Cosmology – Lecture I – Theory

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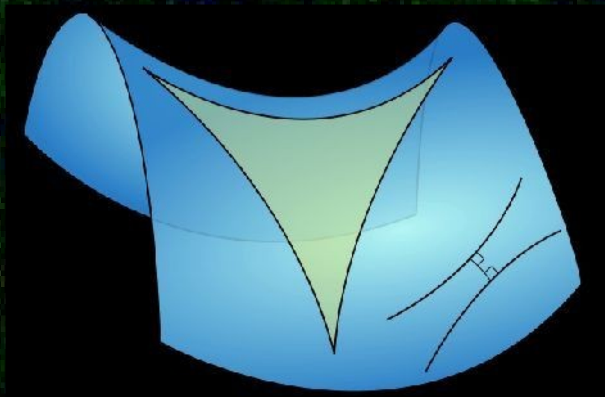
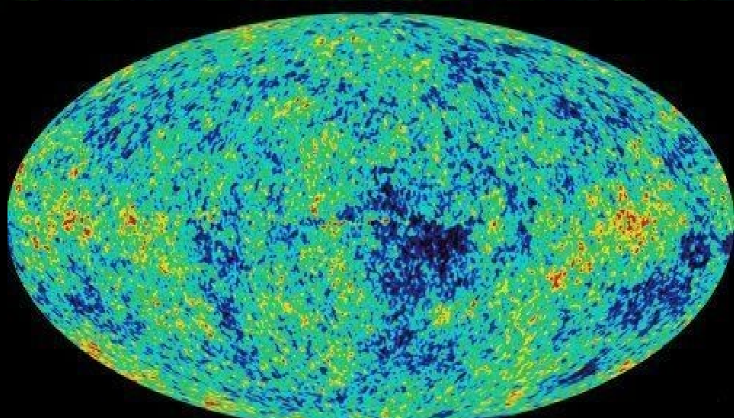
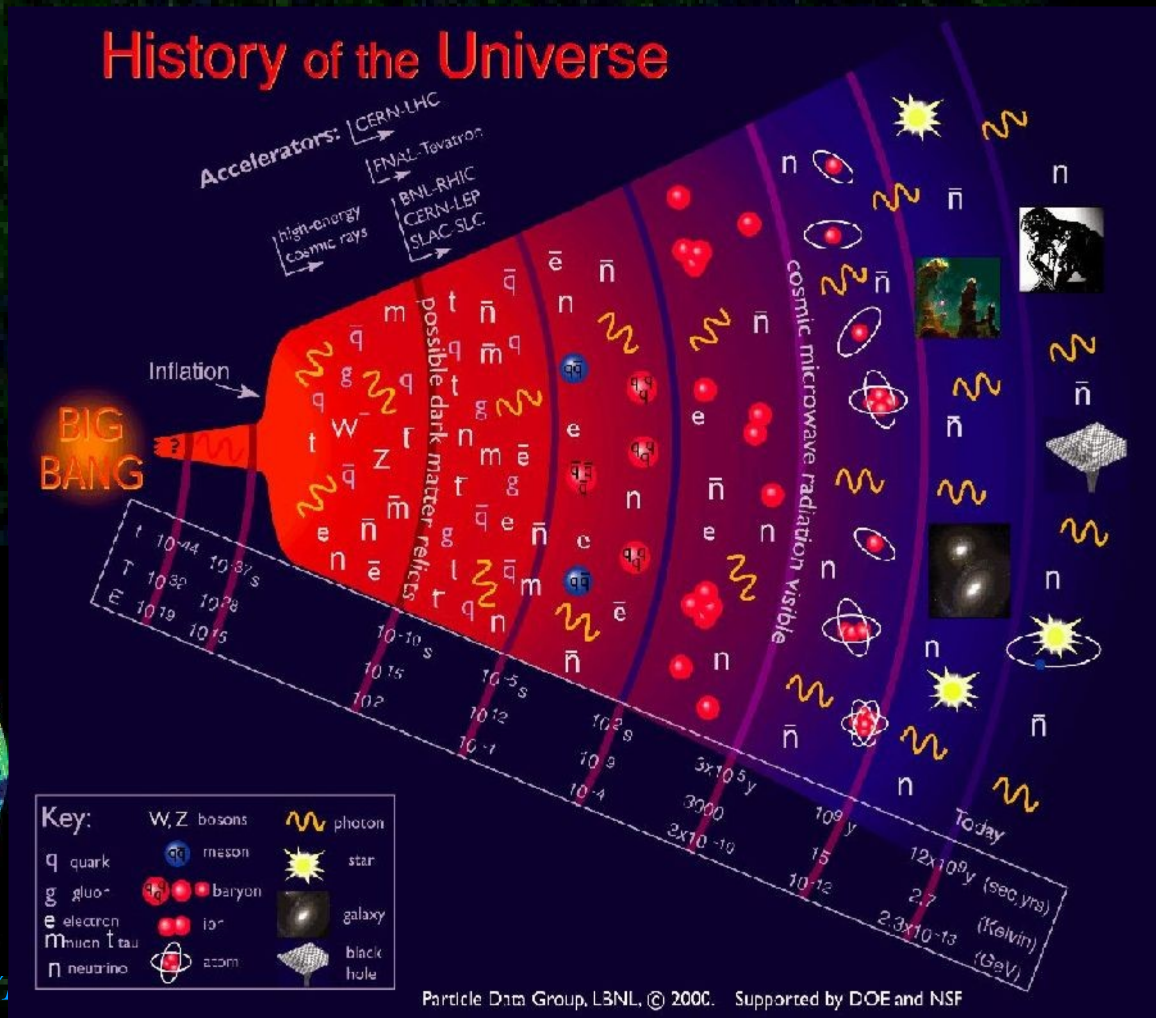




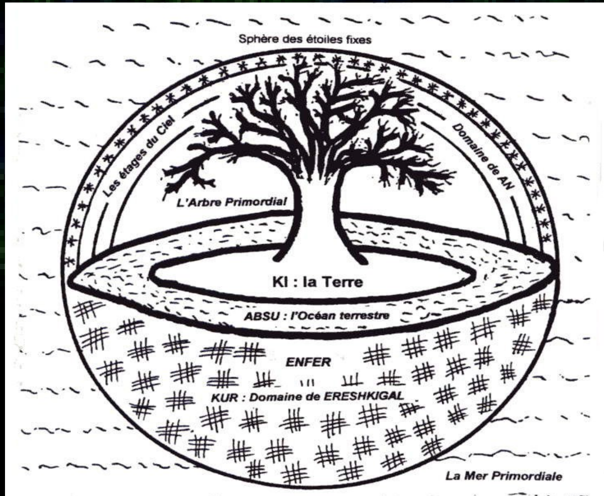


General Introduction
Theory

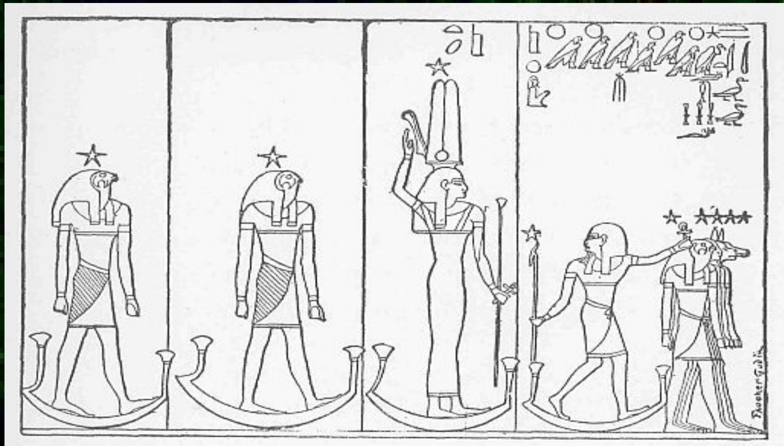
See also talk by Prof. Chanda Prescod-Weinstein (30/11)

What is Cosmology?

- Fundamental questions about the origin and destiny of the Universe:
  - What is the Universe made up of ?
  - How did the matter and structures form in the Universe ?
  - Why is the Universe as we see it ?
  - What is our place in the Universe ?
  - Did the Universe always exists, and if not, what is its age ?
  - How will the Universe evolve / possibly end ?
- Questions that appear in all cultures/religions
- Many different answers across history



La représentation du Monde à l'époque de SUMER

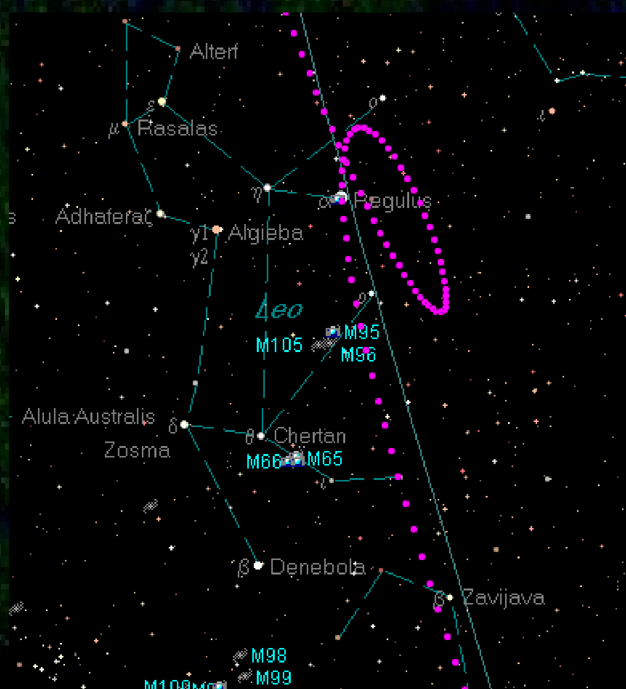


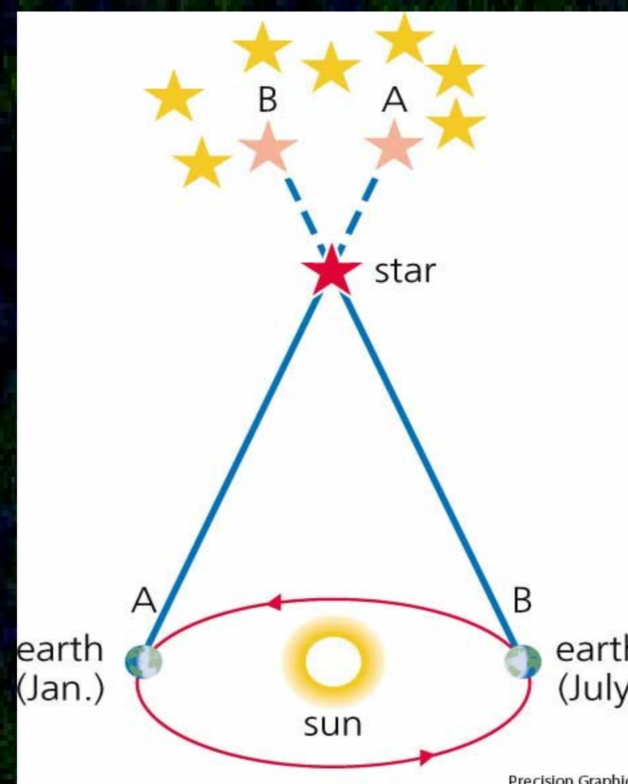
ORION, SOTHIS, AND TWO HORUS-PLANETS STANDING IN THEIR BARKS.[2]

Historical Cosmology

- Movement of the planets & stars:
  - During one night
  - From one night to the other: puzzling retrograde motion
  - From one year to the other: apparent movement of stars
  - From different places on the earth

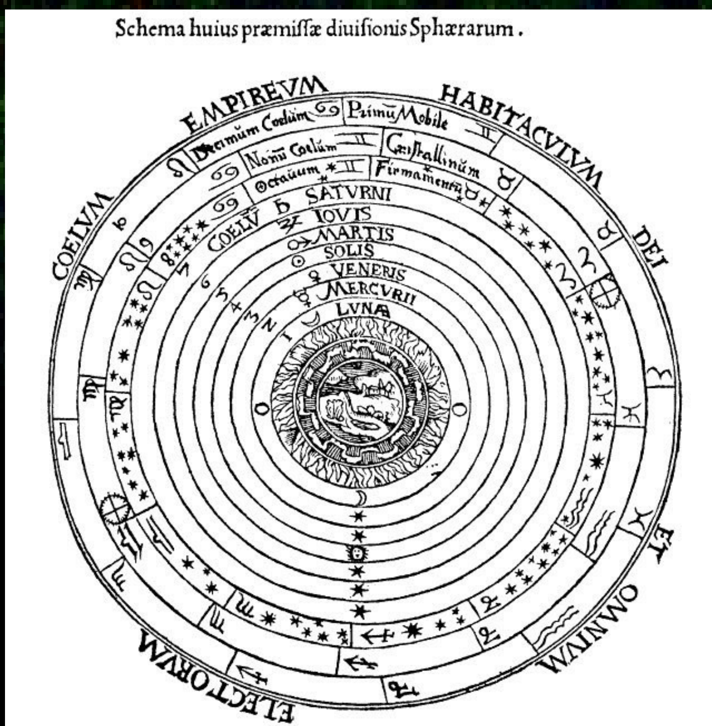
Retrograde motion

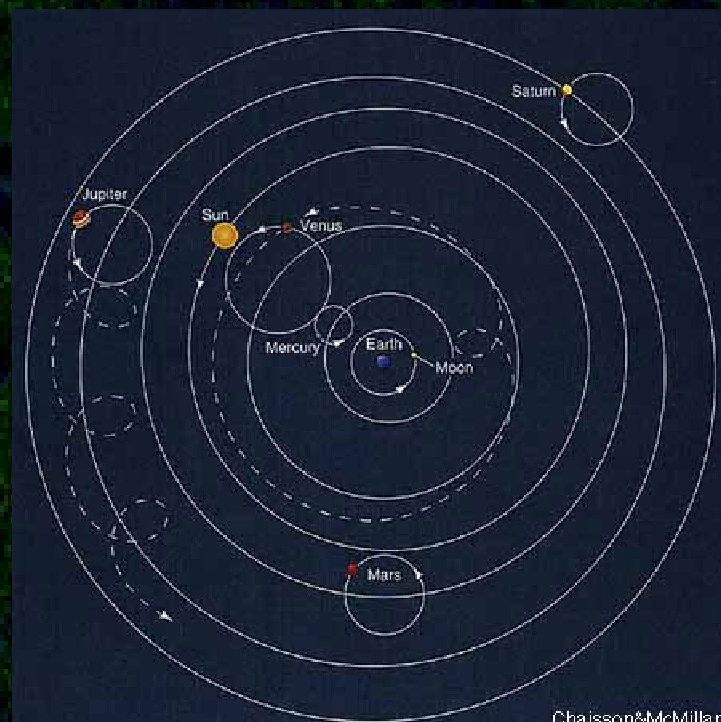


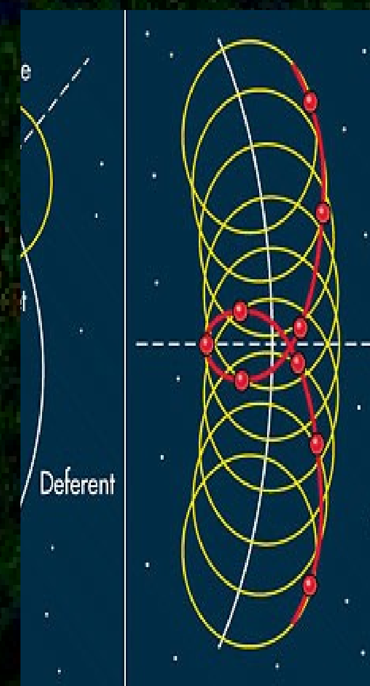


Model of Ptolemy

- Earth at center, fixed stars
- Complicated movement of planets explained by epi-cycles
  - Able to describe this retrograde motion





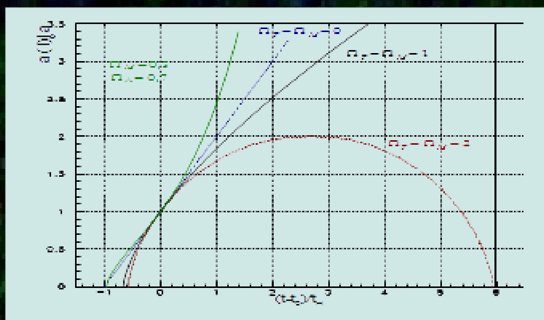


Major Steps in History

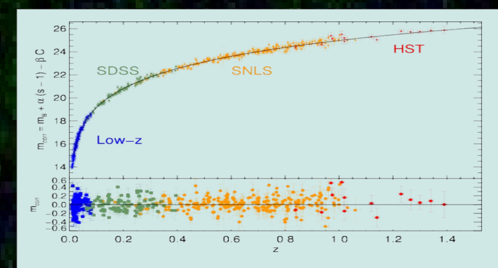
- −3000 : Flat earth, mythological Cosmology (Egypt, …)
- ~100 : Earth at centre (Ptolemy)
- 1520 – 1680 : Sun at centre (Copernic, Newton)
- 1917 : Universe is infinite (Einstein)
- 1922 : Evolving Universe (Friedman – Lemaître)
- 1964 : Discovery of Cosmological Background. Big Bang model (Penzias & Wilson)
- > 2000 : Accelerated expansion (Supernova Ia, …), modern cosmology

Open questions, observables

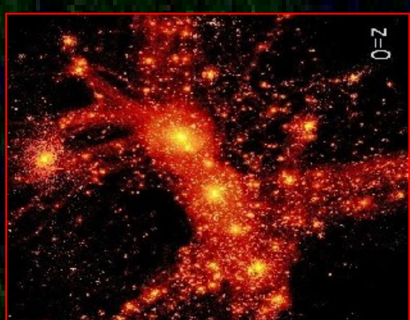
- Evolution of the Universe



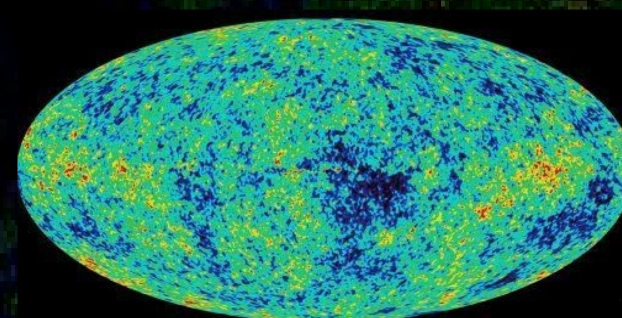
- Supernova 1a: distance versus recession velocity



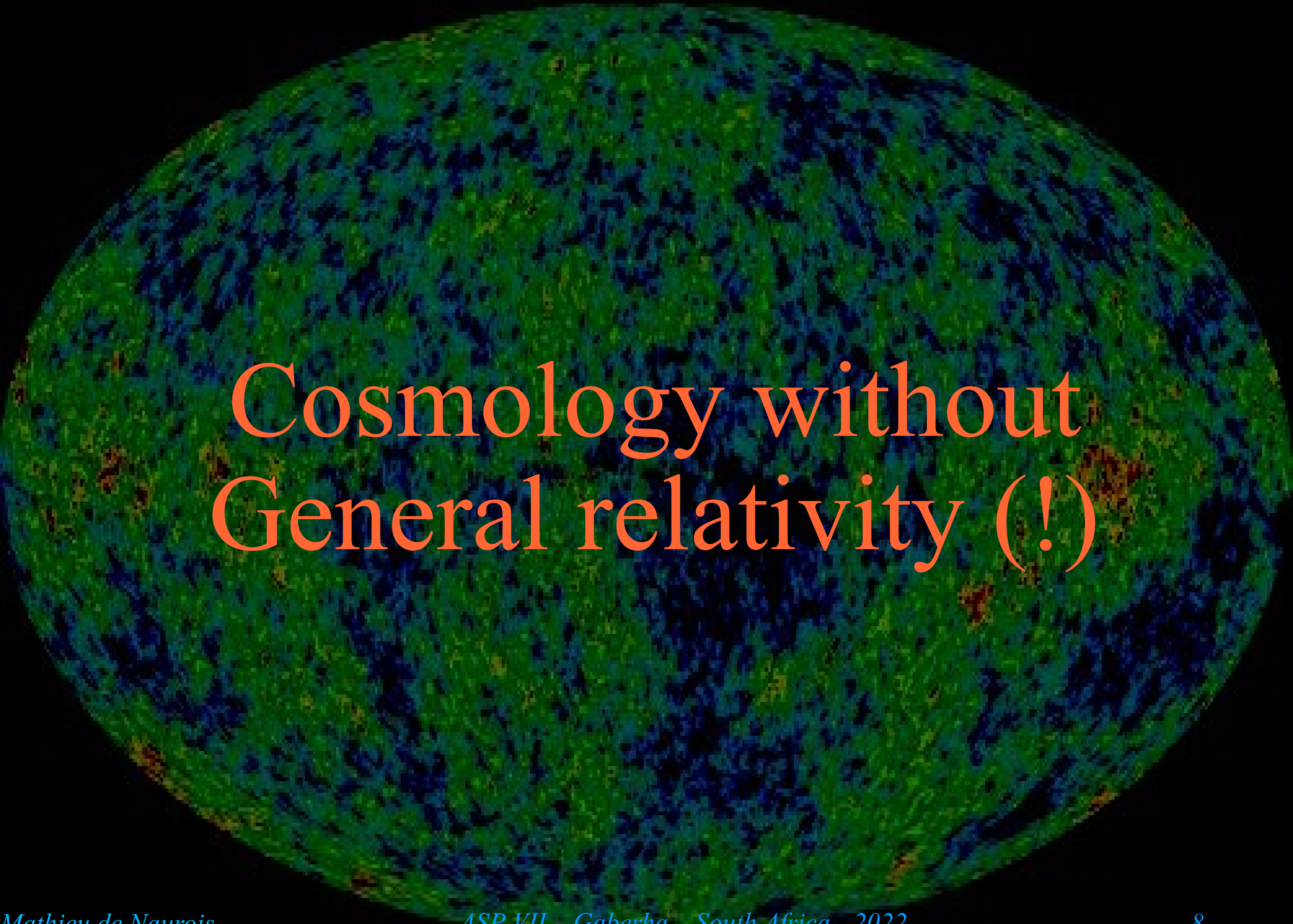
- Formation of structures



- Cosmological Background

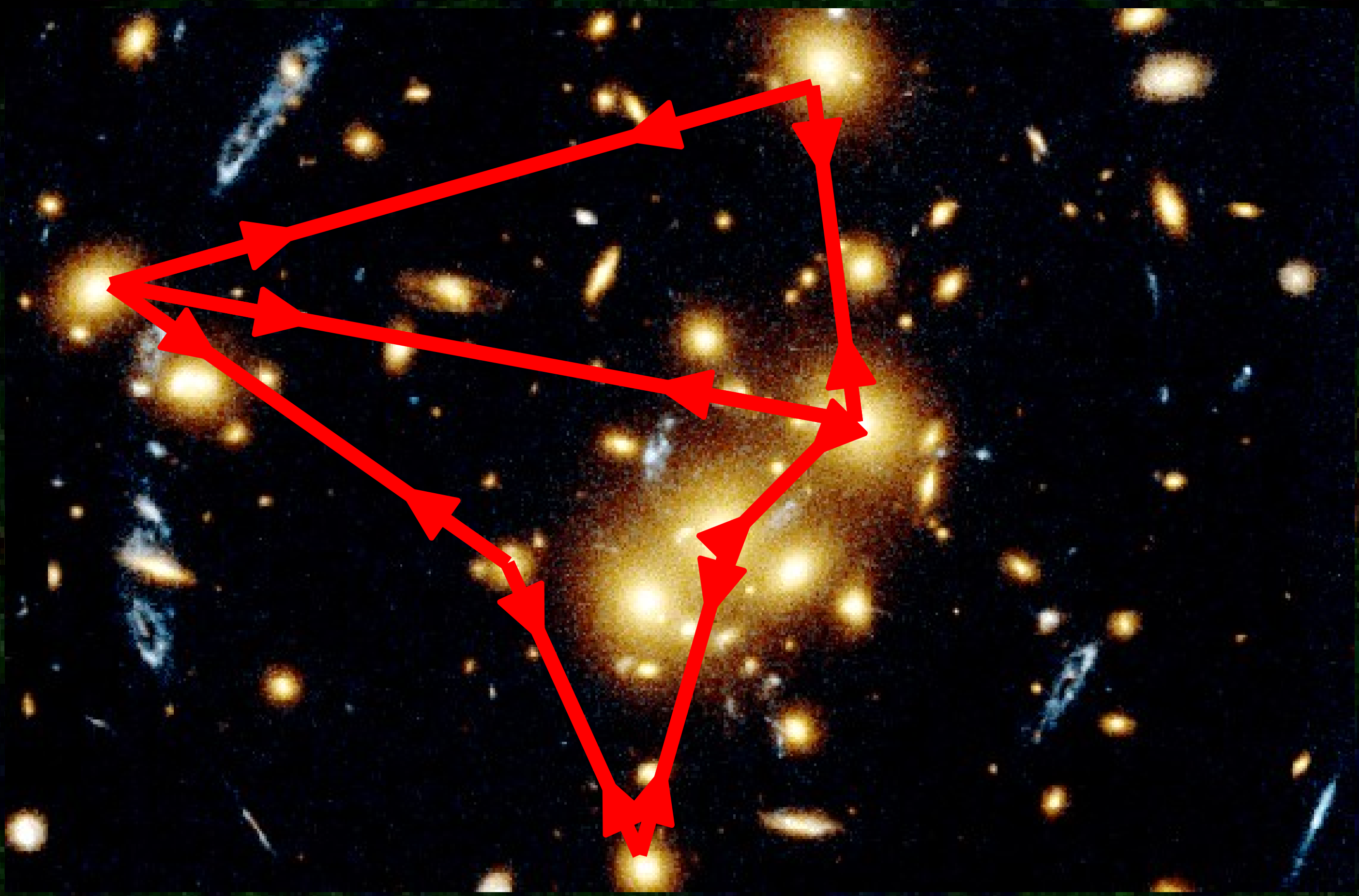


- Big bang Nucleo-synthesis
- Abundances of light elements



Cosmology without General relativity (!)

Is a static Universe possible ?

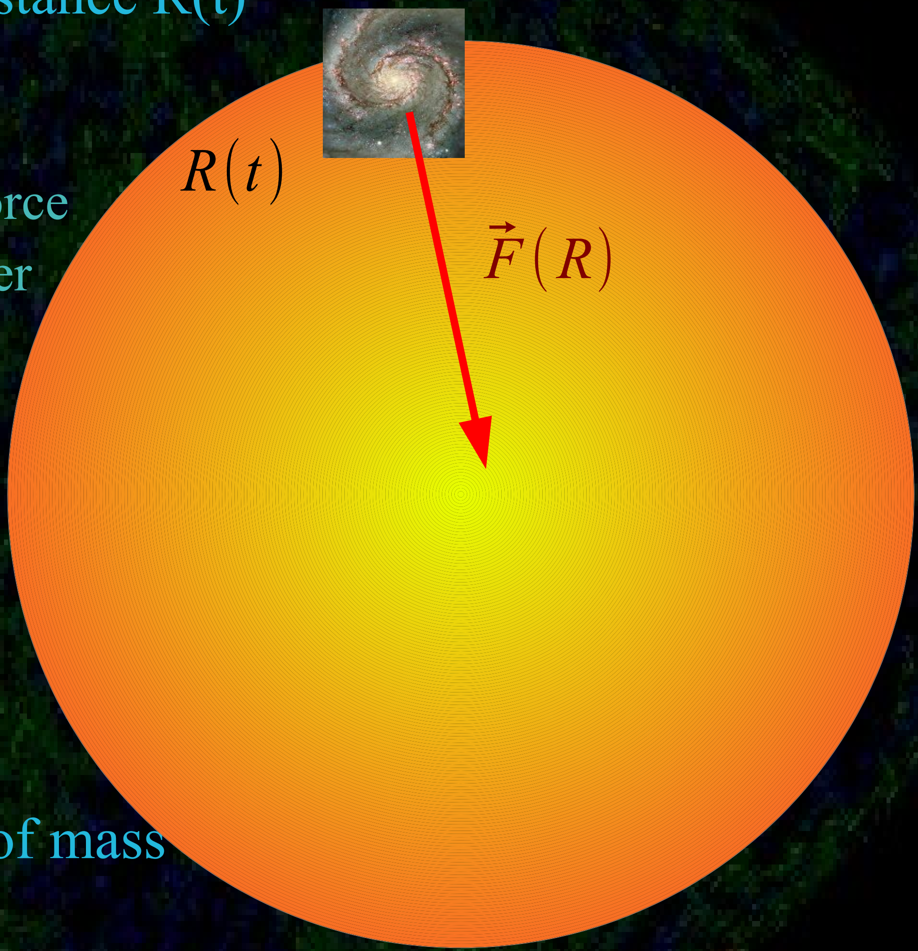


- Take a Universe with many galaxies isotropically distributed
- Gravity force between each pair of galaxies is attractive
- Calculate the evolution in a mean gravitational field

Is a static Universe possible?

- Consider only one Galaxy at distance R(t)
- Forces:
  - Radial by symmetry
  - Isotropic pressure → no net force
  - Radial force due to inner matter (Gauss theorem)

$$\vec{F}(R) = -\frac{G\,M(R)\,m}{R^2}\,\vec{u}_R$$



- Evolution of a "bubble":

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{G\,M(R)}{R^2}$$

- Matter Universe, conservation of mass

$$M(R) = \frac{4}{3}\rho_m(t)\,R^3 = C_{\text{ste}}$$

Evolution of a matter Universe

- Gravitational force

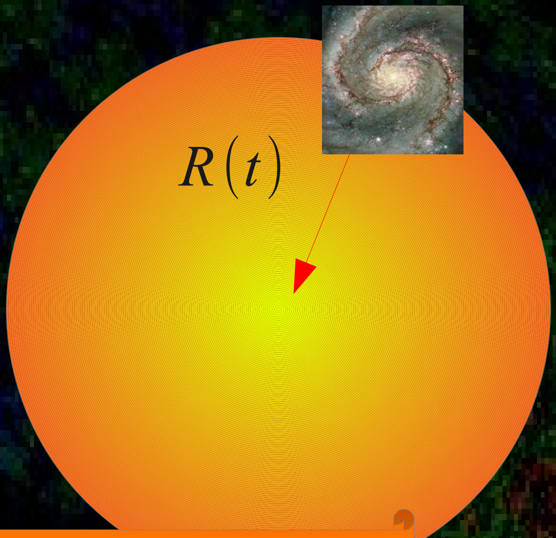
$$\vec{F}(R) = -\frac{G\,M(R)\,m}{R^2}\,\vec{u}_R$$

- Fundamental principle

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{G\,M(R)}{R^2}$$

- Conservation of mass

$$M(R) = \frac{4}{3}\rho_m(t)\,R^3 = C_{\text{ste}}$$



Only for Matter!

- Evolution Equation:

$$\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi}{3}\rho_m G \quad \Rightarrow \quad \dot{R}\ddot{R} = -\frac{4\pi}{3}(\rho_m R^3)\,G\,\frac{\dot{R}}{R^2}$$

$$\Rightarrow \quad \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}(\rho_m R^3)\frac{G}{R^3} + \frac{C}{R^2}$$

Evolution of a matter Universe

- Evolution Equations:

Velocity

$$\left(\frac{\dot{R}}{R}\right)^2 = M_0 \frac{G}{R^3} + \frac{C}{R^2}$$

Acceleration

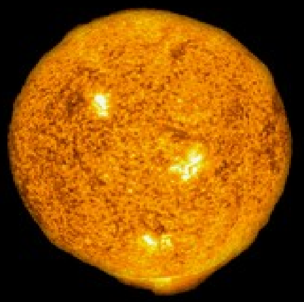
$$q = \left(\frac{\ddot{R}}{R}\right) = -\frac{8\pi G \rho_m}{3} \leq 0$$

- Expansion of the Universe is decelerated by matter content
- C is a constant specific to the Universe (Curvature! - see later)
- This does NOT require general relativity, pure classical mechanics!

NO static massive Universe is possible !!

Interlude – Why no static Universe? Olber's paradox (1758-1840)

- Imagine a infinite, static Universe existing since ever.
- Isotropic distribution of Galaxies
- Light received by a galaxy at distance R scales as $1/R^2$
- Number of galaxies at distance [R, R+dR] scales a $R^2\,dR$
- Each slice contribute to ~ same value, integration leads to infinity



The night sky must be White!

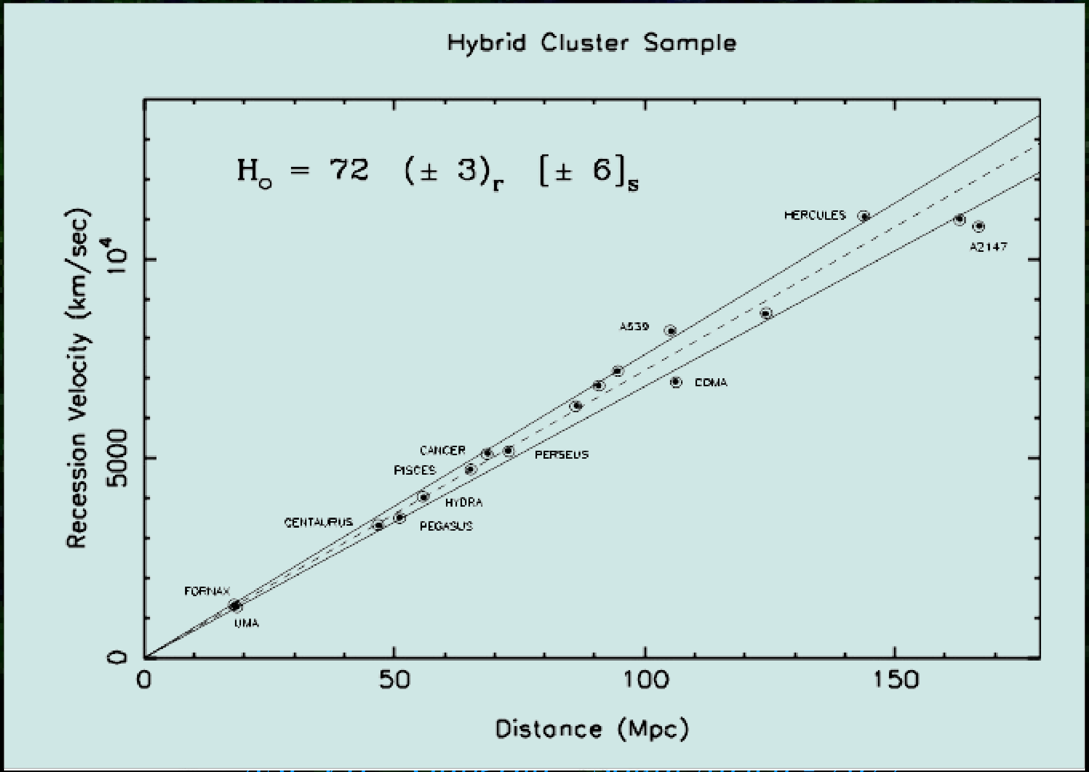
Observation – Hubble Law

- Galaxies are separating apart at a speed proportional to their distance

$$\frac{dR}{dt} = H_0 R + v_p \quad \Rightarrow \quad H_0 = \left\langle \frac{\dot{R}}{R} \right\rangle_{t=t_0}$$

Hubble flow

Proper Motion



Evolution of a matter Universe

- Rewriting the evolution equations with current value

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\left(\rho_m R^3\right)\frac{G}{R^3} + \frac{C}{R^2} \quad \Rightarrow \quad H_0^2 = \frac{8\pi}{3}\left(\rho_m^0 G\right) + \frac{C}{R_0^2}$$

- Critical density $\rho_c = \frac{3H_0^2}{8\pi G}, \quad \Omega_m = \frac{\rho}{\rho_c}$

Matter

(Curvature)

- Dimensionless evolution equation:

$$\frac{1}{H_0^2}\left(\frac{\dot{R}}{R}\right)^2 = \left(\Omega_m\left(\frac{R_0}{R}\right)^3 + \left(1-\Omega_m\right)\left(\frac{R_0}{R}\right)^2\right)$$

- Slowdown of expansion driven by matter:

$$\left(\frac{\ddot{R}}{R}\right) = -\frac{4\pi G\rho_m}{3} = -\frac{\Omega_m}{2}H_0^2$$

Evolution of a matter Universe

- $\Omega_m = 0$, monotonic expansion

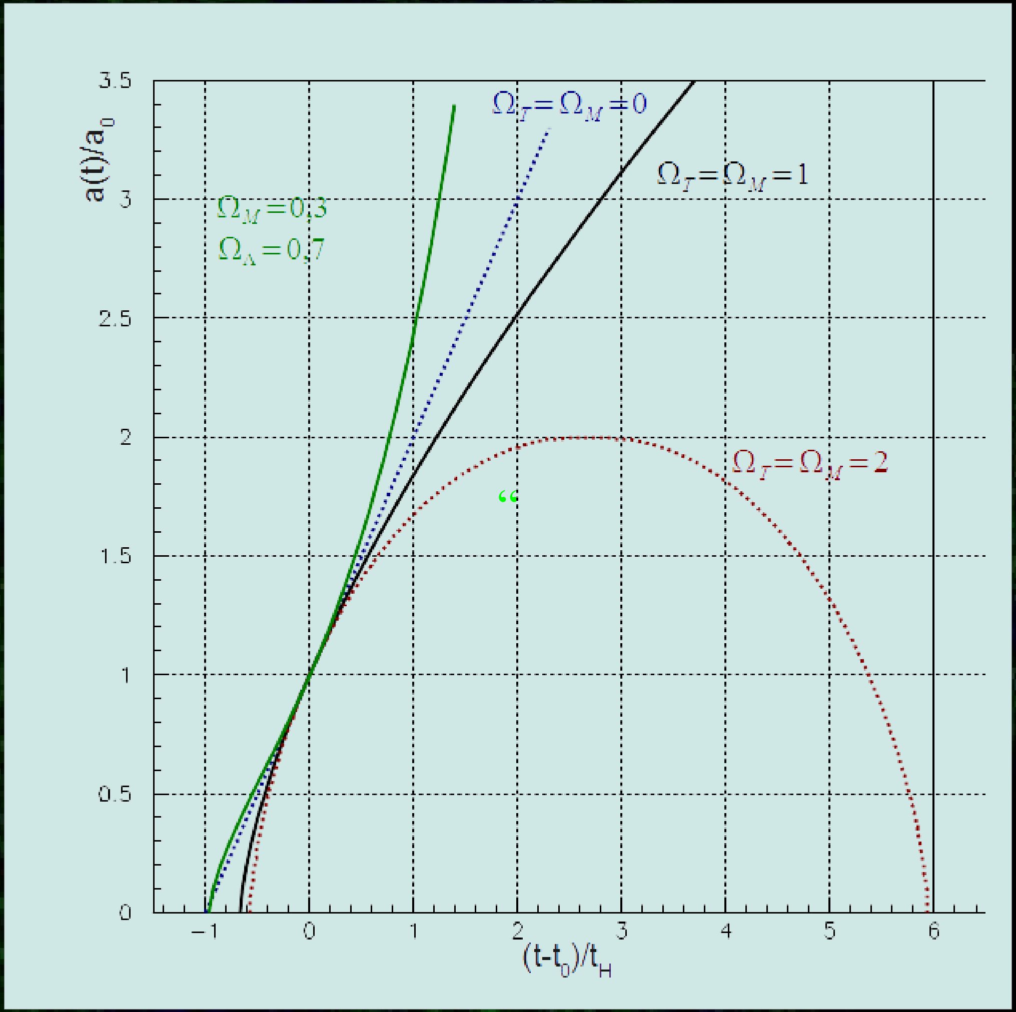
$$R(t) = R_0 H_0 \times t$$

- $\Omega_m = 1$ (critical Universe)
Decelerating expansion

$$R(t) = R_0 \left(\frac{3}{2} H_0 \times t \right)^{2/3}$$

- $\Omega_m > 1$ (critical Universe)
Collapsing Universe

$$R_{max} = R_0 \frac{\Omega_m}{(\Omega_m - 1)}$$



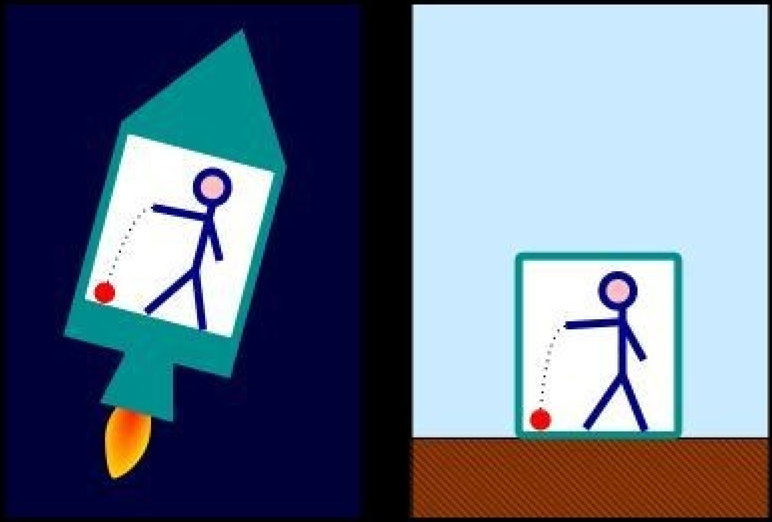
$$\frac{1}{H_0^2} \left(\frac{\dot{R}}{R} \right)^2 = \left(\Omega_m \left(\frac{R_0}{R} \right)^3 + (1 - \Omega_m) \left(\frac{R_0}{R} \right)^2 \right)$$

A (tiny)-bit of General relativity

Equivalence Principle - A. Einstein

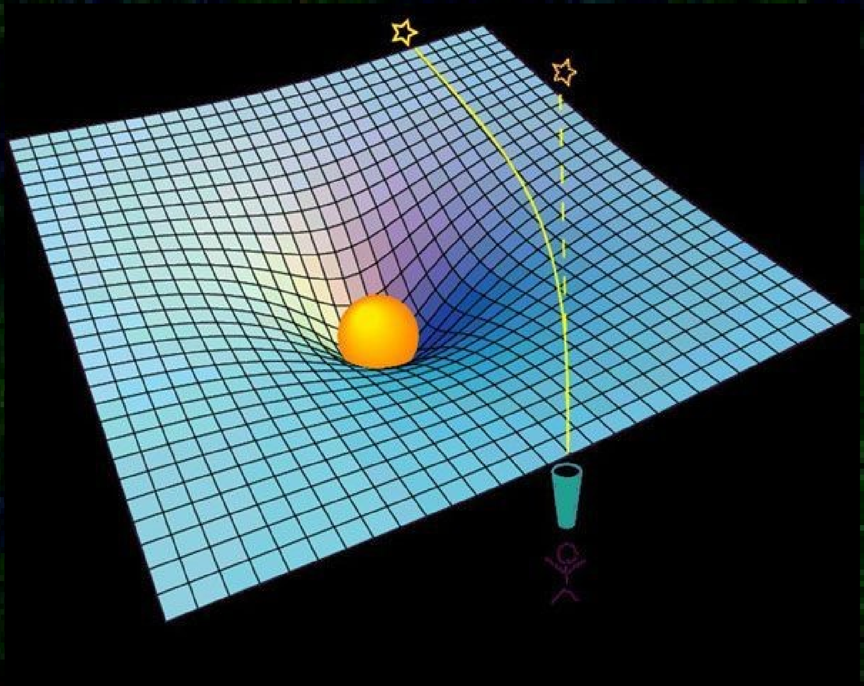
- No difference could be found between inertial mass (in acceleration) and gravitational mass (in gravity forces)
  ⇒ Implies that acceleration of a body in a gravitational field is independent of the nature of the body
  - Tested extensively in vacuum tower

- Thus there is no way to distinguish between a free-fall movement in gravity field from a accelerated movement in absence of field
  ⇒ Gravity can be understood as a property of space and not of the falling body



General Relativity vs Newtonian

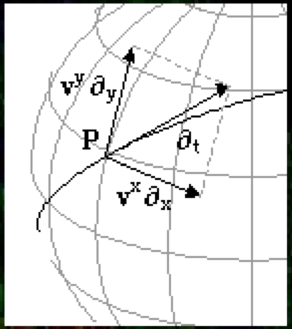
- Newtonian Gravity: Universe is flat and immuable, trajectories are curved due to a force (non-inertial movement)
- General relativity: Gravity is a geometric property of space, not a force. Trajectories are always inertial (geodesics) in a curved space
- Major conclusion: massless particles (light) are also affected, confirmed by measure of deflection of stars (Eddington, 1919)



Evolving Universe – Tensor Algebra

- We consider a space time, in which we have a base of vectors $\{\vec{e}_\mu\}$
- The metric is defined by the cross-product of vectors:

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$$



- Any vector can be decomposed on the base: $\vec{x} = x^\mu \vec{e}_\mu$

Covariant coordinates

- Several bases can describe the same Universe, transformation given by

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial y^\nu} \mathrm{d}y^\nu = \Lambda^\mu{}_\nu \mathrm{d}y^\nu, \quad \vec{e}_\mu = \Lambda_\mu{}^\nu \vec{f}_\nu$$

- Tensors are objects of higher rank (2, 3, ….) which transform in a similar manner

$$T^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T'^{\alpha\beta}$$

Norm & Invariants

- Scalar are invariant by change of coordinate, for instance:

$$A = U^{\mu} \cdot V_{\mu} = g^{\mu\nu} U_{\mu} V_{\mu}$$

- The elementary distance, defining the metric, can be expressed as:

$$ds^2 = dx^{\mu} \cdot dx_{\mu} = g^{\mu\nu} dx_{\mu} dx_{\mu}$$

Units where c = 1 !

And is invariant by coordinate changes (such as the scalar product)

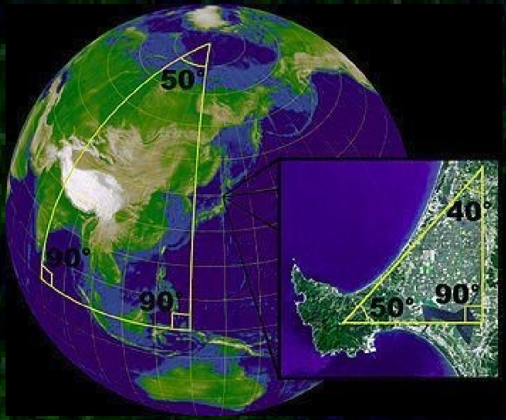
Tensor Algebra is the recipe to ensure that equations are Lorentz invariant, i.e. that equivalence principle is satisfied.

Curved Universe

- In a flat Universe, the metric can be expressed in a diagonal form.
  e.g. Minkowski space (flat space-time)

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- This is not the case any more in curved Universe
- The "**curvature**" is a mathematical concept that is obtained from derivatives of the metric:
  - Ricci tensor $R_{\mu\nu}$
  - Scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$

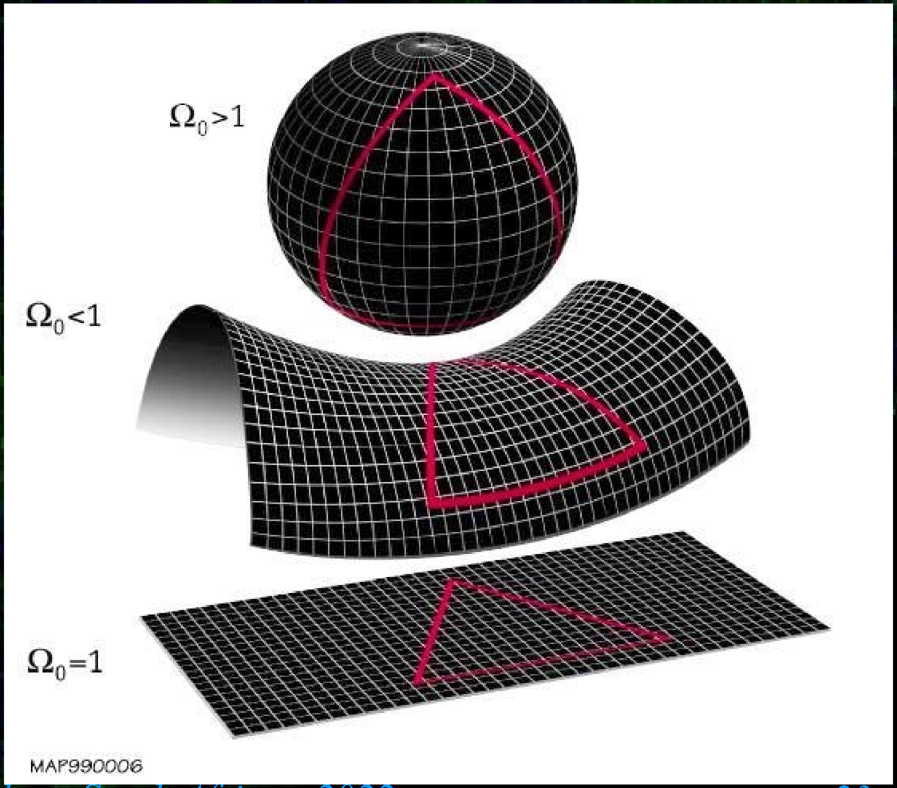


Uniform, Isotropic Universe

- A uniform, isotropic universe can be described by the Friedman-Lemaitre-Robertson-Walker metric

$$ds^2 \equiv dx^\mu dx_\mu = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]$$

  - a(t) is a "scale factor" giving the size of a bubble of Universe
    The grid itself is expanding, not the content!
  - $k = 1$: Spherical space (Sum of angles > π)
  - $k = -1$: Hyperbolic space (Sum of angles < π)
  - $k = 0$: Euclidean space (Sum of angles = π)



Einstein Equation – I

- General idea: find the minimum covariant formalism compatible with Newton gravity
- Start for the Poisson equation for gravitational potential

Field $\nabla^2 \Phi_p = -4\pi\rho_g$ **Matter Content**

- Construct a Lorentz-invariant (Covariant) version

Covariant Derivative $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu = 4\pi\, j^\mu$ **Matter Quadri-current (Density is NOT Lorentz invariant)**

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Curvature of Universe **Energy Content**

Energy Momentum Tensor?

- Need a covariant (Lorentz invariant) formulation of energy conservation
- In special relativity Energy & Momentum are coupled

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0 \quad \longrightarrow \quad \nabla_\mu T^\mu{}_\nu = 0$$

- Energy momentum tensor for a perfect fluid (Lorentz Invariant)

$$T_{\mu\nu} = n(\tilde{x}) \frac{p_\mu p_\nu}{E} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu)$$

u_μ is the four velocity

- In the rest frame of fluid, $u^\mu=(1,0,0,0)$ and thus:

$$T_{\mu\nu} = \begin{pmatrix} \rho(t) & & & \\ & -P(t) & & \\ & & -P(t) & \\ & & & -P(t) \end{pmatrix}$$

Energy Momentum Tensor

Energy Density — Energy Flux

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Viscosity

Pressure

Momentum Density — Momentum Flux

Einstein Equation – II

- Minimum Covariant Equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Curvature of Universe **Energy Content**

- Energy Content:

$$T_{\mu\nu} = \sum_{\text{species}} \left(\rho u_\mu u_\nu + P\left(g_{\mu\nu} + u_\mu u_\nu\right)\right)$$

- One can add a Cosmological Constant to force a static universe (Compensates for matter), no classical equivalent

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

General relativity in Friedman-Lemaitre-Robertson-Walker metric

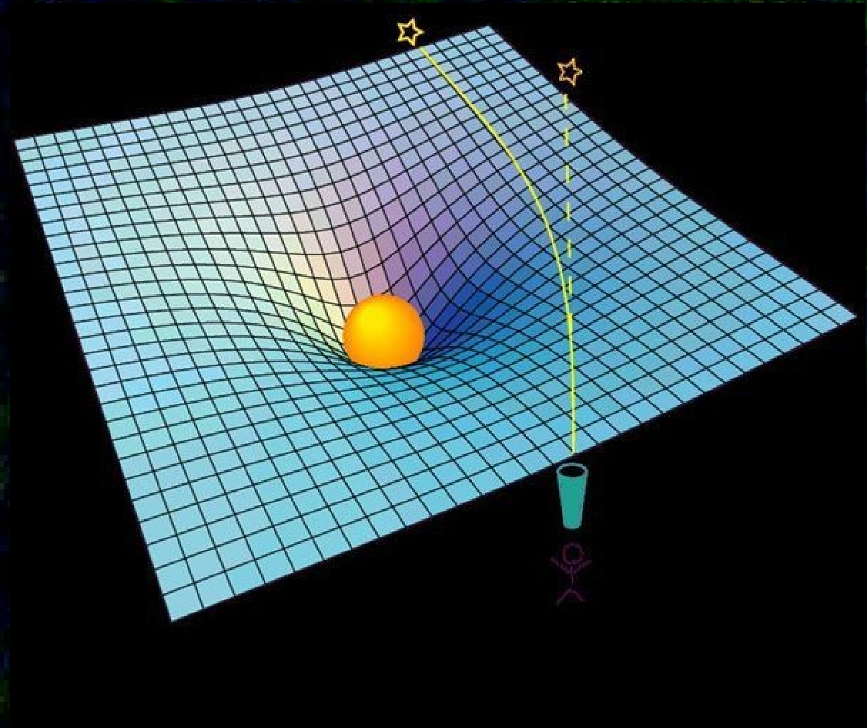
- Einstein Equation (Isotropic Uniform Universe)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}$$

- Acceleration

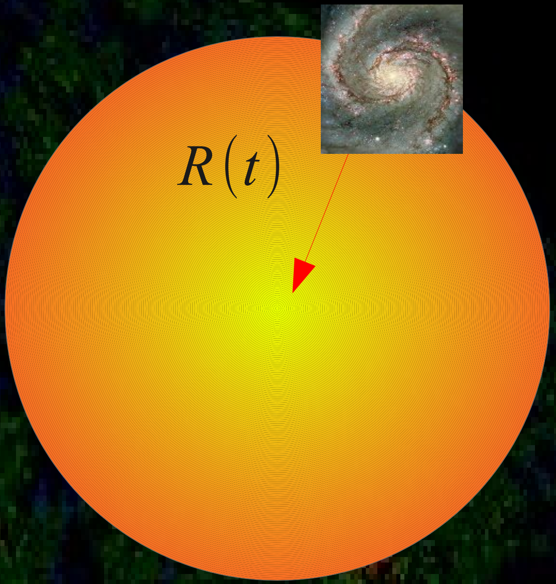
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3\, p_i)$$

- So we still need:
  - Relation between pressure & density (equation of state)
  - Corresponding evolution of density with time

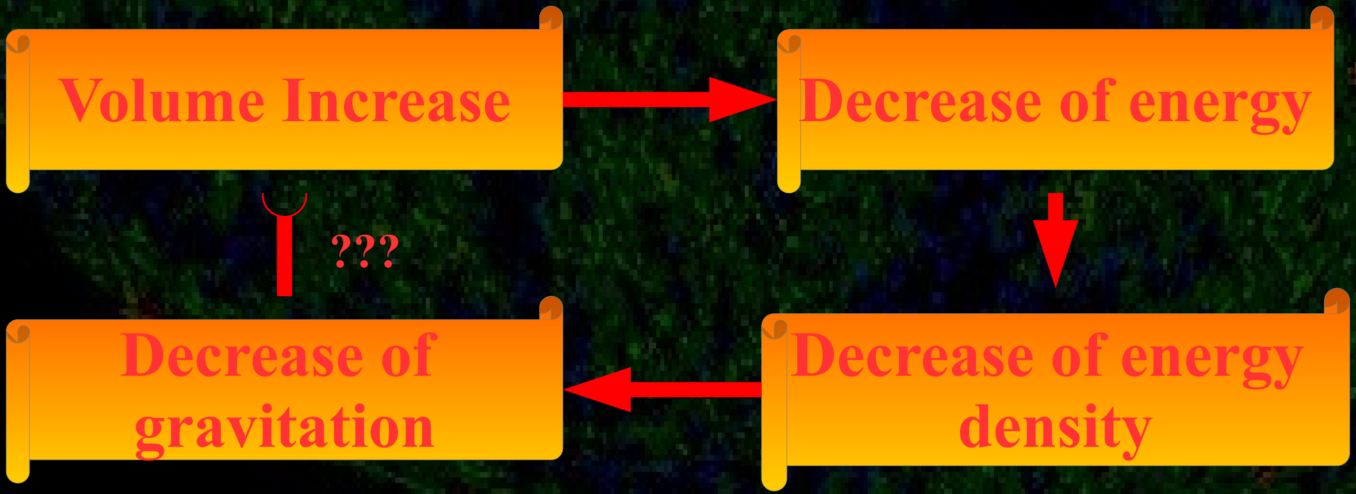


Why pressure?

- Gravitation depends on energy content
- But what if the size of the Universe changes?
- Thermodynamics never lies and says:



$$\mathrm{d}E = \delta W = -p\,\mathrm{d}V$$



Thermodynamics – Evolution of density

- Work of pressure: $$\mathrm{d}E = \delta W = -p\,\mathrm{d}V$$
- Expression of energy: $$E = \rho V$$ $$\frac{\mathrm{d}E}{\mathrm{d}t} = \rho\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}\rho}{\mathrm{d}t} = -p\frac{\mathrm{d}V}{\mathrm{d}t}$$
- Evolution of density: $$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -(p+\rho)\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = -3\frac{\dot{a}}{a}(p+\rho)$$
- In particular, $$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \quad \Leftrightarrow \quad p = -\rho$$

Negative pressure?

- Using equation of state: $$P = w\rho \quad \Rightarrow \quad \rho(t) = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Equation of state – Matter (cold)

- Normal matter:
  - Energy Density

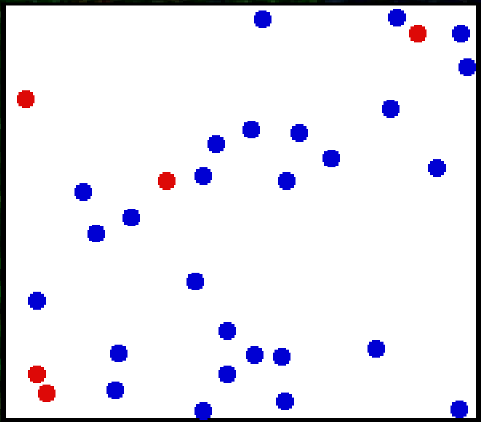
$$\frac{E}{V} = \rho_m \left(c^2 + \frac{1}{2} v^2 \right) \approx \rho_m c^2$$

  - Pressure is related to kinetic energy (internal energy)

$$P = \frac{n\,R\,T}{V} = \frac{2}{3} \frac{\langle E_c \rangle}{V} \approx \frac{2}{3} \frac{\langle v^2 \rangle}{c^2} \times \frac{E}{V} \ll \frac{E}{V}$$

  - For normal matter kinetic energy is negligible compared to mass energy

$$P = 0 = w\,\rho \quad \text{with} \quad w = 0$$

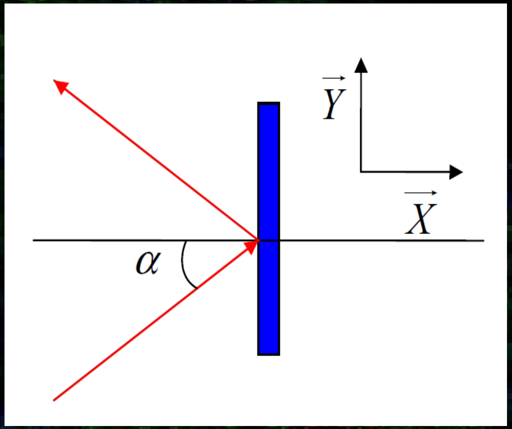


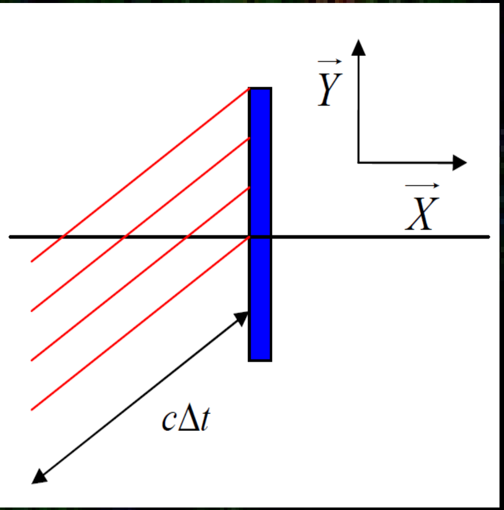
Equation of state – Radiation

- Radiation:
  - Energy Density $\frac{E}{V}=\frac{N}{V}\times pc$
  - Simple calculation (reflection of photons with momentum transfer) shows

$$P=\frac{N}{V}\times pc\int \cos^2\alpha \, d\cos\alpha$$
$$=\frac{1}{3}\frac{E}{V}$$

$$P=w\rho \quad \text{with} \quad w=\frac{1}{3}$$





Equation of state – Cosmological constant

- Cosmological constant is characterized by constant density

$$\rho = \text{constant}$$

- Thus

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}(p+\rho) = 0$$

- This implies

$$P = -\rho = w\rho \quad \text{with} \quad w = -1$$

- Strange fluid with negative pressure!
  ⇒ Volume increase lead to energy increase!

Cosmological Constant

- Introduced by Einstein to allow for a static Universe (counteracting the mass)
- Positive energy density, independent of size, implying negative pressure
- Kind of "vacuum energy"
- But in 1929 Edwin Hubble showed that the Universe is in expansion

Much later, when I was discussing cosmological problems with Einstein, he remarked that the introduction of the cosmological term was the biggest blunder of his life.

-- George Gamow, My World Line, 1970

General relativity in Friedman-Lemaitre-Robertson-Walker metric

- Einstein Equation (Isotropic Uniform Universe)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}$$

- Acceleration

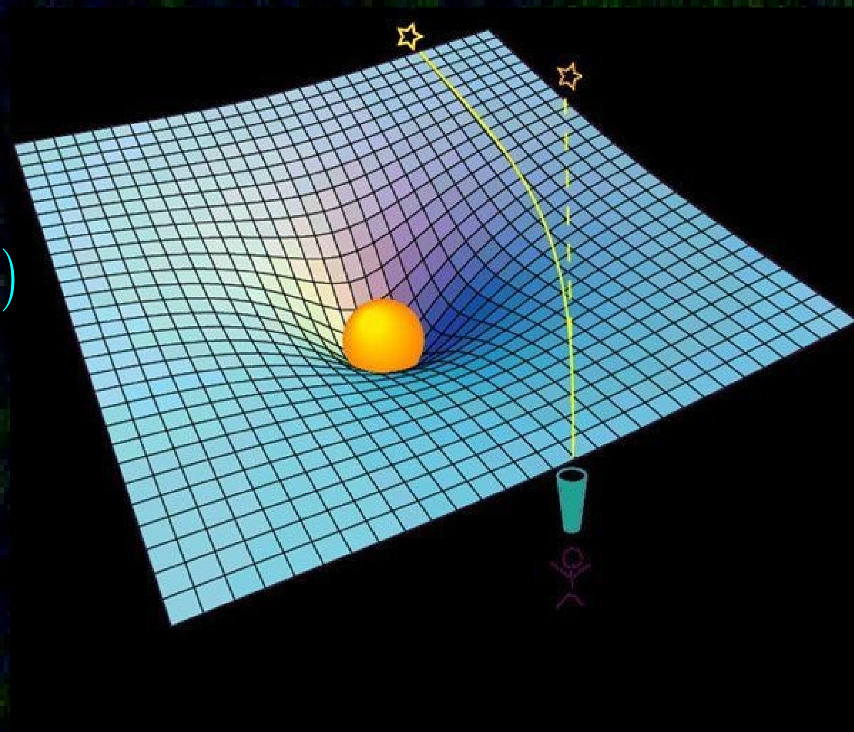
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i)$$

- Evolution of density

$$\frac{d\rho}{dt} = -(p+\rho)\frac{1}{V}\frac{dV}{dt} = -3\frac{\dot{a}}{a}(p+\rho)$$

- Equation of state

$$P = w\rho \quad \Rightarrow \quad \rho(t) = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$



Matter, radiation, ...

Content	State Equation	Dilution Law	Evolution
Cold Matter	$p \approx 0$	$\rho \propto a(t)^{-3}$	$a(t) \propto t^{2/3}$
Hot Radiation	$p = \frac{\rho}{3}$	$\rho \propto a(t)^{-4}$	$a(t) \propto t^{1/2}$
Curvature		$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2}$	$a(t) \propto t$
Cosmological constant	$p = -\rho$	$\rho = C_{ste} = \frac{\Lambda}{8\pi G_N}$	$a(t) \propto e^{H \times t}$
Generic	$p = w\rho$	$\rho \propto a(t)^{-3(1+w)}$	$a(t) \propto t^{1/3(1+w)}$

Evolution of the Universe

$$\left(\frac{H}{H_0}\right)^2 = \Omega_m^0\left(\frac{a_0}{a}\right)^3 + \Omega_r^0\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \left(1-\Omega_{tot}^0\right)\left(\frac{a_0}{a}\right)^2$$

- (Cold) Matter: $\Omega_m^0 \left(a_0/a\right)^3$
- (Hot) Radiation: $\Omega_r^0 \left(a_0/a\right)^4$ Dominates in the early Universe
- Curvature: $\left(1-\Omega_{tot}\right)\left(a_0/a\right)^2$
- Cosmological Constant: Ω_Λ

- Were we used dimensionless densities: $\Omega_i^0 = \frac{\rho_i^0}{\rho_{\text{critic}}} = \frac{8\pi G}{3H_0^2}\rho_i^0, \quad \Omega_k^0 = \frac{-k}{a_0^2 H_0^2}$

Deceleration parameter

- Deceleration parameter

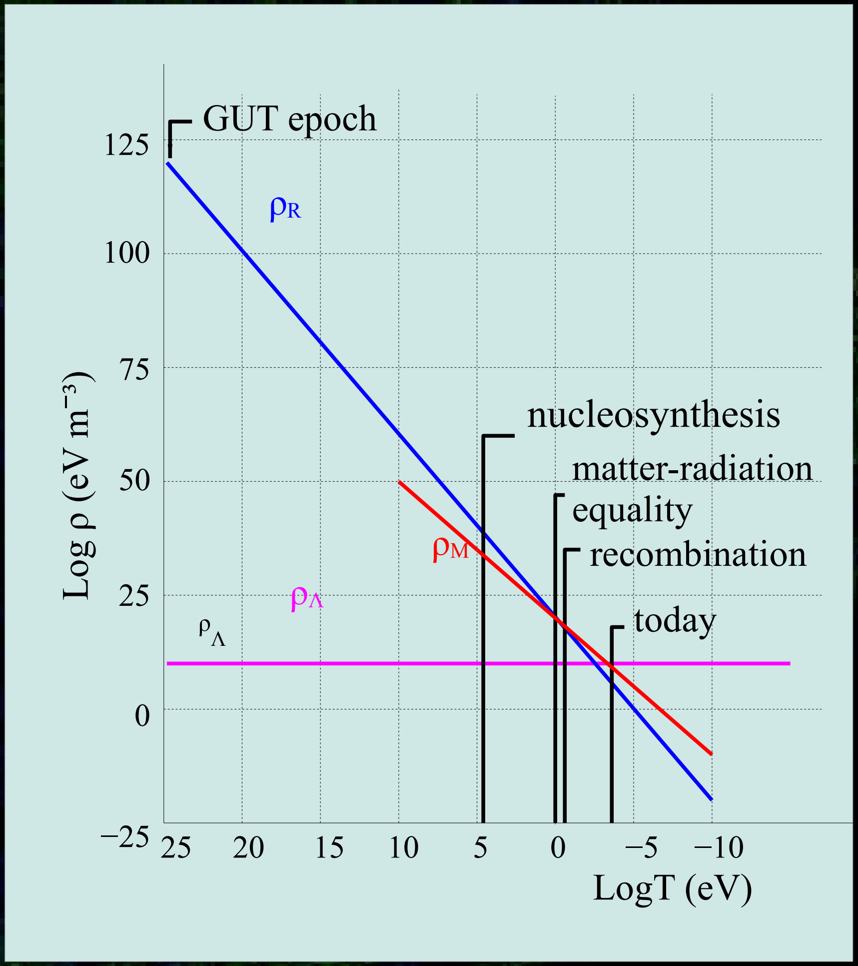
$$q = -\frac{1}{H^2}\left[\frac{\ddot{a}}{a}\right] = \frac{\Omega_m}{2} + \Omega_r - \Omega_\Lambda$$

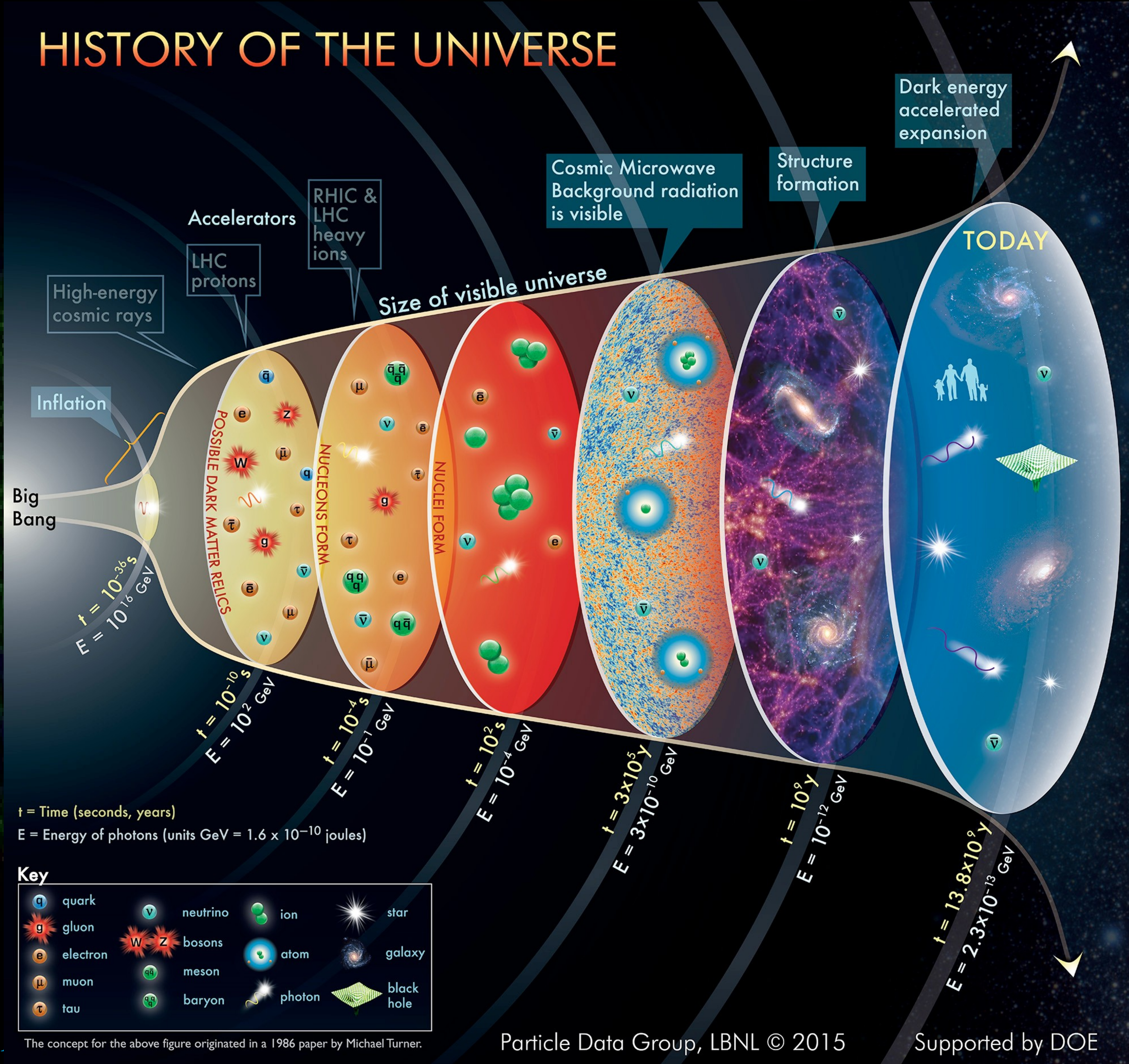
- Matter and radiation decelerates expansion
- Cosmological constants accelerates expansion
- Curvature is neutral
- Null deceleration (static universe) if

$$\Omega_m + 2\,\Omega_r = 2\,\Omega_\Lambda$$

Epochs and Fate

- Universe starts by a radiation dominated era
- After some times, matters dominates over the radiation and expansion slows down
- If $\Omega > 1$ and $\Omega_\Lambda \sim 0$, the Universe re-collapses and radiation dominates again
- If $\Omega < 1$ and $\Omega_\Lambda \sim 0$, the Universe ends in free expansion governed by curvature
- If $\Omega < 1$ and $\Omega_\Lambda > 0$, the Universe ends in accelerated exponential expansion governed by cosmological constant




HISTORY OF THE UNIVERSE
Dark energy accelerated expansion
Structure formation
Cosmic Microwave Background radiation is visible
RHIC & LHC heavy ions
Accelerators
LHC protons
High-energy cosmic rays
Inflation
Big Bang
Size of visible universe
TODAY
POSSIBLE DARK MATTER RELICS
NUCLEONS FORM
NUCLEI FORM
t = 10⁻³⁶ s
E = 10¹⁶ GeV
t = 10⁻¹⁰ s
E = 10² GeV
t = 10⁻⁴ s
E = 10⁻¹ GeV
t = 10² s
E = 10⁻⁴ GeV
t = 3x10⁵ y
E = 3x10⁻¹⁰ GeV
t = 10⁹ y
E = 10⁻¹² GeV
t = 13.8x10⁹ y
E = 2.3x10⁻¹³ GeV
t = Time (seconds, years)
E = Energy of photons (units GeV = 1.6 x 10⁻¹⁰ joules)
Key
quark
gluon
electron
muon
tau
neutrino
bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
The concept for the above figure originated in a 1986 paper by Michael Turner.
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