NELSON MANDELA UNIVERSITY
The 7th Biennial African School of Fundamental Physics and Applications
28 November - 9 December 2022

Quantum Computing

Lecture 1 : Introduction to Quantum Information
Lecture 2 : Quantum Mechanics for Quantum Information
Lecture 3 : Quantum Gates and Quantum Circuits
Lecture 4 : Quantum Computing Algorithms

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5 December 2022

Acknowledgements

- This series of lectures is based on the course of Quantum Information I'm giving at University of Tunis El Manar (Faculty of Science of Tunis, Departement of Physics) for 2nd year (M2) master candidates in Nanophysics and Nanotechnology.

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كلية العلوم بتونس
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- It is an updated version of the lectures given at the 2020 edition of ASP.
- Most of graphics, especially those of quantum circuits are taken from IBM Qiskit website www.qiskit.org

Qiskit

Context

The Nobel Prize in Physics 2022

Ill. Niklas Elmehed © Nobel Prize Outreach

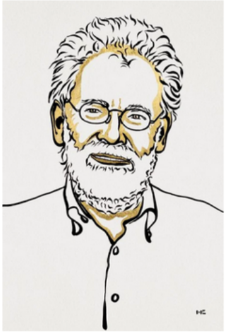
Alain Aspect

Prize share: 1/3

Ill. Niklas Elmehed © Nobel Prize Outreach

John F. Clauser

Prize share: 1/3



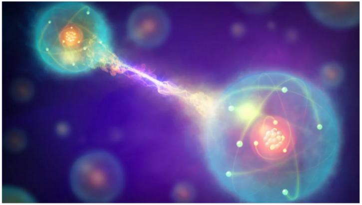
Ill. Niklas Elmehed © Nobel Prize Outreach

Anton Zeilinger

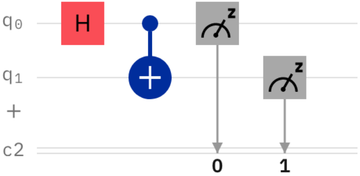
Prize share: 1/3

Nobel Prize in Physics 2022

The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with **entangled photons**, establishing the violation of Bell inequalities and pioneering **quantum information science**"



(Image credit: MARK GARLICK/SCIENCE PHOTO LIBRARY via Getty Images)



Quantum computing landscape 2022

source : The Quantum Insider September 2022

IBM Osprey Quantum Processor 2022

IBM

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IBM Unveils 400 Qubit-Plus Quantum Processor and Next-Generation IBM Quantum System Two

Company Outlines Path Towards Quantum-Centric Supercomputing with New Hardware, Software, and System Breakthrough

Nov 9, 2022

Dario Gil, Jay Gambetta and Jerry Chow holding the new 433 qubit 'IBM Osprey' processor

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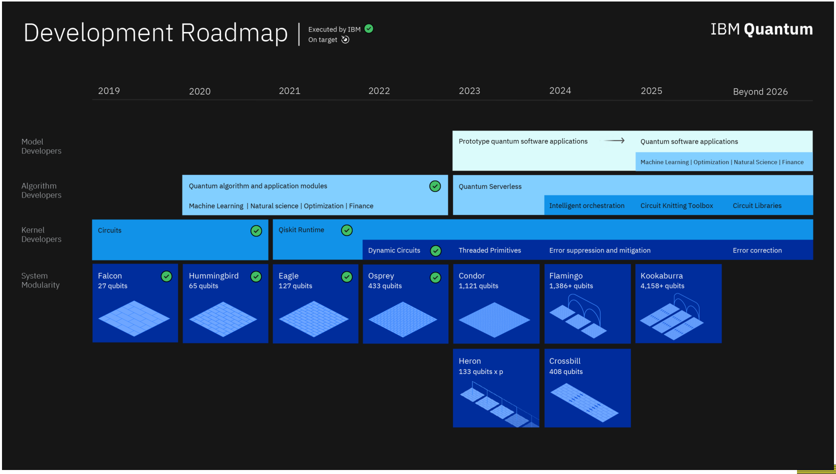
IBM development roadmap 2022



Quantum computing to test entanglement

MIT News
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MIT researchers use quantum computing to observe entanglement

Researchers at the Center for Theoretical Physics lead work on testing quantum gravity on a quantum processor.

Julia C. Keller | School of Science
December 1, 2022

PRESS INQUIRIES

A representation of a traversable wormhole with quantum information passing through.

Image: A. Mueller/inqnet

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Outline

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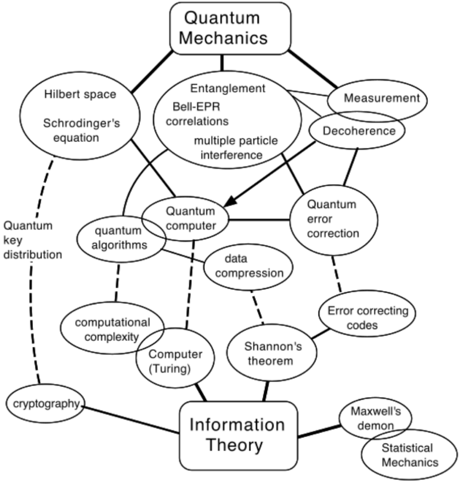
Quantum Mechanics and Information Theory



[1]

[1] Andrew Steane 1998 Rep. Prog. Phys. 61 117

Shannon Information Theory

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- Demonstrated mathematically that even in a noisy channel with a low bandwidth, essentially perfect, error-free communication could be achieved by keeping the transmission rate within the channel's bandwidth and by using error-correcting schemes (redundancy)

Noiseless and noisy Shannon theorems

[2]

- Noiseless channel case:

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[2] S. Barnett, Les Houches Summer School lectures 2009

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Shannon Theory

For more details about Shannon Information Theory :

From Classical to Quantum Shannon Theory

Mark M. Wilde
Hearne Institute for Theoretical Physics
Department of Physics and Astronomy
Center for Computation and Technology
Louisiana State University
Baton Rouge, Louisiana 70803, USA

July 16, 2019

Introduction to Quantum Information

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- The hardware part operates according to the rules of quantum mechanics (semiconductors, transistors, etc.), but the quantum properties at the fundamental scale (superposition, entanglement, non-locality, etc.) are not fully exploited.
- The purpose of Quantum Information Theory is precisely to take advantage of these properties in order to perform tasks which are impossible to realize with classical computers.
- The most known applications of quantum information are Quantum Computing, with the focus on the physical implementation of a universal quantum computer, and Quantum Cryptography for the secure transmission of information.

Elements of binary logic

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- Each integer n is written on binary basis as the sum of terms to the power of 2. For example:

$$17 = 16 + 1 = 2^4 + 1 = 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

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- The binary code of the number 17 is the sequence of bits 0 or 1 which are the coefficients multiplying the power of 2 in the previous writing:

$$17 = 10001$$

where we see that we needed 5 bits to encode this number.

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- For example with 3 bits, we can code 8 numbers: the integers going from 0 to 7.

Integer	Binary code
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Elements of Binary Algebra

- To encode the integers 8 and 9 and thus be able to encode all the digits of the decimal system, we need at least 4 bits:

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- In addition, the 4-bit register makes it possible to encode 6 other symbols.

hexadecimal System

- The hexadecimal system is the one where these symbols are A, B, C, D, E, F respectively:

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- Here the symbols A, B, C, D, E, F do not represent letters of the alphabet, but numerical values in the hexadecimal system corresponding respectively to the integers 10, 11, 12, 13, 14 and 15.

ASCII Code

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- See Appendix 1 for more details

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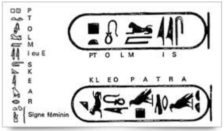
  - Inversion: which changes the value of the bit : $0 \longrightarrow 1; 1 \longrightarrow 0$

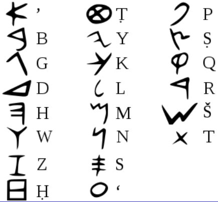
Information storage

- In information processing, you have to be able to write this information on a medium and be able to read it in order to use it.
- Since the beginning of mankind, this need has been implemented on physical media that have evolved throughout history.
- The prehistoric man painted on the walls of caves sketchy drawings that today we call cave paintings (hands, animals, human beings, ...). Even today, we can "read" this information and understand it.

Evolution of information media

- More evolved civilizations have used other techniques (such as engraving) and other supports (stone, marble, ...) for different representations.
- The complexity of life has prompted mankind to invent information encoding systems that allow it to express ideas in a more efficient manner.
- Thus, we created two fundamental tools: numbers and letters (or cryptograms)
- Here is for example the Phoenician alphabet, of which the Punic alphabet (that of Carthage) is a variant. Note the analogies with the current Latin alphabet.





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- This is the system that we use today all over the world.
- Extended to real numbers and augmented by scientific notation (mantissa and exponent), the decimal system satisfies almost all of our needs for expressing any quantity used in a conventional system of units, such as the international SI system. For example $h = 6.62607015 \times 10^{-34} J.s$ (exact value since May 20, 2019).

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- See Appendix 2 for other number systems (binary, ternary, senary, octal, nonary, decimal, duodecimal, hexadecimal, vigecimal, sexagesimal)

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- The other strong point is the ease of performing the operation of adding two numbers, with the introduction of the carry-over rule.
- The operation of multiplying two numbers is more complicated, but its rules are simple and its manual realization, for reasonable numbers remains within reach.

Decimal System

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- To convince yourself, try the following two operations :

$$
\begin{array}{rr}
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+ & \\
 & 15296325750265236 \\
 & \_\_\_\_\_\_\_\_\_
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$$

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- Start the stopwatch at the start and note the time it took you to complete each operation. Surprised ?

Back to the binary system

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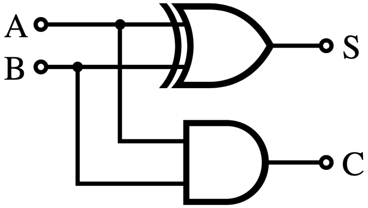
- We recalled above how we code integers in binary. For addition, the rules are as follows for 2 bits:

bit 1	bit 2	sum	carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

- Thanks to the developments of electronics in the 20th century, it was possible to build physical supports (circuits) which make it possible to carry out the operations of storage (writing) and reading, as well as the various mathematical operations in binary code (we will come back to this later).

Half-adder Circuit

- The half-adder circuit is built from an XOR gate and a AND gate (we will come back to this)



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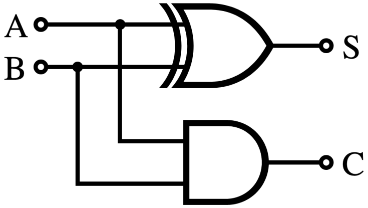

- It takes two bits E_1 and E_2 as input and delivers 2 outputs, the sum S and the carry R,

Hal-adder Quantum Circuit

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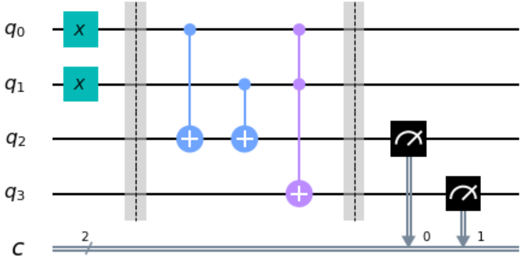

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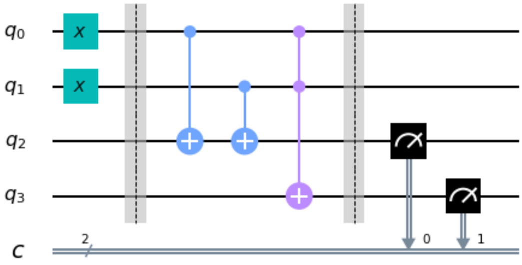

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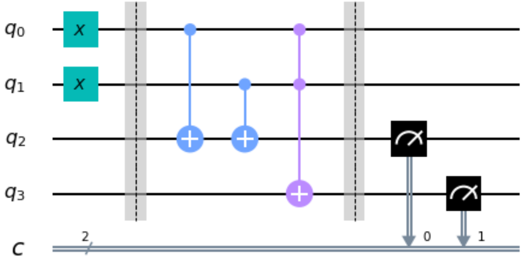
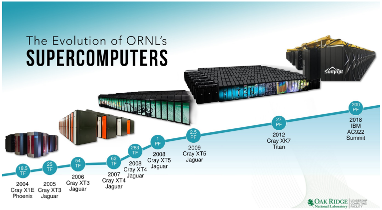


- For now, it seems complicated, but at the end of this course, this kind of circuit will become clear.
- But already know at this level that this circuit is formed by quantum gates X, $CNOT$ and $Toffoli$!

Classical Supercomputers

- This example gives a little idea about the practical implementation of a simple computation operation with logic circuits. The operation of digital calculators and computers is based on the same principle, obviously with a high degree of complexity. but the principle is the same.



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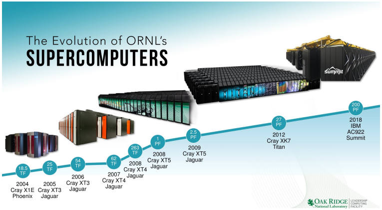
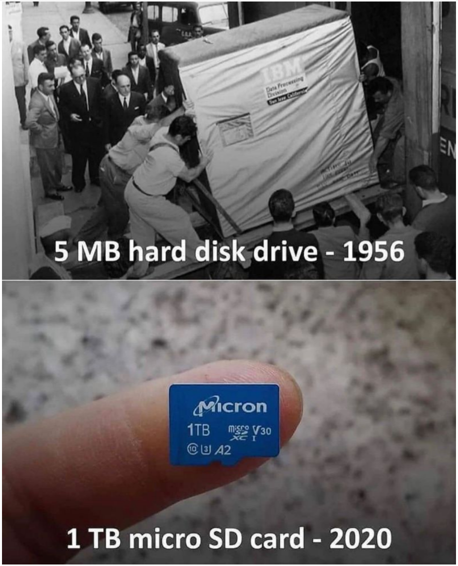


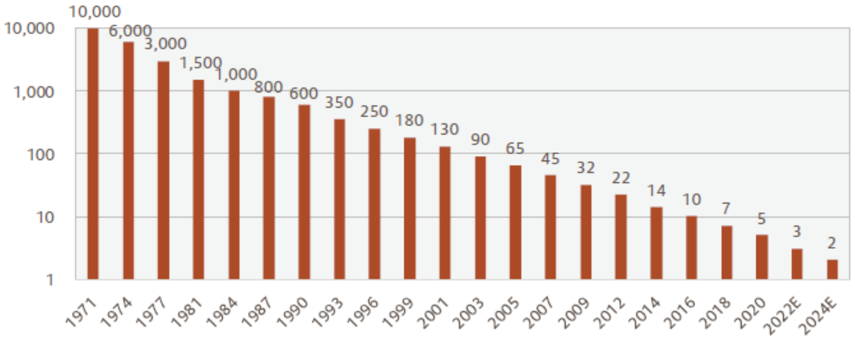
- Today there are very powerful computers like *IBM Summit*, capable of performing up to 2×10^{17} operations / second.

Evolution of storage capacities



Limits of miniaturization: Moore's Law

Figure 2: The incredible shrinking universe (device size in nm, log scale)



Source: PC Magazine, Epoch Investment Partners
Note: For reference, most atoms are 0.1 to 0.5 nm in diameter

Epoch perspectives, 11 February 2021

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- Are today's computers sufficient to perform the calculations we need ?
- Yes, to a certain extent.
- However, for some problems we reach a limit. For example, the factorization of large numbers, used to encrypt messages (RSA protocol) and ensure the security of communications and transactions related to e-commerce, electronic signatures, etc.

FACULTE DES SCIENCES DE TUNIS

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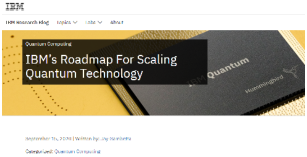
- Some prototypes with a few dozen qubits already exist and work.
- However, to show how fast this field is changing, here are two recent announcements (September 2020):

Dwave has announced the launch of a 5000+ qubit quantum computer:

IBM has published a roadmap announcing a 1000+ qubit computer for 2023.



Quantum Supremacy

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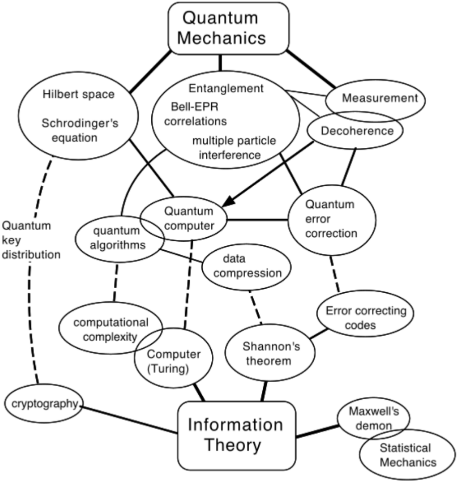
Article | Published: 23 October 2019

Quantum supremacy using a programmable superconducting processor

Frank Arute, Kunal Arya, [...] John M. Martinis

Nature **574**, 505–510(2019) | Cite this article

Quantum Mechanics and Information Theory



[3]

[3] Andrew Steane 1998 Rep. Prog. Phys. 61 117

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- The key used to encrypt is accompanied by a large integer, the product of two large primes kept secret (of the order of 200 digits, see RSA numbers). To calculate the decryption key, the only known method requires knowing the two prime factors.

RSA Protocole

- The security of the RSA system is based on the fact that it is easy to find two large prime numbers (using primality tests) and multiply them between them, but that it would be difficult for an attacker to find these two numbers. . This system also allows the creation of digital signatures, and has revolutionized the world of cryptography.

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- To do this, Alice and Bob need to agree on a code that allows them and no one else to piece together the messages.

Prime numbers

- A prime number is a natural number that admits exactly two distinct positive and integer divisors. These two divisors are 1 and the number considered, since any number has as divisors 1 and itself (as shown by the equality $n = 1 \times n$)

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- This allows the use of very large numbers like $RSA2048$ for public key cryptography protocols.

RSA-2048

25195908475657893494027183240048398571429282126204
03202777713783604366202070759555626401852588078440
69182906412495150821892985591491761845028084891200
72844992687392807287776735971418347270261896375014
97182469116507761337985909570009733045974880842840
17974291006424586918171951187461215151726546322822
16869987549182422433637259085141865462043576798423
38718477444792073993423658482382428119816381501067
48104516603773060562016196762561338441436038339044
14952634432190114657544454178424020924616515723350
77870774981712577246796292638635637328991215483143
81678998850404453640235273819513786365643912120103
97122822120720357

See Appendix 4 for more details about RSA numbers.

Complexity classes

- To better understand the difficulty of the factoring problem, here are some examples of mathematical problems as well as the scale laws of the number of operations n with the number of bits (or digits) as well as the complexity classes:

Problem	Operations	Class
Addition of 2 numbers of n bits	n	P
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- Current computer architectures are unable to deal with complex problems due to a lack of efficient algorithms.

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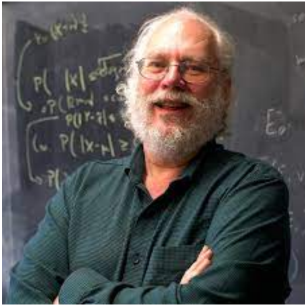
- Quantum information gives hope, and it is reasonable to think that we will be able to solve the problem of factorization of large numbers and many others within a reasonable period of time.
- A major breakthrough was made in 1994 by Peter Shor, who developed a quantum factoring algorithm.

- 

Peter Shor
(ICTP Dirac medal 2017)

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- Physicists made a first rough estimate for RSA-2048 and found that with a quantum computer formed of 10,000 logical qubits and 10 million physical (superconducting) qubits, spaced 1 cm apart for the wiring, which would cost "only" 100 billion USD, and using a modest electric power of 10 MW, would get the job done in 16 hours !!! (J. Preskill 2012)

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- *Letters (65–90 / 97–122)*: divided into two blocks, upper and lower case.

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- A ternary system (base 3) :
- The quinary system (base 5): of which traces remain until the 20th century in African languages, but also, partially, in Chuvash, Suzhou, Roman and Mayan notations. The name of the numbers 6, 7, 8 and 9 in many languages testify to this quinary system: they are said to be 5 + 1, 5 + 2, 5 + 3 and 5 + 4 in Wolof (language of the Niger-Congolese family) , in Khmer (Austro-Asiatic language), in Nahuatl (Uto-Aztec language), and, in many Austronesian languages such as Lote or Ngadha (in partial form). The quinary base appears as a sub-base of the decimal base and the vigesimal base.

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- The duodecimal system (base 12): is used in Nepal by the Chepang people. It is found, because of its advantages in terms of divisibility (by 2, 3, 4, 6), for a certain number of currencies and current account units in Europe in the Middle Ages, partially in Anglo-Saxon countries. in the imperial system of units, and in commerce. It is also used to count months, hours, flowers, oysters and eggs.

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- The sexagesimal system (base 60): was used for Babylonian numeration, as well as by Indians and Arabs in trigonometry. It is currently used in measuring time and angles.

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- When Bob receives the message he calculates $b^d \mod N = a$, which allows him to find Alice's message.

Appendix 3 : RSA Protocole

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- So, in the end, Bob decoded the message and reconstructed the original message a sent by Alice.
- The key to the success of the RSA protocol is the great difficulty of factoring N and finding the two prime numbers p and q, as soon as N is large enough.

Appendix 3 : Fermat's Little Theorem

- Statement 1: "Let p be a prime number, and a be a prime integer with p. Then a^{p-1} has for remainder 1 in the division by p. "

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- Statement: "The equation $x^n + y^n = z^n$ where $x, y, z \in \mathbb{N}$ and $n \geq 3$, does not admit solutions other than $(0, 0, 0)$.

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- Finally Bob extracts the message $a = 4$.

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- the question is therefore: Is it that difficult to find p and q knowing N?
- the answer is yes, it is a really difficult problem, as soon as N is big enough.
- Obviously, in the example chosen $N = 21$, the factorization $21 = 3 \times 7$ is trivial, but the situation gets complicated very quickly when N becomes large.

Appendix 4 : RSA numbers

- RSA-200 : made up of 200 digits in decimal

  27997833911221327870829467638722601621070446786955
  42853756000992932612840010760934567105295536085606
  18223519109513657886371059544820065767750985805576
  13579098734950144178863178946295187237869221823983

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- The calculation carried out on a network of computers required a CPU calculation time equivalent to 75 years on an Opetron processor running at 2.2GHz.

Appendix 4 : RSA numbers

- The current record, dating from 2009, for the largest factored number (RSA-768): formed by 232 digits in decimal

$$
\begin{aligned}
&12301866845301177551304949583849627207728535695953\\
&34792197322452151726400507263657518745202199786469\\
&38995647494277406384592519255732630345373154826850\\
&79170261221429134616704292143116022212404792747377\\
&\hfill 94080665351419597459856902143413
\end{aligned}
$$

- The calculation carried out on a network of computers required approximately two years of calculation, that is to say a CPU calculation time equivalent to 2000 years on an Opetron processor running at 2.2GHz.

Appendix 4 : RSA-2048

25195908475657893494027183240048398571429282126204
03202777713783604366202070759555626401852588078440
69182906412495150821892985591491761845028084891200
72844992687392807287776735971418347270261896375014
97182469116507761337985909570009733045974880842840
17974291006424586918171951187461215151726546322822
16869987549182422433637259085141865462043576798423
38718477444792073993423658482382428119816381501067
48104516603773060562016196762561338441436038339044
14952634432190114657544454178424020924616515723350
77870774981712577246796292638635637328991215483143
81678998850404453640235273819513786365643912120103
97122822120720357

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- With current technology, it is estimated that the time required, on a single processor, to factor a number of "only" 500 digits would be greater than the age of the universe !!!
- This time can be reduced by resorting to parallelization. If we accept a calculation period of 10 years, we would have to use a cluster of computers that would cover the surface of Tunisia several times, which would cost 10^{18} USD and would require an electric power of 10^{12} megawatt, which would exhaust all the world's fossil fuel resources in one day !!! (J. Preskill 2012)

Outline

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