Quantum Field Theory and the Electroweak Standard Model

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Outline of Lectures

Lecture 1:

- What you should know
  - QFT motivation
  - Classical Field Theory
  - Quantization for scalars, fermions.
  - Interactions and Feynman rules
  - ...

- Abelian Gauge Theories

- Non Abelian Gauge Theories

Outline of Lectures

Lecture 2:

- Building the Electroweak Standard Model as a Gauge Theory
- The Mass Problem in the SM: Spontaneous Symmetry Breaking
- The Higgs Boson and its interactions

Lecture 3:

- Tests of the Electroweak Standard Model
- Precision Tests of the Quantum SM
- Conclusions and Outlook

Complementary References

QFT I: http://fma.if.usp.br/~burdman/QFT1/qft1index.html

QFT II: http://fma.if.usp.br/~burdman/QFT2/qft2index.html

Each a semester long course with detailed notes and references

Referenced throughout the lectures. For instance as

I.L7 = QFT I, Lecture 7

Lecture 1

What you should already know

Classical Field Theory (I.L2)

Lagrangian is function of field $\phi(x)$ and its derivatives $\dfrac{\partial\phi(x)}{\partial x^\mu} = \partial_\mu\phi(x)$

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x))\ ,$$

Equations of Motion from $\delta S = 0$

$$\Rightarrow \qquad \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

Euler-Lagrange Eqns.

Example 1: Free real scalar field

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,m^2\,\phi^2$$

$$\Rightarrow \qquad (\partial^2 + m^2)\,\phi(x) = 0$$

Klein-Gordon equation

Example 2: Free Dirac fermion

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x)$$

$$\Rightarrow \begin{cases} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \\ \text{and} \\ \bar{\psi}(x)\,(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 \end{cases}$$

Dirac equation

Continuous Symmetries and Noether's Theorem

We consider an infinitesimal shift in the fields

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \epsilon\, \Delta\phi$$

This results in an infinitesimal shift in the Lagrangian

$$\mathcal{L} \longrightarrow \mathcal{L} + \epsilon\, \Delta\mathcal{L}$$

Noether's Theorem: the current defined as

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \Delta\phi$$

is conserved, i.e. satisfies $\partial_\mu\, j^\mu = 0$ if we use the equations of motion

Example: Complex Scalar Field

The Lagrangian $\mathcal{L} = \partial_\mu \phi^* \, \partial^\mu \phi - m^2 \, \phi^* \phi$ is invariant under the symmetry transformations

$$\left.\begin{aligned} \phi(x) &\longrightarrow e^{i\alpha} \, \phi(x) \\ \phi^*(x) &\longrightarrow e^{-i\alpha} \, \phi^*(x) \end{aligned}\right\}$$

with α a real constant (i.e. this is a global symmetry)

Then the current is

$$j^\mu = i \left\{ (\partial^\mu \phi^*) \, \phi - (\partial^\mu \phi) \, \phi^* \right\}$$

which satisfies $\partial_\mu \, j^\mu = 0$ as long as we impose the KG eqns.

$$(\partial^2 + m^2)\phi^* = 0 \; , \qquad (\partial^2 + m^2)\phi = 0$$

Field Quantization (I.L3)

Similarly as in QM: Field $\phi(x)$ and its conjugate momentum $\pi(x)$ defined as

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

satisfy the equal time commutation relation

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\, \delta^{(3)}(\vec{x} - \vec{x'})$$

just as x and p in QM satisfy

$$[x, p] = i\, \hbar$$

$\Rightarrow$ $\phi(x)$ and $\pi(x)$ are now operators

Expand field and conjugate momentum in most general solution of KG eqn. In momentum space

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left\{a_p e^{-ip_\mu x^\mu} + b_p^\dagger e^{ip_\mu x^\mu}\right\}$$

$\Longrightarrow$ a_p are $b_p^\dagger$ annihilation and creation operators

$$[a_p, a_p^\dagger] = 1 = [b_p, b_p^\dagger]$$

The operator $\phi(x)$ annihilates a particle of momentum $\mathbf{p}$

creates an anti-particle of momentum $\mathbf{p}$

The operator $\phi^\dagger(x)$ creates a particle of momentum $\mathbf{p}$

annihilates an anti-particle of momentum $\mathbf{p}$

Quantization of Fermion Fields (See I.L4 to L6)

Solutions of the Dirac equation $\psi_a(\vec{x}, t)$ are spinors. $a = 1, 2, 3, 4$

Spinors: objects that transform in a certain way so as to keep the Dirac eqn. Lorentz invariant.

Conjugate momentum of spinor:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

Expansion of fermion in momentum space solutions of Dirac eqn.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s \, u^s(\mathbf{p}) \, e^{-iP\cdot x} + b_p^{s\dagger} \, v^s(\mathbf{p}) \, e^{+iP\cdot x} \right)$$

Where $u_a^s(\mathbf{p}),\ v_a^s(\mathbf{p})$ are spinor solutions of the Dirac eqn. in momentum space

and $s = 1, 2$ are the spinor helicities

But now the quantization condition requires *anti-commutation* !

$$\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{x}',t)\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta_{ab}$$

This is necessary to make the Hamiltonian bounded from below. Otherwise, creating particles lowers H !

Equivalent to

$$\{a_p^r, a_k^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs} \qquad \{b_p^r, b_k^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{k})\,\delta^{rs}$$

for the particle and anti-particle annihilation and creation operators.

$\Rightarrow$

- $\psi(x)$ annihilates fermions or creates anti-fermions
- $\psi^\dagger(x)$ creates fermions or annihilates anti-fermions

A consequence of the anti-commutation rule for fermions is Pauli's Exclusion Principle:

E.g. consider a two particle state

$$|1_p^s \, 1_k^r\rangle = a_p^{s\dagger} \, a_k^{r\dagger} |0\rangle$$

Anti-commutation of the creation operators implies

$$a_p^{s\dagger} \, a_k^{r\dagger} = - \, a_k^{r\dagger} \, a_p^{s\dagger} \qquad \Longrightarrow \qquad |1_p^s \, 1_k^r\rangle = -|1_k^r \, 1_p^s\rangle$$

Which means that if all quantum numbers are the same $(s = r, \; \mathbf{p} = \mathbf{k})$ then

$$|1_p^s \, 1_p^s\rangle = -|1_p^s \, 1_p^s\rangle = 0$$

$\Longrightarrow$ State does not exist

Fermion Number Conservation

From the Dirac eqns. for $\psi(x)$ and $\bar{\psi}(x)$ we know that the fermion current is

$j^\mu = \bar{\psi}\gamma^\mu\psi$ and is conserved. I.e. $\partial_\mu j^\mu = 0$

$$Q = \int d^3x\, j^0(x) = \int d^3x\, \bar{\psi}(x)\gamma^0\psi(x) = \int d^3x\, \psi^\dagger(x)\,\psi(x)$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left\{a_p^{s\dagger}\, a_p^s - b_p^{s\dagger}\, b_p^s\right\} = N_{\rm particles} - N_{\rm anti-particles}$$

Fermion number $\left\{\begin{array}{l}\text{Particles have “charge" +1} \\ \text{Anti-particles have “charge" -1}\end{array}\right\}$ is a global charge

Fermion number conservation comes from a symmetry of the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x)$$

$\mathcal{L}$ is invariant under the symmetry transformations

$$\left.\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha}\,\psi(x) \\ \bar{\psi}(x) &\longrightarrow e^{-i\alpha}\,\bar{\psi}(x) \end{aligned}\right\} \quad \text{With } \alpha \text{ a real constant}$$

$\Longrightarrow$ This is a Global Symmetry

When α is a function of the spacetime point x^μ, i.e. $\alpha(x)$, the symmetry becomes local or “gauged".

Interactions and Perturbation Theory (See I.L10 to L14)

Quadratic terms in the fields associated with propagation

Terms with 3 or more fields $\longrightarrow$ Interactions

Example: Real scalar with self interactions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad \text{or} \qquad \mathcal{L}_{\text{int.}} = -\frac{\lambda}{4!}\phi^4$$

Other examples:

Fermion-scalar (Yukawa) $\quad \mathcal{L}_{\text{int.}} = -g\,\bar{\psi}\psi\phi$

Fermion-Gauge Boson (QED) $\quad \mathcal{L}_{\text{int.}} = -e\,A_\mu\bar{\psi}\gamma^\mu\psi$

From Correlation Functions to Amplitudes (I.L12)

N-point Correlation Functions

$$G^{(n)}(x_1, \cdots, x_n) = \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$$

contain all the information of a Quantum Field Theory

We can relate them to the momentum space amplitude for a given process by the LSZ formula

$$\mathcal{A}_{fi}(p_1, \ldots, p_n) = \mathcal{O}p(x_1, \cdots, x_n) \times G^{(n)}(x_1, \ldots, x_n)$$

with $\mathcal{O}p(x_1, \cdots, x_n)$ a differential operator acting on the external spacetime positions

and depending on the appropriate equations of motion operators.

Perturbation Theory

- In the absence of interactions all correlation functions can be computed exactly

  They are products of propagators (2-point correlation functions) with the internal positions integrated over

  In the functional integral approach, this can be understood as a result of the integrability of quadratic forms

- But interactions involve more than 2 powers of the field !

  So to implement the calculation of the correlation functions need to implement a controlled approximation

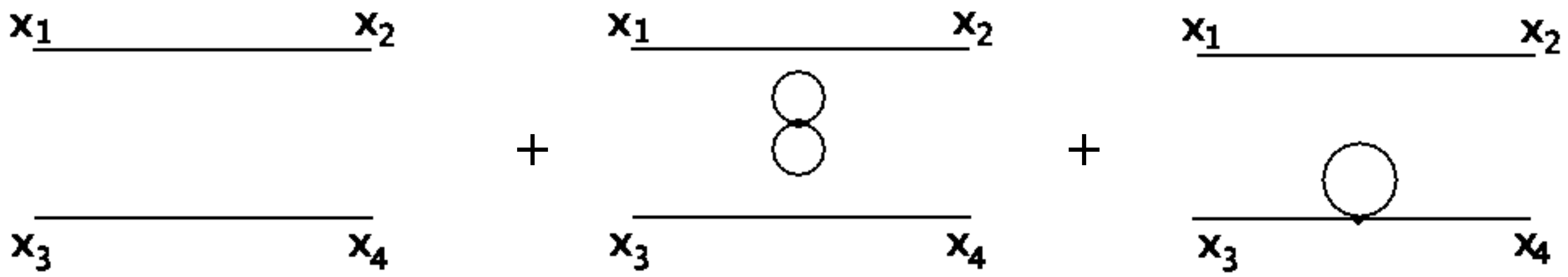
$$G^{(n)}(x_1,\ldots,x_n) = N \int \mathcal{D}\phi\, e^{i\int d^4x\{\mathcal{L}_0+\mathcal{L}_{\rm int.}\}}\ \phi(x_1)\ldots\phi(x_n)$$

$$G^{(n)}(x_1,\ldots,x_n) = N \int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}_0}\,\phi(x_1)\ldots\phi(x_n)\ \times \left(1+i\int d^4y\mathcal{L}_{\rm int.}[\phi(y)]\ +\ldots\right)\ ,$$

Each additional power of $\mathcal{L}_{\rm int.}$ corresponds to another power of the coupling constant

The functional integrals weighted by $\mathcal{L}_0$ result in products of propagators! (Wick's theorem)

Example: 4 point function in $\mathcal{L}_{\text{int.}} = -\frac{\lambda}{4!}\phi^4$



disconnected

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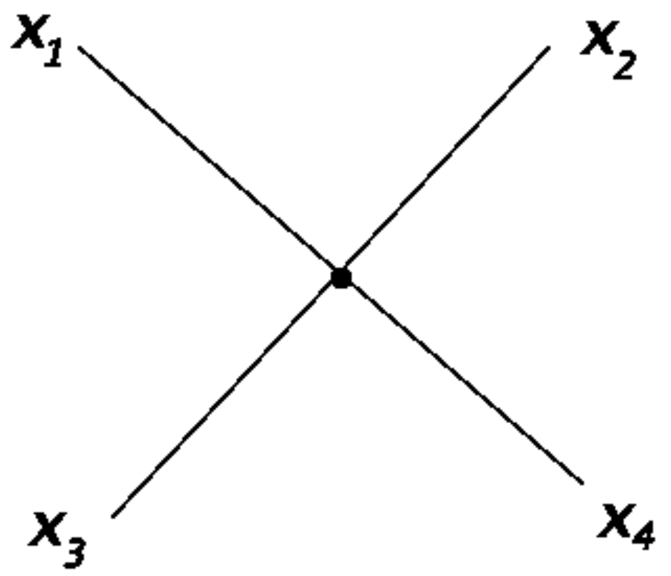
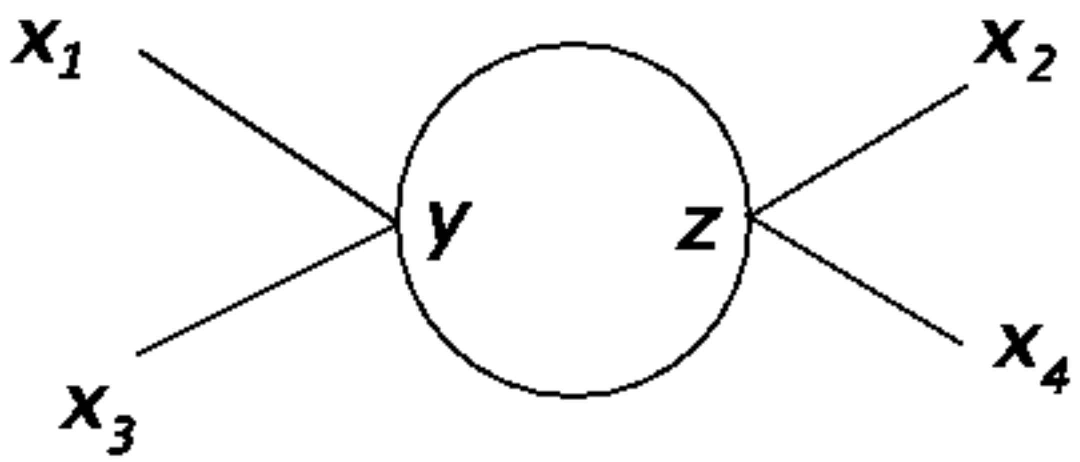


= $(-i\lambda)$

+



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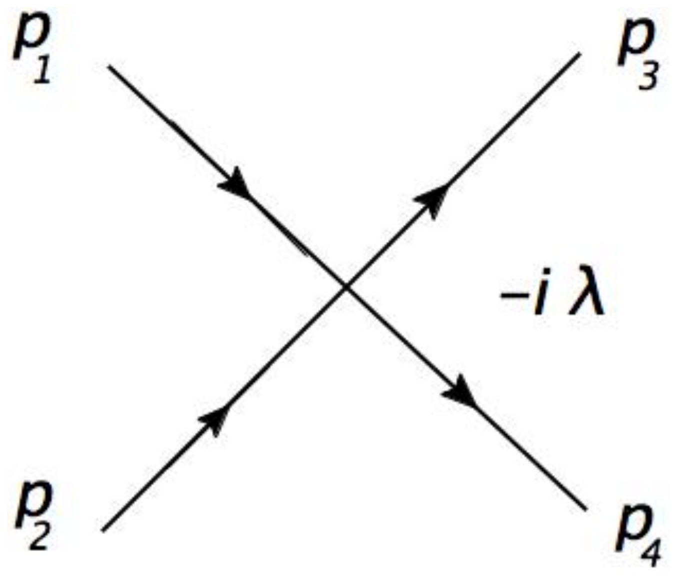
LSZ Formalism takes us to Amplitudes in Momentum Space

(See I.L12)



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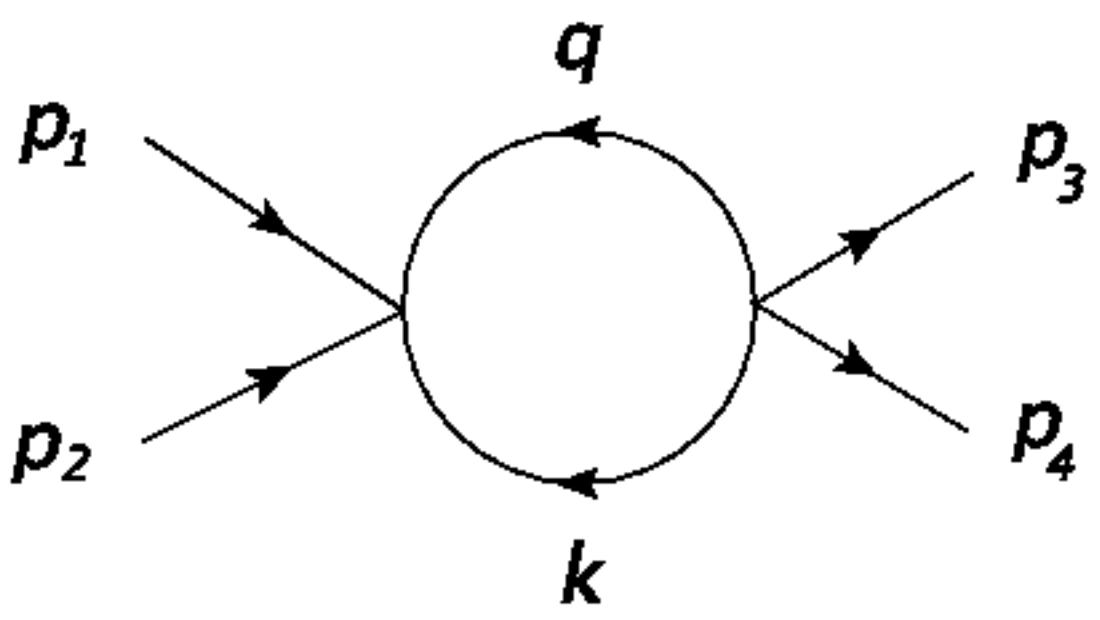

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Feynman Rules in Perturbation Theory

Contributions to amplitudes computed to a given order in perturbation theory

(Small) coupling expansion and loop $(\hbar)$ expansion

In momentum space:

- Draw all diagrams contributing to the process up to the desired order in PT
- Insert a factor of the coupling at each vertex (scalar theories). Typically the correct normalization comes from $i\mathcal{L}$ up to combinatoric factors. E.g. $i\lambda$
- Momentum conservation at each vertex results in overall factor of $(2\pi)^4\,\delta^{(4)}(p_3 \ldots p_n - p_i)$
  (Reflects the fact that interactions are local !)
- Loop integration: for each undetermined momentum p^μ add a factor of

$$\int \frac{d^4p}{(2\pi)^4}$$

(1 loop diagrams have 1 of these, 2 have 2, etc.)

- Divide by the appropriate symmetry factors

- Propagators:
  - For each internal scalar line of momentum p^μ $$\frac{i}{p^2 - m^2}$$
  - For each internal fermion line $$\frac{i}{\not{p} - m}$$
  - For each gauge boson internal line $$\frac{-ig_{\mu\nu}}{p^2} \times \text{(gauge dep. factors)}$$
- External fermion of momentum p^μ :
  - $u^s(p)$ $(\bar{u}^s(p))$ for each incoming (outgoing) fermion
  - $\bar{v}^s(p)$ $(v^s(p))$ for each incoming (outgoing) anti-fermion
- Multiply by (-1) each closed fermion loop (consequence of anti-commutation rules)

- External gauge boson of momentum p^μ :
  - Factor of the polarization $\epsilon^\mu(p)$

Feynman Rules $\longrightarrow$ Amplitude $\longrightarrow$ Cross sections, Decay rates,…

Gauge Theories (See I.L16)

Symmetries

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$ Dirac Lagrangian for a free fermion

is invariant under the continuous field transformations

$$\left.\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha}\,\psi(x) \\ \bar{\psi}(x) &\longrightarrow e^{-i\alpha}\,\bar{\psi}(x) \end{aligned}\right\}$$ With α a real constant

This is a global U(1) symmetry

Under it $\left\{ \begin{array}{l} \text{fermions have global charge +1} \\ \text{anti-fermions have global charge -1} \end{array} \right.$

The conserved current is $$j^\mu = \bar{\psi}\gamma^\mu\psi$$

But if we want the transformation to be *local* , i.e. $\alpha = \alpha(x)$

$$\psi(x) \longrightarrow e^{i\alpha(x)}\,\psi(x)$$

$$\bar{\psi}(x) \longrightarrow e^{-i\alpha(x)}\,\bar{\psi}(x)$$

$$\Longrightarrow \qquad \mathcal{L} \to \mathcal{L}' = \mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \partial_\mu\alpha(x)\,\bar{\psi}\gamma^\mu\psi \neq \mathcal{L}$$

The Lagrangian is not invariant under this local symmetry. It needs to be modified. The problem is clearly with the derivative. We define the *covariant derivative* acting on the fermion field D_μ such that now

$$\mathcal{L} = \bar{\psi}(iD\!\!\!\!/ - m)\psi$$

is invariant under the local (gauge) transformation. For this lagrangian to be invariant we need

$$D_\mu\psi(x) \longrightarrow e^{i\alpha(x)}\,D_\mu\psi(x)$$

Covariant derivative must transform just as $\psi(x)$

To cancel the term containing $\partial_\mu \alpha(x)$ we write the covariant derivative as

$$D_\mu \psi(x) = (\partial_\mu + ieA_\mu(x))\psi(x)$$

such that the field $A_\mu(x)$ transforms under the symmetry as

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu \alpha(x)$$

This guarantees $D_\mu \psi(x) \longrightarrow e^{i\alpha(x)}\, D_\mu \psi(x)$ (Exercise)

And the invariance of $\mathcal{L}$ under the transformations of $\psi(x)$ and $A_\mu(x)$

To summarize:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$$

with

$$D_\mu \psi(x) = (\partial_\mu + ieA_\mu(x))\psi(x)$$

is invariant under the transformations

$$\left.\begin{aligned} \psi(x) &\longrightarrow e^{i\alpha(x)}\,\psi(x) \\ \bar{\psi}(x) &\longrightarrow e^{-i\alpha(x)}\,\bar{\psi}(x) \\ A_\mu(x) &\longrightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu \alpha(x) \end{aligned}\right\}$$

Local (gauge) U(1) transformations

QED (I.L18)

Quantum Electrodynamics is a QFT of electromagnetism

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi = \bar{\psi}\left(i\not{\partial} - m\right)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi$$

The Dirac lagrangian with the covariant derivative is *gauge invariant.*

But we have a new field, $A_\mu(x)$, which needs a gauge invariant kinetic term

It must have:
- Two power of derivatives of the field ($\sim p^2$)
- Be invariant under $A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\,\partial_\mu\alpha(x)$

The tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is gauge invariant by itself

$\Longrightarrow F^{\mu\nu}F_{\mu\nu}$ Fits all the requirements for a gauge field kinetic term

The coefficient is fixed asking for $F^{\mu\nu}$ to match the electromagnetic field tensor of electromagnetism

Then, the full QED lagrangian is

$$\mathcal{L} = \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}$$

$$= \bar{\psi}\,(i\not{\partial} - m)\,\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}$$

↑

Interaction of fermion conserved current with the gauge boson field (photon)

Fermions that are charged under the U(1) gauge symmetry do transform under the gauge transformations

Charge in units of e: +1 (protons, anti-charged leptons), -1 (charged leptons), 2/3 (up quarks), etc.

Note: A mass term for the gauge boson is forbidden by gauge invariance:

$M_A^2 A_\mu A_\mu$ Is not gauge invariant $\Rightarrow M_A = 0$

<u>Note</u>: Other terms involving $\psi(x)$ and $A_\mu(x)$ that are both gauge and Lorentz invariant are possible

But they all are higher dimensional operators (HDO) (more about this in Lecture 3)

E.g.

$$F^{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi$$

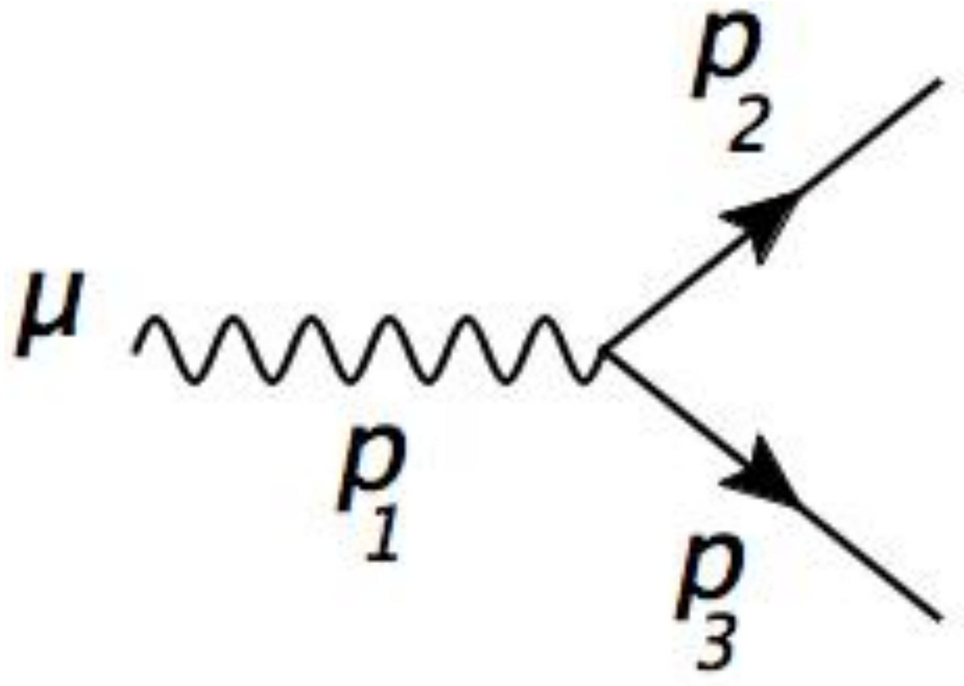
with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

Operator responsible for (g-2). If the theory is renormalizable, HDO generated by loops are finite.

QED : Feynman Rules

- Vertex

$$\mathcal{L}_{\text{int.}} = -e\, A^{\mu}\, \bar{\psi}\gamma_{\mu}\psi \qquad \Rightarrow$$



$(-ie)\gamma_{\mu}$

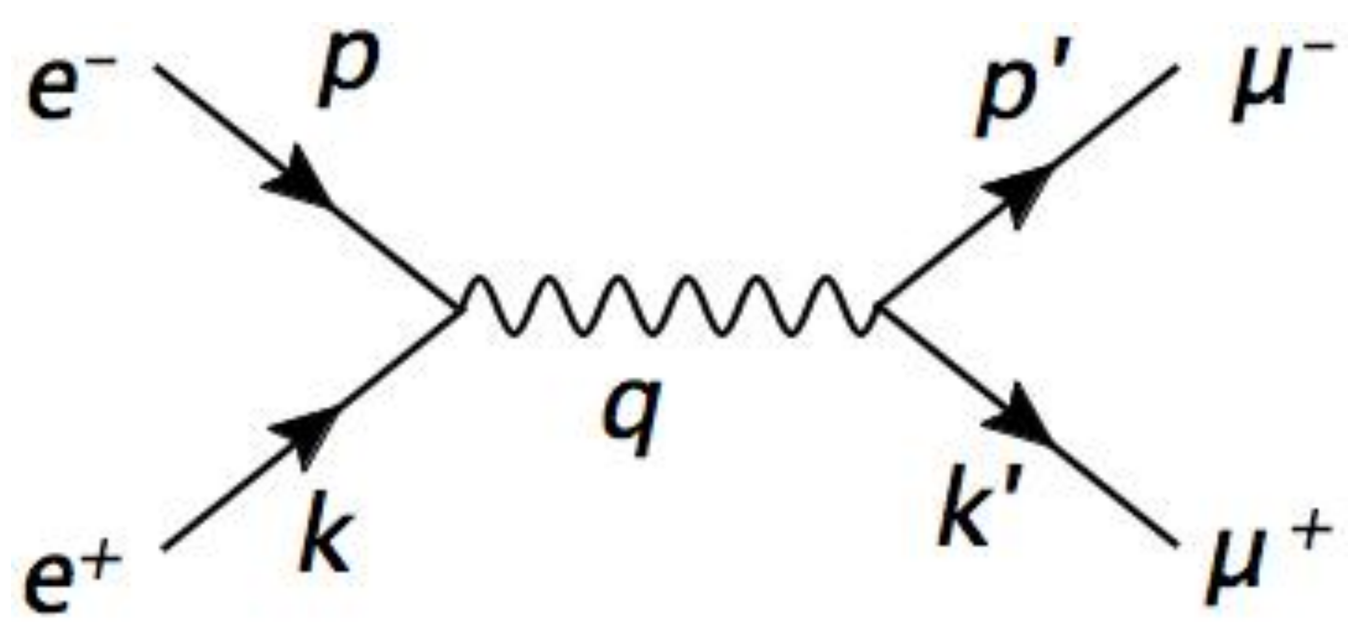
- Photon propagator (in momentum space)

$$\tilde{D}_{F\mu\nu}(k) = -\frac{i}{k^2}\left[g_{\mu\nu} - (1-\xi)\,\frac{k_{\mu}k_{\nu}}{k^2}\right]$$

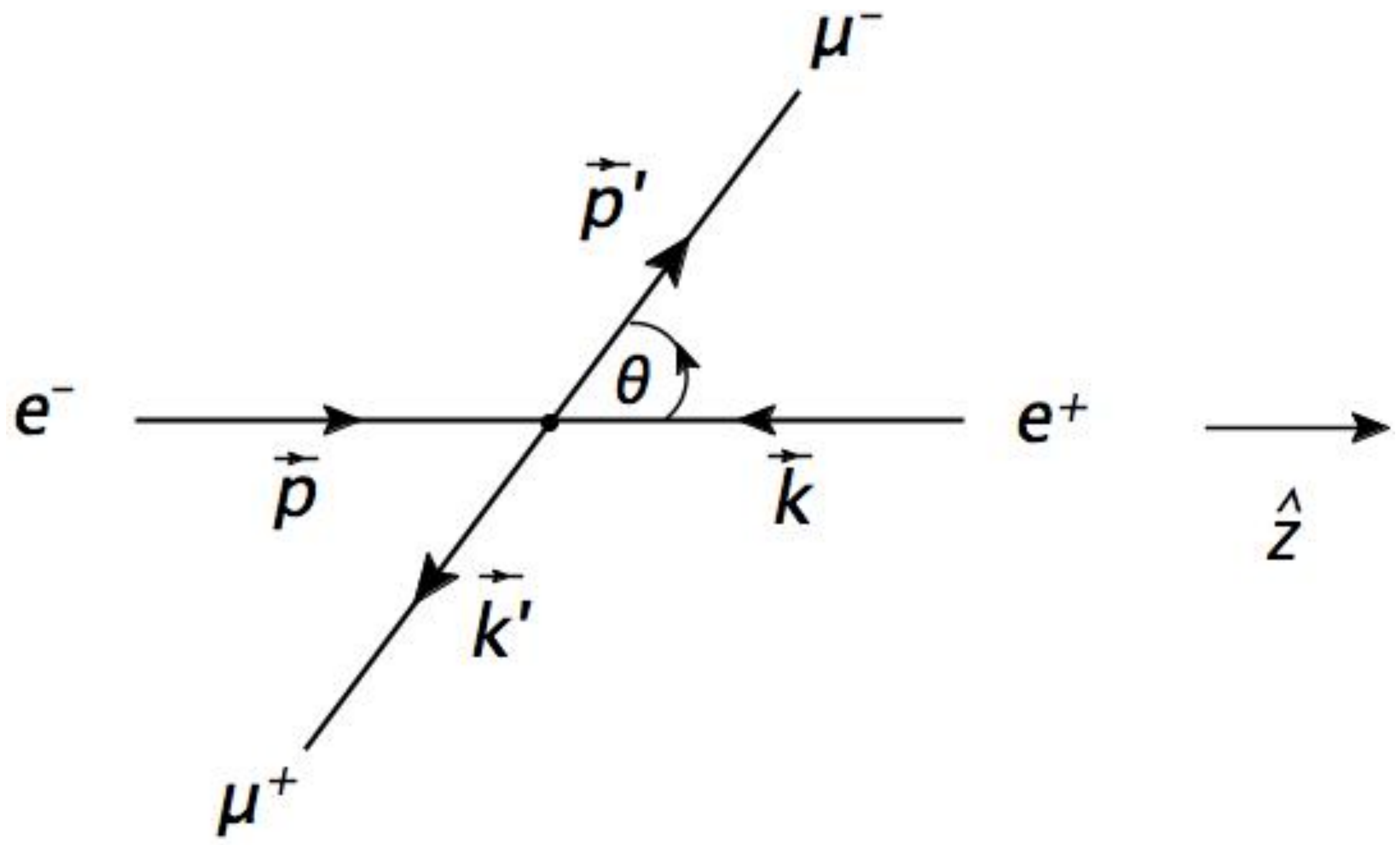
Gauge fixing parameter ξ

E.g. $\xi = 1$ Feynman gauge

Example: $e^- e^+ \to \mu^- \mu^+$

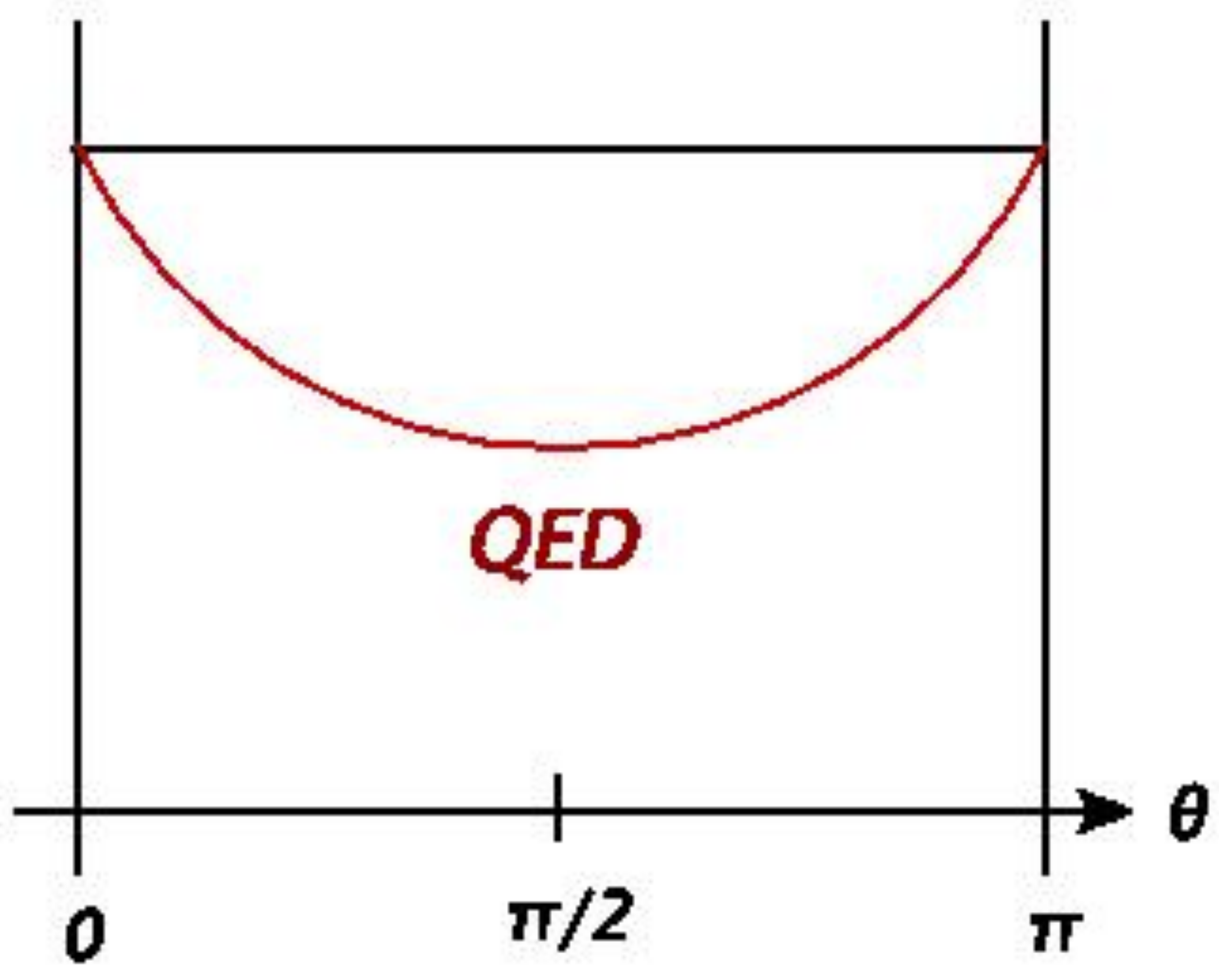


In the Center of Mass



Angular Distribution

$$\frac{d\sigma}{d\cos\theta} \sim$$



Number of events "forward" same as number of events "backwards". QED preserves parity!

$$\mathcal{L}_{\text{int.}} = -e\, A^{\mu}\, \bar{\psi}\gamma_{\mu}\psi = -e\, A^{\mu}\, (\bar{\psi}_L\gamma_{\mu}\psi_L + \bar{\psi}_R\gamma_{\mu}\psi_R)$$

Photon couples the same to left-handed and right-handed fermions ! (QED is a vector theory)

Non Abelian Gauge Theories

Continuous groups of interest are ***Lie Groups*** (See II.L14)

Some basic facts about Lie groups

$$g \in G \qquad \Rightarrow \quad g(\alpha) = 1 + i\alpha^a t^a + \mathcal{O}(\alpha^2)$$

with the α^a real infinitesimal parameters and the t^a the generators of G

Imposing basic properties on the group elements $g(0) = 1\,,\; g^{-1}\,,$ multiplication

$$\Rightarrow \quad [t^a, t^b] = i\, f^{abc}\, t^c$$ Algebra of G

With f^{abc} G dependent constants (structure constants)

In general, for non infinitesimal α^a's $\qquad g(\alpha) = e^{i\alpha^a t^a}$

Lie Groups of Interest:

- SU(N): Unitary transformations of N-dimensional vectors

If u and v are N-dim. vectors an element $g \in SU(N)$ must preserve

$$u^\dagger v$$

$$\Rightarrow \qquad u \to g\,u\ , \qquad v \to g\,v\ ,$$

Then $u^\dagger v \to u^\dagger g^\dagger\, g v = u^\dagger v$ requires

$$g^\dagger = g^{-1}$$

So we conclude that g must be a unitary matrix

But then, we can write

$$g = e^{iH}$$

With H a *hermitian* matrix

Remembering that $$g = e^{i\alpha^a t^a} = 1 + i\alpha^a t^a + \dots$$

The generators t^a must be hermitian matrices

But so far, this describes a group called U(N). This is because it includes as one group element just a phase transformation

$$u \to e^{i\theta I}\, u$$ where I is the NxN identity

This element constitutes a U(1) subgroup of U(N). If we want to separate it, we can demand that

$$det[g] = 1 \quad \Longrightarrow \quad e^{i\mathrm{Tr}[H]} = 1 \qquad \text{or} \qquad \mathrm{Tr}[H] = 0$$

This removes the identity as a generator since $\mathrm{Tr}[I] = N \neq 0$

In sum: SU(N) generators are NxN *traceless*, *hermitian* matrices

Or we can say that SU(N) is U(N) with the identity removed as generator

$$\Rightarrow \qquad U(N) = SU(N) \times U(1)$$

$\Rightarrow$ The number of generators of SU(N) is $N^2 - 1$

Other Lie groups:

- SO(N): Orthogonal transformations on N-dimensional vectors (i.e. rotations in N-dim space)

  Transformations (rotations) must preserve the scalar product $\vec{u} \cdot \vec{v}$

  $\frac{N(N-1)}{2}$ independent generators (angles!) 1 for N=2 (plane), 3 for N=3 (3D space), etc.

- Sp(N): Symplectic transformations on N-dimensional vectors. They must preserve

$$u \cdot v = u_a \, \epsilon_{ab} \, v_b$$

- Exceptional Groups: G_2, F_4, E_6, E_7

The Standard Model is a gauge theory using SU(3), SU(2) and a U(1)

- The generator of U(1) is always proportional to the identity
- SU(2) generators: 3 traceless, 2x2, hermitian matrices. They are

$$t^a = \frac{\sigma^a}{2}$$

with the Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

} Electroweak Standard Model

- SU(3) generators: 8 traceless, 3x3, hermitian matrices. They are

$$t^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0\ 0 & 0 \end{pmatrix} \quad t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \cdots \quad t^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Non Abelian Gauge Theories (II.L15)

Just as for the U(1) case, we consider the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi + \ldots$$

and demand that it be invariant under the SU(N) local transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha^a(x)t^a}\,\psi(x) = g(x)\,\psi(x)$$

This requires the covariant derivative acting on the fermion to be

$$D_\mu\psi(x) = \left(\partial_\mu - ig\,A^a_\mu(x)\,t^a\right)\psi(x)$$

as many gauge bosons A^a_μ as generators t^a

$N^2 - 1$ in SU(N)

and the gauge fields to transform like

$$A_\mu(a) \to g(x)\left(A_\mu(x) + \frac{i}{g}\,\partial_\mu\right)\,g^\dagger(x)$$

Where we defined the gauge boson matrix by $A_\mu(x) \equiv A^a_\mu(x)\,t^a$

The covariant derivative

$$D_\mu \psi(x) = \left(\partial_\mu - ig\, A^a_\mu(x)\, t^a\right) \psi(x)$$

$\Rightarrow \qquad D_\mu \psi(x) \to g(x)\, D_\mu \psi(x)$ so $\bar{\psi}\,(i\not{D} - m)\,\psi$ is gauge invariant

To complete $\mathcal{L}$ we need the kinetic terms for the gauge bosons

Need 2 derivatives of the gauge fields. We start by considering the following “differential operator”

$$[D_\mu, D_\nu]\psi(x) \equiv -ig\, F_{\mu\nu}\, \psi(x)$$

Writing it out

$$[D_\mu, D_\nu]\psi(x) = -ig\, \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \psi(x) - g^2\, [A_\mu, A_\nu]\, \psi(x)$$

Then $[D_\mu, D_\nu]$ not really a differential operator!

From the definition we see that the tensor matrix is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$$

Or, in components

$$F_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\, t^a - ig\, A^a_\mu A^b_\nu\, [t^a, t^b]$$

It is useful to define

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\, t^a$$

So the non Abelian field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Gauge Boson Kinetic Term

Since under a gauge transformation $\psi(x) \to g(x)\,\psi(x)$ and also $D_\mu\psi(x) \to g(x)\,D_\mu\psi(x)$

then

$$[D_\mu, D_\nu]\psi(x) \to g(x)\,[D_\mu, D_\nu]\psi(x)$$

which means that

$$[D_\mu, D_\nu] \to g(x)\,[D_\mu, D_\nu]g^\dagger(x)$$

or equivalently

$$F_{\mu\nu} \to g(x)\,F_{\mu\nu}\,g^\dagger(x)$$

$$\Rightarrow \qquad \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \to \mathrm{Tr}[g(x)F_{\mu\nu}g^\dagger(x)g(x)F^{\mu\nu}g^\dagger(x) = \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$$ is gauge invariant

Choosing the standard normalization (which reduces to the QED one for the U(1) case) and using that

$$\mathrm{Tr}[t^a t^b] = \frac{\delta^{ab}}{2}$$

We arrive at the Gauge invariant Lagrangian of a non Abelian gauge theory

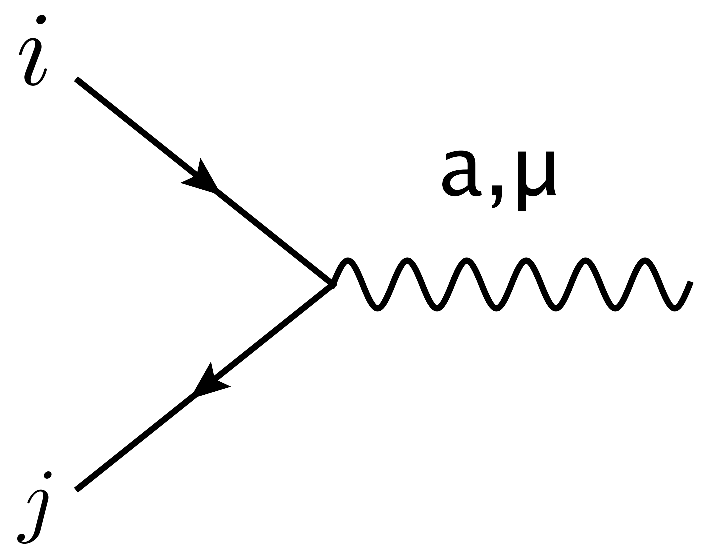
$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}\mathrm{Tr}[F_{\mu\nu}\, F^{\mu\nu}] \\ &= \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}\, F^{a\mu\nu}\end{aligned}$$

Careful: form of gauge boson kinetic term is deceivingly simple! It's not just the sum of $N^2 - 1$ photons

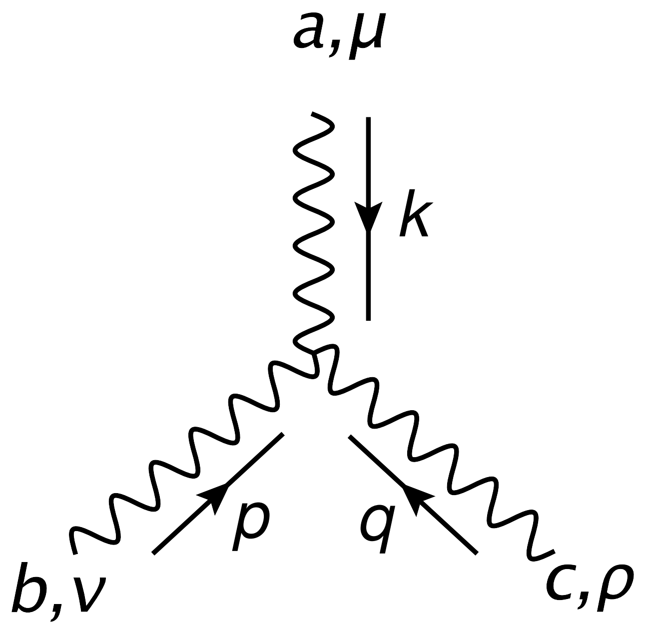
$$\left(F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\right)^2$$

Contains triple and quartic gauge bosons interactions in addition to the squared derivatives

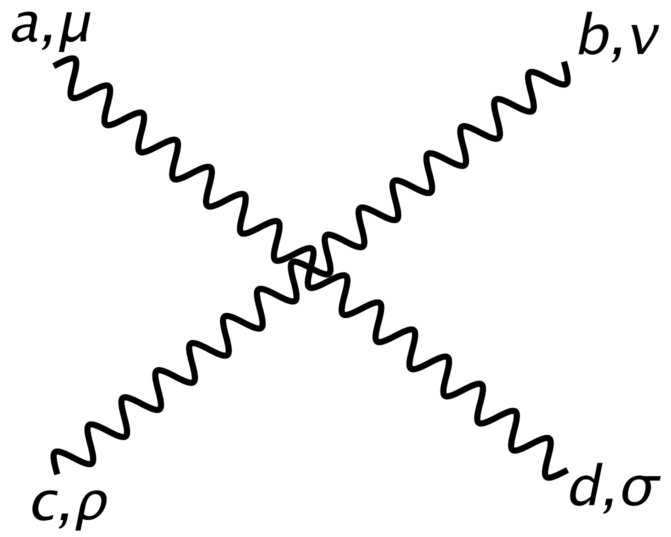
Feynman Rules in Non Abelian Gauge Theories (II.L16)



$$i\,g\gamma^\mu\,t^a_{ij} \qquad (i,j=1,\ldots,N) \qquad (a=1,\ldots,N^2-1)$$



$$g\,f^{abc}\,[g^{\mu\nu}(k-p)^\rho+g^{\nu\rho}(p-q)^\mu+g^{\rho\mu}(q-k)^\nu]$$



$$\begin{aligned}-ig^2\;\Big[&f^{abe}f^{cde}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\&+f^{ace}f^{bde}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})\\&+f^{ade}f^{bce}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big]\end{aligned}$$