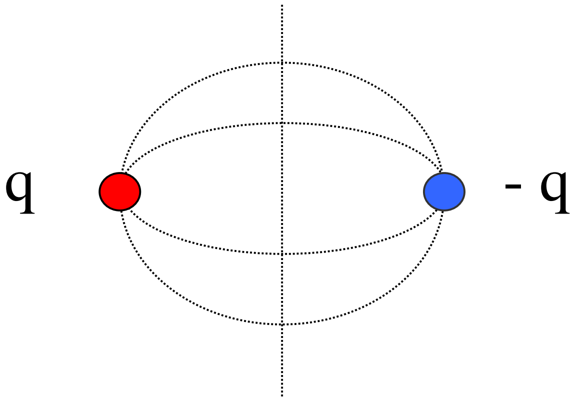
Static Fields: conducting or magnetic screens

Let us consider a point charge q close to a **conducting screen**.

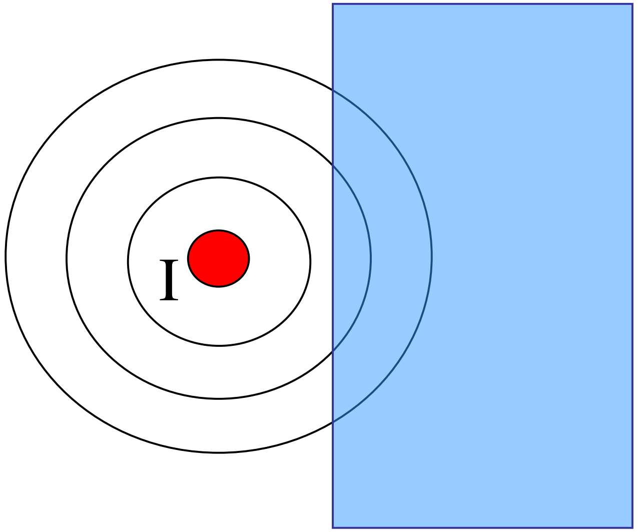
The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen





A constant current in the free space produces a circular magnetic field.

If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.

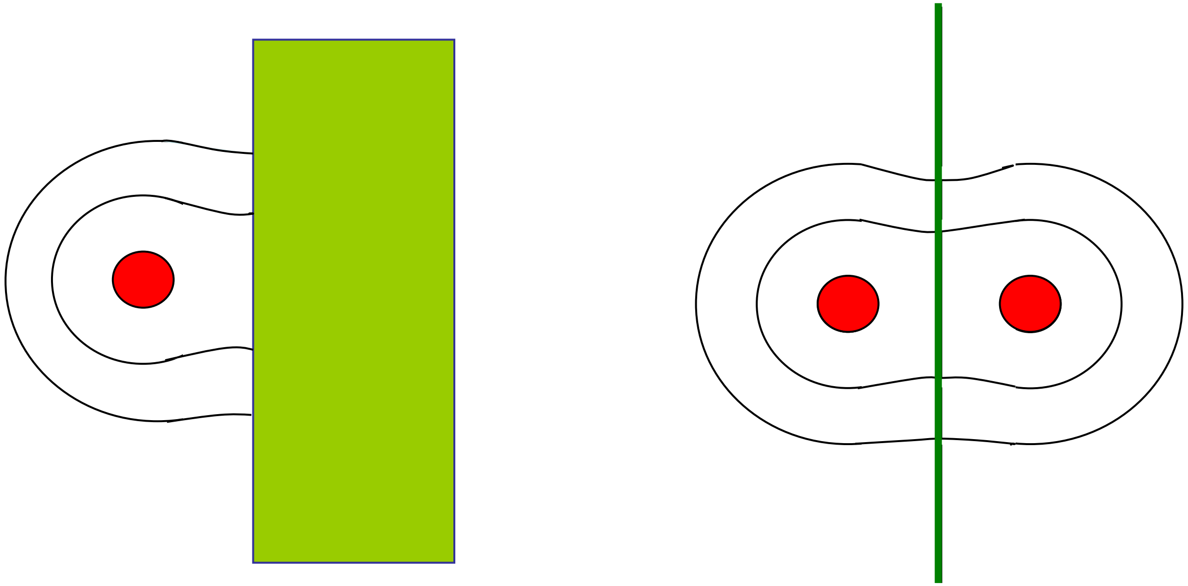


For **ferromagnetic materials**, with **$\mu_r>>1$**, the very high magnetic permeability makes the tangent magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.

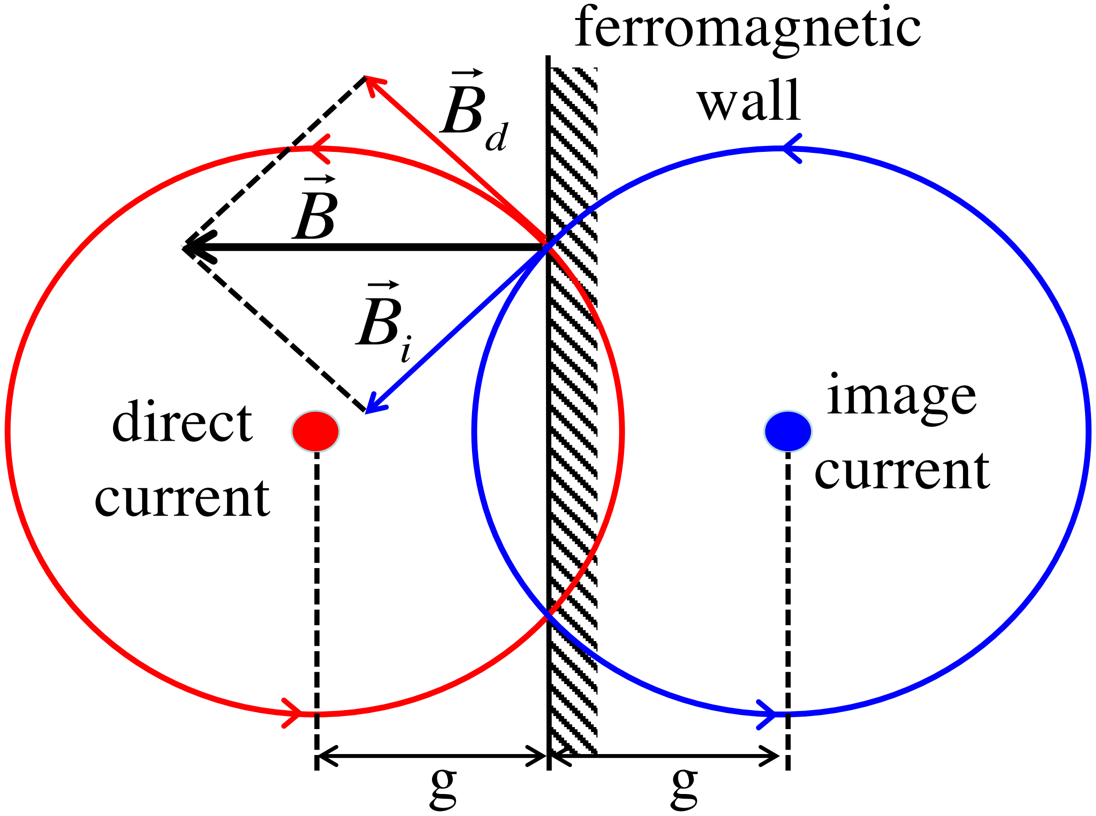
Law of refraction of magnetic field lines:

$$\frac{\tan\theta_1}{\tan\theta_2}=\frac{\mu_{r1}}{\mu_{r2}}$$

$$\boxed{\mu_{r2}\to\infty \xrightarrow{\text{yields}} \theta_1\to 0}$$



In analogy with the image method we get the magnetic field, in the region outside of the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.



Satisfying a magnetic boundary condition by an image current.

Time-varying fields

Static electric fields vanish inside a conductor for any finite conductivity, while magnetic fields pass through unless of high permeability.

This is no longer true for time changing fields, which can penetrate inside the material in a region characterized by a quantity called skin depth δ_w. Inside the conducting material we write the following Maxwell's equations:

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \end{cases}$$

$$\begin{cases} \vec{B} = \mu\vec{H} \\ \vec{D} = \varepsilon\vec{E} \\ \vec{J} = \sigma\vec{E} \end{cases} \Rightarrow$$

Constitutive relations

Copper $\sigma = 5.8\ 10^{7}\ (\Omega\text{m})^{-1}$
Aluminium $\sigma = 3.5\ 10^{7}\ (\Omega\text{m})^{-1}$
Stainless steel $\sigma = 1.4\ 10^{6}\ (\Omega\text{m})^{-1}$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad\qquad \nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

Consider a plane wave (H_y , E_x) propagating in the material

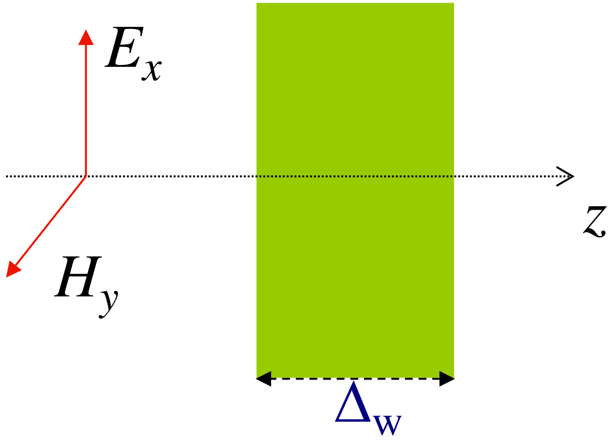
$$\frac{\partial^2 E_x}{\partial z^2} - \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2} - \mu\sigma\frac{\partial E_x}{\partial t} = 0$$

(the same equation holds for H_y). Assuming that fields propagate in the z-direction with the law:

$$E_x = \tilde{E}_0 e^{i(kz-\omega t)}$$

$$H_y = \tilde{H}_0 e^{i(kz-\omega t)}$$

$$(-k^2 + \varepsilon\mu\omega^2 + i\sigma\mu\omega)\tilde{E}_0 e^{i(kz-\omega t)} = 0$$

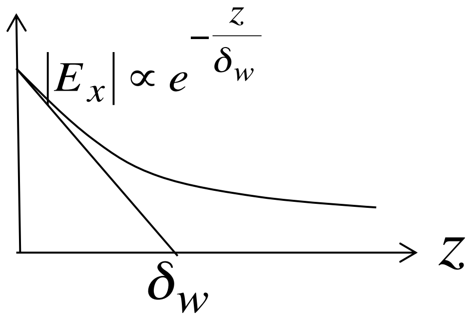


We say that the material behaves like a conductor if $\sigma \gg \omega\varepsilon$ thus:

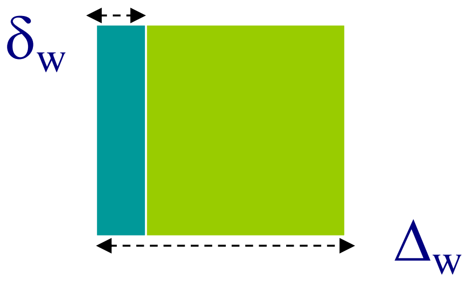
$$k \simeq (1+i)\sqrt{\frac{\sigma\mu\omega}{2}} \qquad \mathrm{Im}[k] = \sqrt{\frac{\sigma\mu\omega}{2}}$$

⇨ Exponential decay

Fields propagating along “z” are attenuated.
The attenuation constant measured in meters is called skin depth δ_w:



$$\delta_w = \frac{1}{\mathrm{Im}[k]} = \sqrt{\frac{2}{\sigma\mu\omega}}$$

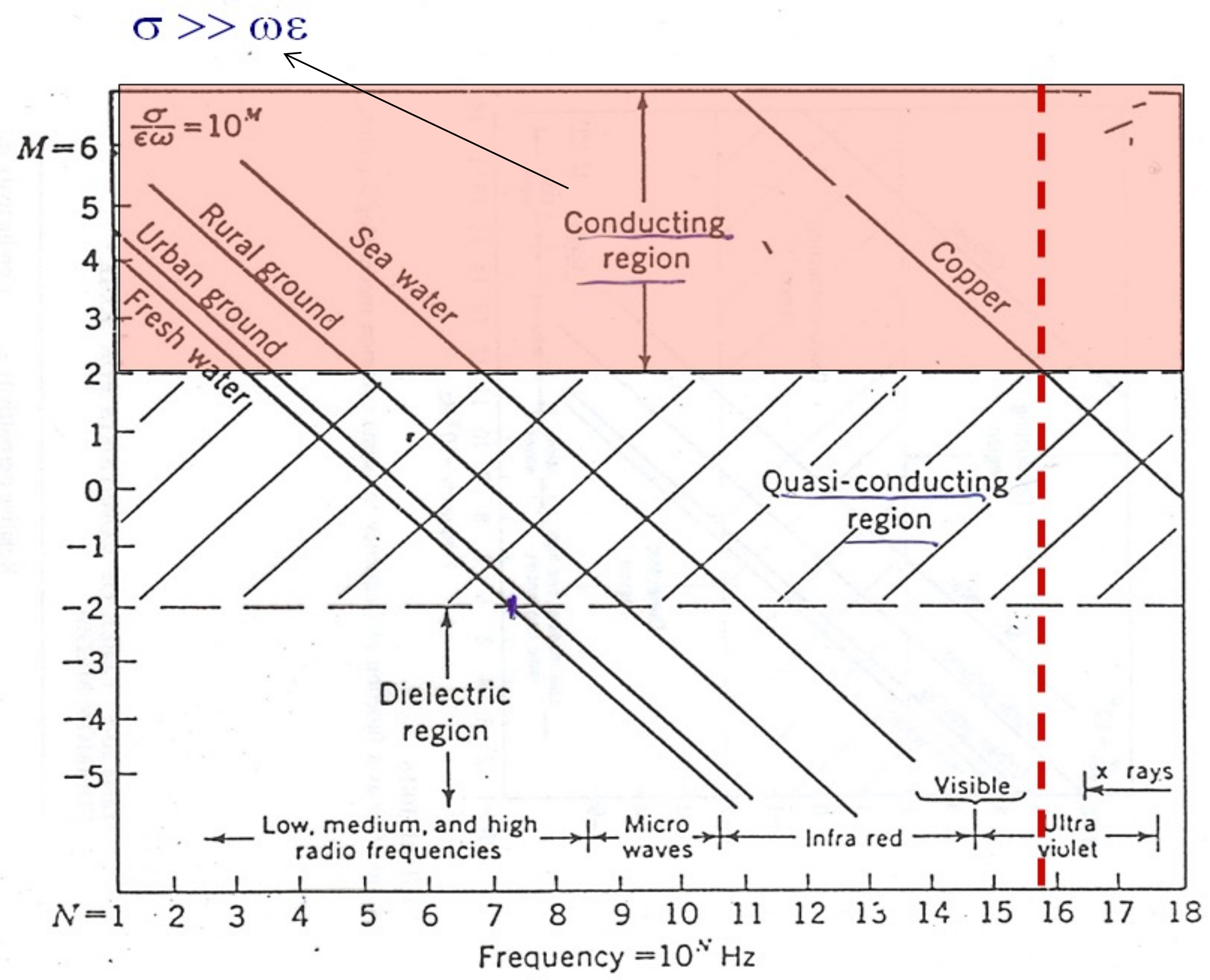


The skin depth depends on the material properties and on the frequency. Fields pass through the conductor wall if the skin depth is larger than the wall thickness Δ_w. This happens at relatively low frequencies.

At higher frequencies, for a good conductor $\delta_w << \Delta_w$ and both electric and magnetic fields vanish inside the wall.

For the copper $$\delta_w \simeq \frac{6.6}{\sqrt{f\,(\mathrm{Hz})}}\ (\mathrm{cm}); \quad \omega = 2\pi f$$

For a pipe 2mm thick, the fields pass through the wall up to 1 kHz.
(Skin depth of Aluminium is larger by a factor 1.28)



Ratio σ/ωε as a function of frequency f for some common media (log-log plot)

Note that copper behaves like a conductor at frequencies far above the microwave region. On the other hand, fresh water acts like a dielectrics at frequencies above about 10MHz

$$M = \log\frac{\sigma}{\varepsilon\omega} =$$

$$= \log\frac{\sigma}{\varepsilon 2\pi} - \log f$$

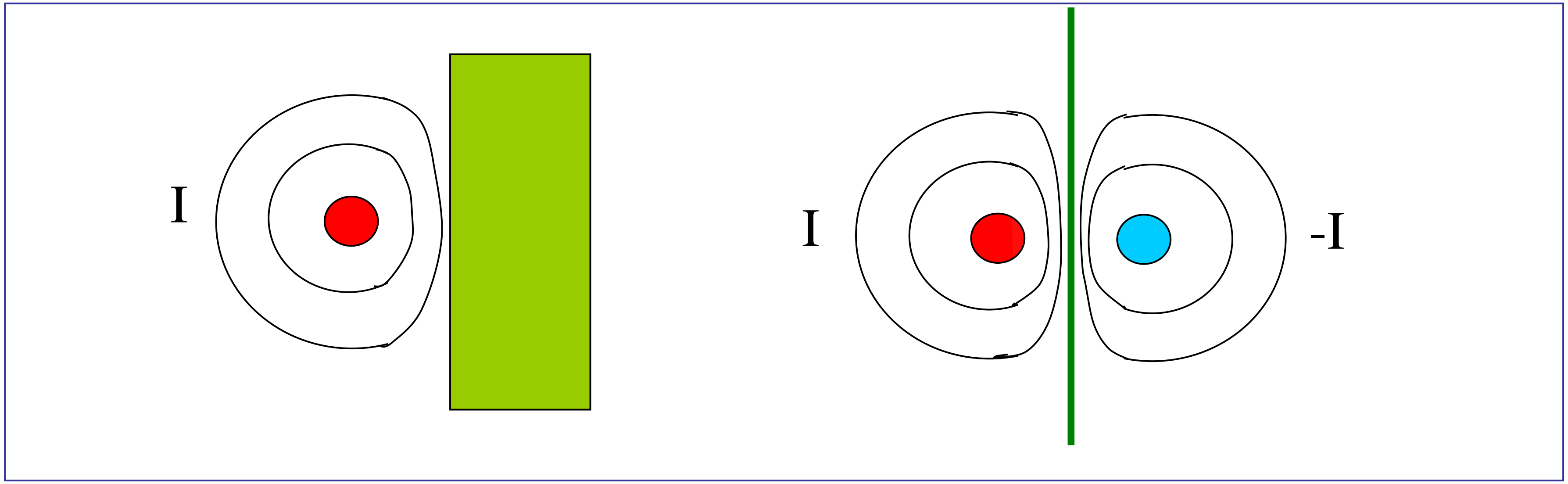
$$= \log\frac{\sigma}{\varepsilon 2\pi} - N$$

Method to use for time-varying fields

- Compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

- If the fields penetrate and pass through the material, they can interact with bodies in the outer region.

- If the skin depth is very small (rapidly varying fields), fields do not penetrate, the electric field lines are perpendicular to the wall, as in the static case, while the magnetic field lines are tangent to the surface.



Quiz # 8 - 10

When is it convenient to use the method of image charges?

1) Any time we have a conducting material
2) When we have two or more particles, as in a beam
3) When we have a charge close to a conducting plane
4) I don't know what the method of image charges is

Can a uniform, infinite beam in the free space produce an electromagnetic wave?

1) Yes, always
2) No, since it only produces an electric field
3) No, since it produces static electric and magnetic fields

The skin depth

1) Exists only in a perfect conductor
2) It depends on the frequency of the electromagnetic wave
3) It is a quantity of no interest for a particle accelerator

CONTENTS (SPACE CHARGE)

- Introduction, self induced fields
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- Boundary conditions: image charges and currents
- Examples: beam in a circular pipe, in parallel plates
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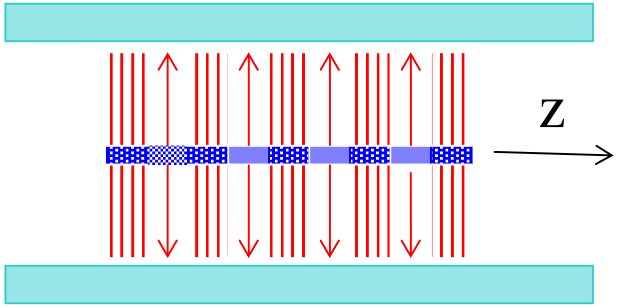
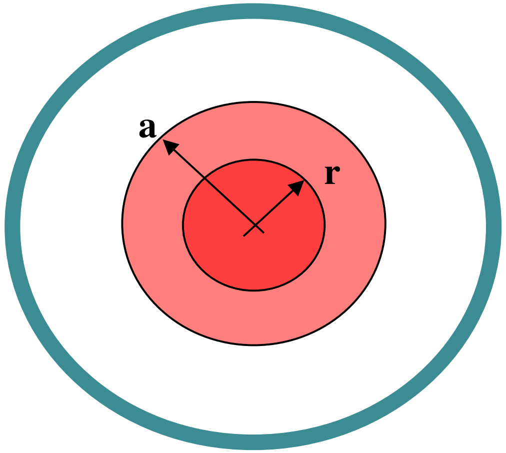
Example 2: Circular Perfectly Conducting Pipe

(Uniform Beam at Center)



If we take the previous uniform cylindrical beam and enclose it into a cylindric perfectly conducting pipe, the field lines are not perturbed because the electric ones are already radial and then perpendicular to the pipe, and the magnetic ones remain circular. The presence of the pipe does not affect the fields.

In the case of cylindrical charge distribution, with γ>>1, the electric field lines can be considered perpendicular to the direction of motion. The intensities of the transverse fields can be computed as in the static case, applying the Gauss's and Ampere's laws.



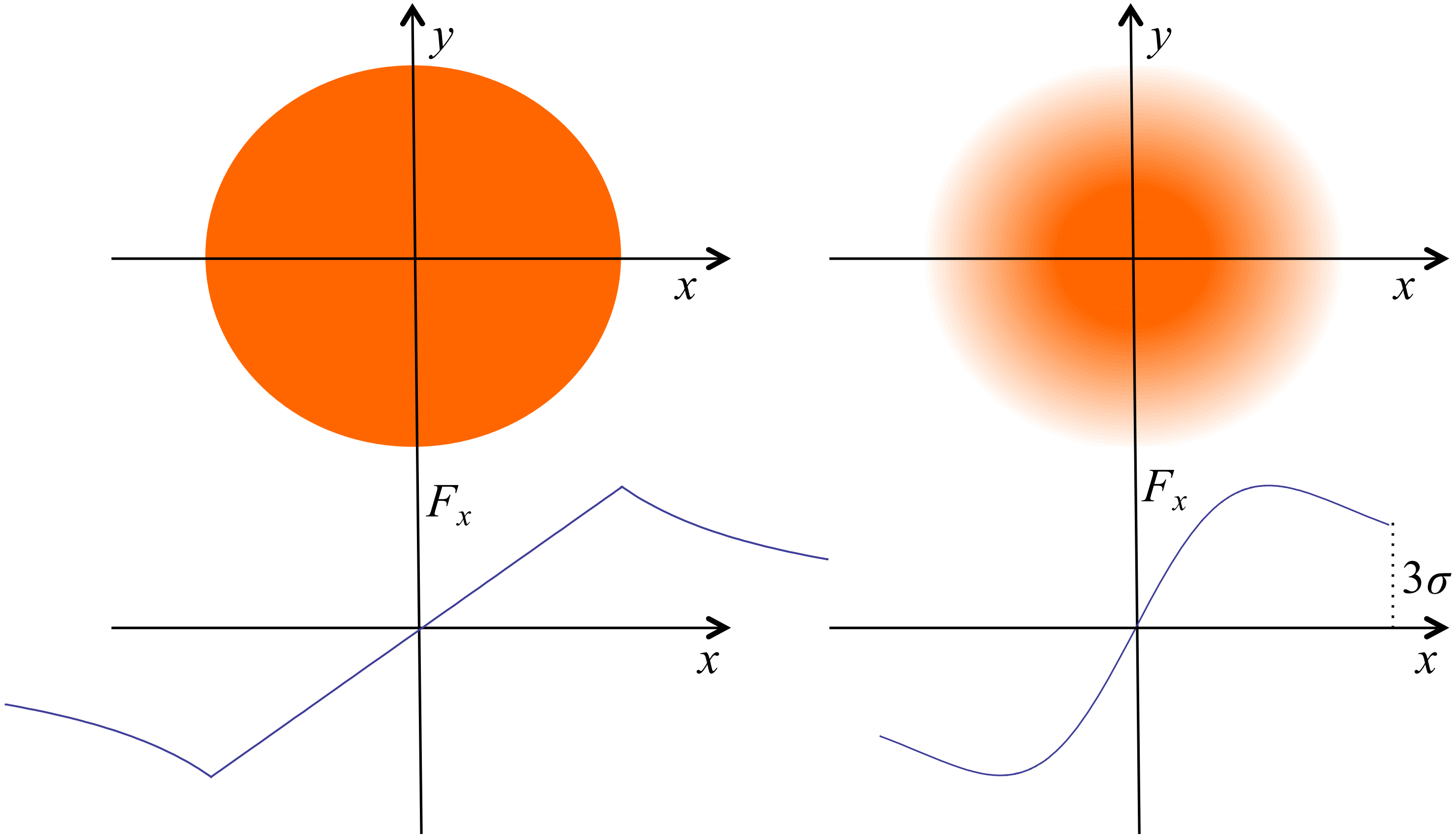
$$E_r = \frac{\lambda_0}{2\pi\varepsilon_0 a^2} r; \qquad B_\theta = \frac{\beta}{c} E_r = \frac{\lambda_0 \beta}{2\pi\varepsilon_0 a^2 c} r$$

$$F_r(r) = e(E_r - vB_\theta) = \frac{e}{\gamma^2} \frac{\lambda_0}{2\pi\varepsilon_0 a^2} r$$

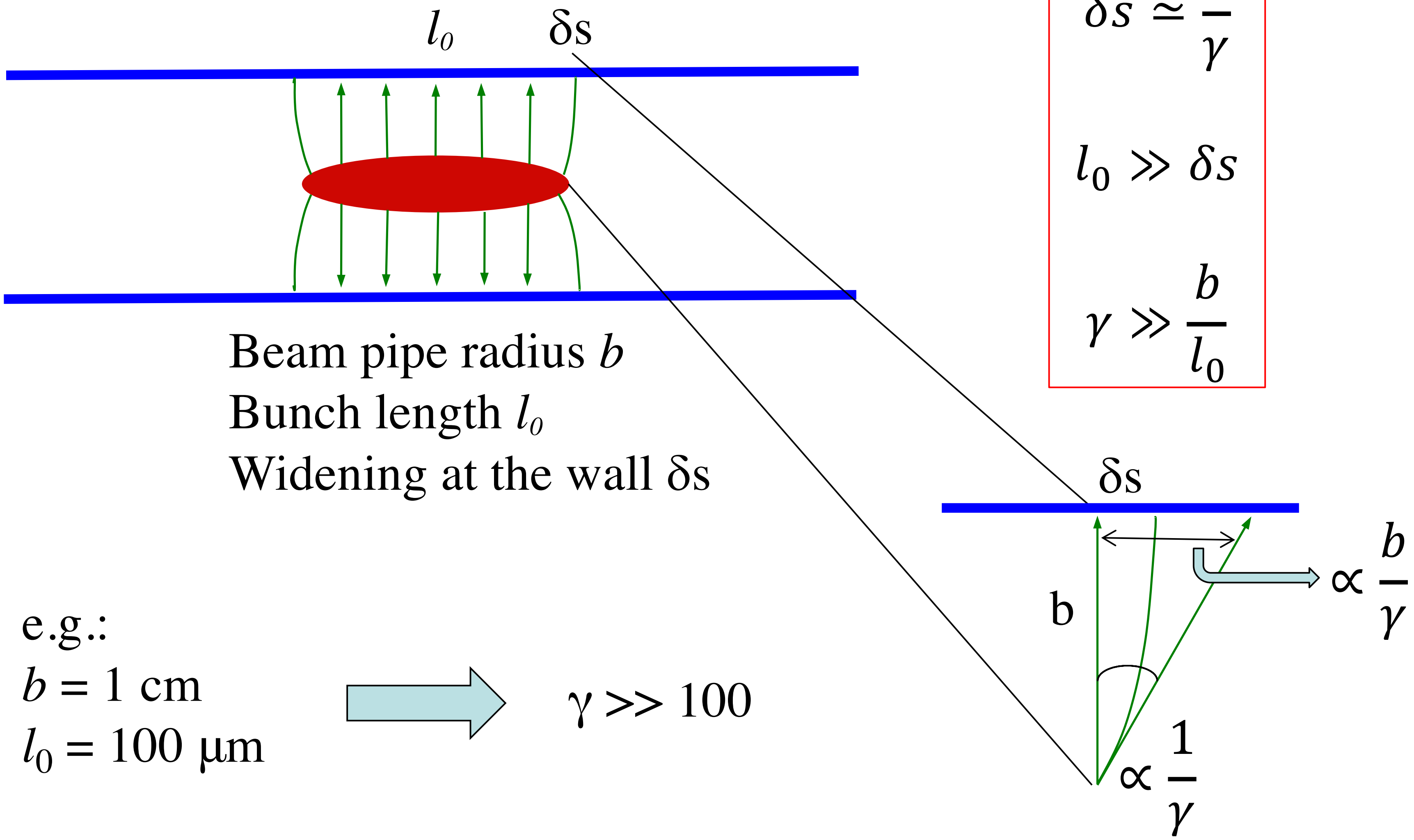
This direct space charge force does not depend on the longitudinal position along the beam. If λ is not constant, one should consider the local charge density λ(z) (some examples in the exercises).

If the transverse distribution is not uniform, we can still apply Gauss's and Ampere's laws (example in the exercises).

Defocusing transverse self-induced forces produced by direct space charge in case of uniform (left) and Gaussian (right) distributions.



Relativistic Uniform Cylindrical Beam – finite length



$$\delta s \simeq \frac{b}{\gamma}$$

$$l_0 \gg \delta s$$

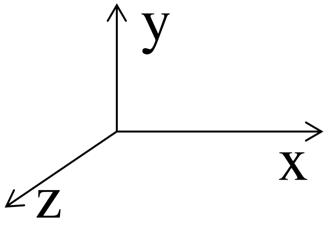
$$\gamma \gg \frac{b}{l_0}$$

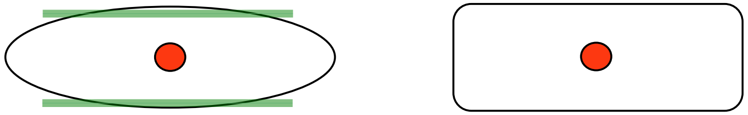
Beam pipe radius *b*
Bunch length l_0
Widening at the wall δs

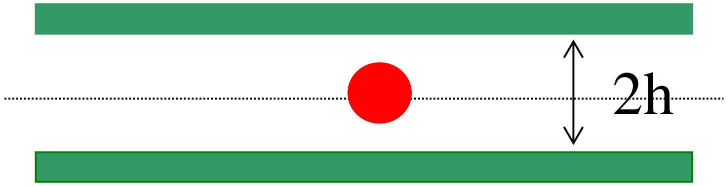
e.g.:
$b = 1$ cm
$l_0 = 100$ μm

⇒ $\gamma >> 100$

Parallel Plates (Beam at Center)

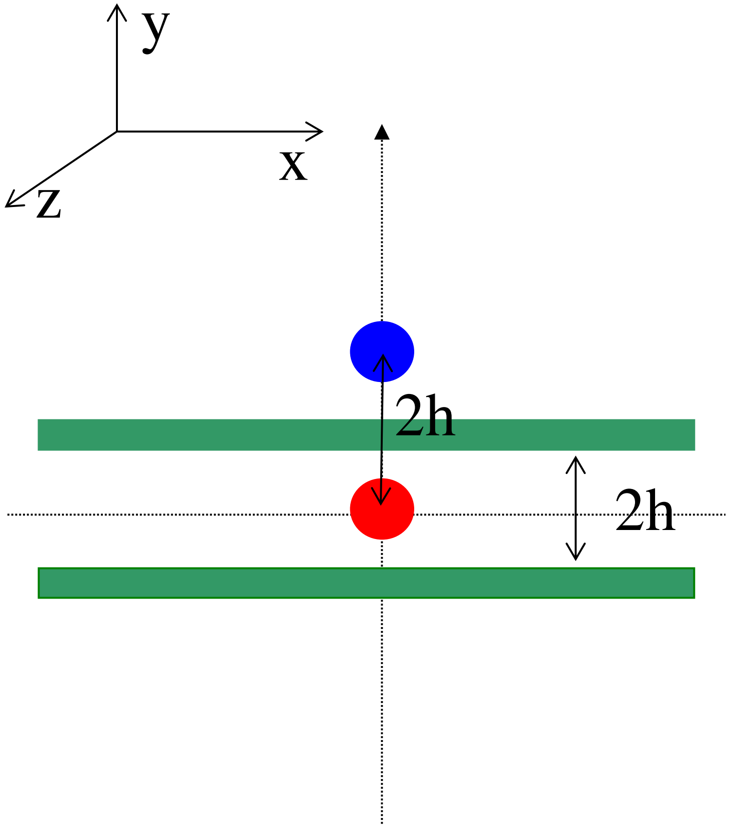


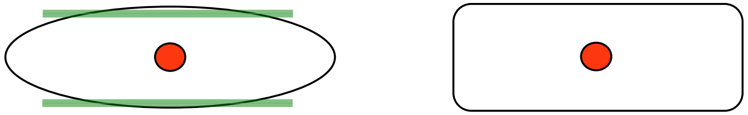




In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius a between two conducting plates $2h$ apart. By applying the superposition principle, we get the total image field at a position y inside the beam.

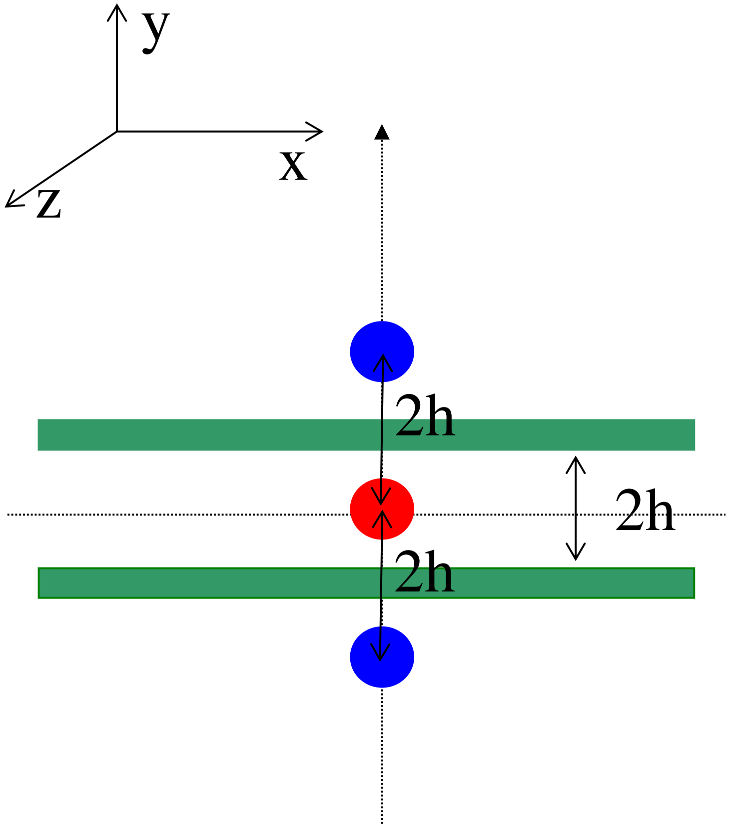
Parallel Plates (Beam at Center)

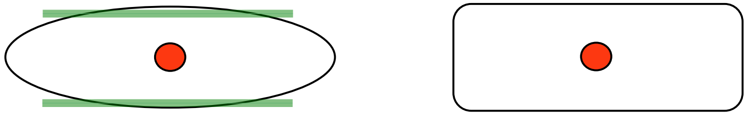




In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle, we get the total image field at a position *y* inside the beam.

Parallel Plates (Beam at Center)





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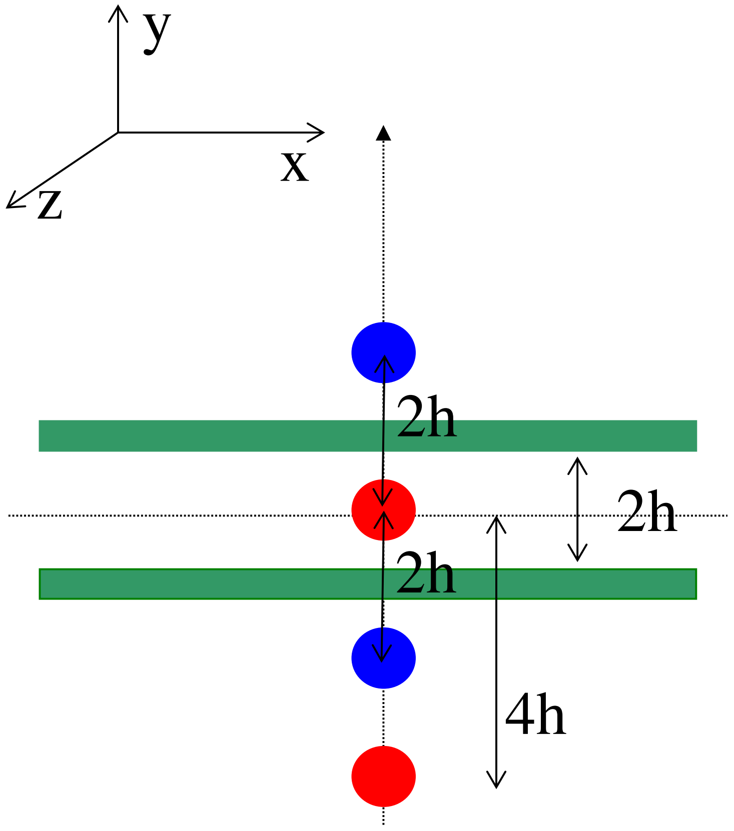

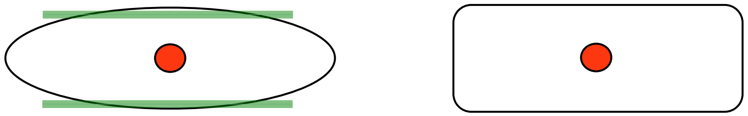

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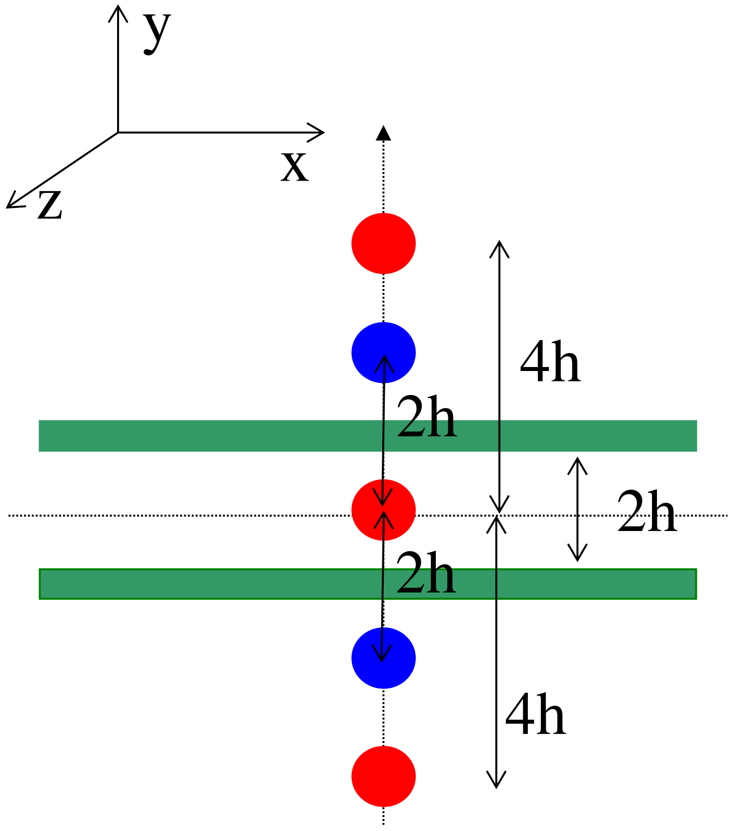
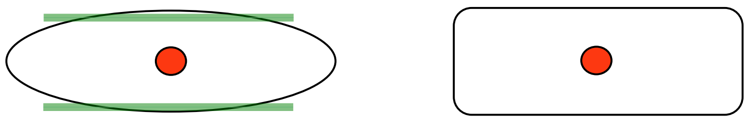
Parallel Plates (Beam at Center)





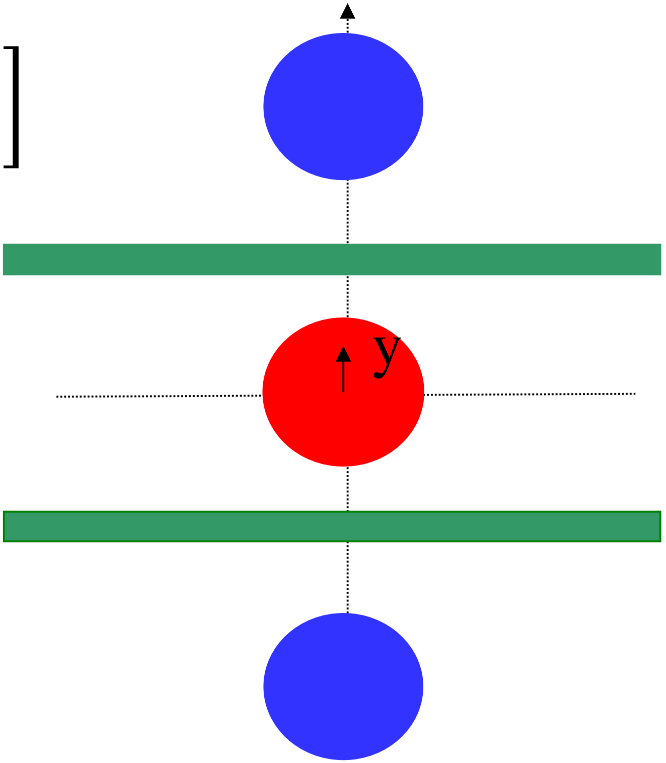
In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle, we get the total image field at a position *y* inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_0}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_0}\sum_{n=1}^{\infty}(-1)^n\frac{-2y}{(2nh)^2 - y^2}$$

$$E_y^{im}(z,y) \simeq \frac{\lambda(z)}{4\pi\varepsilon_0 h^2}\frac{\pi^2}{12}y$$

Where we have assumed h $\gg a \geq y$.



For d.c. or slowly varying currents, the boundary conditions imposed by the conducting plates do not affect the magnetic field.

There is no magnetic field which can compensate the electric field due to the "image" charges.

$$\boxed{F_y(y) = \frac{e}{\gamma^2}E_y^{dir} + eE_y^{im} = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{y}{a^2} + \frac{e\lambda(z)}{4\pi\varepsilon_0 h^2}\frac{\pi^2}{12}y}$$

From the divergence equation $\left(\nabla \cdot \vec{E}^{im} = \frac{\rho^{im}}{\varepsilon_0} = 0\right)$ we derive also the other transverse component:

$$\frac{\partial}{\partial x} E_x^{im} = -\frac{\partial}{\partial y} E_y^{im} \rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\varepsilon_0 h^2} \frac{\pi^2}{12} x$$

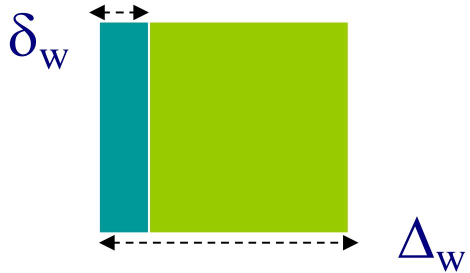
Including also the direct space charge force, we get:

$$F_x(z,x) = \frac{e\lambda(z)x}{\pi\varepsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$F_y(z,y) = \frac{e\lambda(z)y}{\pi\varepsilon_0}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

Parallel Plates (Beam at Center) a.c. currents



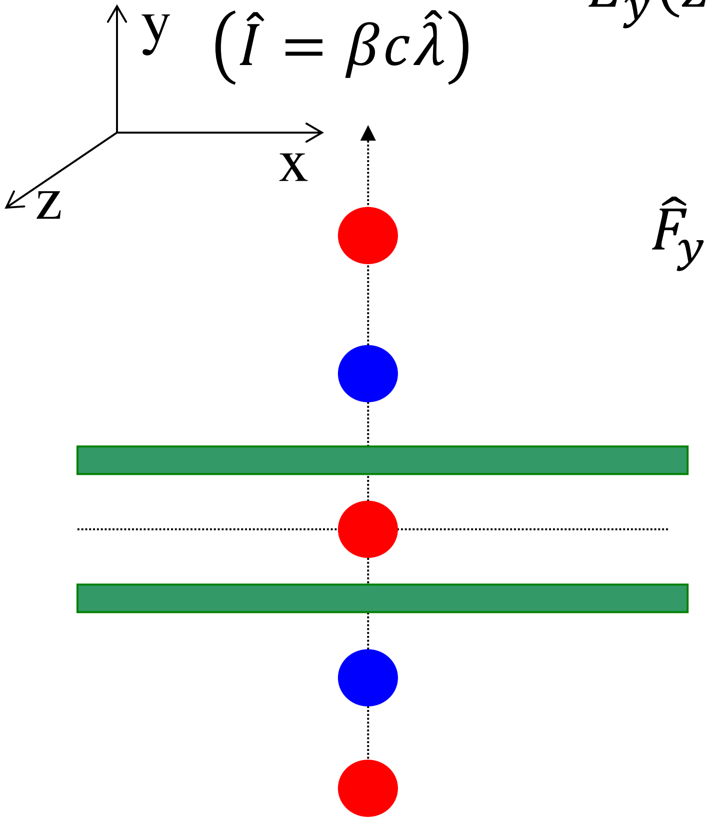
Usually, the frequency spectrum of a beam is quite rich of harmonics, especially for bunched beams.

To simplify our study it is convenient to decompose the current into a d.c. component, I, and an a.c. component, Î, for which we consider $\delta_w << \Delta_w$.

The d.c. component of the magnetic field is not affected by the presence of the material, and only the 'image' electric field must be considered.

The a.c. component of the magnetic field must be tangent to the pipe wall, and it can be obtained by using an infinite sum of image currents with alternating directions as we did for the electric field.

We can see that this magnetic field is able to cancel the effect of the electric force.



$$\left(\hat{I} = \beta c \hat{\lambda}\right)$$

$$\hat{E}_y(z,y) = \frac{\hat{\lambda}(z)y}{\pi\varepsilon_0}\frac{\pi^2}{48h^2}; \qquad \hat{B}_x(z,y) = -\frac{\beta}{c}\hat{E}_y(z,y)$$

$$\hat{F}_y(z,y) = e\hat{E}_y(z,y)\left(1 - v\frac{\beta}{c}\right) = \frac{e\hat{\lambda}(z)y}{\pi\varepsilon_0\gamma^2}\frac{\pi^2}{48h^2}$$

$$\hat{F}_y(z,y) = \frac{e\hat{\lambda}(z)y}{\pi\varepsilon_0\gamma^2}\left(\frac{1}{2a^2} + \frac{\pi^2}{48h^2}\right)$$

$$\hat{F}_x(z,x) = \frac{e\hat{\lambda}(z)x}{\pi\varepsilon_0\gamma^2}\left(\frac{1}{2a^2} - \frac{\pi^2}{48h^2}\right)$$

For the a.c. current there is cancellation of the electric and magnetic forces.

Quiz # 11 - 13

Is there an indirect space charge effect for an infinite cylindrical beam in the centre of a circular, perfectly conducting pipe? (yes always, no never, it depends on the beam energy)

When is the direct space charge force linear with the distance from beam centre?

1) When the transverse beam distribution is uniform

2) Only very close to the centre, even with uniform transverse distribution

3) We did not afford this question in the lecture

1) Direct space charge force in free space for a dc beam
2) Indirect space charge force in free space for a dc beam
3) Direct space charge force in a circular perfectly conducting pipe for a dc beam
4) Indirect space charge force in a circular perfectly conducting pipe for a dc beam
5) Indirect space charge force in parallel plates for a dc beam

6) - 10): as 1) - 5) for the ac component of a bunched beam

Up to this point of the lecture which points have we studied?

Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, considering also the presence of ferromagnetic materials in dipoles, we can write the expression of the force as:

$$F_u = \frac{e}{\pi\varepsilon_0}\left[\frac{1}{\gamma^2}\left(\frac{1}{2a^2} \mp \frac{\pi^2}{48h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{48h^2} + \frac{\pi^2}{24g^2}\right)\bar{\lambda}\right]u$$

where λ is the total current divided by βc, $\overline{\lambda}$ its d.c. part, g the gap in a dipole, and we take the sign (+) if u=y, and the sign (–) if u=x.

It is interesting to note that these forces are linear in the transverse displacement x and y.

-L. J. Laslett, LBL Document PUB-616, 1987, vol III

Space Charge Force - General expression

One often finds the space charge force written in terms of the Laslett form factors f_0, f_1 and f_2

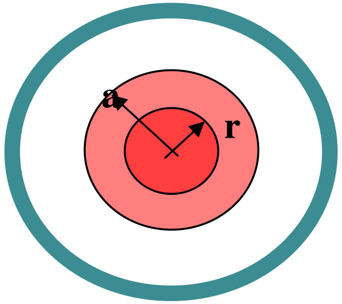
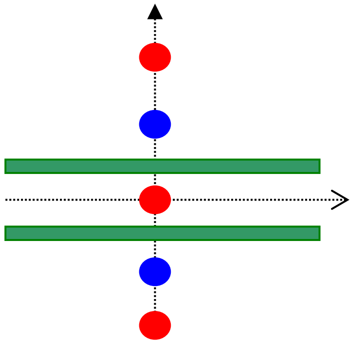
$$F_u = \frac{e}{\pi\varepsilon_0}\left[\frac{1}{\gamma^2}\left(\frac{f_0}{a^2} \mp \frac{f_1}{h^2}\right)\lambda \mp \beta^2\left(\frac{f_1}{h^2} + \frac{f_2}{g^2}\right)\bar{\lambda}\right]u$$

where the Laslett form factors have been obtained for different pipe geometries (as elliptical and rectangular, with more complicated reasoning).

For example, for our case of parallel plates, we have:

$$f_0 = 1/2, \quad f_1 = \pi^2/48, \quad f_2 = \pi^2/24$$

$$\lambda(z) = \lambda_0 + \hat{\lambda}\cos(k_z z)\,; \qquad k_z = 2\pi/l_w$$

	D.C.	A.C. $(\delta_w \ll \Delta_w)$
	$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$	
	$$F_x(z,x) = \frac{e\lambda(z)x}{2\pi\varepsilon_0}\left(\frac{1}{a^2\gamma^2} - \frac{\pi^2}{24h^2}\right)$$ $$F_y(z,y) = \frac{e\lambda(z)y}{2\pi\varepsilon_0}\left(\frac{1}{a^2\gamma^2} + \frac{\pi^2}{24h^2}\right)$$	$$\hat{F}_x(z,x) = \frac{e\hat{\lambda}(z)x}{2\pi\varepsilon_0\gamma^2}\left(\frac{1}{a^2} - \frac{\pi^2}{24h^2}\right)$$ $$\hat{F}_y(z,y) = \frac{e\hat{\lambda}(z)y}{2\pi\varepsilon_0\gamma^2}\left(\frac{1}{a^2} + \frac{\pi^2}{24h^2}\right)$$

Space charge effects in circular accelerators

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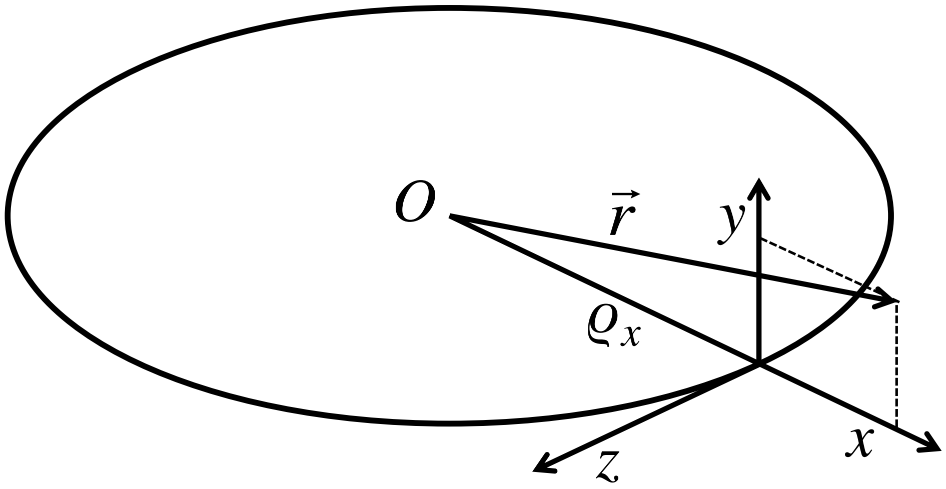
Self fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have:

$$\frac{d(m\gamma\vec{v})}{dt} = \vec{F}^{ext}(\vec{r}) + \vec{F}^{self}(\vec{r}) \qquad \frac{d(\vec{v})}{dt} = \frac{\vec{F}^{ext}(\vec{r}) + \vec{F}^{self}(\vec{r})}{m\gamma}$$



Following the same steps of the "transverse dynamics" lectures, we write:

$$\vec{r} = (\rho_x + x)\hat{e}_x + y\hat{e}_y$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \omega_0(\rho_x + x)\hat{e}_z$$

$$\vec{a} = [\ddot{x} - \omega_0^2(\rho_x + x)]\hat{e}_x + \ddot{y}\hat{e}_y + [\dot{\omega}_0(\rho_x + x) + 2\omega_0\dot{x}]\hat{e}_z$$

For the motion along x:

$$\ddot{x} - \omega_0^2(\rho_x + x) = \frac{1}{m_0\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

which is rewritten with respect to the azimuthal position $s = v_z t$:

$$\ddot{x} = v_z^2 x'' = \omega_0^2(\rho_x + x)^2 x''$$

$$x'' - \frac{1}{\rho_x + x} = \frac{1}{m_0 v_z^2 \gamma}\left(F_x^{ext} + F_x^{self}\right)$$

We assume a small transverse displacement *x* so that:

$$x'' - \frac{1}{\rho_x} + \frac{1}{\rho_x^2} x = \frac{1}{m_0 v_z^2 \gamma} \left(F_x^{ext} + F_x^{self} \right)$$

The external force is due to the magnetic guiding fields. We suppose to have only dipoles and quadrupoles, or, equivalently, we expand the external guiding fields in a Taylor series up to the quadrupole component:

$$-F_x^{ext} = q v_z B_y = q v_z B_{y0} + q v_z \left(\frac{\partial B_y}{\partial x} \right) x + \cdots$$

the dipolar magnetic field B_{y0} is responsible of the circular motion along the reference trajectory of radius ρ_x according to the equation :

$$q v_z B_{y0} = \frac{m_0 \gamma v_z^2}{\rho_x}$$

We finally get:

$$x'' + \left[\frac{1}{\rho_x^2} + \frac{q}{m_0 v_z \gamma}\left(\frac{\partial B_y}{\partial x}\right)\right]x = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}$$

which can also be written as:

$$x'' + \left[\frac{1}{\rho_x^2} - k\right]x = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}$$

where we have introduced the normalized gradient

$$k = \frac{g}{p/q} = -\frac{q}{m_0 v_z \gamma}\left(\frac{\partial B_y}{\partial x}\right)$$

with g the quadrupole gradient in [T/m] and p the charge momentum

Both the curvature radius and the normalized gradient depend on the azimuthal position ‘*s*’. By using the focusing constant $K_x(s)$ we then can write:

$$x''(s) + K_x(s)x(s) = \frac{1}{m_0 v_z^2 \gamma} F_x^{self}(x, s)$$

In absence of self fields, the solution of the free equation, known as Hill’s equation gives the betatron oscillations.

Putting $v_z = \beta_z c \simeq \beta c$ (small beam divergence), we get

$$x''(s) + K_x(s)x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x, s)$$

where E_0 is the particle energy.

- In the analysis of the motion of the particles in presence of the self fields, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

- This is the case for which the focusing term is constant along the machine. Although this condition is never fulfilled in a real accelerator, it provides a reliable model for the description of the effects due to the self fields .

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x,s)$$

Free betatron motion:

$$x''(s) + K_x x(s) = 0$$

$$\begin{cases} x(s) = A_x \cos\left[\sqrt{K_x} s - \varphi_x\right] \\ \lambda_\beta = \dfrac{2\pi}{\sqrt{K_x}} \\ Q_{x0} = \dfrac{2\pi\rho_x}{\lambda_\beta} = \dfrac{2\pi\rho_x\sqrt{K_x}}{2\pi} = \rho_x\sqrt{K_x} \\ K_x = \left(\dfrac{Q_{x0}}{\rho_x}\right)^2 \end{cases}$$

Perturbed motion:

$$x''(s) + \left(\frac{Q_{x0}}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x,s)$$

Quiz # 14 - 15

You have already studied the previous equations in the lectures of:

1) Longitudinal beam dynamics
2) Transverse beam dynamics
3) Special relativity and electromagnetism
4) Classic literature

The goal of this part of the lecture was to:

- Obtain the space charge force
- Obtain the transverse single particle equation of motion with the inclusion of the space charge force
- Obtain the longitudinal single particle equation of motion with the inclusion of the space charge force
- Evaluate the betatron tune
- I don’t see a goal in this part of the lecture

CONTENTS (SPACE CHARGE)

Transverse incoherent effects

We take the linear term of the self induced transverse force in the betatron equation:

$$F_x^{self}(x,s) = F_x^{s.c.}(x,s) \simeq \left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \Longrightarrow x''(s) + \left(\frac{Q_{x0}}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x$$

$$x''(s) + \left[\left(\frac{Q_{x0}}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}\right] x = 0 \Longrightarrow x''(s) + \left(\frac{Q_{x0} + \Delta Q_x}{\rho_x}\right)^2 x = 0$$

$$x''(s) + \left(\frac{Q_{x0}^2 + 2Q_{x0}\Delta Q_x + \cancel{\Delta Q_x^2}}{\rho_x^2}\right) x = 0 \Longrightarrow x''(s) + \left(\frac{Q_{x0}^2}{\rho_x^2} + \frac{2Q_{x0}\Delta Q_x}{\rho_x^2}\right) x = 0$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}}$$

Transverse incoherent effects

$$\boxed{\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}}$$

The shift of betatron wave number (tune shift) is negative since the space charge forces are defocusing on both planes (the betatron wavelength increases). Remember that the space charge force, and then the tune shift, is, in general, function of "z", $\lambda(z)$, therefore this expression represents a tune spread inside the beam. This is why we call it incoherent. This conclusion is generally true also for more realistic non-uniform transverse beam distributions, which are characterized by a tune shift dependent also on the betatron oscillation amplitude. When ΔQ_x is not constant in the beam, instead of tune shift the effect is called tune spread.

Example: incoherent betatron tune shift for a uniform electron beam of radius *a*=100µm, length *l*$_o$=100µm, inside a circular perfectly conducting pipe (energy E$_0$=1GeV, N=10^{10}, ϱ$_x$=20m, Q$_{xo}$=4.15)

$$\boxed{\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{e\lambda_0 x}{2\pi\varepsilon_0\gamma^2 a^2}\right) = \frac{e\lambda_0}{2\pi\varepsilon_0\gamma^2 a^2}}$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N e^2}{4\pi\varepsilon_0 a^2\beta^2\gamma^2 E_0 Q_{x0} l_0}}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2} \quad (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{x0} l_0} \simeq \qquad}$$

Remember that for real bunched beams the space charge forces depend on the longitudinal and radial position of the charge => tune spread.

ΔQ as function of beam emittance and filling factor of the ring

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{x0} l_0}}$$

$$a^2 = \varepsilon_x \beta_x$$

$$\beta_x = \frac{\lambda_\beta}{2\pi} = \frac{1}{\sqrt{K_x}}$$

$$K_x = \left(\frac{Q_{x0}}{\rho_x}\right)^2$$

$$Q_{x0} = \rho_x \sqrt{K_x} = \frac{\rho_x}{\beta_x}$$

$$\boxed{\begin{aligned} \Delta Q_x &= -\frac{\rho_x^{\cancel{2}} N r_{e,p}}{\varepsilon_x \cancel{\beta_x} \beta^2 \gamma^3 \dfrac{\cancel{\rho_x}}{\cancel{\beta_x}} l_o} \\ \Delta Q_x &= -\frac{N r_{e,p}}{2\pi\varepsilon_x \beta^2 \gamma^3}\left(\frac{2\pi\rho_x}{l_o}\right) \end{aligned}}$$

This expression is valid also in the general case of non-uniform focusing along the accelerator for a uniform beam inside a circular pipe.

General expression of the incoherent tune shift

We have seen that the equation of the betatron oscillations in presence of self forces is linearized as

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x) = \frac{1}{\beta^2 E_0} \left(\frac{\partial F_x^{self}}{\partial x} \right)_{x=0} x$$

which we can also write as

$$x''(s) + \left[K_x - \frac{1}{\beta^2 E_0} \left(\frac{\partial F_x^{self}}{\partial x} \right)_{x=0} \right] x(s) = 0$$

From beam optics, it is possible to demonstrate that, having a circular machine with design quadrupole stength $K_x(s)$ and gradient errors $\Delta K_x(s)$ distributed along the machine, these errors lead to a tune shift of

$$\Delta Q_x = \frac{1}{4\pi} \oint \beta_x(s) \Delta K_x(s) ds = -\frac{1}{4\pi\beta^2 E_0} \oint \beta_x(s) \left(\frac{\partial F_x^{self}}{\partial x} \right)_{x=0} ds$$

General expression of the incoherent tune shift

$$\Delta Q_x = \frac{1}{4\pi}\oint \beta_x(s)\Delta K_x(s)ds = -\frac{1}{4\pi\beta^2 E_0}\oint \beta_x(s)\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0} ds$$

This is a general expression. The only assumption is the linearization of the self field forces. For a uniform transverse distribution, $\beta_x(s) = a^2/\varepsilon_x$

$$\Delta Q_x = -\frac{1}{4\pi\beta^2\varepsilon_x E_0}\oint a^2\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0} ds$$

$$\Delta Q_x = -\frac{2\pi\rho_x}{4\pi\beta^2\varepsilon_x E_0}\left\langle a^2\left(\frac{\partial F_x^{self}}{\partial x}\right)_{x=0}\right\rangle$$

where $\langle \quad \rangle$ means averaged over the machine circumference. With this expression we can account for the variation of the beam dimensions along the machine due to the betatron function.

Quiz # 16 - 17

What is the dependence of the incoherent transverse tune with the particle transverse displacement in the linear approximation?

1) I do not understand the question
2) Independent
3) Linear
4) Inversely proportional

The linearization of the space charge force:

- Cannot be used in the study of beam dynamics because its motion is not linear
- Allows to obtain the amplitude of the single particle transverse motion
- Allows to obtain the tune of single particle transverse motion
- Allows to obtain the single particle betatron tune shift
- More than one answer is correct

CONTENTS (SPACE CHARGE)

- Introduction, self induced fields
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- Boundary conditions: image charges and currents
- Examples: beam in a circular pipe, in parallel plates
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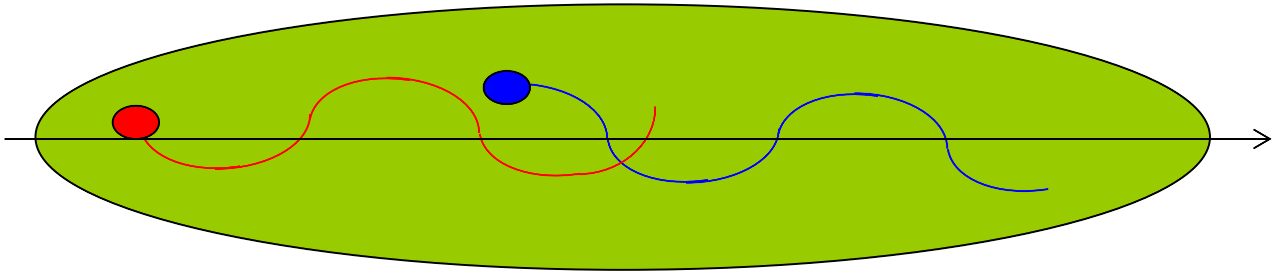
- Longitudinal space charge forces
- Simulation codes

Shift and spread of the incoherent tunes

If the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass (coherent), but they change the trajectory of individual charges in the beam (incoherent).



These forces may have a complicate dependence on the charge position. Our simple analysis is done by considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

The consequences are a shift and a spread of the incoherent tunes.

Consequences of the space charge tune spreads

In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable. The spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances, the stability requires

$$|\Delta Q_u| < 0.5^*$$

If the tune spread exceeds this limit, it is possible to reduce the effects of space charge tune spread, e.g. by increasing the injection energy or the transverse beam size.

The incoherent tune spread produces also a beneficial effect, called Landau damping, which can cure the coherent instabilities, provided that the coherent tune remains inside the incoherent spread.

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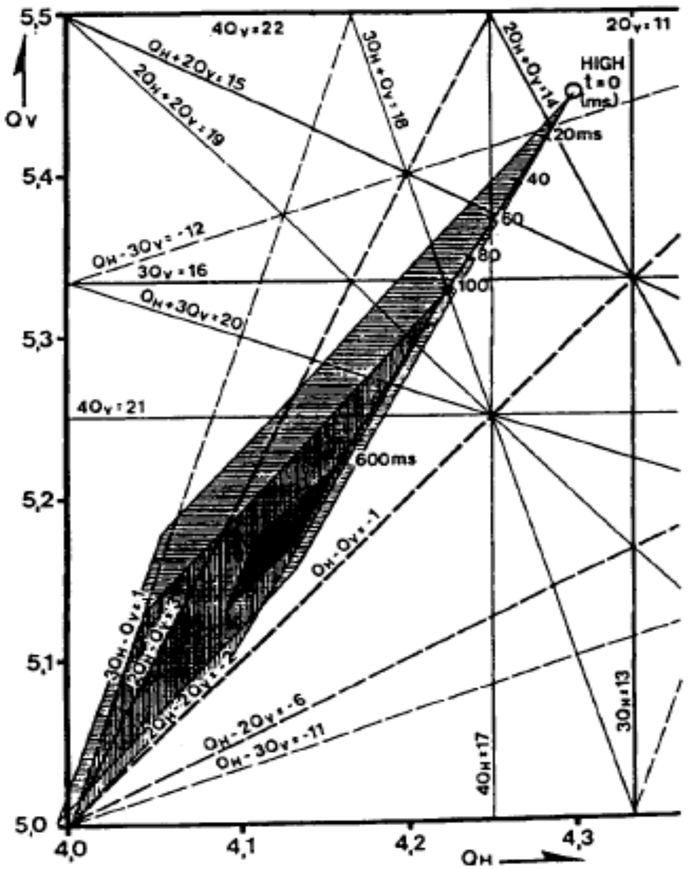
[*] See, for example, J. Rossbach, P. Schmüser, ‘Basic course on accelerator optics’, CAS Jyväskylä 1992, CERN 94–01, p. 76.
J. P. Delahaye, et al., Proc. 11th Int. Conf. on High Energy Accelerators, Geneva, 1980, p. 299.

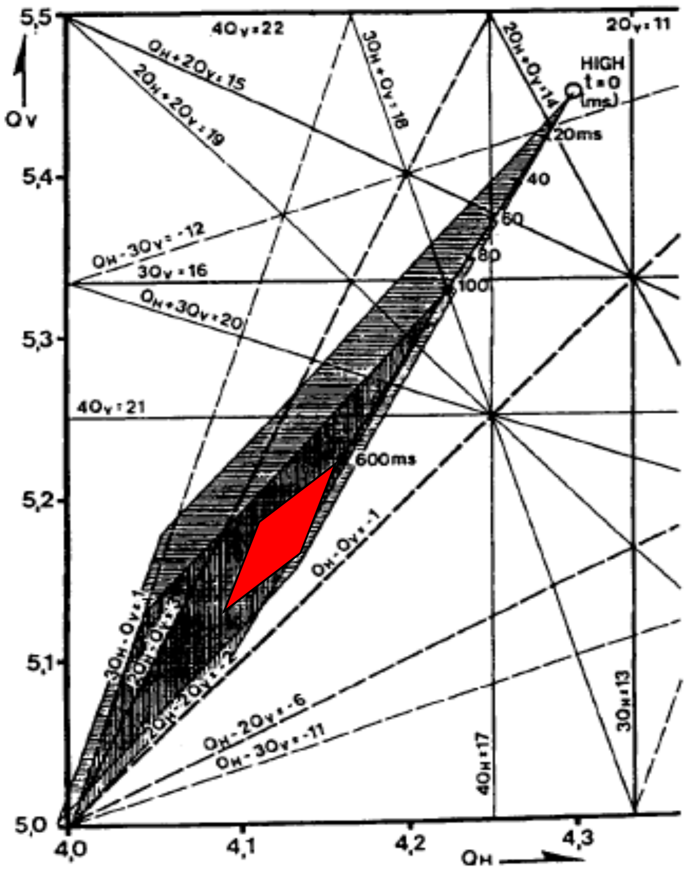
Example from A. Hofmann in CAS 1992 (General Course - Jyväskylä Finland)



CERN PS Booster accelerated proton bunches from 50 to 800 MeV in about 0.6 s. The tunes occupied by the particles are indicated in the diagram by the shaded area. As time goes on, the energy increases and the space charge tune spread gets smaller covering at t=100 ms the tune area shown by the darker area. The point of highest tune corresponds to the particles which are least affected by the space charge. This point moves in the Q diagram since the external focusing is adjusted such that the reduced tune spread lies in a region free of harmful resonances.

The small red area shows the situation at t=600 ms when the beam has reached the energy of 800 MeV. The tune spread reduction is lower than expected with the energy increase ($1/\gamma^3$) dependence since the bunch dimensions also decrease during the acceleration.

Example from A. Hofmann in CAS 1992 (General Course - Jyväskylä Finland)



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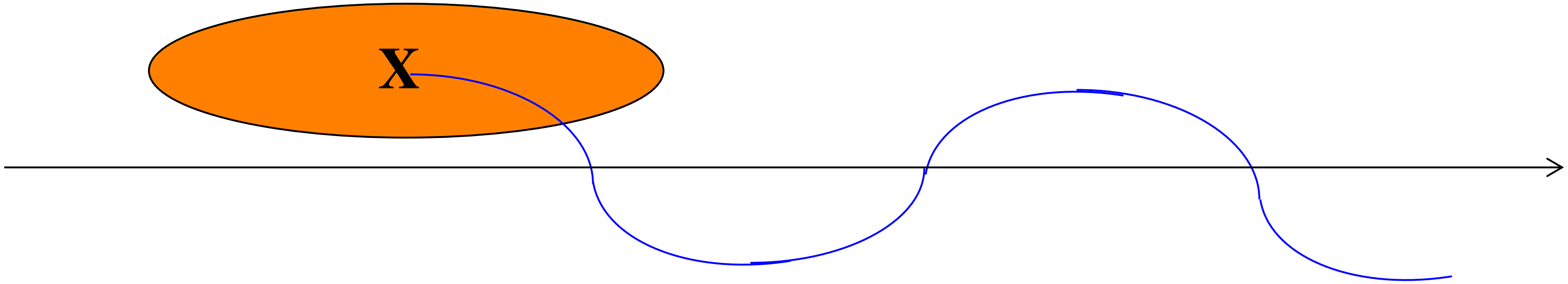
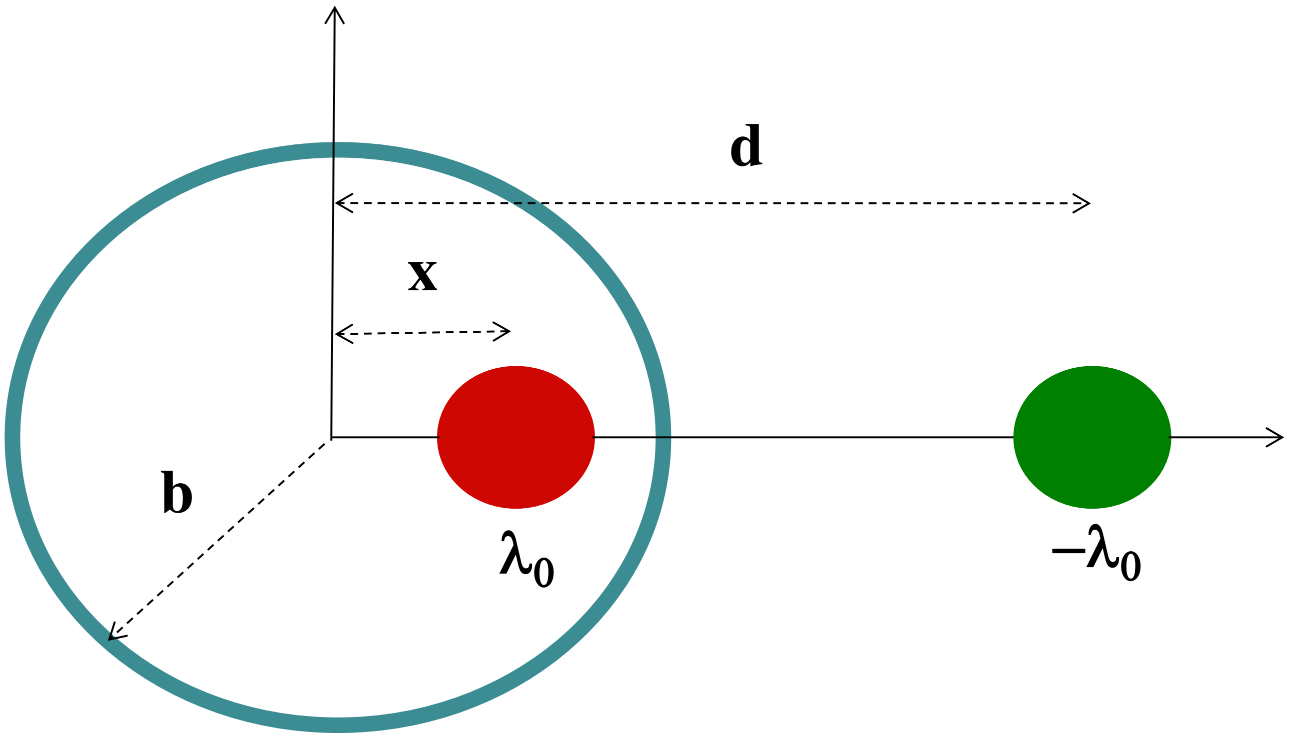
CONTENTS (SPACE CHARGE)

Transverse coherent effects

If the beam experiences a transverse deflection kick, it starts to perform betatron oscillations as a whole. The beam, source of the space charge fields moves transversely inside the pipe, but its centre of mass (X), due to simmetry, cannot be affected by the direct space charge. Only image space charge can affect its motion.

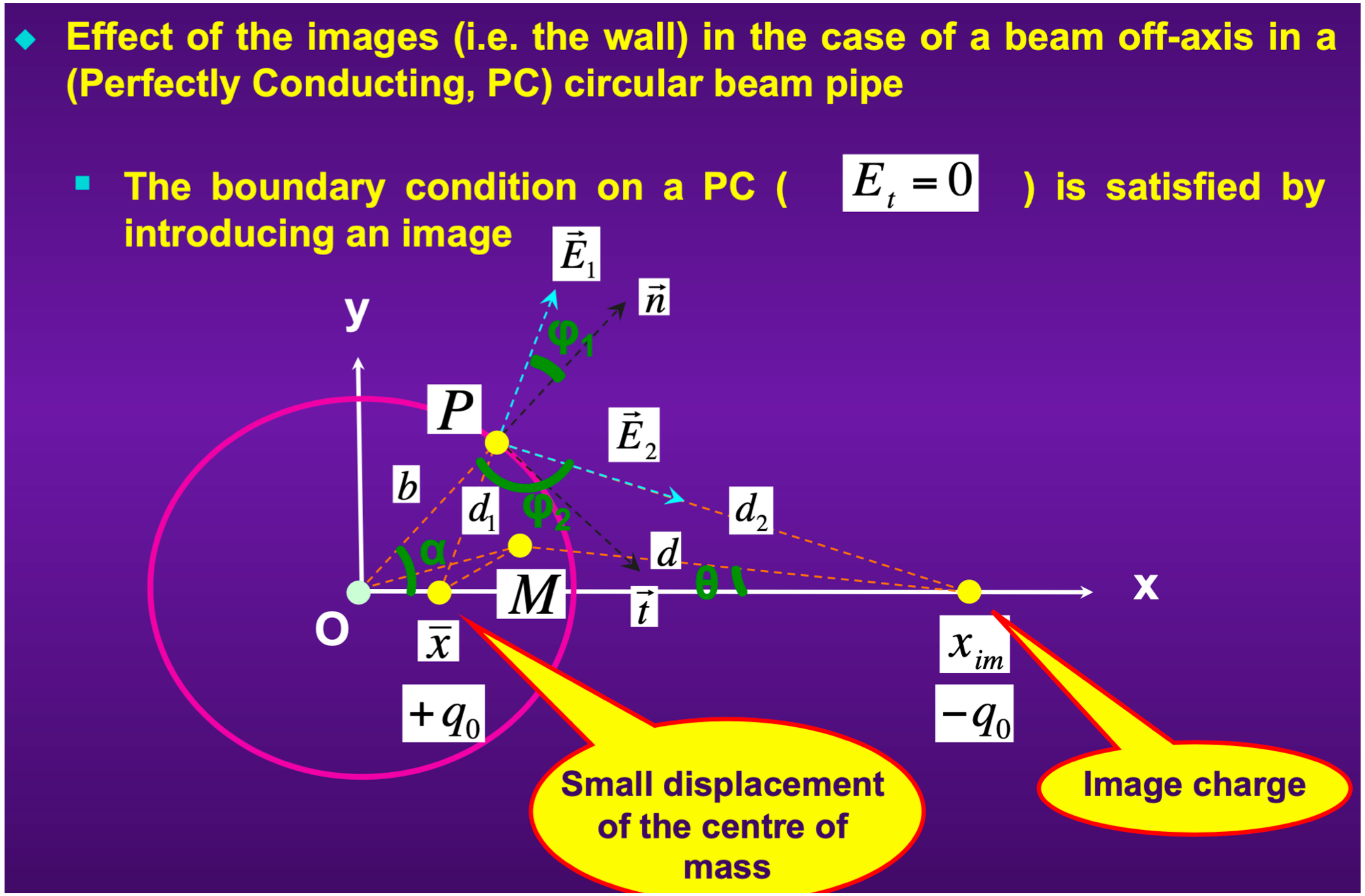




$$\boxed{d = \frac{b^2}{x}}$$

The image charge is at a distance “d” such that the pipe surface is at constant voltage, and pulls the beam away from the center of the pipe.

Courtesy of E. Métral

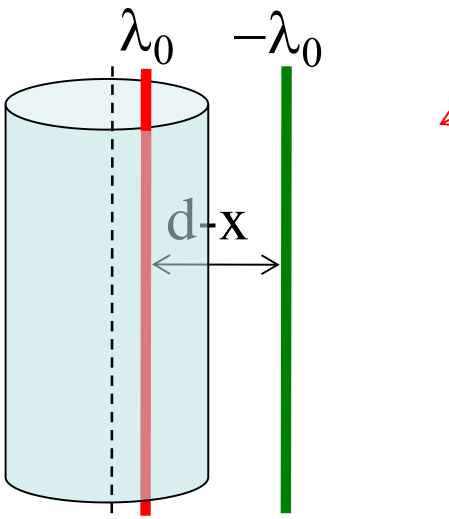


- Effect of the images (i.e. the wall) in the case of a beam off-axis in a (Perfectly Conducting, PC) circular beam pipe
  - The boundary condition on a PC ($E_t = 0$) is satisfied by introducing an image

Small displacement of the centre of mass

Image charge

The effect is defocusing: the horizontal electric image field E and the horizontal force F are:



$$E_{xc}(x)=\frac{\lambda_0}{2\pi\varepsilon_0}\frac{1}{d-x}\cong\frac{\lambda_0}{2\pi\varepsilon_0}\frac{1}{d}=\frac{\lambda_0}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$F_{xc}(x)=\frac{e\lambda_0}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$\Delta Q_{xc}=-\frac{\rho_x^2}{2\beta^2E_0Q_{x0}}\left(\frac{\partial F_{xc}}{\partial x}\right)=-\frac{\rho_x^2}{2\beta^2E_0Q_{x0}}\frac{e\lambda_0}{2\pi\varepsilon_0b^2}$$

$$r_{e,p}=\frac{e^2}{4\pi\varepsilon_0m_0c^2}$$

$$\boxed{\Delta Q_{xc}=-\frac{r_e\rho_x^2}{\beta^2\gamma Q_{x0}}\frac{N}{b^2l_0}}$$

Example 4: coherent betatron tune shift for a uniform electron beam of length l_o=100μm, inside a circular perfectly conducting pipe of radius b=14cm, (energy E_0=1GeV, N=10^{10}, ϱ_x=20m, Q_{xo}=4.15)

$$\Delta r_e = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2} = 2.82\times10^{-15}\ \mathrm{m}$$

$$\Delta Q_{xc} = -\frac{r_e \rho_x^2}{\beta^2 \gamma Q_{x0}} \frac{N}{b^2 l_0} \approx ?$$

Quiz # 18 - 19

Is the coherent tune shift due to the direct or indirect space charge?

1) Direct
2) Indirect
3) Both
4) None of the two

What is the difference between incoherent and coherent tune shift?

1) They are the same since they both refer to the particles inside a beam
2) The first one refers to the particles inside the beam, the second one to the centre of mass
3) The coherent tune shift can produce a tune spread, while the incoherent one no
4) I do not have any idea

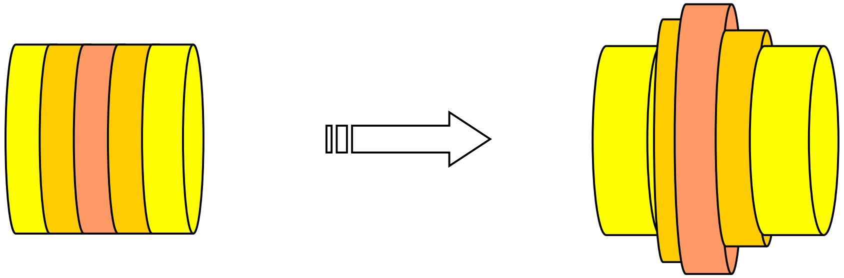
CONTENTS (SPACE CHARGE)

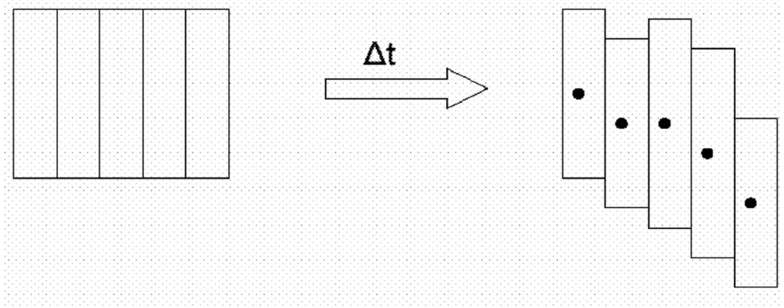
Consequences of the space charge forces on LINACS

In a LINAC or a beam transport line, the space charge forces cause energy spread and perturb the equilibrium beam size.

They can also lead to a significant longitudinal-transverse correlation of the bunch parameters, which may produce mismatch with the focusing and accelerating devices, thus contributing to emittance growth.

The dynamics can be studied by considering the beam as an ensemble of longitudinal slices, for each of which it is possible to write a differential equation giving the behaviour of the transverse dimension along the machine (**envelope equation**).





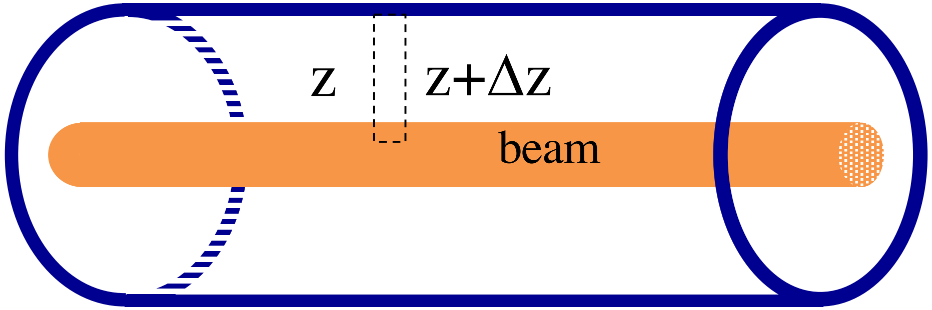
For the stability it is required anyway that the defocusing space charge forces must not be larger than the external focusing forces.

LONGITUDINAL FORCES

Longitudinal forces can be obtained from the knowledge of the transverse ones.

In order to derive the relationship between the longitudinal and transverse forces inside a beam, let us consider the case of cylindrical symmetry and ultra-relativistic bunches. We know from Faraday's law of induction that a varying magnetic field produces a rotational electric field:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \hat{n} dS$$



We choose as path a rectangle going through the beam pipe and the beam, parallel to the axis.

$$E_z(r,z)\Delta z + \int_r^b E_r(r',z+\Delta z)dr' - E_z(b,z)\Delta z - \int_r^b E_r(r',z)dr' =$$

$$-\Delta z \frac{\partial}{\partial t}\int_r^b B_\theta(r',z)dr'$$

$$E_r(r',z+\Delta z) - E_r(r',z) = \frac{\partial E_r(r',z)}{\partial z}\Delta z$$

$$E_z(r,z) = E_z(b,z) - \int_r^b \left[\frac{\partial E_r(r',z)}{\partial z} + \frac{\partial B_\theta(r',z)}{\partial t}\right]dr'$$

$$\boxed{dz = -vdt}$$

$$E_z(r,z) = E_z(b,z) - \frac{\partial}{\partial z}\int_r^b [E_r(r',z) - vB_\theta(r',z)]dr'$$

$$\boxed{B_\theta = \frac{\beta E_r}{c}}$$

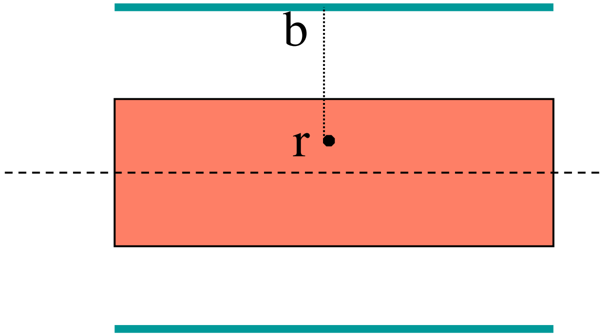
$$E_z(r,z) = E_z(b,z) - \frac{\partial}{\partial z}\int_r^b [E_r(r',z) - \beta^2 E_r(r',z)]dr'$$

$$\boxed{E_z(r,z) = E_z(b,z) - (1-\beta^2)\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'}$$

where $(1-\beta^2)=1/\gamma^2$. For perfectly conducting walls $E_z=0$.

$$\boxed{E_z(r,z) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'}$$

Transverse uniform beam in a circular p.c. pipe.



$$\boxed{F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}}$$

Longitudinal self fields and synchrotron motion

Longitudinal equations of motion for constant energy and circular machine, ignoring radiation damping

$$\frac{d\phi}{dt} = -\frac{2\pi h\eta}{\beta^2 T_0}\frac{\Delta E}{E_0} \qquad \frac{d(\Delta E)}{dt} = \frac{qV_{rf}}{T_0}(\sin\phi - \sin\phi_s) \qquad \eta = \frac{1}{\gamma^2} - \alpha_c$$

ϕ is the RF phase, h the harmonic number, η the slippage factor, ΔE the energy difference with respect to the synchronous particle

$$\Delta\phi'' + \left(\frac{Q_s}{\rho_x}\right)^2 \Delta\phi = 0 \qquad Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{e\eta h V_{rf}\cos\phi_s}{2\pi\beta^2 E_0}}$$

$\Delta\phi$ is the phase difference with respect to the synchronous particle. Including longitudinal space charge forces the equation becomes:

$$\Delta\phi'' + \left(\frac{Q_s}{\rho_x}\right)^2 \Delta\phi = \frac{h\eta F_z^{self}}{\rho_x\beta^2 E_0}$$

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Numerical Analysis - 1

There are numerical codes used to evaluate the space charge effects in circular accelerators and Linacs:

ORBIT: Objective Ring Beam Injection and Tracking Code

https://oraweb.cern.ch/pls/hhh/code_website.disp_code?code_name=ORBIT

ORBIT is a computer code designed for beam dynamics calculations in high-intensity rings. Its intended use is the detailed simulation of realistic accelerator problems, although it is equally applicable to idealized situations. ORBIT is a particle-in-cell tracking code in 6D phase space that transports bunches of interacting particles through a series of nodes representing elements, dynamic effects, or diagnostics that occur in the accelerator lattice. It can be used in combination with PTC, a 6D integrator as tracker.

GPT: General Particle Tracer

http://www.pulsar.nl/gpt/

GPT is based on full 3D particle tracking techniques, providing a basis for the study of all 3D and non-linear effects of charged particles dynamics in electromagnetic fields. All built-in beam line components and external 2D/3D field-maps can be arbitrarily positioned and oriented to simulate a complicated setup-up and study the effects of misalignments. An embedded fifth order Runge-Kutta driver with adaptive stepsize control ensures accuracy while computation time is kept to a minimum. GPT provides various 2D and 3D space-charge models.

Numerical Analysis - 2

PARMELA: Phase and Radial Motion in Electron Linear Accelerators

http://laacg.lanl.gov/laacg/services/serv_codes.phtml#parmela

PARMELA is a multi-particle beam dynamics code used primarily for electron-linac beam simulations. It is a versatile code that transports the beam, represented by a collection of particles, through a user-specified linac and/or transport system. It includes several space-charge calculation methods. Particle trajectories are determined by numerical integration through the fields. This approach is particularly important for electrons where some of the approximations used by other codes (e.g. the "drift-kick" method commonly used for low-energy protons) would not hold.

PARMILA: Phase And Radial Motion in Ion Linear Accelerators

http://www.lanl.gov/projects/feynman-center/technologies/software/parmila.php

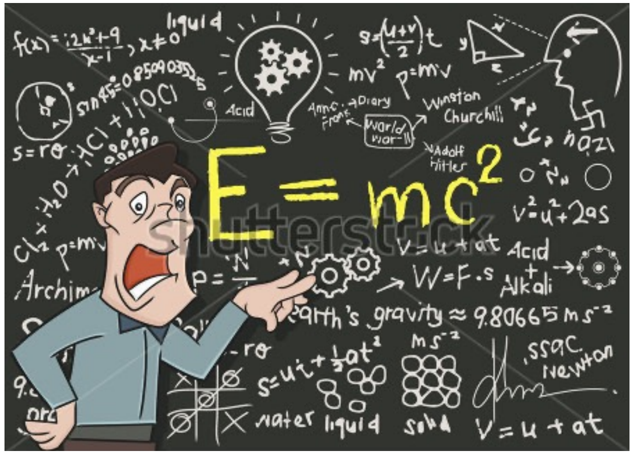
Parmila has been the standard code for the design of RF linacs for many years. The enhanced, second generation, PARMILA 2 program is utilized in the PBO Lab PARMILA-2 Module. The Module is ideally suited for the design of complex ion accelerator components such as drift tube linacs (DTLs), coupled cavity linacs (CCLs), coupled-cavity drift tube linacs (CC-DTLs) and superconducting linacs (SCLs). The program offers two different multi-particle space charge algorthims which permits comparing different high beam current modeling approximations. The PARMILA-2 Module is also useful for the simulation of intense beams in transport channels and for studying beam loss, misalignments, cavity mispowering, and similar off-nominal operation.

Last quiz on space charge

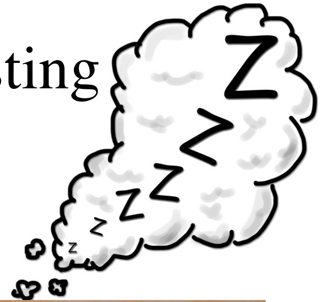
This first part of the lecture on space charge was:

1) too easy

2) too difficult



3) too interesting



4) I don't know because I didn't understand anything

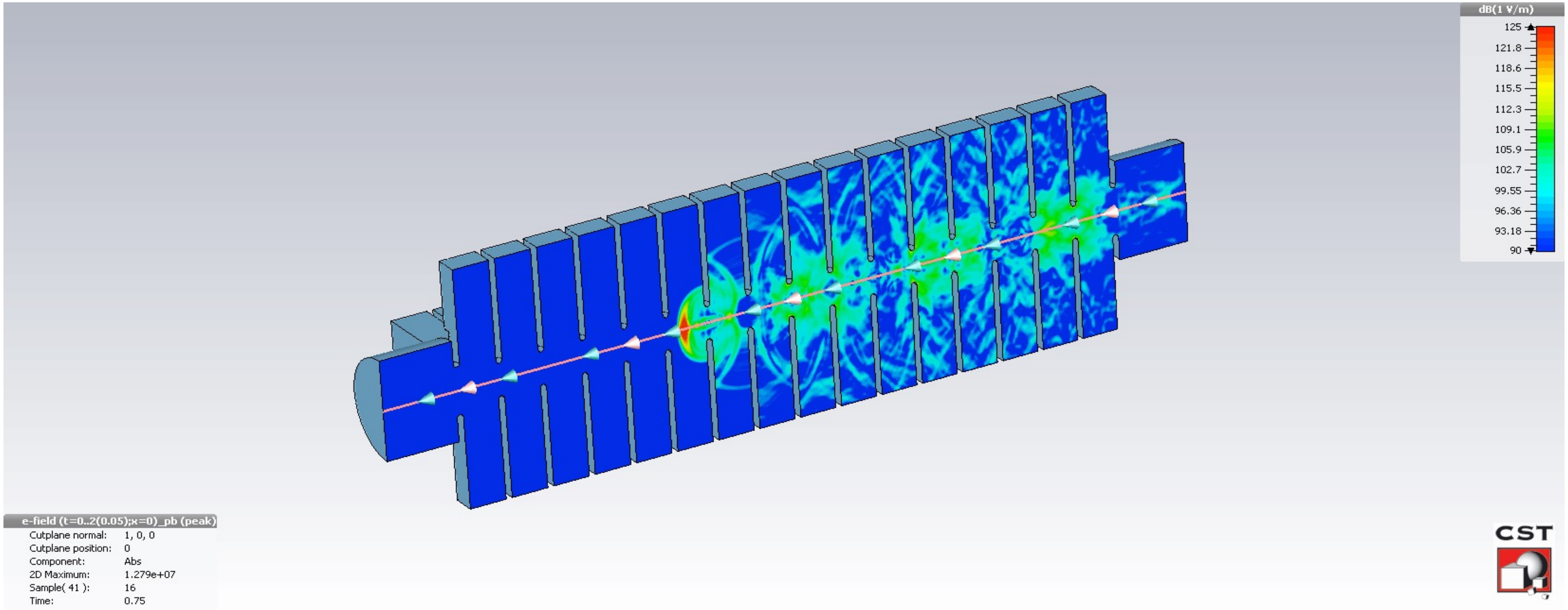
5) I prefer not to answer

Wakefields and Instabilities

Mauro Migliorati

LA SAPIENZA - *Università di Roma and INFN*

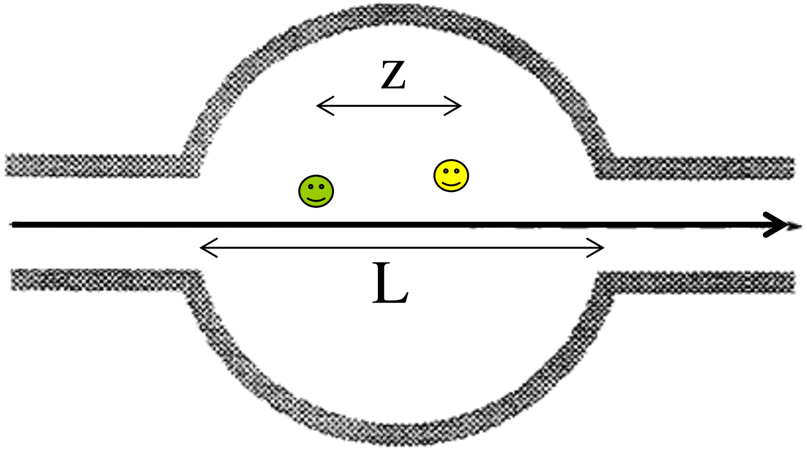
- **Introduction to wakefields/potentials**
- **Instability mechanism**
- **Instability in Linacs**
- **Instability in Circular Accelerators**
- **Other effects not discussed here**



e-field (t=0..2(0.05);x=0)_pb (peak)

Cutplane normal: 1, 0, 0
Cutplane position: 0
Component: Abs
2D Maximum: 1.279e+07
Sample(41): 16
Time: 0.75

Wakefields and Wake Potentials



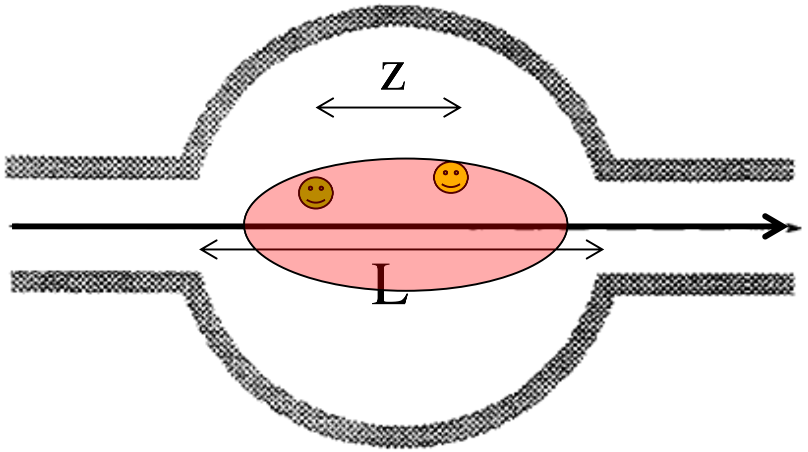
$$\vec{F} = q[E_z\hat{z} + (E_x - cB_y)\hat{x} + (E_y + cB_x)\hat{y} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

This force depends on the longitudinal and transverse position of the two particles. It is useful to distinguish two effects on the **test charge** :

1) a longitudinal force which changes its energy,

2) a transverse force which deflects its trajectory.

Two approximations



At high energies, the particle beam is rigid and two approximations apply:

1) The rigid beam approximation, which says that the beam traverses the discontinuity of the vacuum chamber rigidly and the electromagnetic field is a perturbation that does not affect the motion of the beam during the traversal of the discontinuity. *This implies that the distance ‘z’ between the two charges does not change.*

2) The impulse approximation: although the test charge sees a force coming from the electromagnetic field all along the structure, what it cares is the impulse

$$\boxed{\Delta\vec{p} = \int_{-\infty}^{\infty} \vec{F}\,dt}$$

as the charge completes the traversal through the discontinuity at its fixed velocity v.

If we consider a device of length L, we can perform the integral of the force acting on the test charge along the longitudinal path and get:

the Energy Gain (J):

$$\boxed{U(r, r_0, z) = \int_0^L F_{\parallel} ds \simeq U(z)}$$

the Transverse Deflecting Kick (N · m):
(dipolar)

$$\boxed{\vec{M}(r, r_0, z) = \int_0^L \vec{F}_{\perp} ds \simeq r_0 \vec{M}(z)}$$

These quantities are both function of the distance z between the two particles. The transverse deflecting kick depends also on r_0, the transverse position of the source charge.

Note that the integration is performed over a given path of the trajectory.

These quantities, normalised to the charges, are called ***wakefields***

Longitudinal wakefield
(Volt/Coulomb)

$$\boxed{w_{\parallel}(z) = -\frac{U(z)}{q^2}}$$

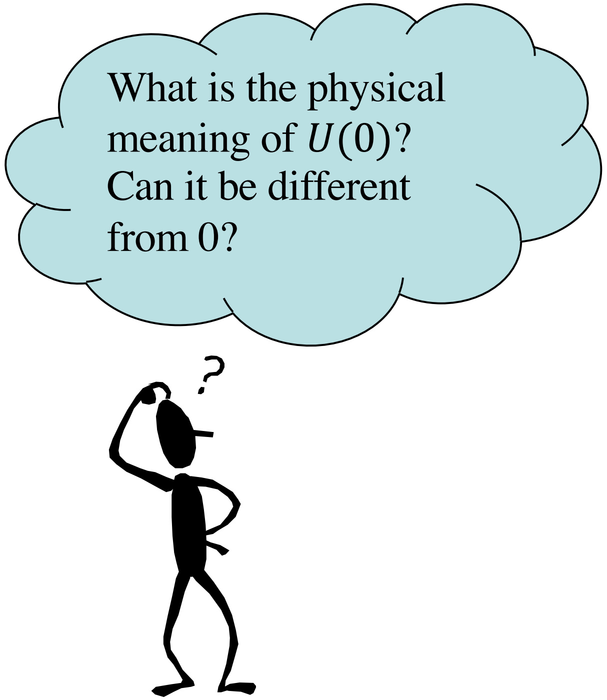
Transverse dipole wakefield
(Volt/Coulomb/meter)

$$\boxed{\vec{w}_{\perp}(z) = \frac{\vec{M}(z)}{q^2}}$$

The minus sign in the longitudinal wakefield means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

The wakefields are the important quantities to study the beam dynamics.



Longitudinal wakefield

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The minus sign in the longitudinal wakefield means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

The wakefields are the important quantities to study the beam dynamics.

Quiz # 20

The longitudinal wakefield gives

- The force acting on the test charge due to the source one
- The transverse deflecting kick of the test charge due to the source one
- The energy lost by the test charge due to the electromagnetic field of the leading one normalized by the two charges
- Wakefield is a cathedral city and the administrative center of the City of Wakefield district in West Yorkshire, England

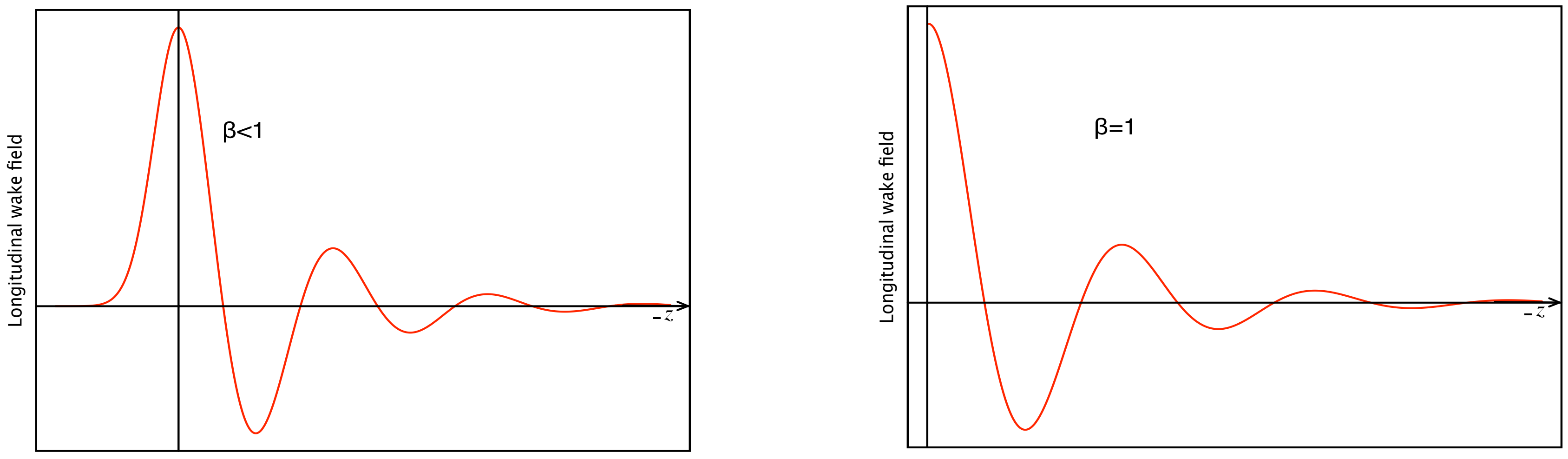
The loss factor

We know that em fields transport energy. Who does supply this energy to the fields? When a charge interacts with the surrounding environment generating these fields, the particle loses a bit of its energy.

It is therefore useful to define the ***loss factor*** k as the energy lost by the **source charge q** due to the em fields of the charge itself normalised to q^2. From the definition of wakefield we have then

$$k = -\frac{U(z=0)}{q^2} = w_{\parallel}(z=0)$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the speed of light, the longitudinal wakefield is discontinuous at z=0



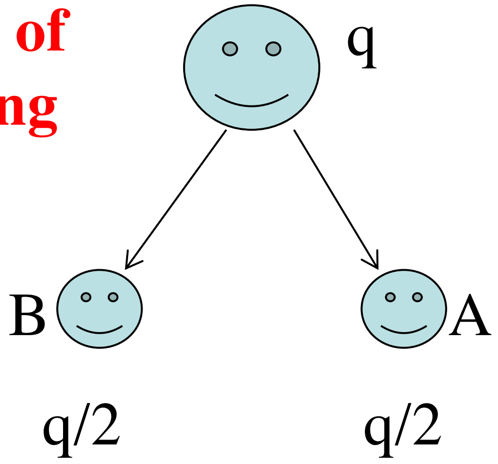
Causality requires that the longitudinal wakefield of a charge travelling with the speed of light is discontinuous in the origin.

The exact relationship between k and $w_{\parallel}(z \rightarrow 0)$ is given by the ***beam loading theorem*****:**

$$\boxed{k = \frac{w_{\parallel}(z \rightarrow 0^{-})}{2}}$$

NB: this is true only in the longitudinal plane.

Demonstration of the beam loading theorem



$$U_A = q_A^2 k = \frac{q^2}{4} k$$

$$U_B = q_B^2 k + q_A q_B w_{//}(z)$$

$$= \frac{q^2}{4} k + \frac{q^2}{4} w_{//}(z)$$

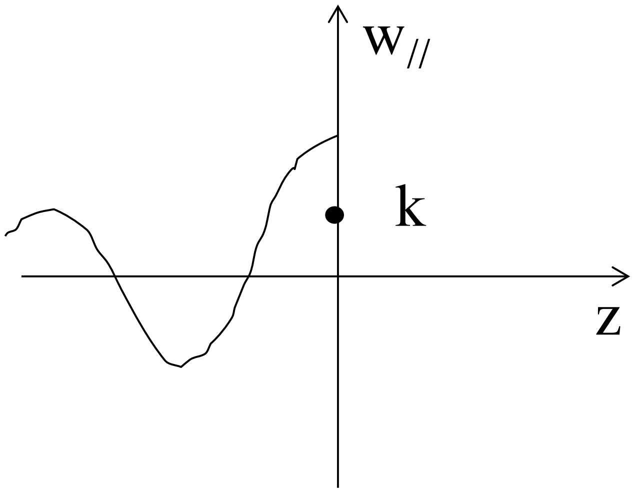
$$\boxed{U_A + U_B = \frac{q^2}{2} k + \frac{q^2}{4} w_{//}(z)}$$

$$\boxed{z \longrightarrow 0 \qquad U_A + U_B = q^2 k}$$

$$\frac{q^2}{2} k + \frac{q^2}{4} w_{//}(0) = q^2 k$$

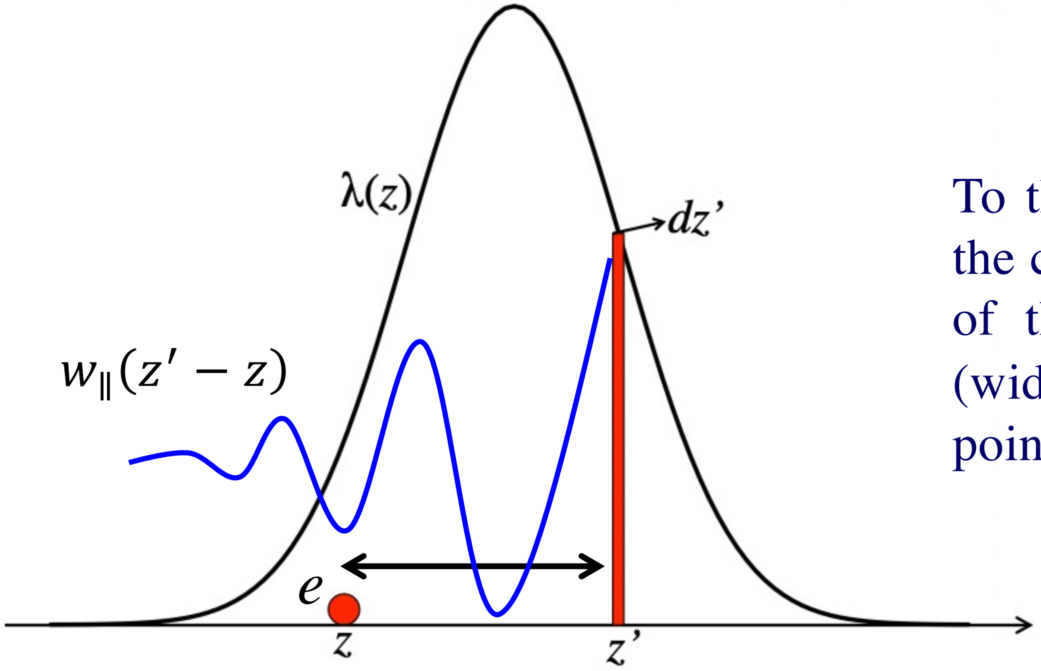
$$\frac{w_{//}(0)}{4} = \frac{k}{2}$$

$$\boxed{k = \frac{w_{//}(0)}{2}}$$



Wake potential and energy loss of a bunched distribution

When we have a bunch with longitudinal charge density $dq/dz = \lambda(z)$, we may want to get the amount of energy lost or gained by a single charge *e* in the beam.



To this end let us evaluate the effect on the charge *e* in a posizion *z* due to a slice of the bunch in a position *z*' so thin (width *dz*') that it can be considered as a point charge:

$$dU(z) = -edq(z')w_{\parallel}(z - z') = -ew_{\parallel}(z - z')\lambda(z')dz'$$

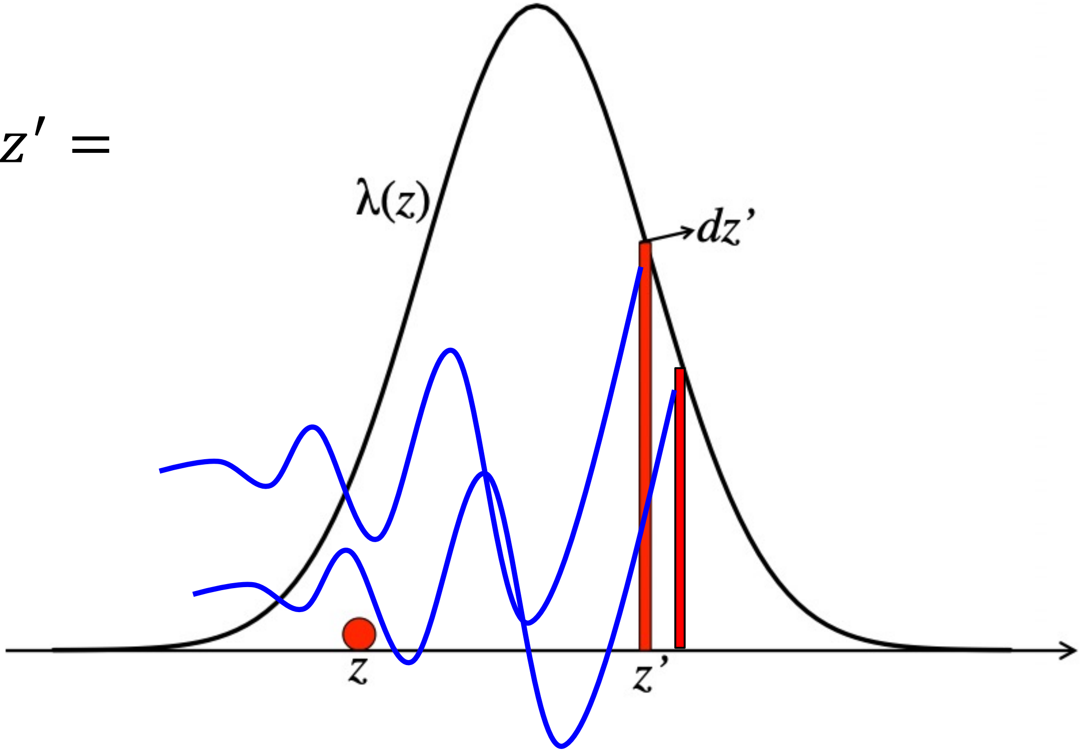
We now use the superposition principle to obtain the energy lost or gained by the charge *e* due to the entire distribution.

Wake potential and energy loss of a bunched distribution

$$U(z) = -e\int_{-\infty}^{\infty} w_{\parallel}(z-z')\lambda(z')dz' =$$

$$= -e\int_{z}^{\infty} w_{\parallel}(z-z')\lambda(z')dz'$$

NB: we have $q = \int_{-\infty}^{\infty} \lambda(z)dz$



The energy lost allows to define the ***longitudinal wake potential of a distribution***

$$W_{\parallel}(z) = -\frac{U(z)}{qe} = \frac{1}{q}\int_{-\infty}^{\infty} w_{\parallel}(z-z')\lambda(z')dz'$$

The total energy lost by the bunch is computed summing up the losses of all the particles:

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U(z)\lambda(z)dz = -q\int_{-\infty}^{\infty} W_{\parallel}(z)\lambda(z)dz$$

Some comments on the wake potential

$$W_{\parallel}(z) = -\frac{U(x)}{qe} = \frac{1}{q}\int_{-\infty}^{\infty} w_{\parallel}(z - z')\lambda(z')dz'$$

- Observe that if we know the wakefield, we can obtain the wake potential of any distribution, but if we know the wake potential, we are limited to a particular beam distribution.
- In a LINAC, with particles moving at the speed of light, the longitudinal distribution does not change, and the wake potential can be used to evaluate the energy variation of particles inside the bunch (energy spread). In this situation, the knowledge of the wake potential can be sufficient to study the beam dynamics.
- In a circular accelerator the longitudinal position of a charge depends on its energy through the slippage factor, and this energy is modified by the wake potential. As a consequence, the wake potential changes the longitudinal distribution which, on its turn, changes the wake potential. In this case we have to study the beam dynamics in a self consistent way, and the knowledge of the wake potential is not sufficient.

Numerical Analysis

The study of the em fields requires to solve the Maxwell's equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the em design of accelerator devices, and new ones are developed. Examples of codes: **CST STUDIO SUITE, GDFIDL, ACE3P, ECHO(2D, 3D), ABCI, …**

However, the result of the codes is a wake potential and not a wakefield …

Theoretical Analysis

The wake potentials given by numerical codes depend on the particular charge distribution of the beam. It is therefore desirable to know what is the effect produced by a single charge, i.e. **find the Green function** (wakefield), in order to reconstruct the fields produced by any charge distribution.

Quiz # 21

What is the difference between the wakefield and the wake potential?

- They are the same: they both refer to the electromagnetic field produced by a source on a test charge divided by both the charges
- I do not understand the difference
- They are different: the wakefield refers to two charges, the wake potential to the entire bunch acting on a charge
- I don’t know because I was reading comics during this part of the lecture

Coupling Impedance

The wakefields are generally useful to study the beam dynamics in the time domain (for example instabilities in a LINAC). If we take the equation of motion in the frequency domain (a trick generally used to study instabilities in circular accelerators), we need the Fourier transforms of the wakefields. Since these quantities have ohms units they are called ***coupling impedances***:

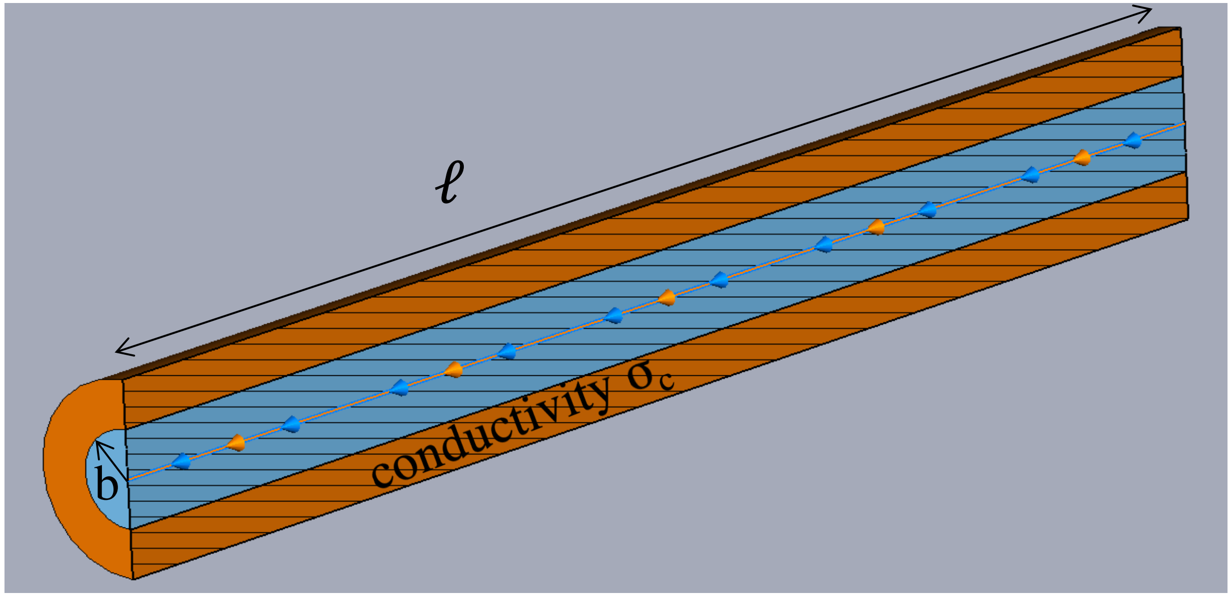
Longitudinal impedance (Ω)

$$Z_{\parallel} = \frac{1}{v}\int_{-\infty}^{\infty} w_{\parallel}(z) e^{-i\frac{\omega z}{v}} dz$$

Transverse dipolar impedance (Ω/m)

$$\vec{Z}_{\perp} = \frac{-i}{v}\int_{-\infty}^{\infty} \vec{w}_{\perp}(z) e^{-i\frac{\omega z}{v}} dz$$

Example of longitudinal wakefield and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



Hp: high conductivity such that the skin depth is much smaller than the wall thickness and

$$c\chi\,/b \ll \omega \ll c\chi^{-1/3}/b$$

$$\chi^{1/3}b \ll z \ll b/\chi$$

with $\chi = \dfrac{1}{Z_0\sigma_c b}$

Example: aluminum σ_c=3.5x10^7 [Ωm]$^{-1}$, b=5 cm:

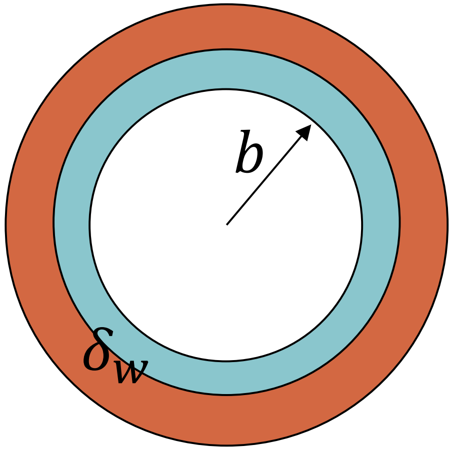
$9 \ll \omega \ll 5.2\times10^{12}$ [rad/s] $\quad 5.7\times10^{-5} \ll z \ll 3.3\times10^{7}$ [m]

$$\boxed{Z_{\parallel}(\omega) = [1-i]\frac{\ell}{2\pi b}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}}$$

$$\boxed{w_{\parallel}(z) = \frac{\ell c}{4\pi b}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{1}{|z|^{3/2}}}$$

Example of longitudinal wakefield and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)

$$\boxed{Z_{\parallel}(\omega) = [1-i]\frac{\ell}{2\pi b}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}}$$

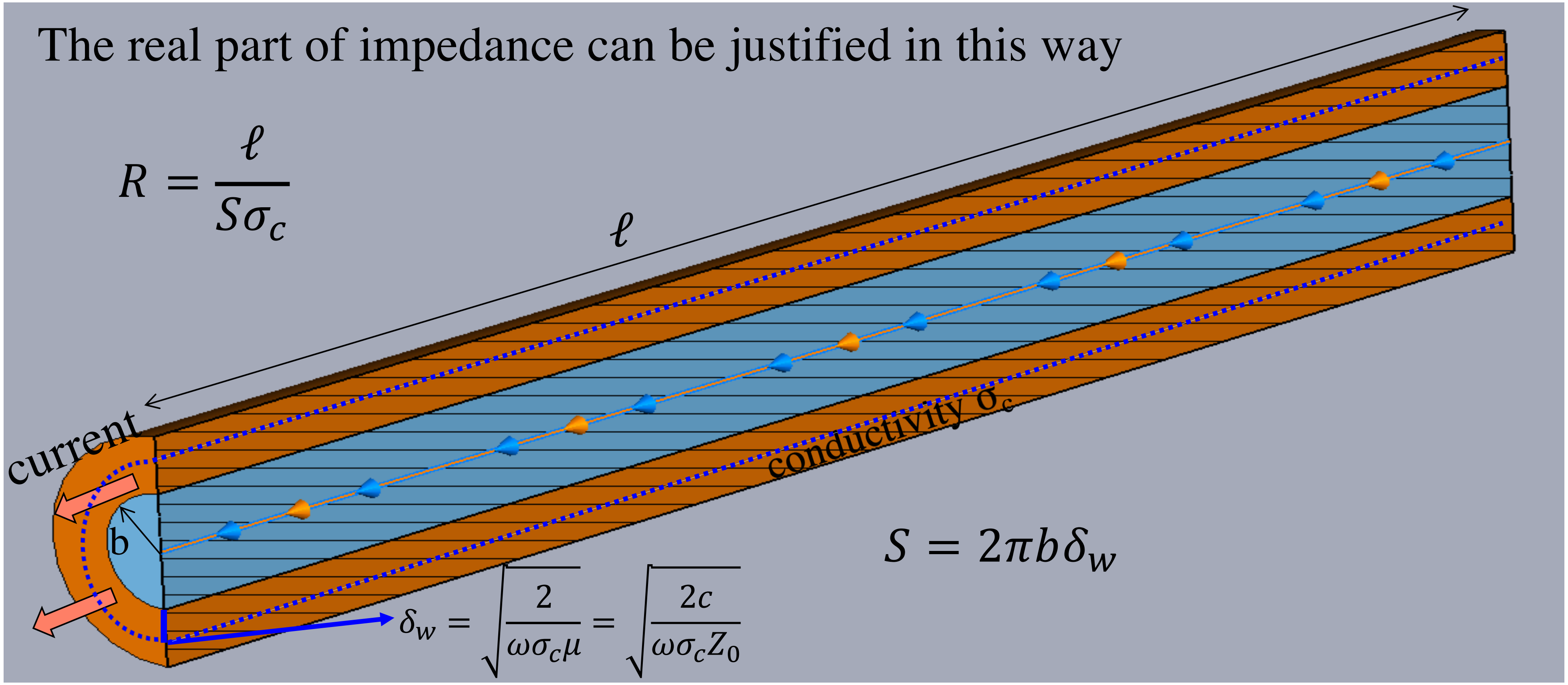


For a good conductor we have defined the skin depth

$$\delta_w = \sqrt{\frac{2}{\sigma_c\mu\omega}} \rightarrow Z_{\parallel}(\omega) = \frac{\ell}{2\pi b}\boxed{[1-i]\frac{1}{\delta_w\sigma_c}}$$

Surface impedance

Example of longitudinal wakefield and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



The real part of impedance can be justified in this way

$$R = \frac{\ell}{S\sigma_c}$$

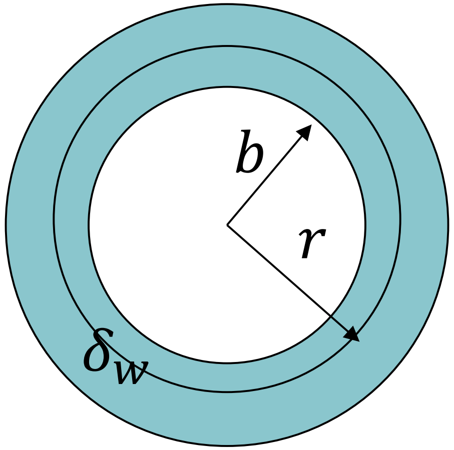
$$S = 2\pi b\delta_w$$

$$\delta_w = \sqrt{\frac{2}{\omega\sigma_c\mu}} = \sqrt{\frac{2c}{\omega\sigma_c Z_0}}$$

$$\Rightarrow R = \frac{\ell}{2\pi b\delta_w\sigma_c} = \frac{\ell}{2\pi b}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}$$

$$\boxed{Z_{\parallel}(\omega) = [1-i]\frac{\ell}{2\pi b}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}}$$

Example of longitudinal wakefield and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



The imaginary part of impedance can be justified in this way: the current is flowing through the area of thickness $\delta_w \ll b$. From the Ampere's law $(2\pi r \simeq 2\pi b)$

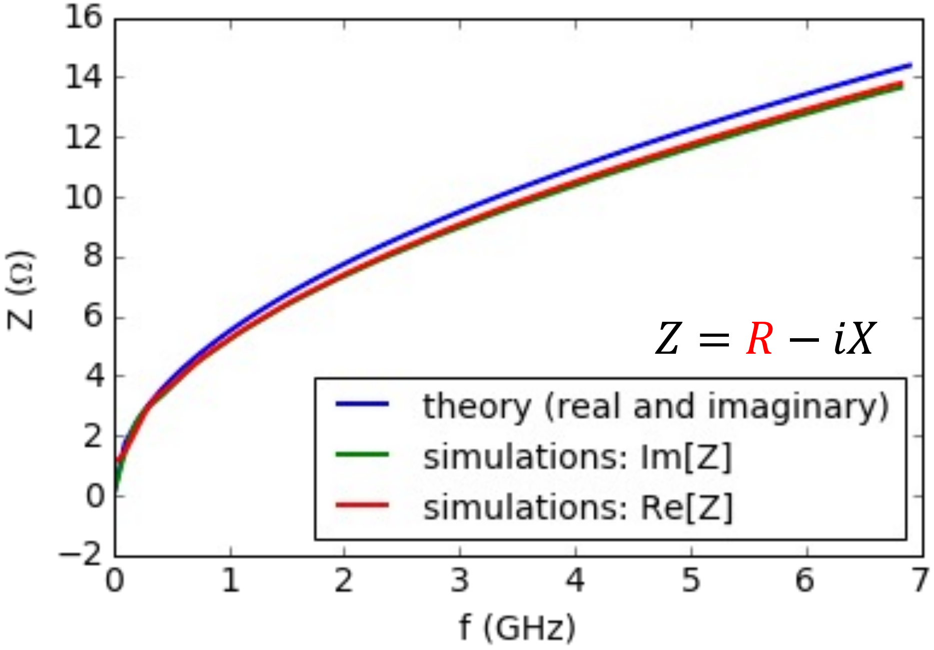
$$2\pi b B = \mu J 2\pi b(r-b) \quad \rightarrow \quad B = \mu J(r-b)$$

$$\Phi(B) = \ell\mu J\int_b^{b+\delta_w}(r-b)dr = \frac{\ell\mu J}{2}(r-b)^2\Big|_b^{b+\delta_w} = \frac{\ell\mu J}{2}\delta_w^2$$

$$I = J2\pi b\delta_w \quad \rightarrow \Phi(B) = \frac{\ell\mu I\delta_w}{4\pi b} \qquad\qquad L = \frac{\Phi}{I} = \frac{\ell\mu\delta_w}{4\pi b}$$

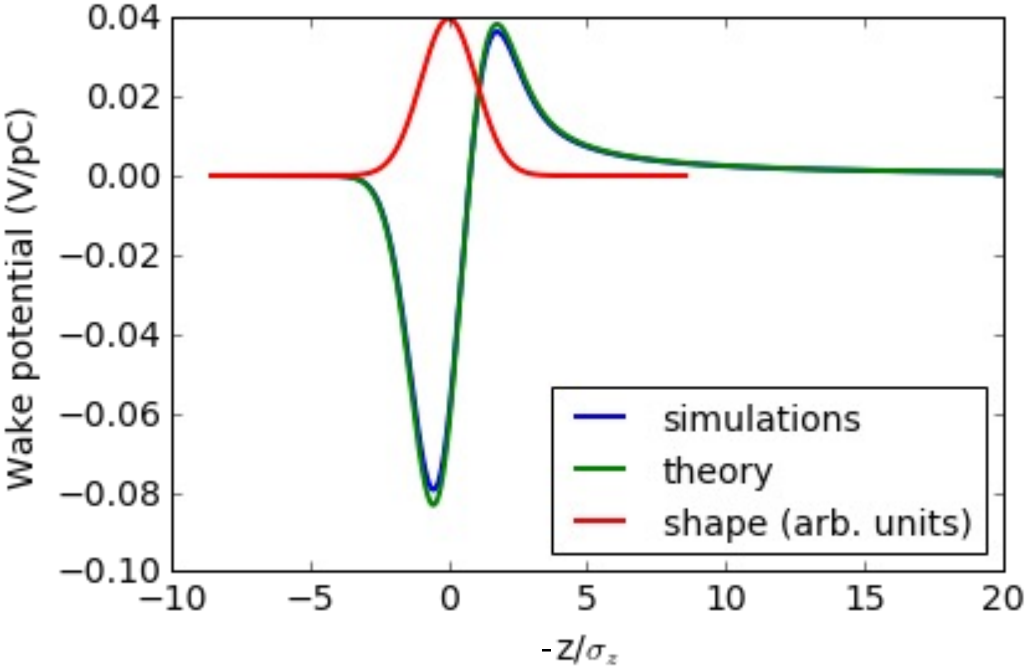
$$-Z_{im} = \omega L = \frac{\ell}{2\pi b}\frac{\omega\mu\delta_w}{2} = \frac{\ell}{2\pi b}\sqrt{\frac{2\mu^2\omega^2}{4\omega\sigma_c\mu}} = \boxed{\frac{\ell}{2\pi b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}}$$

Example of longitudinal wakefield and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



Impedance comparison

Wake potential comparison



Quiz # 22

The resistive wall impedance

- Is a circuit inserted in the accelerator generating a resistance and an inductance
- Is the Fourier transform of the wakefield in frequency domain due to the finite conductivity of the pipe wall
- Can be known analytically in a certain frequency range only in case of perfectly conducting beam pipe

Example of longitudinal wakefield and coupling impedance: space charge

Even if in the ultra-relativistic limit with $\gamma \longrightarrow \infty$, we have seen that there is no space charge effect, we can still define a wakefield by considering a moderately relativistic beam with γ>>1 but not infinite. It turns out that the space charge forces can fit into the definition of wakefield, and when that is done, we find that the wake depends on beam properties such as the transverse beam radius a and the beam energy γ. Let us consider a relativistic beam with cylindrical symmetry and uniform transverse distribution. We have already obtained the longitudinal force acting on a charge of the beam travelling inside a cylindrical pipe of radius b:

$$F_{\parallel}(r,z) = \frac{-q}{4\pi\varepsilon_0\gamma^2}\left(1 - \frac{r^2}{a^2} + 2\ln\frac{b}{a}\right)\frac{\partial}{\partial z}\lambda(z)$$

Example of longitudinal wakefield and coupling impedance: space charge

We can define the longitudinal wakefield of a piece of pipe of length ℓ with constant radius. Assuming r→0 (particle on axis), and a charge line density given by $\lambda(z) = q_0\delta(z)$, we obtain

$$w_{\parallel}(z) = -\frac{1}{qq_0}\int_0^L F_{\parallel} ds \qquad w_{\parallel}(z) = \frac{\ell}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{d}{dz}\delta(z)$$

$$Z_{\parallel}(\omega) = \frac{1}{v}\int_{-\infty}^{\infty} w_{\parallel}(z)e^{-i\frac{\omega}{v}z}dz = \ell\frac{1 + 2\ln(b/a)}{v4\pi\varepsilon_0\gamma^2}\int_{-\infty}^{\infty} e^{-i\frac{\omega}{v}z}\left(\frac{d}{dz}\delta(z)\right)dz$$

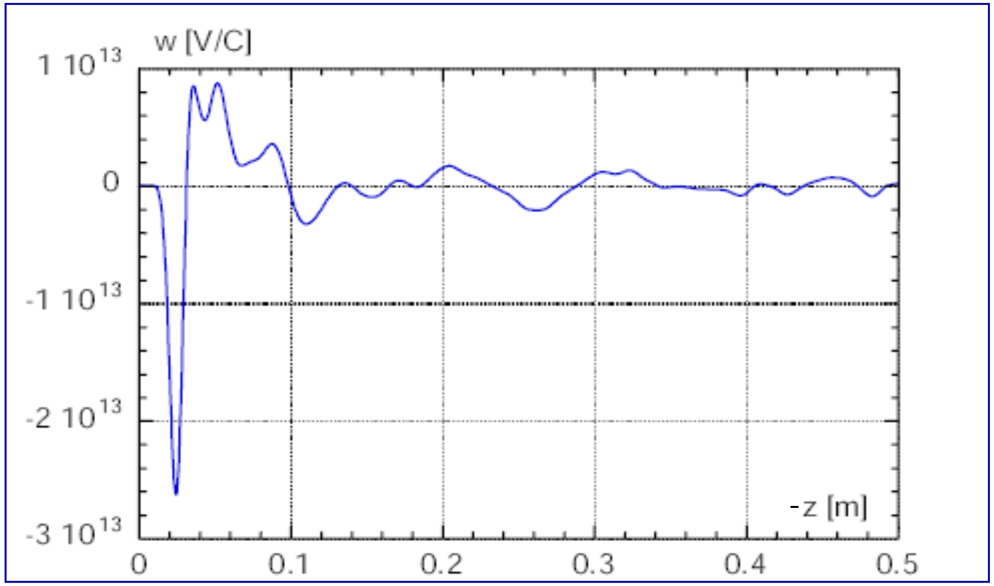
$$\boxed{Z_{\parallel}(\omega) = \ell\frac{i\omega Z_0}{4\pi c\beta^2\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)}$$

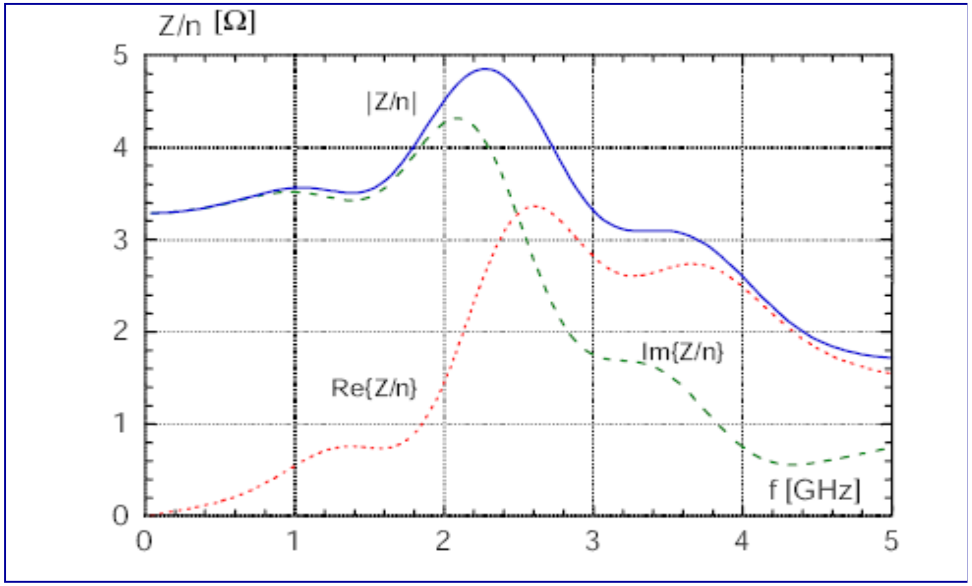
What happens when we consider an entire machine?

In many cases we can ignore the em interference between different devices, which can then be evaluated individually.

The impedance (and wakefield) of the machine is simply the sum of that of all the elements

Example of wake potential and longitudinal coupling impedance for DAΦNE accumulator



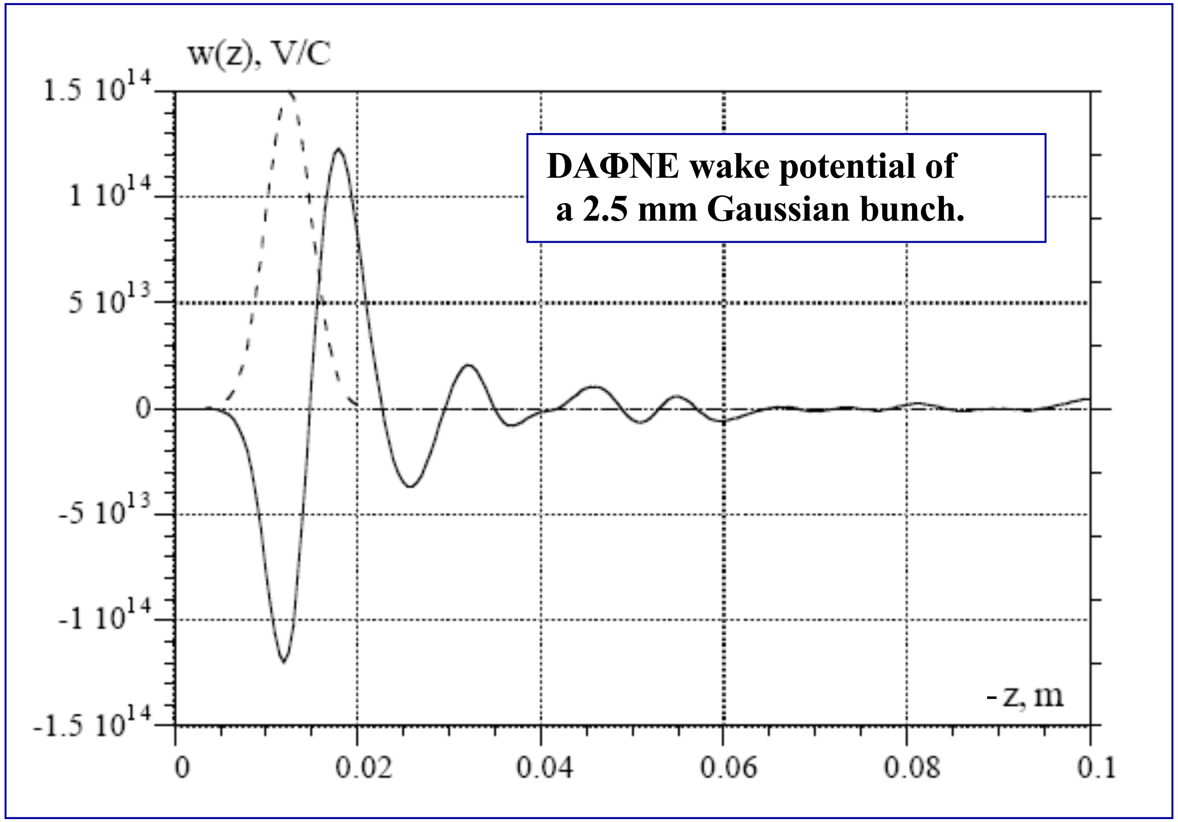


DAΦNE accumulator wake potential of a 2.5 mm Gaussian bunch.

$$\frac{Z_{\|}(\omega)}{n}=\frac{Z_{\|}(\omega)}{\omega/\omega_o}$$

Another important necessary distinction is between short and long range wakefield/potential

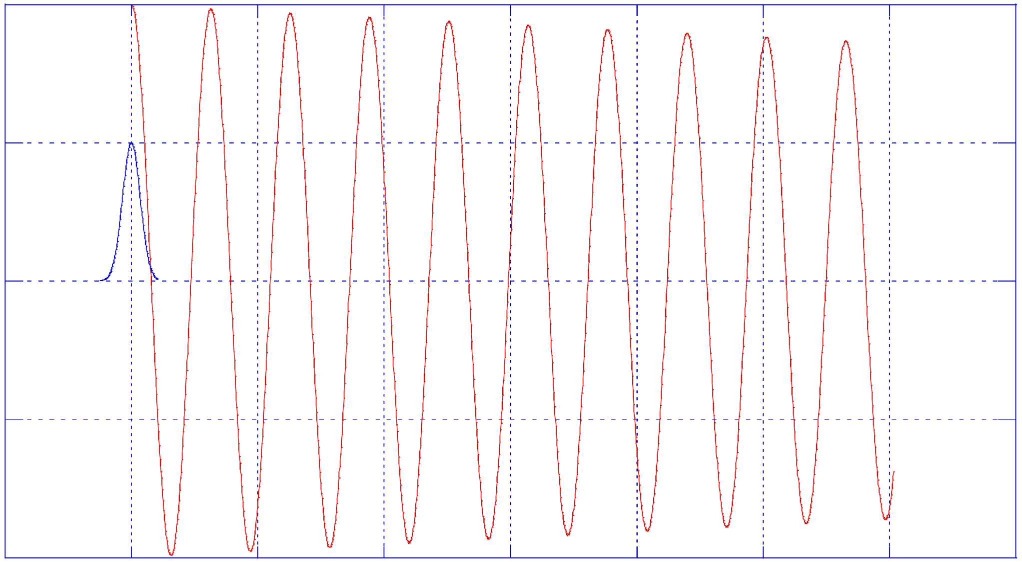
Short range wakefield/potential acts over the bunch length

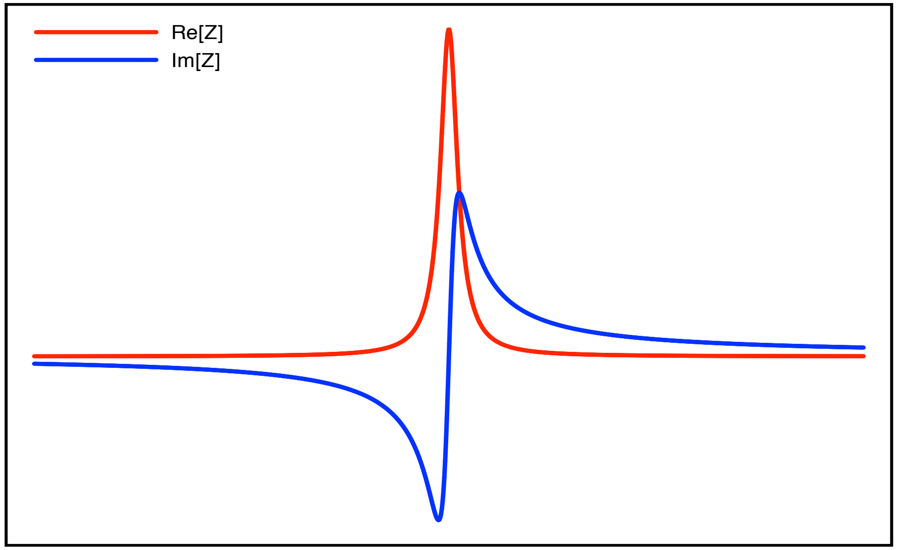


DAΦNE wake potential of a 2.5 mm Gaussian bunch.

- Vanishes after a distance of few bunch lengths
- Influences the single bunch beam dynamics
- Poor frequency resolution of Fourier transform of coupling impedance => broad band impedance

Long range wakefield/potential acts on many bunches/multi-turn

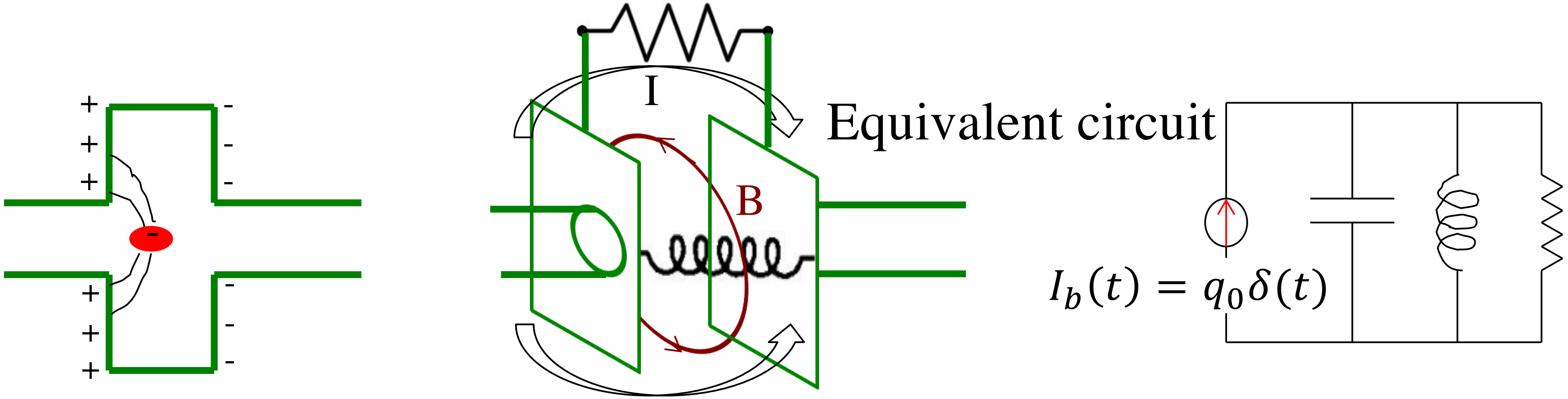




- Field oscillates over long distances
- High peak impedance
- Produced by high quality resonant modes
- Described by only 3 parameters: Q, ω_r and R_s

Longitudinal wakefield of a resonant mode

When a charge crosses a resonant structure, as an RF cavity, it excites resonant modes (fundamental and HOMs).



Equivalent circuit

$I_b(t) = q_0 \delta(t)$

Each mode can be treated as an electric RLC circuit loaded by an impulsive current. Just after the charge passage, the capacitor is charged with a voltage $V_o = q_o$ /C and the electric field is $E_{so} = V_o / l_o$.

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount V_0. This potential will oscillate and decay producing a current flow in the resistor and inductance.

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0$$

For $t > 0$ the potential satisfies the following equations and initial conditions:

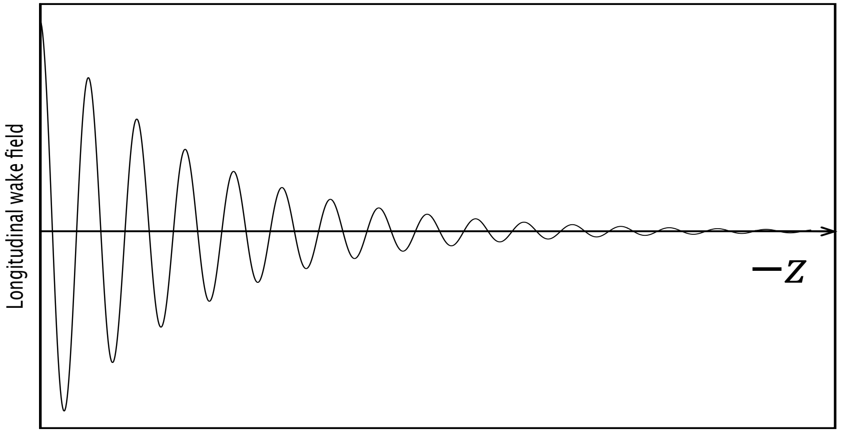
$$V(t) = V_0 e^{-\gamma t}\left[\cos(\omega_n t) - \frac{\gamma}{\omega_n}\sin(\omega_n t)\right]$$

$$V(t = 0^+) = \frac{q_0}{C} = V_0$$

$$\omega_n^2 = \omega_r^2 - \gamma^2 \qquad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\gamma = \frac{1}{2RC}$$

$$\dot{V}(t = 0^+) = \frac{\dot{q}}{C} = -\frac{I(0^+)}{C} = -\frac{V_0}{RC}$$



putting z = -ct (z is negative behind the source charge),

$$w_0 = \frac{1}{C}$$

$$\boxed{w_{\|}(z) = \frac{V(z)}{q_0} = w_0 e^{\frac{\gamma z}{c}}\left[\cos\left(\frac{\omega_n z}{c}\right) + \frac{\gamma}{\omega_n}\sin\left(\frac{\omega_n z}{c}\right)\right]H(-z)}$$

Coupling impedances of a resonant mode

Longitudinal Impedance:

$$Z_{\parallel}(\omega) = \frac{R_s}{1 - iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

The parameters R_s, Q and ω_r, that can be evaluated by computer codes, can be related to the parameters *RLC* of the parallel circuit

shunt resistance: $R_s = R = \frac{w_0}{2\gamma}$

quality factor: $Q = \frac{\omega_r}{2\gamma}$

Transverse wakefield and impedance of a resonant mode:

$$\boxed{w_{\perp}(z) = \frac{R_{\perp}\omega_r}{Q} e^{\frac{\gamma z}{c}} \sin\left(\frac{\omega_n z}{c}\right)}$$

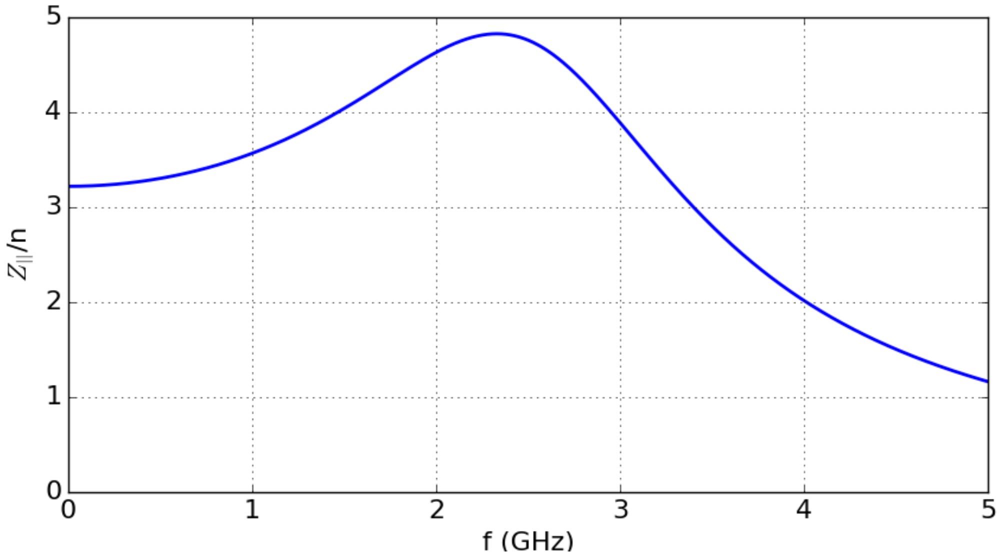
$$Z_{\perp}(\omega) = \frac{\omega_n}{\omega} \frac{R_{\perp}}{1 - iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

Some remarks on the longitudinal impedance of a resonant mode

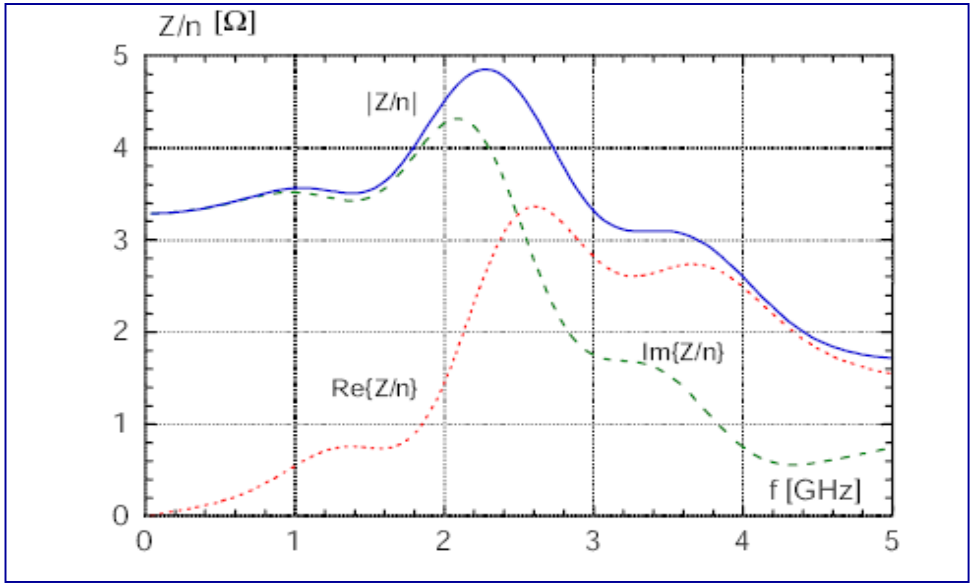
$$Z_{\parallel}(\omega) = \frac{R_s}{1 - iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

This impedance can be also used as a simplified impedance model of a whole machine for the short range wakefields assuming Q ~ 1 (it is called **Broad Band Impedance Model)**

Broad Band Resonator Model



DAΦNE Accumulator Impedance



Another Broad Band Impedance model

Another simple **Broad Band Impedance Model** is obtained by a phenomenological expansion over $\sqrt{\omega}$ of the different contributions to a machine impedance. By considering only the first two terms of the expansion, we have the so called RL impedance model

$$Z_{\parallel}(\omega) = R - i\omega L$$

The resistive term R takes into account the losses of the beam, and the second term, which represents an inductive impedance, gives the low frequency behaviour typical of tapers, shielded bellows and vacuum ports, small discontinuities as slots, shallow cavities in flanges …

The wakefield of the resistive impedance is just proportional to the Dirac delta function $w_{\parallel}(z) = cR\delta(z)$, while that of the inductance is similar to what we have obtained for the space charge.

Quiz # 23

How can we obtain the wakefield of a resonant mode?

- From the voltage across the capacitor of an RLC circuit excited by an impulsive current
- Only by using an electromagnetic solver (as CST Microwave Studio)
- From the broad band R-L impedance model of an accelerator
- None of the above answers is correct

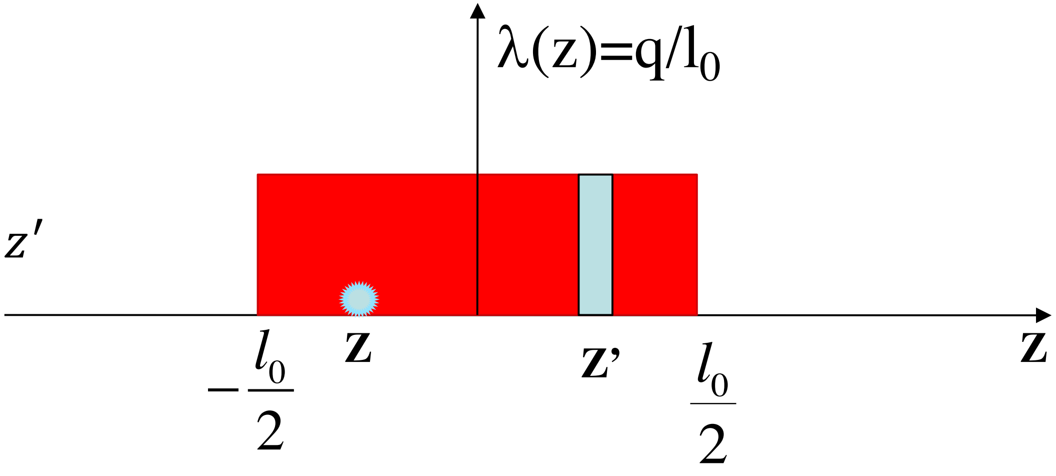
Wakefields effects in LINACS

Example: Energy lost by a finite uniform beam due to a resonant mode

$$w_{\parallel}(z) = w_0 e^{\frac{\gamma z}{c}} \left[\cos\left(\frac{\omega_n z}{c}\right) + \frac{\gamma}{\omega_n} \sin\left(\frac{\omega_n z}{c}\right)\right] H(-z) \simeq w_0 \cos\left(\frac{\omega_r z}{c}\right) H(-z)$$

$$U(z) = -e \int_{-\infty}^{\infty} w_{\parallel}(z - z')\lambda(z')dz'$$

$$U(z) = -\frac{eqw_0}{l_0} \int_z^{\frac{l_0}{2}} \cos\left[\frac{\omega_r}{c}(z - z')\right] dz'$$



$$(z - z') = x$$

$$U(z) = \frac{eqw_0}{l_0} \int_0^{z-\frac{l_0}{2}} \cos\left(\frac{\omega_r}{c} x\right) dx =$$

$$= \frac{eqw_0}{l_0} \left[\frac{\sin\left(\frac{\omega_r}{c} x\right)}{\left(\frac{\omega_r}{c}\right)}\right]_0^{z-\frac{l_0}{2}}$$

$$U(z) = -\frac{eqw_0}{2} \left[\frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\right]$$

Example: Energy lost by a finite uniform beam due to a resonant mode

$$w_{\parallel}(z) = w_0 e^{\frac{\gamma z}{c}} \left[\cos\left(\frac{\omega_n z}{c}\right) + \frac{\gamma}{\omega_n} \sin\left(\frac{\omega_n z}{c}\right)\right] H(-z) \simeq w_0 \cos\left(\frac{\omega_r z}{c}\right) H(-z)$$

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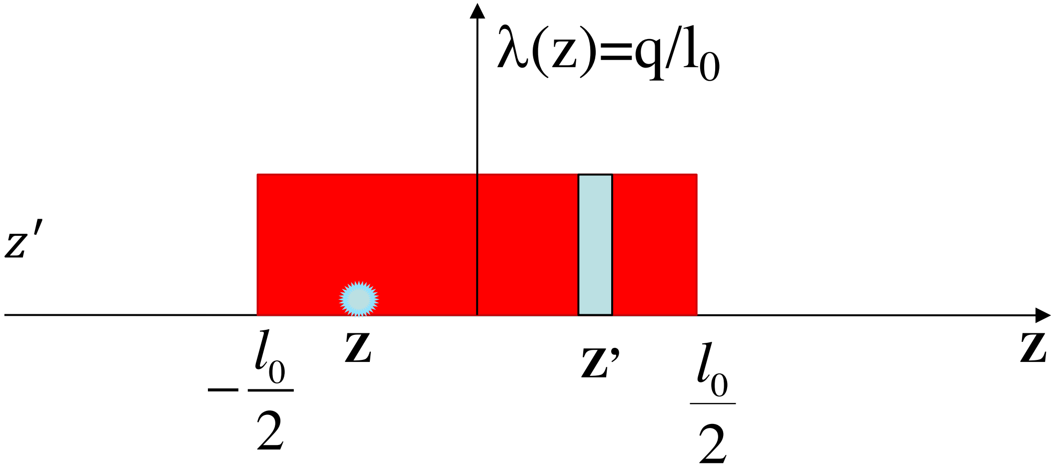
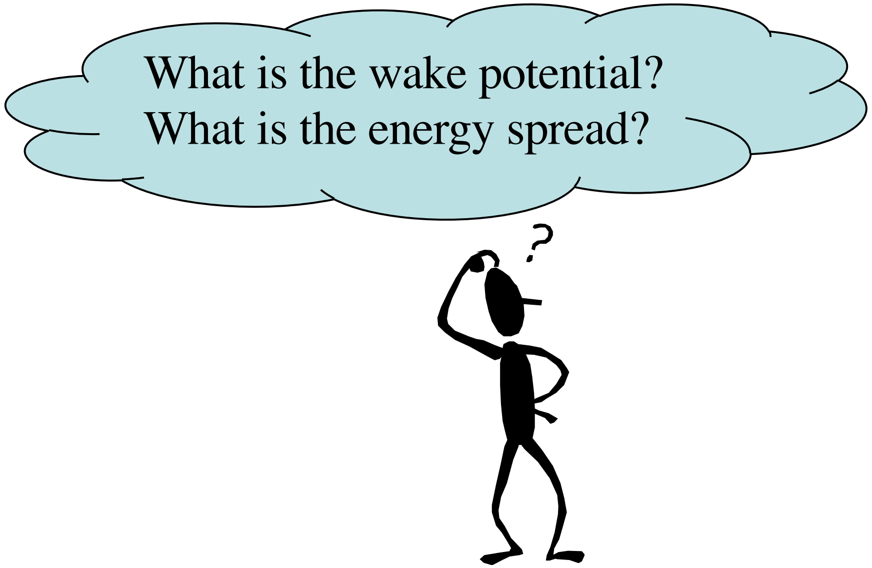


$$(z - z') = x$$

$$U(z) = \frac{eqw_0}{l_0} \int_{0}^{z-\frac{l_0}{2}} \cos\left(\frac{\omega_r}{c} x\right) dx =$$

$$= \frac{eqw_0}{l_0} \left[\frac{\sin\left(\frac{\omega_r}{c} x\right)}{\left(\frac{\omega_r}{c}\right)}\right]_0^{z-\frac{l_0}{2}}$$

$$U(z) = -\frac{eqw_0}{2} \left[\frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\right]$$



What is the wake potential?
What is the energy spread?

Energy loss

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U(z)\lambda(z)dz \simeq \frac{-q^2 w_0}{2l_0\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\int_{-\frac{l_0}{2}}^{\frac{l_0}{2}} \sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2}-z\right)\right]dz$$

$$U_{bunch} = \frac{-q^2 w_0 c}{\omega_r l_0^2}\left|\frac{-\cos\left[\frac{\omega_r}{c}\left(\frac{l_0}{2}-z\right)\right]}{-\frac{\omega_r}{c}}\right|_{-\frac{l_0}{2}}^{\frac{l_0}{2}}$$

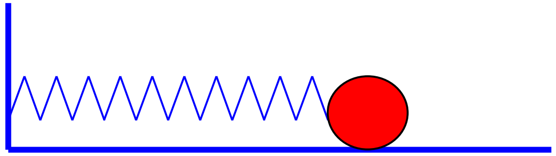
$$U_{bunch} = \frac{-q^2 w_0 c^2}{\omega_r^2 l_0^2}\left[1-\cos\left(\frac{\omega_r l_0}{c}\right)\right] = -\frac{2q^2 w_0 c^2}{\omega_r^2 l_0^2}\sin^2\left(\frac{\omega_r l_0}{2c}\right)$$

$$U_{bunch} = -\frac{q^2 w_0}{2}\frac{\sin^2\left(\frac{\omega_r l_0}{2c}\right)}{\left(\frac{\omega_r l_0}{2c}\right)^2}$$

$$\lim_{l_0\to 0} U_{bunch} = -q^2 k = ?$$

Instability: driven oscillator

Consider a harmonic oscillator with natural frequency ω and with an external excitation at frequency Ω. Instead of time, let us use, as independent variable, $s = ct$:



$$x'' + \frac{\omega^2}{c^2} x = A\cos\left(\frac{\Omega s}{c}\right)$$

General solution:

$$x(s) = x^{free}(s) + x^{driven}(s)$$

$$\cos\left(\frac{\Omega s}{c}\right) \Longrightarrow e^{i\frac{\Omega s}{c}}$$

$$x^{free}(s) = \tilde{x}_m^f e^{i\frac{\omega s}{c}}$$

$$x^{driven}(s) = \tilde{x}_m^d e^{i\frac{\Omega s}{c}}$$

substitution in the diff. equation:

$$(\omega^2 - \Omega^2)\, \tilde{x}_m^d e^{i\frac{\Omega s}{c}} = c^2 A e^{i\frac{\Omega s}{c}}$$

$$x^{driven}(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)} e^{i\frac{\Omega s}{c}}$$

The general solution has to satisfy the initial conditions at s=0. In our case we assume that the oscillator is at rest for s=0:

$$x^{free}(s=0) = -x^{driven}(s=0)$$

$$\tilde{x}_m^f = -\frac{c^2 A}{(\omega^2 - \Omega^2)}$$

thus we get:

$$x(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)}\left(e^{i\frac{\Omega s}{c}} - e^{i\frac{\omega s}{c}}\right)$$

taking only the real part:

$$x(s) = \frac{c^2 A}{(\omega^2 - \Omega^2)}\left[\cos\left(\frac{\Omega s}{c}\right) - \cos\left(\frac{\omega s}{c}\right)\right]$$

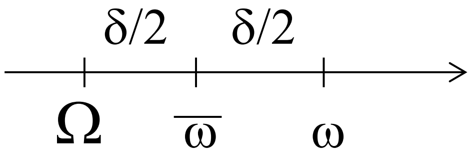
NB: if the initial conditions are different, we just need to add to our solution a sinusoidal term

$$x(s) = X_0 \cos\left(\frac{\omega s}{c} + \theta_0\right) + \frac{c^2 A}{(\omega^2 - \Omega^2)}\left[\cos\left(\frac{\Omega s}{c}\right) - \cos\left(\frac{\omega s}{c}\right)\right]$$

This expression is suitable for deriving the response of the oscillator driven at resonance or at frequency very close:

$$\omega = \Omega + \delta, \qquad \delta \to 0$$

$$\varpi = (\omega + \Omega)/2; \quad \omega = \varpi + \delta/2, \qquad \Omega = \varpi - \delta/2$$



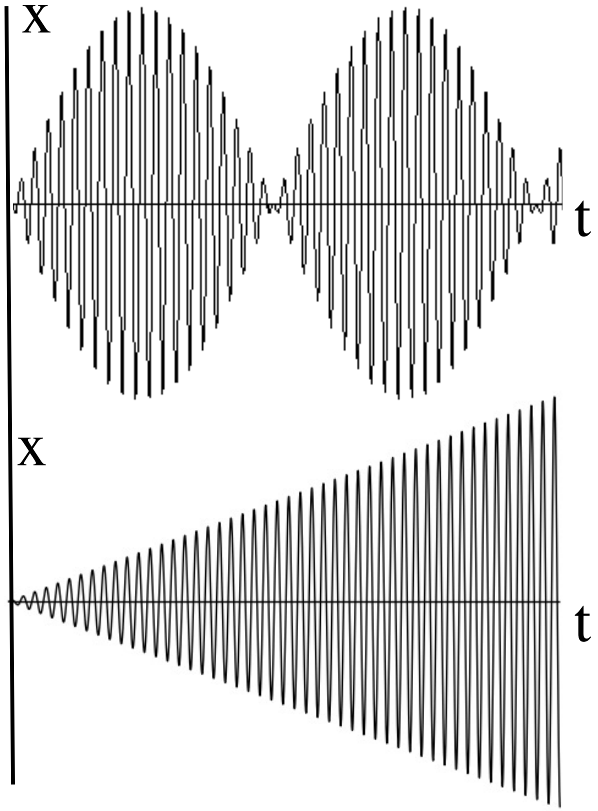
$$\omega^2 - \Omega^2 = (\omega - \Omega)(\omega + \Omega) = \delta 2\varpi$$

$$x(s) = \frac{c^2 A}{2\varpi\delta}\left[\cos\left(\frac{\varpi s}{c}\right)\cos\left(\frac{\delta s}{2c}\right) + \sin\left(\frac{\varpi s}{c}\right)\sin\left(\frac{\delta s}{2c}\right)\right] +$$

$$-\left[\cos\left(\frac{\varpi s}{c}\right)\cos\left(\frac{\delta s}{2c}\right) - \sin\left(\frac{\varpi s}{c}\right)\sin\left(\frac{\delta s}{2c}\right)\right]$$

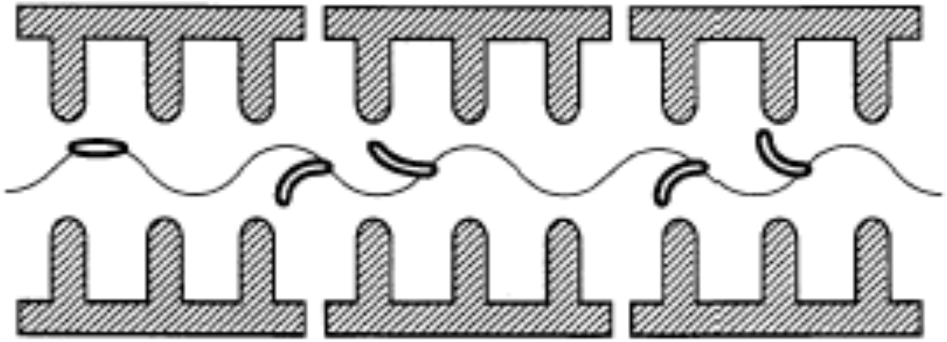
$$x(s) = \overbrace{\frac{c^2 A}{\varpi\delta}\sin\left(\frac{\delta s}{2c}\right)}^{\text{amplitude modulation}}\sin\left(\frac{\varpi s}{c}\right) = \frac{cAs}{2\varpi}\sin\left(\frac{\varpi s}{c}\right)\frac{\sin\left(\frac{\delta s}{2c}\right)}{\frac{\delta s}{2c}}$$

$$\lim_{\delta\to 0} x(s) = \frac{cAs}{2\varpi}\sin\left(\frac{\varpi s}{c}\right)$$

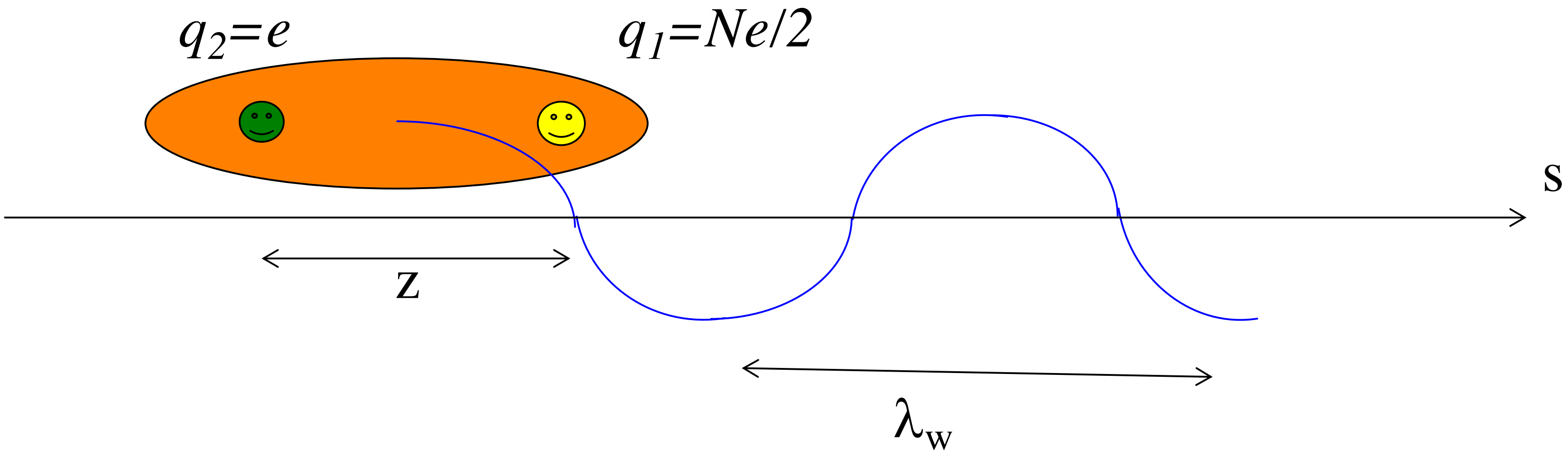


Single Bunch Beam Break Up in Linacs

A beam injected off-centre in a LINAC, because of the focusing quadrupoles, executes betatron oscillations. The displacement produces a transverse wakefield in all the devices crossed during the flight, which deflects the trailing charges.



In order to understand the effect, we consider a simple model with only two charges $q_1 = Ne/2$ (source charge = half bunch) and $q_2 = e$ (test charge = single charge).



the source charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right); \quad \frac{\omega_y}{c} = \frac{2\pi}{\lambda_\beta} = \frac{Q_y}{\rho_x}$$

the test charge, at a distance z behind, over a length L_w experiences a deflecting force proportional to the displacement y_1, and dependent on the distance z:

$$\boxed{w_\perp(z) = \frac{M(z)}{q^2}}$$

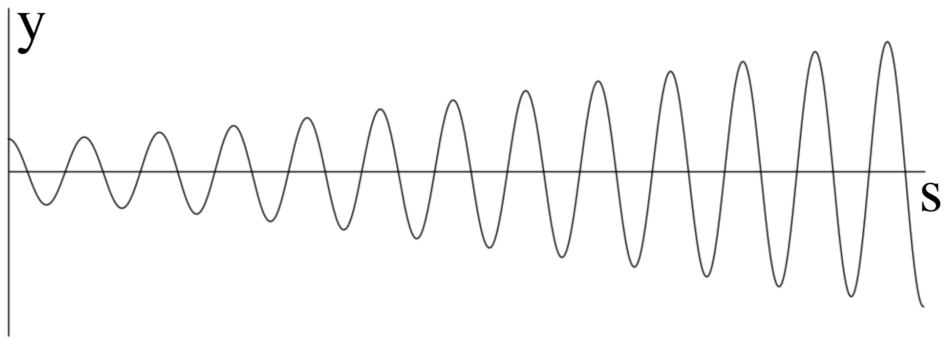
$e \qquad Ne/2$

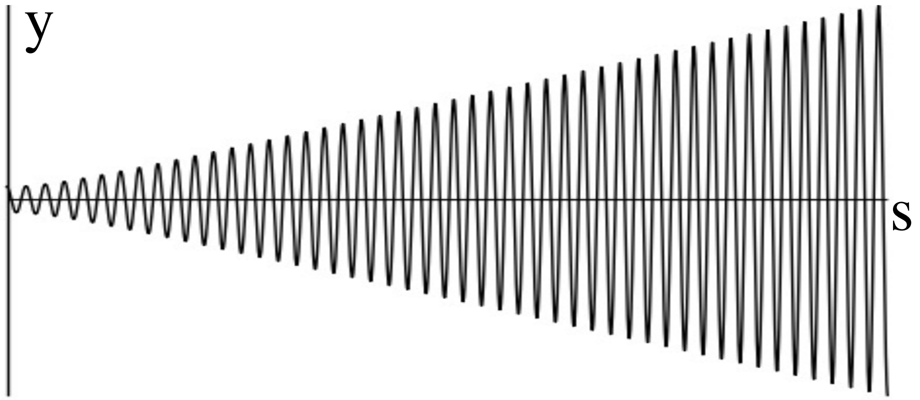
$$r_0 M(z) = \int_0^{L_w} F_\perp ds = \langle F_\perp(r_0, z)\rangle L_w \quad \Rightarrow \quad \langle F_\perp(r_0, z)\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

This force drives the motion of the test charge:

$$\overbrace{y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{1}{\beta^2 E_0}\langle F_\perp(r_0, z)\rangle}^{\textbf{betatron equation of motion with coherent force}} = \frac{Ne^2}{2\beta^2 E_0 L_w} w_\perp(z) \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

This is the typical equation of a harmonic oscillator driven at the resonant frequency. The solution is given by the superposition of the “free” oscillation and a “driven” oscillation, which, being driven at the resonant frequency, grows linearly with s.





$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c} s\right) + y_2^{driven}$$

$$y_2^{driven} = \frac{cNe^2 w_\perp(z) s}{4\omega_y E_0 L_w} \hat{y}_1 \sin\left(\frac{\omega_y}{c} s\right)$$

$(\beta = 1)$

At the end of the LINAC of length L_L, the oscillation amplitude of the tail with respect to the head is grown by $(\hat{y}_1 = \hat{y}_2)$

$$\boxed{\left[\frac{y_2(L_L) - y_1(L_L)}{\hat{y}_1}\right]_{max} = \frac{cNew_\perp(z) L_L}{4\omega_y (E_0/e) L_w}}$$

Balakin-Novokhatsky-Smirnov Damping

The BBU instability can be quite harmful and hard to take under control even at high energy, with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is due to the **“resonant” driving force**.

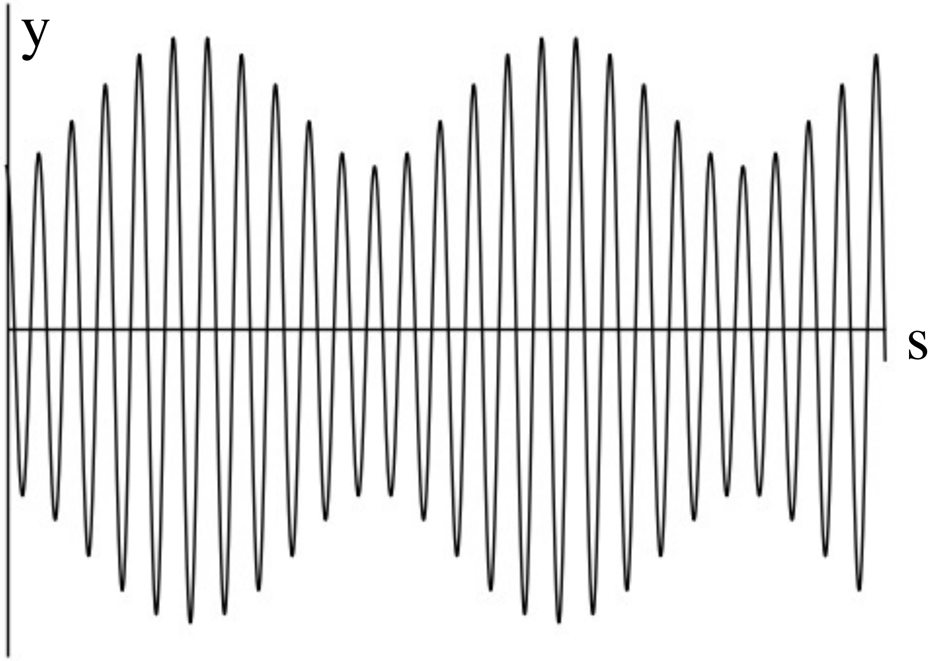
If the tail and the head of the bunch oscillate with different frequencies, this effect can be significantly removed.

Let us assume that the tail oscillates with a frequency $\omega_y + \Delta\omega_y$, the equation of motion becomes:

$$\boxed{y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2}{2\beta^2 E_0 L_w} w_\perp(z)\hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)}$$

the solution of which is $(\hat{y}_1 = \hat{y}_2)$

$$y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + \frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_0 L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y}{c} s\right) - \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right)\right]$$



by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail.

$$\frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_0 L_w} = 1 \qquad \Longrightarrow \qquad \boxed{y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) = y_1(s)}$$

$$\boxed{\Delta\omega_y = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y E_0 L_w}}$$

The extra focusing on the tail can be obtained by:

- Using an RFQ, where head and tail see a different focusing strength.

- Creating a correlated energy distribution along the bunch which, because of the chromaticity, induces a spread in the betatron frequencies. An energy spread correlated with the longitudinal position is attainable with the external accelerating voltage, or with the longitudinal wakefields.

Quiz # 24

What have we studied in this part of the lecture?

- Two effects of wakefields in LINACS: driven oscillator and BNS damping
- Nothing interesting
- The transverse betatron motion of a particle inside a Linac
- Two effects of wakefields in LINACS: energy spread (and loss) and a type of instability (BBU)

Instabilities in Circular Accelerators

Longitudinal effects on beam dynamics

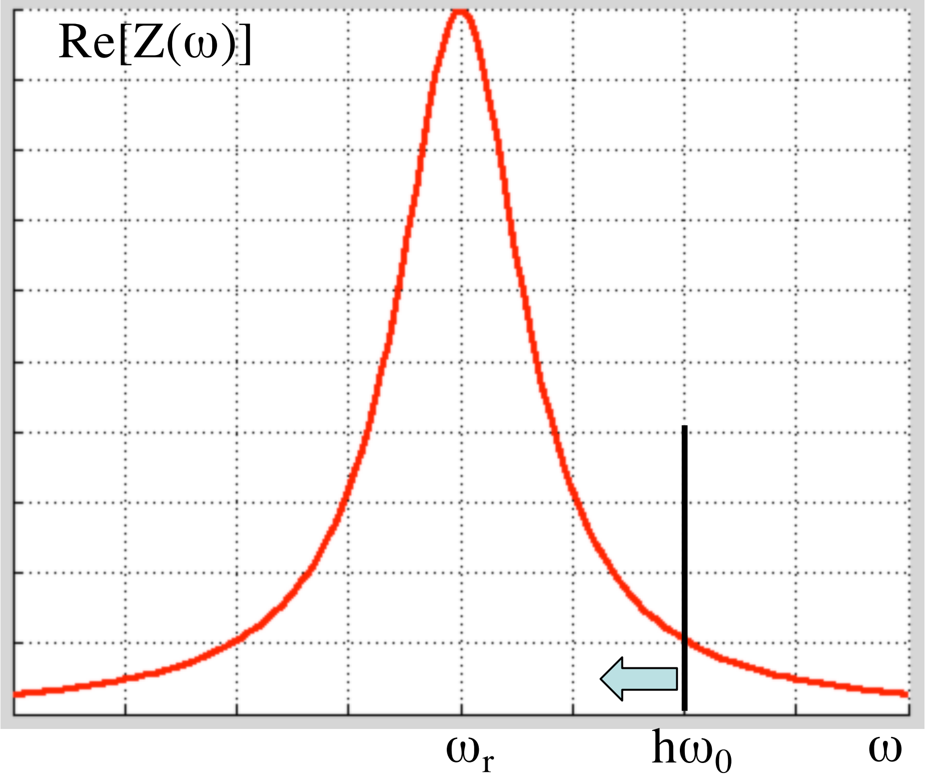
Short range wakefields:

- Potential well distortion ➔ deformation of the longitudinal distribution
- Longitudinal emittance growth, microwave instability

Long range wakefields:

- Robinson instability (RF fundamental mode)
- Coupled bunch instability (HOMs)

Robinson instability of the RF fundamental mode



Let us consider the real part of the RF fundamental mode, and a bunch with revolution period T_0. The bunch spectrum has lines every ω_0 (we suppose the bunch as a point charge), and its lost energy due to the mode is proportional to the real part of the impedance at $h\omega_0$. If the bunch, during the synchrotron oscillations, has an increasing energy, and we are above transition, its revolution period increases and the frequency decreases.

If ($h\omega_0 > \omega_r$), as in the figure, the resistance found by the beam is higher, producing a higer energy loss, which reduces the energy increase giving a stabilizing effect.

Robinson instability of the RF fundamental mode

Longitudinal equations of motion of the bunch centre of mass, for constant energy in a circular machine, ignoring radiation damping

$$\frac{d\phi}{dt} = -\frac{h\eta}{R_0 p_0}\Delta E \qquad\qquad \frac{d\Delta E}{dt} = \frac{qV_{rf}}{T_0}(\sin\phi - \sin\phi_s)$$

Combining the two equations, for small oscillation amplitudes, we obtain a second order linear differential equation

$$\frac{d^2\Delta\phi}{dt^2} + \omega_s^2\Delta\phi = 0 \quad \text{with} \quad \omega_s^2 = \frac{qV_{rf}h\eta c^2\cos\phi_s}{2\pi R_0^2 E_0} \quad \text{and} \quad \eta\cos\phi_s > 0$$

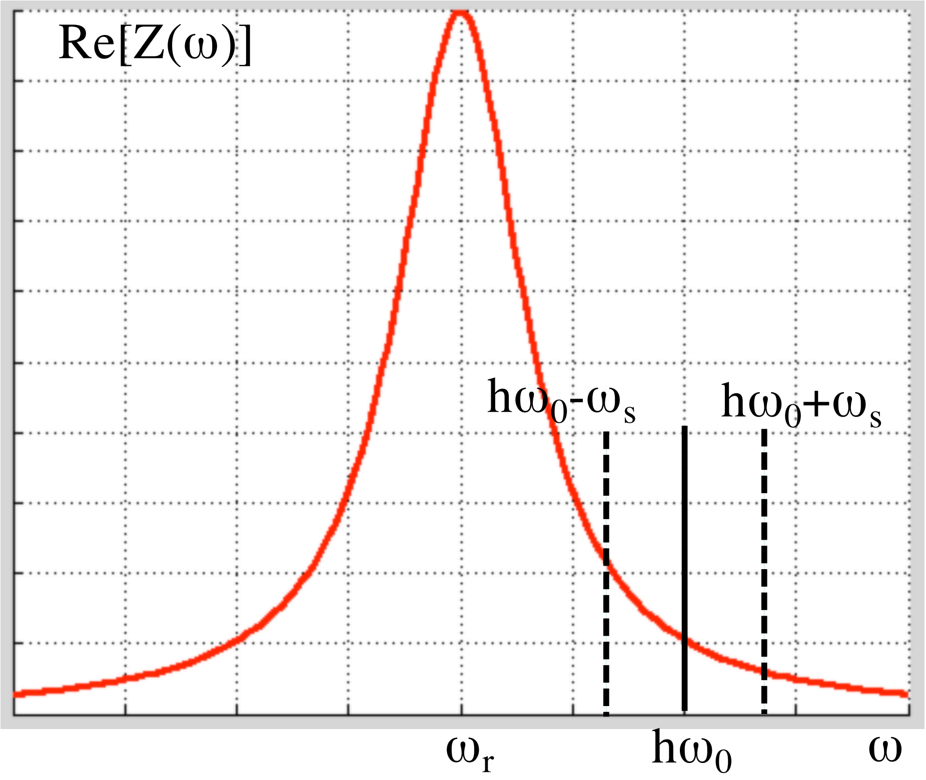
Solution $\Delta\phi = \Delta\phi_{max}\cos(\omega_s t + \theta_0)$

Robinson instability of the RF fundamental mode

By including also the wakefield of the fundamental resonant mode (beam loading effect) the equation of motion becomes

damping/exciting term due to the resonant mode

$$\frac{d^2\Delta\phi}{dt^2}+\boxed{2\alpha_r\frac{d\Delta\phi}{dt}}+\omega_s^2\Delta\phi=0$$



If $\alpha_r<\omega_s \;\rightarrow \omega_n=\sqrt{\omega_s^2-\alpha_r^2}$

Solution

$$\Delta\phi=\Delta\phi_{max}\exp[-\alpha_r t]\cos(\omega_n t+\theta_0)$$

$$\alpha_r=\frac{eN_p\eta h\omega_0}{2\omega_s(E_0/e)T_0^2}\mathrm{Re}[\Delta Z] \qquad \mathrm{Re}[\Delta Z]=\mathrm{Re}[Z(h\omega_0+\omega_s)-Z(h\omega_0-\omega_s)]$$

OTHER EFFECTS NOT DISCUSSED HERE: LANDAU DAMPING

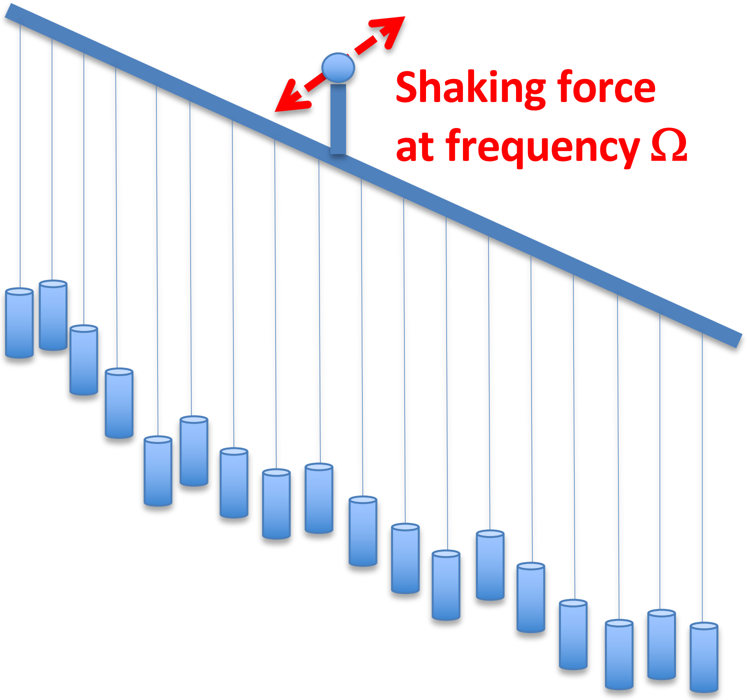
- There is a natural stabilising effect against the collective instabilities called "Landau Damping". The basic mechanism relies on the fact that if the particles in a beam have a spread in their natural frequencies (synchrotron or betatron), their motion can't be coherent for a long time.

- The mechanism is in general triggered when an infinite set of identical systems oscillates at different frequencies, spread over some range of values. Under these conditions, if any periodic force has its frequency within the considered range, the oscillation amplitude, averaged over all the systems, instead of growing as one should expect, remains constant.

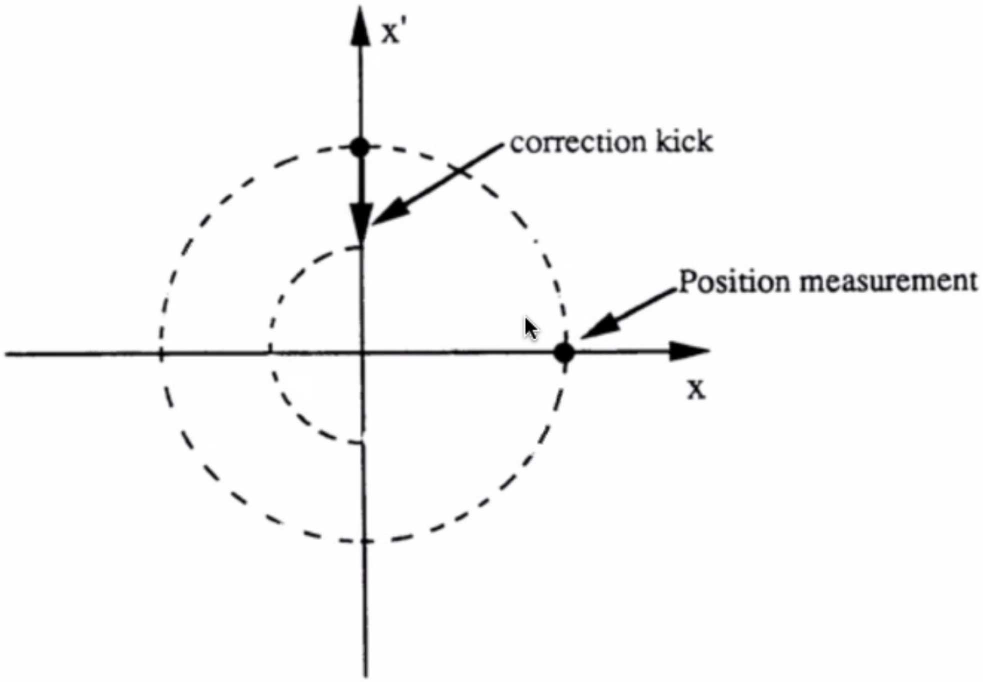
- Even if a periodic force pumps energy into the system, this energy is not converted into an increase of the average oscillation amplitude: the number of particles in resonance with the external force decreases with time, so that the net contribution to the average oscillation amplitude remains constant.

OTHER EFFECTS NOT DISCUSSED HERE:

LANDAU DAMPING



ACTIVE FEEDBACK SYSTEMS



OTHER EFFECTS NOT DISCUSSED HERE

- Vlasov and Fokker-Plank equations

These equations are used to study analytically the collective effects in circular accelerators.

The **Vlasov** equation describes the collective behaviour of a multiparticle system under the influence of electromagnetic forces.
It is valid if we can ignore diffusion or external damping effects, such as, for example, longitudinal and transverse beam dynamics of proton beams.

For electron beams, synchrotron radiation cannot be neglected and we obtain another equation called **Fokker-Planck** equation. It's stationary solution in the longitudinal plane is called **Haissinski** equation.

Vlasov equation is sometimes loosely referred to as the Liouville theorem. However it applies to a system of many particles when collisions among particles are excluded.

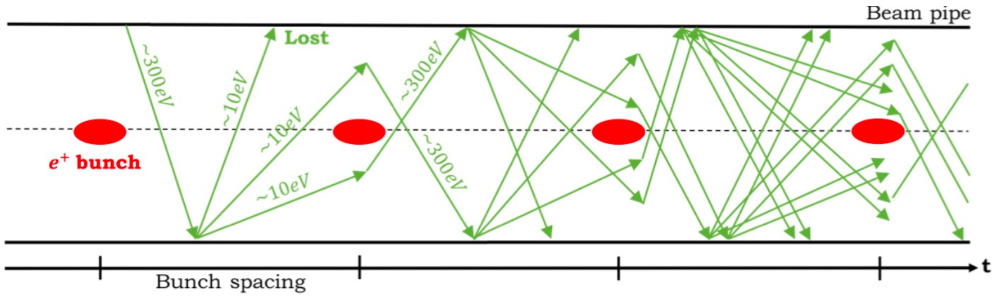
Strictly, the Liouville theorem applies to an ensemble of many systems, each containing many particles. It describes the conservation of density of the ensemble in the 2N-dimensional space and applies to situations much more general than that considered here, such as when collisions among discrete particles are included.

OTHER EFFECTS NOT DISCUSSED HERE

- Touschek effect and intra-beam scattering: they describe the scattering and loss of particles in a circular machine in the collisional regime. (See, e.g. A. Piwinski, in Proceedings of the 9th International Conference on High Energy Accelerators, Stanford, CA, 1974 (SLAC, Stanford, 1974), p. 405)

- Electron cloud

  Positive charges disturb electrons already in the tube, and bounce them into the wall. These electrons can be photo-electrons from synchrotron radiation or electrons from ionized gas molecules. When an electron hits the wall, the wall emits more electrons due to secondary emission. These electrons in turn hit another wall, releasing more and more electrons into the accelerator chamber.



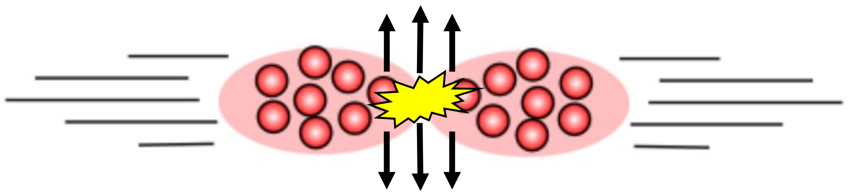
Sketch of the formation of an electron cloud in a section of the machine.

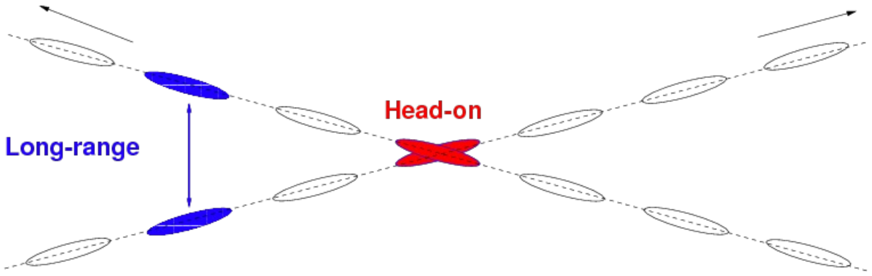
(See, e.g. ELECTRON CLOUD STUDIES FOR CERN PARTICLE ACCELERATORS AND SIMULATION CODE DEVELOPMENT, Doctoral Thesis by G. Iadarola, CERN-THESIS-2014-047)

OTHER EFFECTS NOT DISCUSSED HERE

- Beam-beam

(See, e.g. W. Herr, T. Pieloni, Beam-beam effects, DOI: 10.5170/CERN-2014-009.431)





- Beam Ion Instabilities

(See, e.g. C. Li , S. Tian, N. Wang, H. Xu, Beam-ion instability and its mitigation with feedback system, Phys. Rev. AB 23, 074401 – 2020).

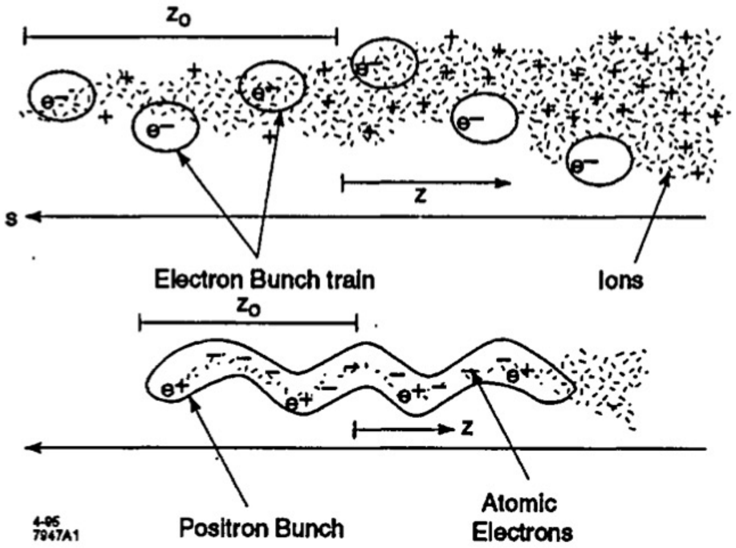


Figure 3: Schematic of fast beam-ion instability which can arise due to ion trapping in electron bunch train or due to trapping of free electrons in single positron bunch.

Quiz # 25

The Landau damping:

- Does not regard the beam dynamics of an accelerator
- Describes the scattering and loss of particles in a circular machine in the collisional regime
- It is a stabilizing effect happening when a beam has a spread in the synchrotron or betatron frequencies
- Is an instability mechanism that can be derived from the Vlasov equation

Appendix

Relationship between transverse and longitudinal forces:

The transverse gradient of the longitudinal force is equal to the longitudinal gradient of the transverse force

"**Panofsky-Wenzel theorem**".

$$\nabla_\perp F_{//} = \frac{\partial}{\partial z}\boldsymbol{F}_\perp$$

$$\nabla_\perp w_{//} = \frac{\partial}{\partial z}\boldsymbol{w}_\perp$$

References

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A. Mosnier - *Instabilities in Linacs* - CAS (Advanced) - 1994

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G. V. Stupakov - *Wake and Impedance* - SLAC-PUB-8683

K. Y. Ng – *Physics of intensity dependent beam instabilities* – US Particle Accelerator School, 2002