Exercises on Space Charge

Exercise 1

Compute the transverse space charge forces and the incoherent tune shifts for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian.

Evaluate also the tune spread (max tune shift – min tune shift) produced by the space charge forces with the same distributions.

parabolic	$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$
sinusoidal modulation	$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \quad ; \quad k_z = 2\pi / \lambda_w$
Gaussian	$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$

$$\boxed{E_r(r) = \frac{\lambda(z)r}{2\pi\varepsilon_0 a^2}; \quad B_\theta(r) = \frac{\lambda(z)\beta r}{2\pi\varepsilon_0 c a^2} \\ F_r(r) = e(E_r - vB_\theta) = \frac{eE_r(r)}{\gamma^2} = \frac{e}{\gamma^2}\frac{\lambda(z)r}{2\pi\varepsilon_0 a^2}}$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0 a^2} = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}}$$

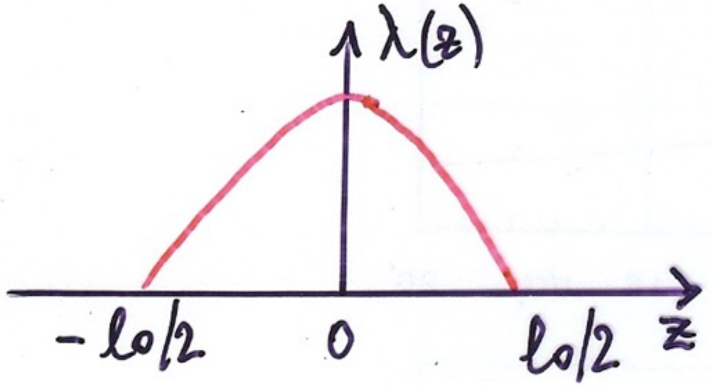
$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$ (electrons: 2.82×10^{-15} m, protons: 1.53×10^{-18} m)

$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

a) Parabolic bunch (q_0 = Ne)

$$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$$



$$\Delta Q_{\max}(\text{at } z = 0) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3 a^2 Q_{x0}}\frac{3N}{2l_0}$$

$$\Delta Q_{\min}\left(\text{at } z = \pm\frac{l_0}{2}\right) = 0$$

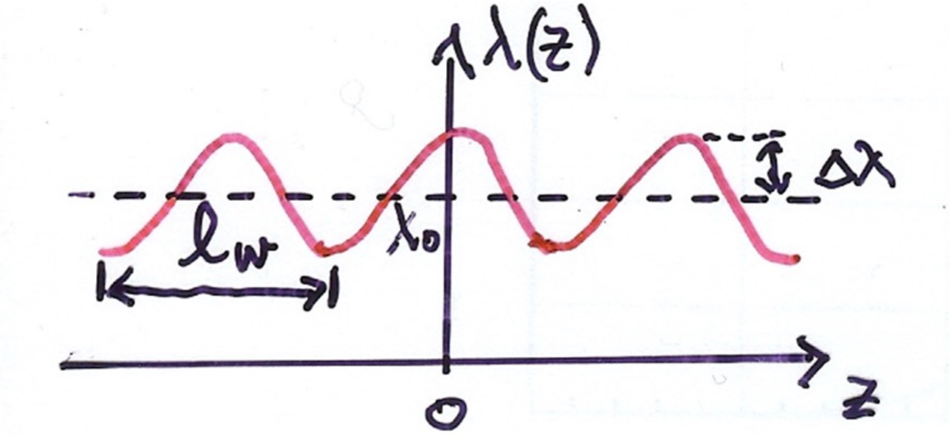
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

b) Sinusoidal modulation (λ_0 =Ne/l_0)

$$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \;\; ; \; k_z = 2\pi / \lambda_w$$



$$\Delta Q_{\max}\left(\text{at } k_z z = 2n\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 + \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$\Delta Q_{\min}\left(\text{at } k_z z = (2n+1)\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 - \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

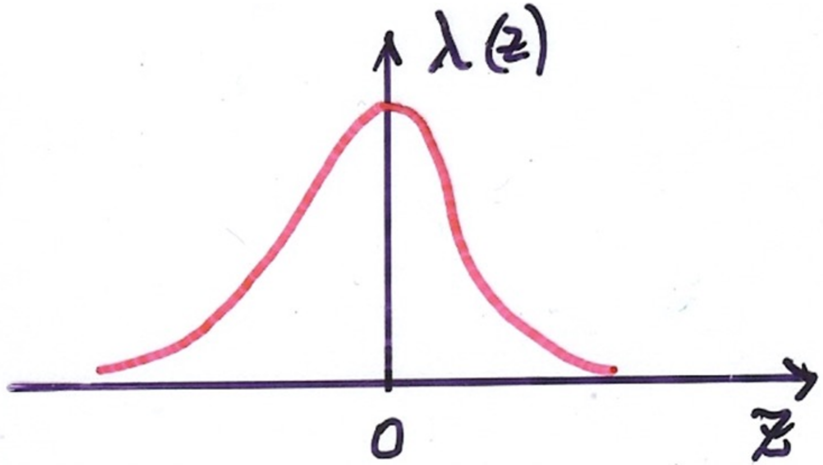
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = -\frac{2\rho_x^2 r_{e,p}\Delta\lambda}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2\lambda(z)r_{e,p}}{e\beta^2\gamma^3a^2Q_{x0}}$$

c) Gaussian bunch (q_0 = Ne)

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

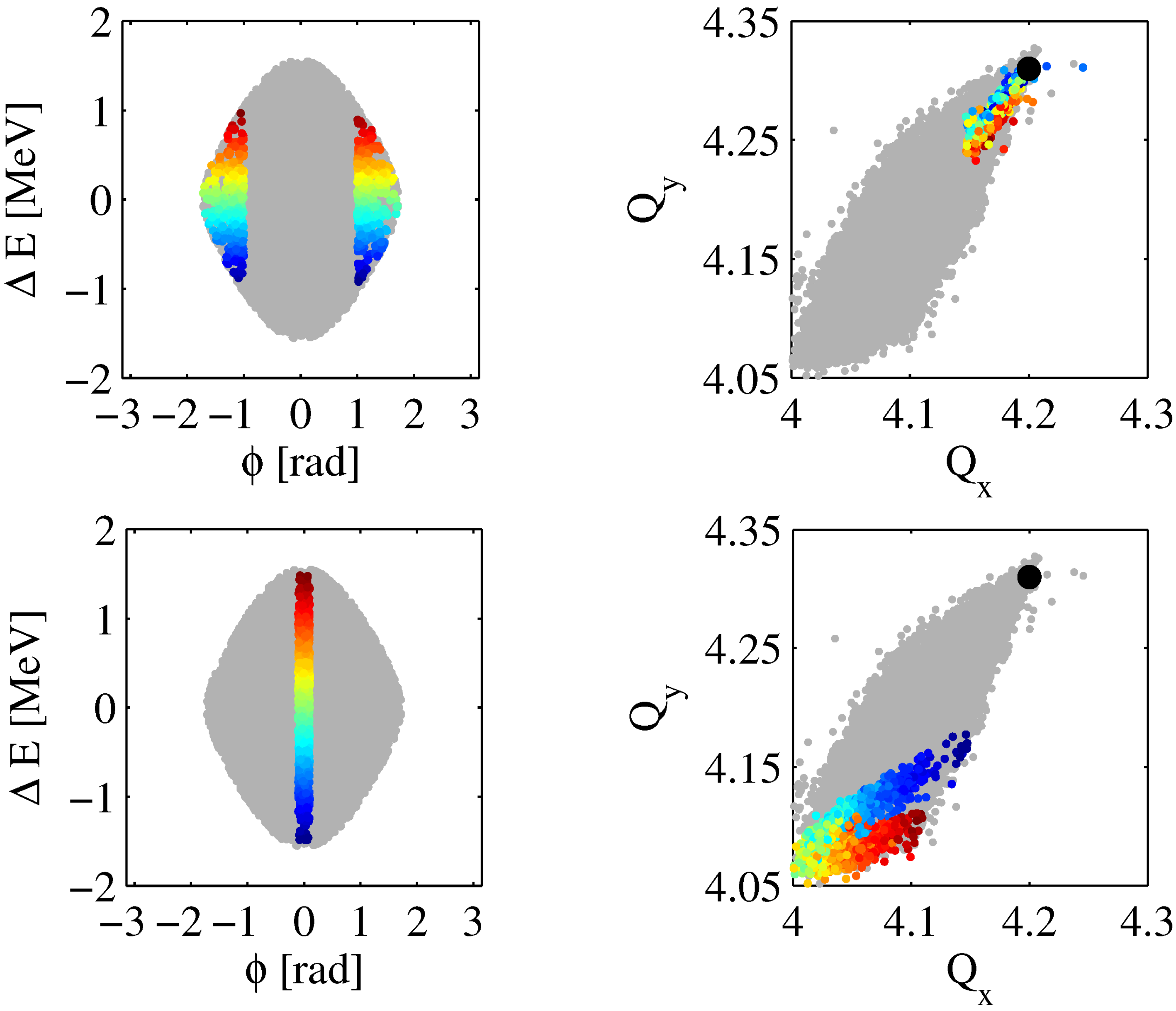


$$\Delta Q_{\max}\left(\text{at } z=0\right) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3a^2Q_{x0}}\frac{N}{\sqrt{2\pi}\sigma_z}$$

$$\Delta Q_{\min}\left(\text{at } z\to\pm\infty\right) = 0$$

$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

Effect of longitudinal distribution

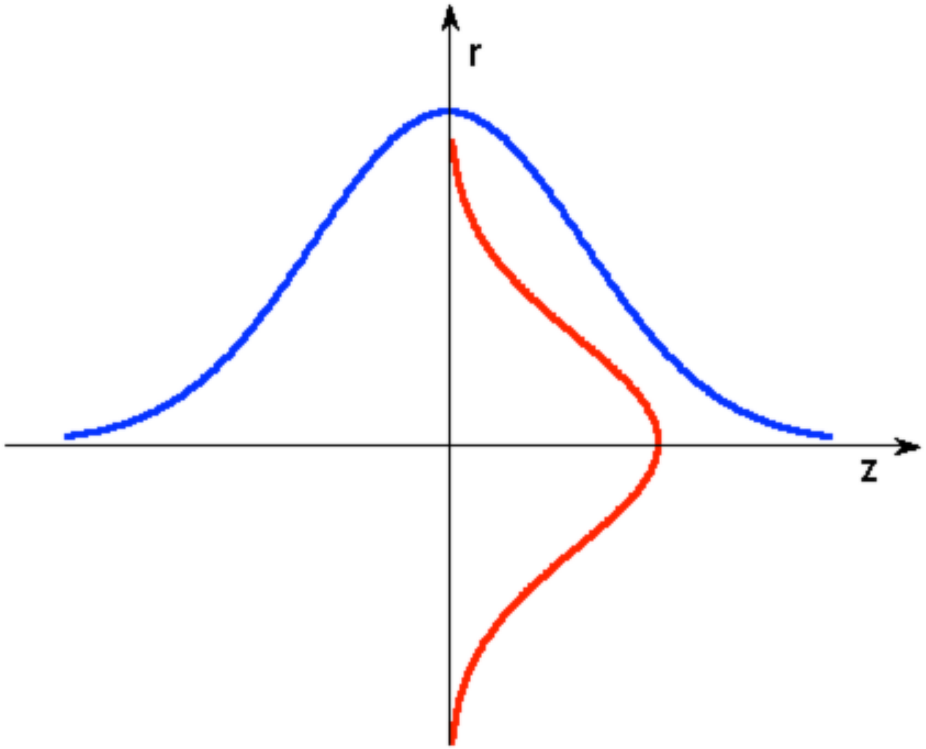


Longitudinal phase-space ($\Delta\phi$, ΔE) scatter plot of the bunch tune footprint. The black dot is the bare tune. Particles at the edges of the bunch have tunes close to the bare tune in the necktie.
Indeed, in this longitudinal region, the beam line density is smaller with respect to the centre of the bunch, therefore also the space charge detuning is small.

(Courtesy of V. Forte, 'Performance of the CERN PSB at 160 MeV with H- charge exchange injection', PhD thesis, Université Blaise Pascal, Clermont-Ferrand, France, 2016)

Exercise 2

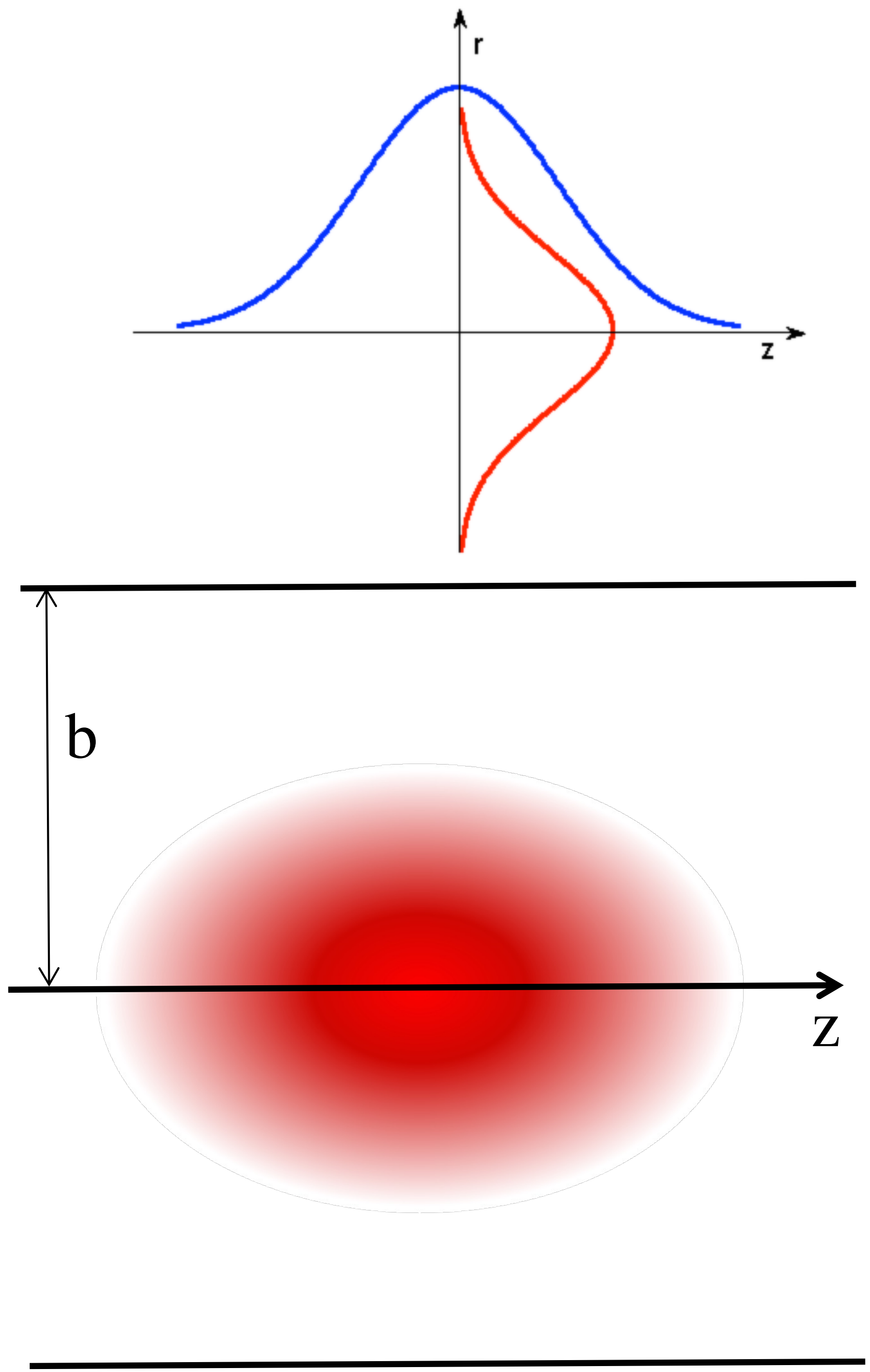
Compute the transverse space charge force and the incoherent tune shift for a cylindrical beam in a circular beam pipe, having a bi-Gaussian longitudinal and transverse distribution.



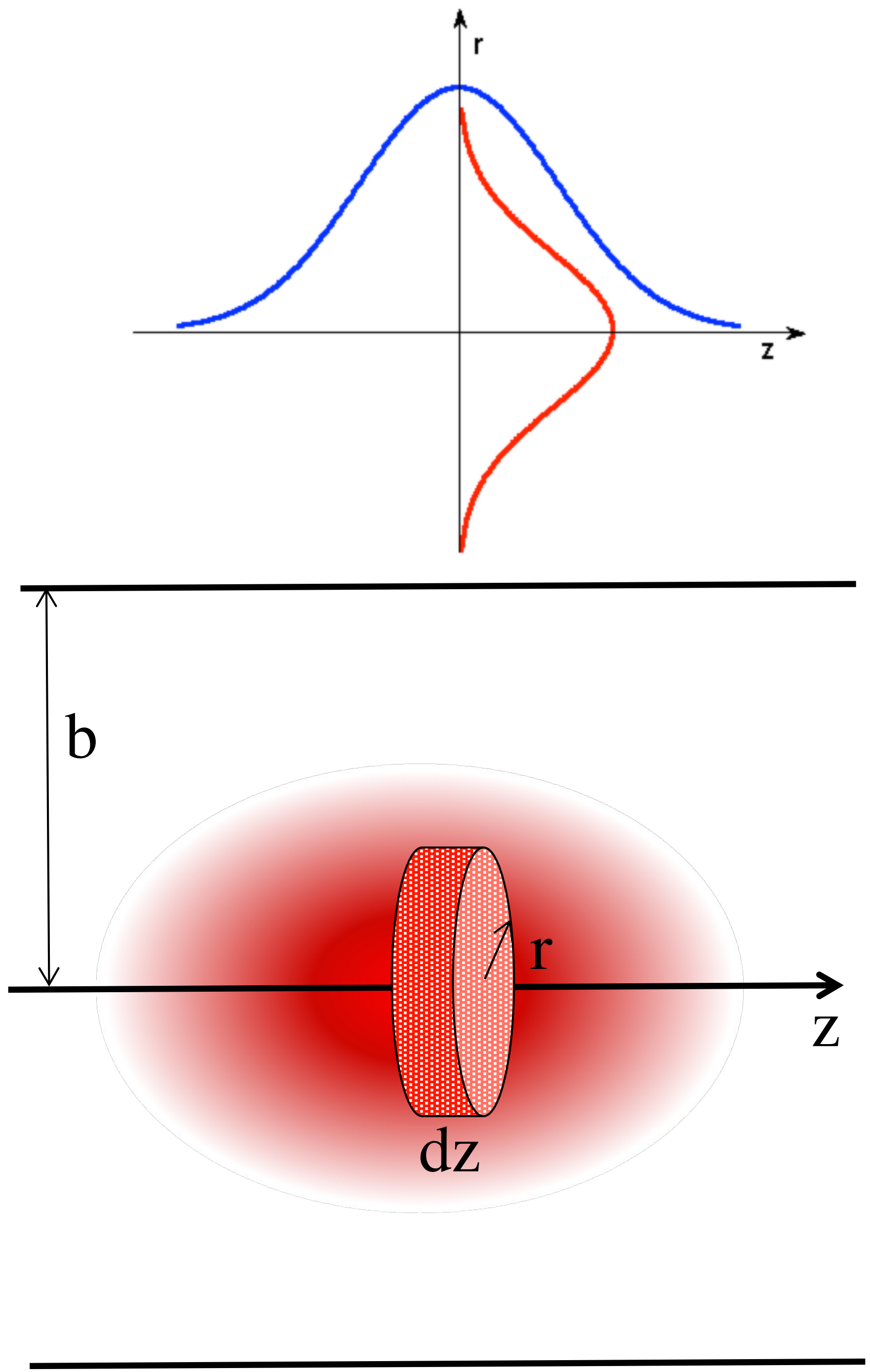
$bi-Gaussian$

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r,z) = \frac{\lambda(z)}{2\pi\sigma_r^2}\exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$



r
z
b
z


r
z
b
r
z
dz

bi − Gaussian

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r,z) = \frac{\lambda(z)}{2\pi\sigma_r^2} \exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

$$\int \vec{E}\cdot\hat{n}\, dS = \frac{q(r)}{\varepsilon_0}$$

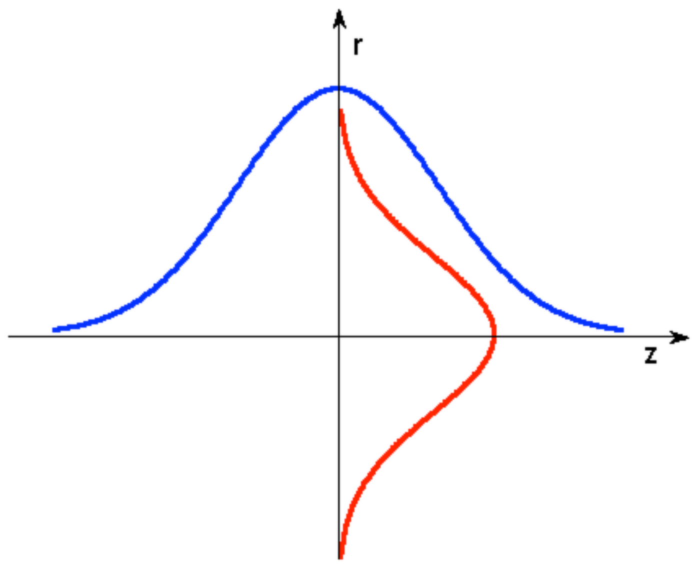
$$\boxed{\left(\gamma >> 1 \qquad E_z \approx 0\right)}$$

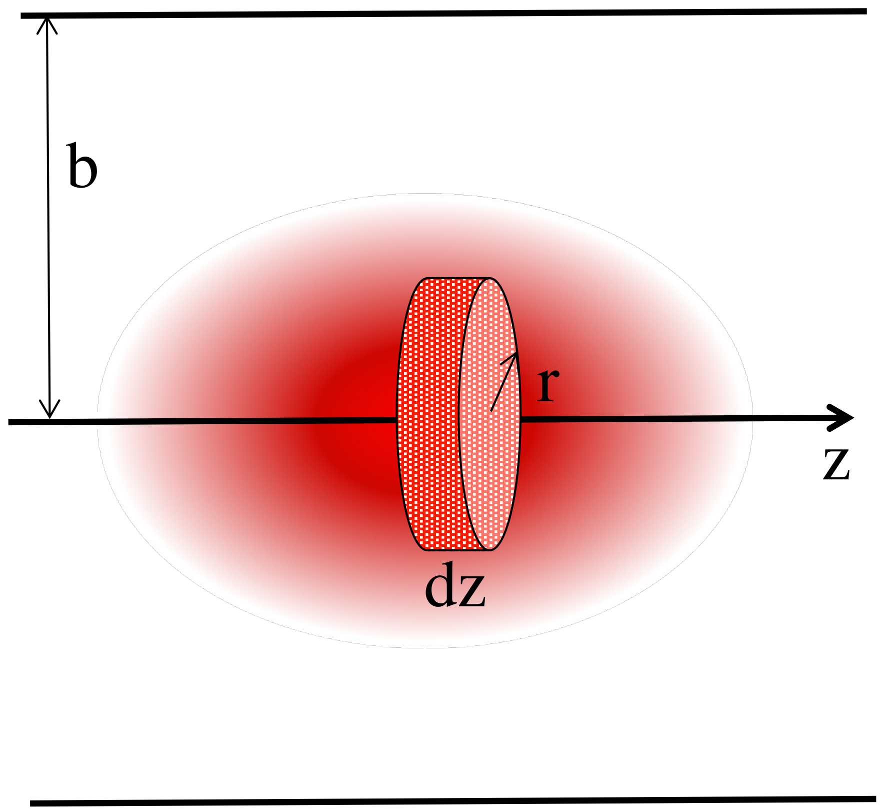
$$E_r(r) 2\pi r dz = \frac{dz}{\varepsilon_0}\int_0^r \rho(r',z) 2\pi r'\, dr'$$

$$E_r(r) = \frac{\lambda(z)}{2\pi\varepsilon_0\sigma_r^2 r}\int_0^r \exp\left(\frac{-r'^2}{2\sigma_r^2}\right) r' dr'$$

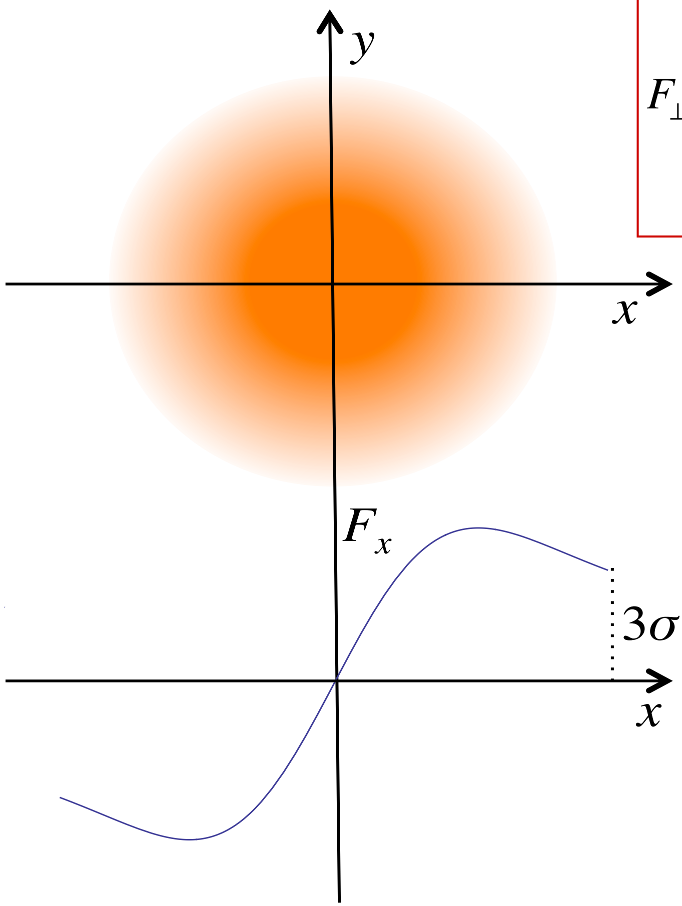
$$E_r(r) = \frac{\lambda(z)}{2\pi\varepsilon_0 r}\left[-\exp\left(\frac{-r'^2}{2\sigma_r^2}\right)\right]_0^r = \frac{\lambda(z)}{2\pi\varepsilon_0}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$

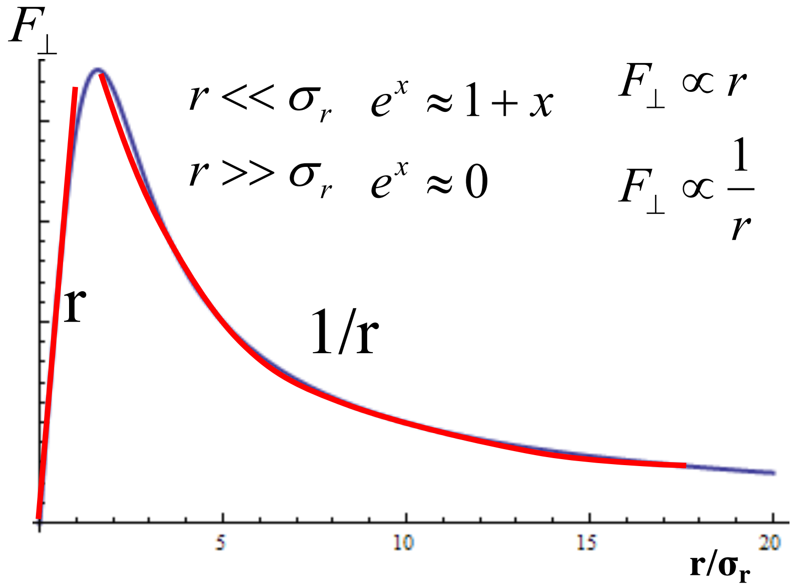
$$F_\perp(r) = \frac{e}{\gamma^2}E_r = \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$





$$F_{\perp}(r) = \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right] \approx \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\frac{r}{2\sigma_r^2} \qquad (r \ll \sigma_r)$$





$(r \ll \sigma_r)$

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\,\varepsilon_o}\frac{1}{2\sigma_x^2}$$

If the charge distribution is Gaussian but with different σ_x and σ_y (not cylindrical geometry), it is still possible to obtain the transverse electric field. The expression is known as Bassetti-Erskine formula: M. Bassetti and G.A. Erskine, "Closed expression for the electrical field of a two-dimensional Gaussian charge", CERN-ISR-TH/80-06 (1980).

$$E_x = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)-e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]}w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)-e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]}w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

with the complex error function w(z) given by

$$w(z) = e^{-z^2}\left[1+\frac{2i}{\sqrt{\pi}}\int_o^z e^{\zeta^2}\,d\zeta\right]$$

NB: here Q is the line density.

In the limit $\sigma_x \to \sigma_y$ the above electric field is the one that we have obtained previously

This complicated expression is highly non-linear. It is however possible to obtain a simple expression in the linear approximation which gives

$$E_x \approx \frac{\lambda(z)}{2\pi\varepsilon_0} \frac{x}{\sigma_x(\sigma_x + \sigma_y)}$$

$$E_y \approx \frac{\lambda(z)}{2\pi\varepsilon_0} \frac{y}{\sigma_y(\sigma_x + \sigma_y)}$$

As for the cylindrical symmetry case, there are also magnetic fields associated with the electric fields, so that the transverse force is

$$F_{x,y} \approx \frac{e}{\gamma^2} E_{x,y}$$

and, as in the previous cases, it is possible to obtain the incoherent tune shift (but remember that we are in the linear approximation).

Exercise 3

Evaluate the dependence of the longitudinal and transverse space charge force with z at fixed r (e.g. $<< \sigma_r$) for the bi-Gaussian distribution

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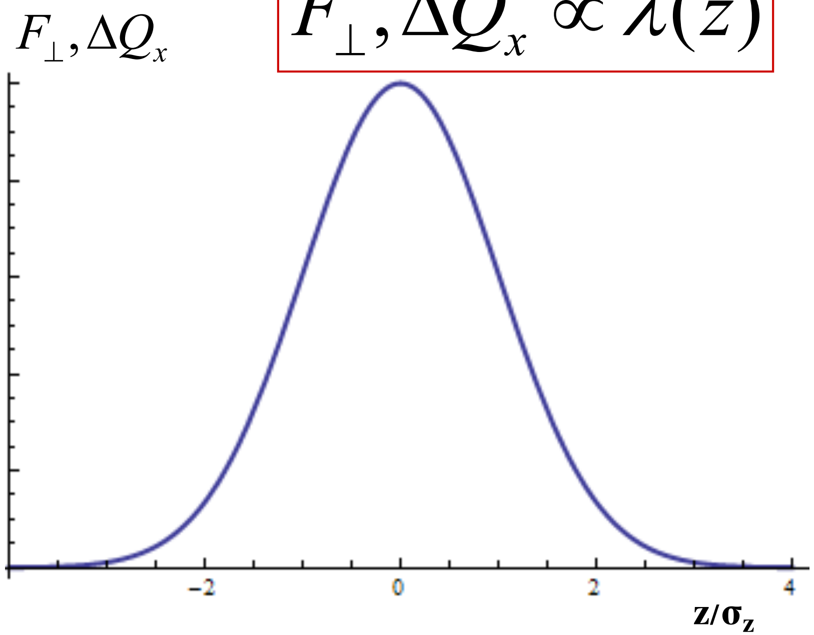
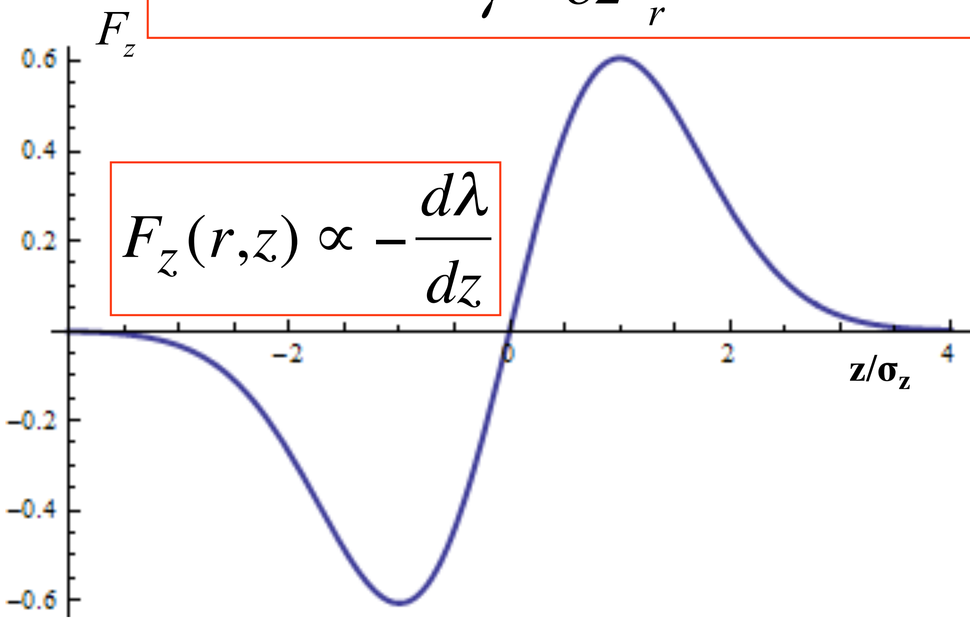
$$\boxed{F_\perp, \Delta Q_x \propto \lambda(z)}$$



$$\boxed{F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'}$$



$$\boxed{F_z(r,z) \propto -\frac{d\lambda}{dz}}$$

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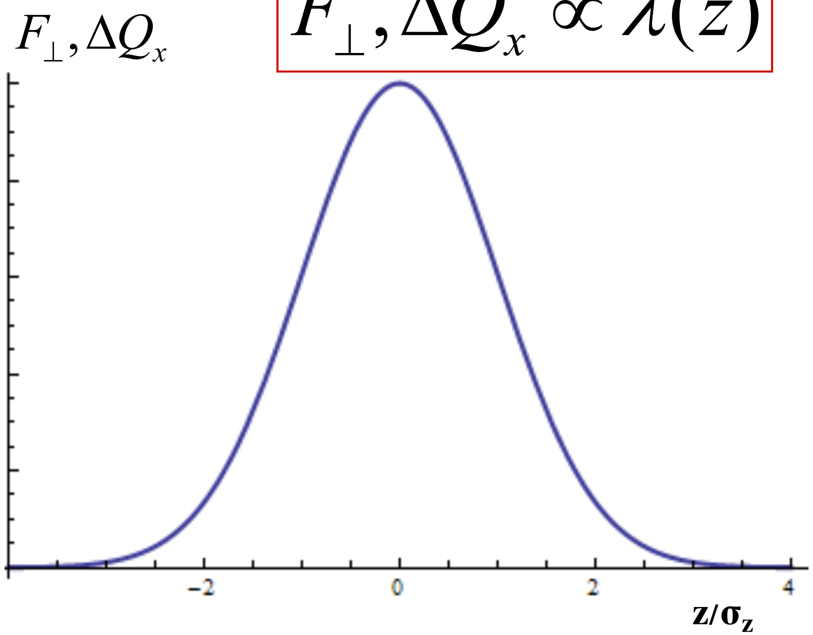

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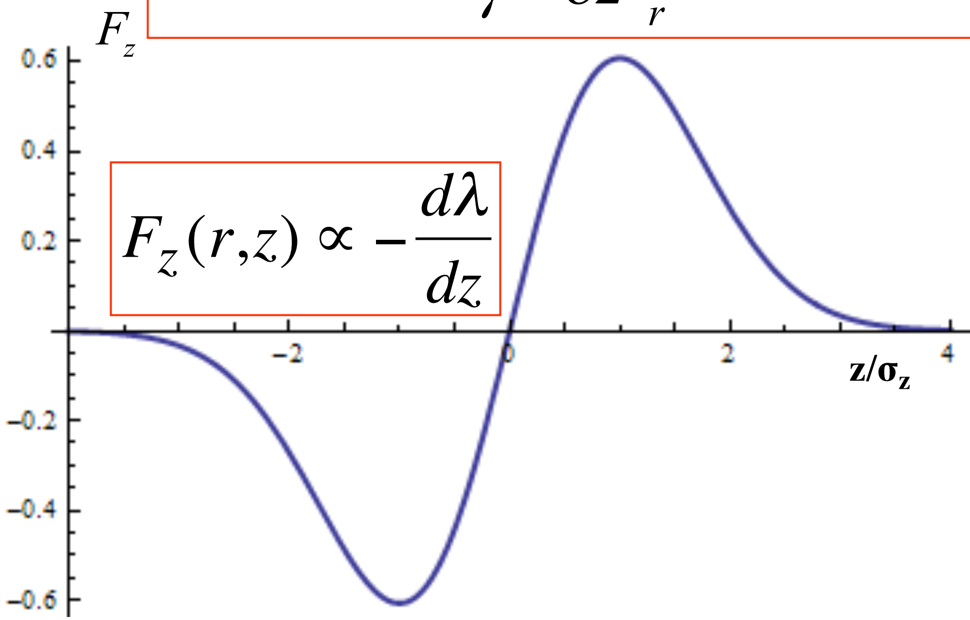

The effect of the longitudinal force is an energy spread: the head of the beam gains energy (positive force) and the tail loses energy. The consequences on beam dynamics depends on the machine

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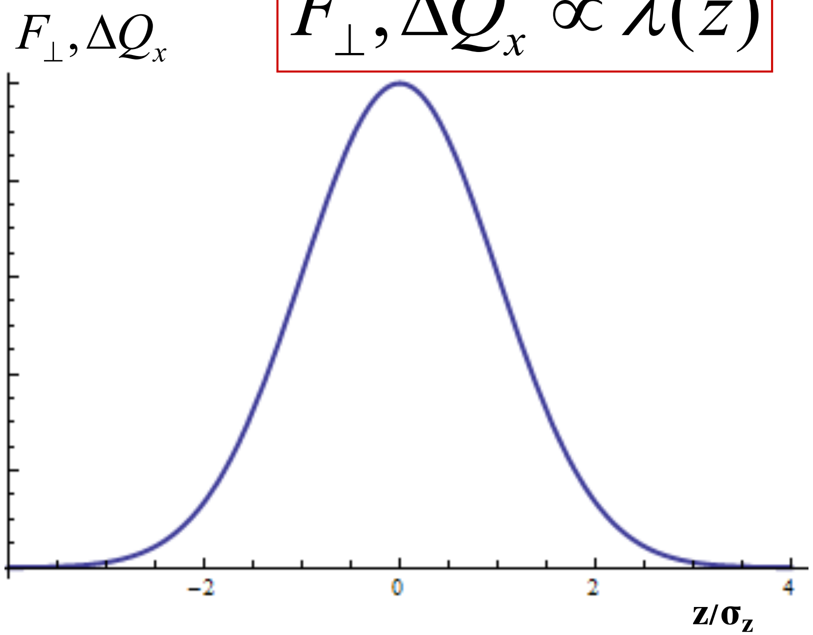
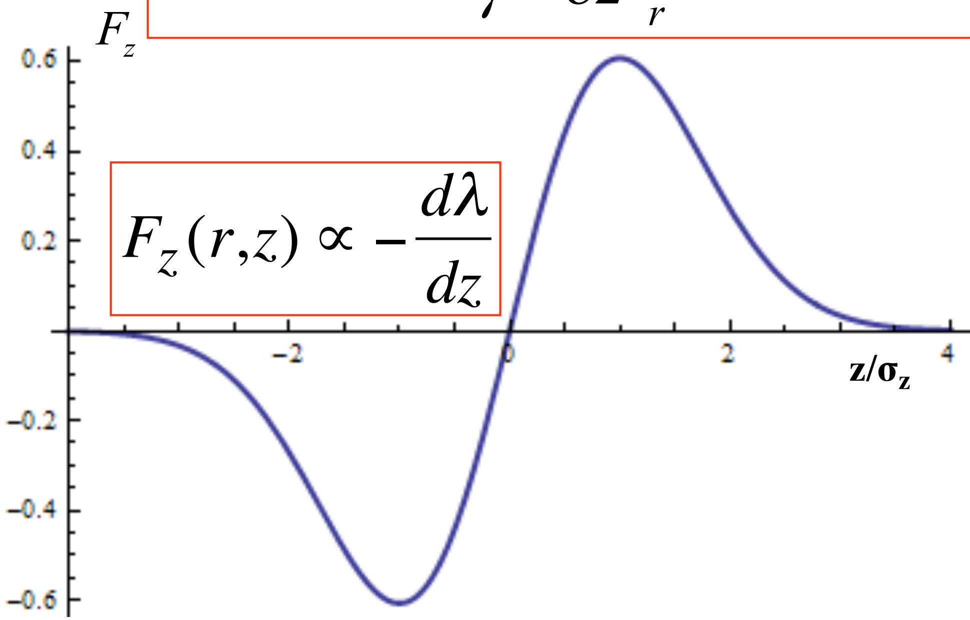


$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)\,dr'$$

$$F_z(r,z) \propto -\frac{d\lambda}{dz}$$



In a LINAC at ultra-relativistic velocity the beam is frozen and there is only an energy spread

In a circular accelerator, it depends on slippage factor η: above transition, higher energy means higher revolution time, the head of the bunch delays, the tail anticipates and the bunch is shortened

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Compute the longitudinal space charge force of a transverse uniform cylindrical beam in a circular perfectly conducting beam pipe

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$$\Rightarrow$$

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$$E_r(r \le a) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\frac{r}{a^2}$$

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$$F_z(r,z) = -\frac{e}{2\pi\varepsilon_0\gamma^2}\left[\int_r^a \frac{r'}{a^2}dr' + \int_a^b \frac{1}{r'}dr'\right]\frac{\partial\lambda(z)}{\partial z}$$

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$$E_r(r \le a) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\frac{r}{a^2} \qquad E_r(r \ge a) = \frac{\lambda(z)}{2\pi\varepsilon_0 r}$$

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$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$

Exercise 5

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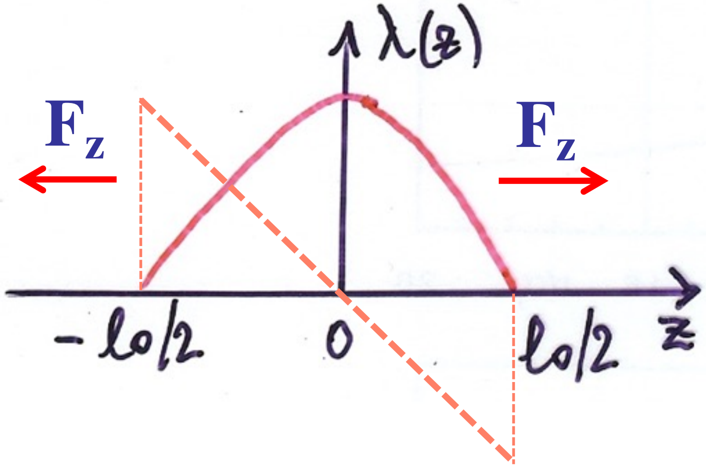
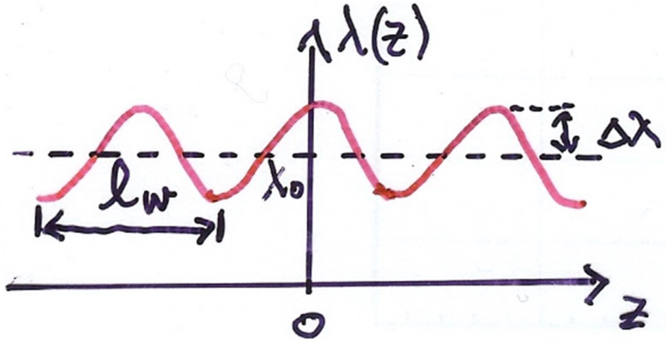
$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$



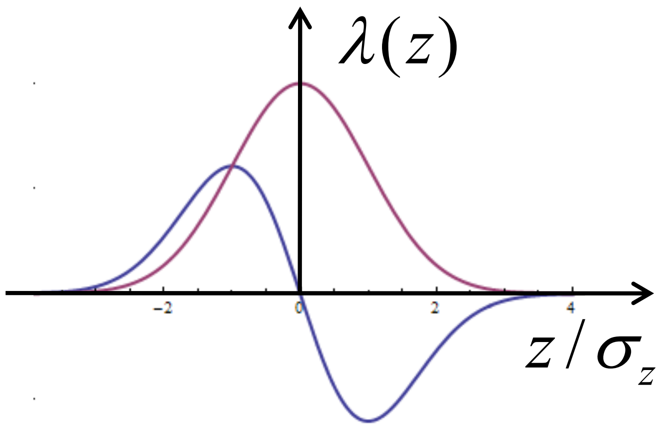
$$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$$

$$\frac{d\lambda(z)}{dz} = -\frac{12Ne}{l_0^3} z$$



$$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \quad ; \; k_z = 2\pi / l_w$$

$$\frac{d\lambda(z)}{dz} = -\Delta\lambda k_z \sin(k_z z)$$



$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\frac{d\lambda(z)}{dz} = -\frac{Ne}{\sqrt{2\pi}\sigma_z^3} z \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

Exercise 6

Compute the incoherent betatron tune shift of a uniform proton beam inside two perfectly conducting parallel plates

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$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial\ F_x^{s.c.}}{\partial x}\right)$$

$$F_x(z,x) = \frac{e\lambda_0 x}{\pi\ \varepsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$\Delta Q_x = -\frac{\rho_x^2 e\lambda_0}{2\pi\ \varepsilon_0\beta^2 E_0 Q_{x0}}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

Exercises on Wake Fields and Instabilities

Exercise 1:

Show that the impedance of an RLC parallel circuit is that of a resonant mode and relate R, L and C to Q, R_s and ω_r

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$$Z(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

Exercise 1:

Show that the impedance of an RLC parallel circuit is that of a resonant mode and relate R, L and C to Q, R_s and ω_r

$$Z_R = R \qquad Z_C = \frac{i}{\omega C} \qquad Z_L = -i\omega L$$

$$\frac{1}{Z} = \frac{1}{R} - i\omega C + i\frac{1}{\omega L} =$$

$$\frac{\omega L - i\omega^2 LCR + iR}{R\omega L} = \frac{1}{R}\left(1 + i\left(\frac{R}{\omega L} - \omega CR\right)\right) =$$

$$\frac{1}{Z} = \frac{1}{R}\left(1 + iR\sqrt{\frac{C}{L}}\left(\frac{1}{\omega\sqrt{CL}} - \omega\sqrt{CL}\right)\right)$$

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Show that the impedance of an RLC parallel circuit is that of a resonant mode and relate R, L and C to Q, R_s and ω_r

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Exercise 2:

Calculate the amplitude of the resonator wake field given R_s = 1 kΩ, ω_r = 5 GHz, $Q = 10^4$

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Calculate the ratio $|Z(\omega_r)| / |Z(2\omega_r)|$ for Q = 1, 10^3, 10^5 $|1 - i\, 3Q/2|$

$Q = 1 \rightarrow 1.8$

$Q = 10^3 \rightarrow 1.5\times10^3$

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Exercise 3: Beam Break Up

Consider a beam in a linac at 1 GeV without acceleration. Obtain the growth of the oscillation amplitude of the tail with respect to the head after 3 km if:

$N = 5e10$, $w_\perp(-1\ \text{mm}) = 63\ \text{V/(pC m)}$, $L_w = 3.5\ \text{cm}$, $k_y = 0.06\ \text{1/m}$

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$$\left[\frac{y_2(L_L) - y_1(L_L)}{\hat{y}_1}\right]_{max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_0/e)L_w} = 180$$

To preserve the beam emittance, it is necessary to have

$$\left[\frac{y_2(L_L) - y_1(L_L)}{\hat{y}_1}\right]_{max} \hat{y}_1 = 180\times\hat{y}_1 \ll \text{transverse beam size}$$

This means that the beam must be injected onto the linac axis with an accuracy better than a fraction of a per cent of the beam size, which is difficult to achieve.

Exercise 4: Beam Break Up (2)

Consider the same beam of the previous exercise being now accelerated from 1 GeV with a gradient g =16.7 MeV/m. Obtain the growth of the oscillation amplitude of the tail with respect to the head. With a constant acceleration, if $E_f = E_0 + gL_L \simeq gL_L$, the expression is the same of that with constant energy multiplied by a factor

$$F = \frac{E_0}{E_f} \ln \frac{E_f}{E_0} < 1$$

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$$F = \frac{E_0}{E_f} \ln \frac{E_f}{E_0} < 1$$

In this case $E_f = 1 + 16.7 * 3000 = 51.1 \simeq 50.1$ so the factor is

$$F = 0.078 \rightarrow \left[\frac{y_2(L_L) - y_1(L_L)}{\hat{y}_1}\right]_{max} = 180 * 0.078 = 14$$

Acceleration is helpful to reduce the instability

Exercise 5: Evaluate the energy lost per unit length by a charge due to the longitudinal wake field of the space charge and compare it with the longitudinal space charge force in r=0

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$$\frac{w_{//}}{L} = \frac{dw_{//}(z)}{ds} = \frac{1}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{\partial}{\partial\, z}\delta(z)$$

$$\frac{dU(z)}{ds} = -e\int_{-\infty}^{\infty}\frac{dw_{//}(z'-z)}{ds}\lambda(z')dz' =$$

$$-\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\int_{-\infty}^{\infty}\frac{\partial}{\partial\, z'}\delta(z'-z)\lambda(z')dz' =$$

$$= -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1+2\ln\frac{b}{a}\right)\frac{\partial\,\lambda(z)}{\partial\, z}$$

Exercise 6: Evaluate the energy spread (U_max-U_min) of a Gaussian bunch of RMS length σ due to the longitudinal wake field of the space charge in a structure of length L

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$$\lambda(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} \qquad \frac{\partial \lambda(z)}{\partial z} = -\frac{z}{\sqrt{2\pi}\sigma^3} e^{-\frac{z^2}{2\sigma^2}}$$

$$U(z) = -\frac{eL}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{\partial\ \lambda(z)}{\partial\ z} = \frac{eL}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{z}{\sqrt{2\pi}\sigma^3} e^{-\frac{z^2}{2\sigma^2}}$$

$$\frac{\partial U}{\partial z} = 0 \Rightarrow z = \pm\sigma$$

$$U_{\max} - U_{\min} = 2U_{\max} = \frac{2eL}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2}}$$

Exercise 7: Evaluate the energy lost by a charge inside a uniform beam of length l_0 due to the longitudinal wake field of a pill box cavity of length g at high frequency ω>>c/b (diffraction model), with a pipe radius b.

$$w_{\parallel}(z) = \frac{Z_0 c\sqrt{2g}}{2\pi^2 b}\frac{1}{z^{1/2}}$$

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$$w_{\parallel}(z) = \frac{Z_0 c\sqrt{2g}}{2\pi^2 b}\frac{1}{z^{1/2}}$$

$$U(z) = -e\int_{-\infty}^{\infty} w_{\parallel}(z'-z)\lambda(z')dz' =$$

$$-\frac{eqZ_0 c\sqrt{2g}}{l_0 2\pi^2 b}\int_{z}^{l_0/2}\frac{dz'}{(z'-z)^{1/2}} = -\frac{2eqZ_0 c\sqrt{g(l_0-2z)}}{l_0 2\pi^2 b}$$