Fundamentals of Statistical Analysis in Physics
Lectures 5, 6: Bayesian Statistical Analysis

African School of Physics 2022

Harrison B. Prosper

Kirby W. Kemper Endowed Professor of Physics
Department of Physics, Florida State University

1 December, 2022

Outline

1. Type1a Supernovae
2. Bayesian Analysis of Type1a Data
3. Summary

Recap

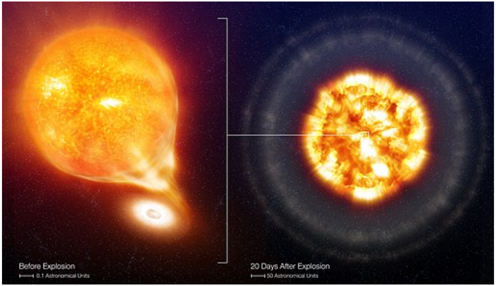
- Last time we illustrated the frequentist approach to statistical analysis by analyzing the DØ top quark discovery data.
- The key ideas were:
  - the method of maximum likelihood to estimate the parameters of the model;
  - the profile likelihood as a way to deal with nuisance parameters, and
  - the profile likelihood ratio statistic as a way to construct confidence intervals and to test hypotheses.

Outline

1. Type1a Supernovae
2. Bayesian Analysis of Type1a Data
3. Summary

Type1a supernovae (1)

A Type 1a supernova is thought to be the detonation of a white dwarf that has breached the Chandrasekhar limit of about $1.4 M_{\odot}$ (solar masses). These explosions are so bright that they release in a matter of weeks more energy than the Sun will release in 10 billion years!

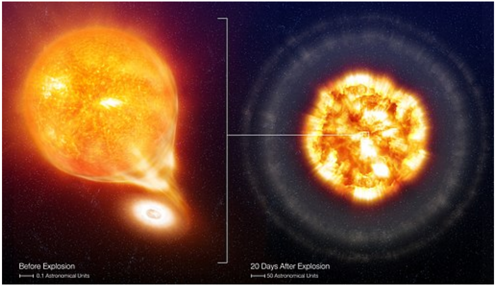


https://pl.wikipedia.org/wiki/SN_2006X

Because each explosion releases about the same amount of energy, the equivalent of $1.4 M_{\odot}$, Type1a supernovae (after correcting their light curves) are excellent standard candles that can be used to determine the distances to galaxies. (A standard candle is a light source whose luminosity is known.)

Type1a supernovae (2)

Consider a light source, of luminosity L joules per second, at a distance d from us. Assume that the energy is emitted isotropically.



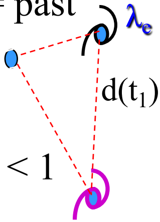
https://pl.wikipedia.org/wiki/SN_2006X

Given the known luminosity L of the light source and a measurement of its apparent brightness f, that is, the energy received per second per unit area, we can infer the distance to the light source using the inverse square law

$$f = \frac{L}{4\pi d^2}.$$

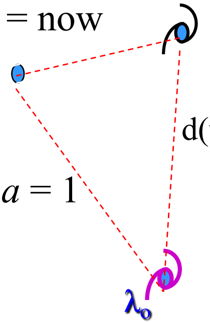
But when applied to distant supernovae, there is a subtlety!

$t_1 = \text{past}$



$a < 1$

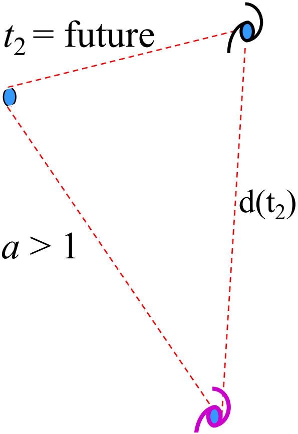
$t_0 = \text{now}$



$a = 1$

$a(t)$ is the scale factor of the Universe

$t_2 = \text{future}$



$a > 1$

$$d(t) = a(t)\, d_0$$

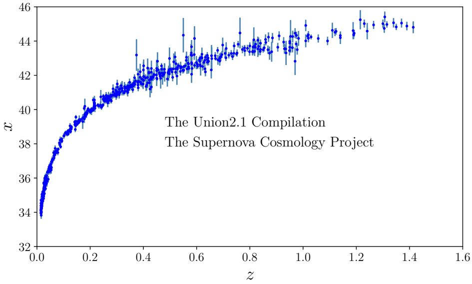
$$\lambda_e = a(t)\, \lambda_o$$

$$z = (\lambda_o - \lambda_e) / \lambda_e$$

$$1 + z = \frac{1}{a(t)}$$

t is cosmic time

The Supernova Cosmology Project created the Union2.1 Compilation of brightness and redshift data for 580 Type 1a supernovae. For each supernova, the following data are provided: (x, σ, z), where x, the distance modulus, is a measure of the brightness of the supernova with larger values corresponding to fainter supernovae. The standard deviation σ is a measure of the uncertainty in the measurement x.



The expansion history of the universe is encoded in the brightness-redshift data. Therefore, by comparing theoretical models of the expansion history with these data one can estimate the model parameters and infer which theoretical models are preferred by the data.

Outline

Bayesian Analysis of Type1a Data

In analyses of the supernovae data $\{(x, \sigma, z)\}$ it is generally assumed that the probability to measure the distance modulus, x, for a supernova can be modeled with a Gaussian distribution with standard deviation σ.

(The uncertainty in the measurement of the redshift z is negligible.)

Let's briefly review a few facts about the Gaussian.

Gaussian Distribution

The probability density function (pdf) of the Gaussian (also known as the normal distribution) with mean μ and variance σ^2 is given by

$$\boxed{\text{Gauss}(x; \mu, \sigma) = \frac{e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}}{\sigma\sqrt{2\pi}}.}$$

Other facts about the Gaussian are the probability contents of various intervals. Let $z = (x - \mu)/\sigma$. Then

$$P(z \in [-1.00, 1.00]) = 0.683$$
$$P(z \in [-1.64, 1.64]) = 0.900$$
$$P(z \in [-1.96, 1.96]) = 0.950$$
$$P(z \in [-3.00, 3.00]) = 0.997$$
$$P(z \in [5.00, \infty)) = 2.7 \times 10^{-7}$$

The Gaussian is the most important distribution in statistics.

Bayesian Inference in One Slide!

In the Bayesian approach to inference, we need to specify the

$$p(x|\theta) : \text{likelihood and the}$$
$$\pi(\theta) : \text{prior},$$

where θ are the parameters of the theoretical model and the prior density $\pi(\theta)$ encodes what is known, or assumed, about the parameters independently of the data. One completes the inference by computing the posterior density, $p(\theta|x)$, which is given by Bayes' theorem,

$$\boxed{p(\theta|x) = \frac{p(x|\theta)\,\pi(\theta)}{p(x)}}, \quad p(x) = \int p(x|\theta)\,\pi(\theta)\,d\theta.$$

If $\theta = \phi, \omega$ and we are interested only in ϕ, the nuisance parameters ω are eliminated by marginalization: $p(\phi|x) = \int p(\phi, \omega|x)\,d\omega$.

Bayesian Analysis of Type1a Data

We shall take the likelihood and prior for the Type1a data to be

$$p(x|\theta) = \prod_{i=1}^{580} \mathrm{Gauss}(x_i; \mu(z_i, \theta), \sigma_i), \tag{1}$$

$$\pi(\theta) = \mathrm{constant}. \tag{2}$$

In this case, the posterior density simplifies to

$$p(\theta|x) = \frac{p(x|\theta)}{\int p(x|\theta)\, d\theta}. \tag{3}$$

In the Standard Model of Cosmology the energy density is given by

$$\Omega(a) = \frac{\Omega_M}{a^3} + \frac{(1 - \Omega_M - \Omega_\Lambda)}{a^2} + \Omega_\Lambda, \tag{4}$$

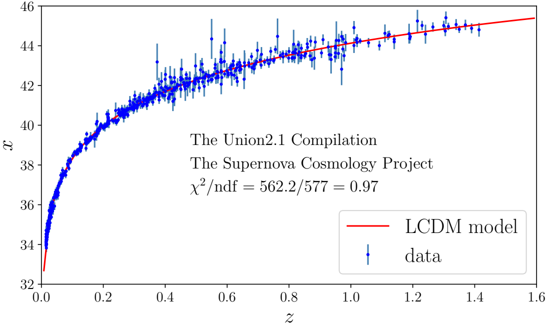
where Ω_M and Ω_Λ are the matter and vacuum energy parameters, respectively. The other important parameter is H_0, the Hubble constant.

Bayesian Analysis of Type1a Data

The inference is completed by computing the posterior density
$p(\Omega_M, \Omega_\Lambda, H_0|x) = p(x|\Omega_M, \Omega_\Lambda, H_0)\,\pi(\Omega_M, \Omega_\Lambda, H_0)\,/\,p(x)$.

Taking the posterior mode as our estimates of the parameters, we find $\Omega_M = 0.28 \pm 0.07$, $\Omega_\Lambda = 0.73 \pm 0.12$, and $H_0 = 69.8 \pm 0.4\ \mathrm{km/s/Mpc}$.

When these values are entered into the theoretical prediction for the distance modulus function $\mu(z, \theta)$ we find the result below.



Outline

- The Bayesian approach requires specification of both a likelihood and a prior. Unfortunately, constructing a sensible prior can be extremely challenging and controversial.
- Nevertheless, the Bayesian approach is widely used in cosmological studies and has been successfully used in many high-profile analyses, including the measurement of the top quark mass by the DØ Collaboration in 1997; the discovery of the accelerating expansion of the universe by the SCP and High Z Collaborations in 1998; the discovery of single top quark production by the DØ Collaboration in 2009, and the in-depth experimental study of a theoretical model called the pMSSM by the CMS Collaboration in 2016.

Thank You!