The Double Cover of the Icosahedral Symmetry Group and Quark Mass Textures

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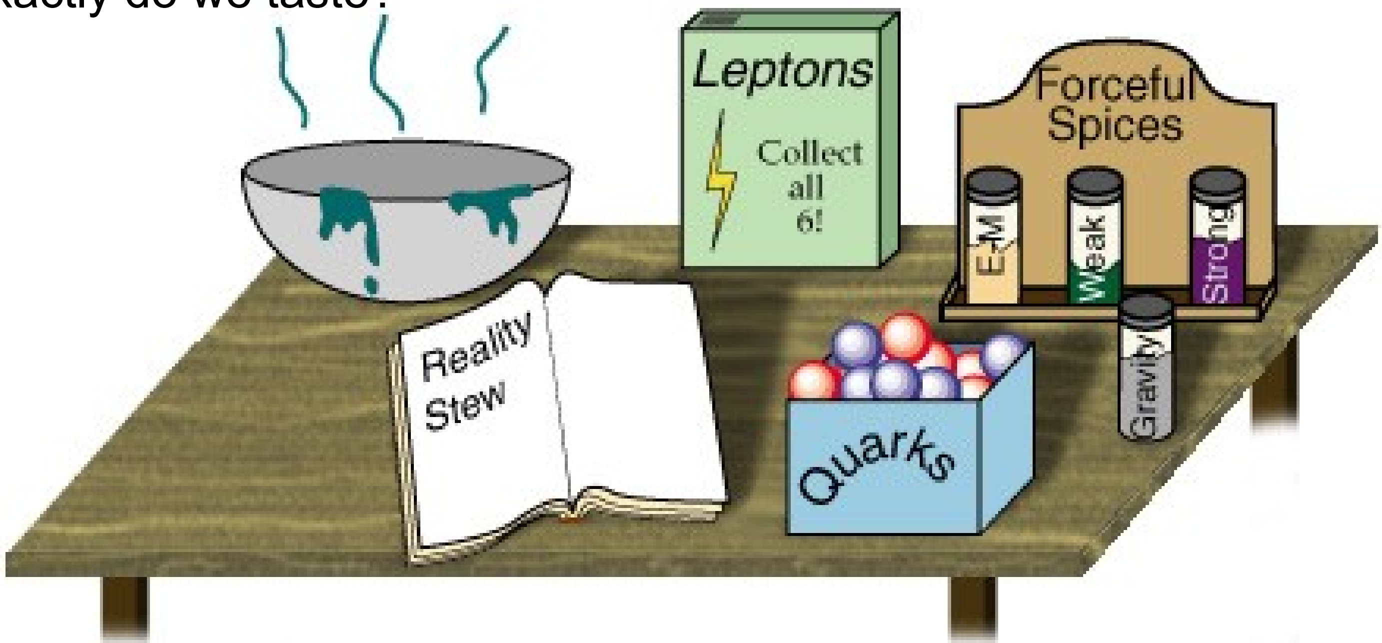
Based on: L. Everett and A. Stuart, Phys.Lett.B698:131-139,2011.
arXiv:1011.4928 [hep-ph]

The Standard Model

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

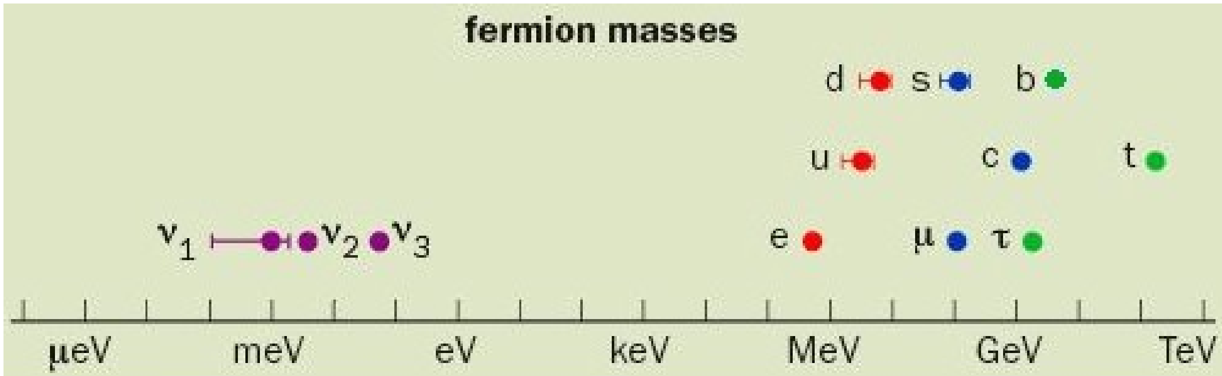
•Marvel of modern science, but incomplete. Fails to predict measured masses and mixings of fermions.

•What exactly do we taste?



http://www.particleadventure.org/frameless/standard_model.html

What We Taste



Quark Mixing Angles

$$\mathcal{U}_{\rm CKM} = \mathcal{R}_1(\theta_{23}^{\rm CKM})\mathcal{R}_2(\theta_{13}^{\rm CKM}, \delta_{\rm CKM})\mathcal{R}_3(\theta_{12}^{\rm CKM})$$

$$\theta_{12}^{\rm CKM} = 13.0^\circ \pm 0.1^\circ$$

$$\theta_{23}^{\rm CKM} = 2.4^\circ \pm 0.1^\circ$$

$$\theta_{13}^{\rm CKM} = 0.2^\circ \pm 0.1^\circ$$

$$\delta_{\rm CKM} = 60^\circ \pm 14^\circ$$

Lepton Mixing Angles

$$\mathcal{U}_{\rm MNSP} = \mathcal{R}_1(\theta_\oplus)\mathcal{R}_2(\theta_{13}, \delta_{\rm MNSP})\mathcal{R}_3(\theta_\odot)\mathcal{P}$$

$$\theta_\odot = 33.9^\circ\,^{+2.4^\circ}_{-2.2^\circ}$$

$$\theta_\oplus = 45^\circ\,^{+10^\circ}_{-10^\circ}$$

$$\theta_{13} < 13^\circ$$

(presently no constraints on CP violation)

Adding Some Spice

(i.e. a discrete flavor symmetry that is spontaneously broken by flavon vevs to generate observed masses and mixings)

Icosahedral Symmetry

$$I \cong A_5$$

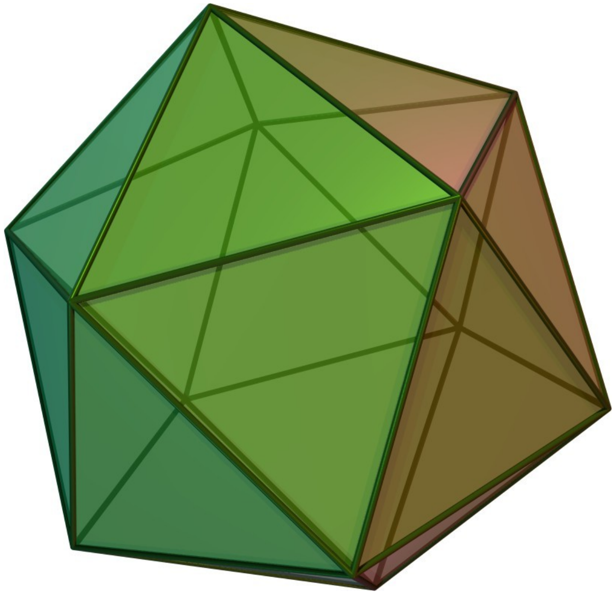
- All over nature!
- Provides "natural" setting to look at $\theta_{sol} = \arctan(\frac{1}{\phi}) = 31.7175^{\circ}$
- Has been applied in this context in arXiv:0812.1057[hep-ph] (L. Everett and A.S.) and arXiv:1101.0393 [hep-ph](F. Feruglio et al.)
- Now let's apply it to the quarks....

What exactly is Icosahedral Symmetry?

The Icosahedral Group, I

- An icosahedron is the Platonic solid that consists of 20 equilateral triangles. → f=20
- 20 triangles each have 3 sides → 60 edges but 2 triangles/edge → 30 edges → e=30
- 20 triangles each have 3 vertices → 60 vertices but 5 vertices/edge → v=12
- Are we right? $\chi(g) = 2 - 2g = v - e + f$
- I consists of all rotations that take vertices to vertices i.e. $0, \pi, \frac{2\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}$

$$A_5 \simeq I \subseteq SO(3)$$



http://upload.wikimedia.org/wikipedia/commons/e/eb/Icosahedron.jpg

Conjugacy Classes of I

(A way to partition a group into disjoint pieces.)

•Rotation by each angle forms its own conjugacy class.

Schoenflies Notation: C_n^k is a rotation by $\frac{2\pi k}{n}$

in front = # of elements in class

So for the icosahedral group we have:

$$I, 12C_5, 12C_5^2, 20C_3, 15C_2$$

Note: $1+12+12+15+20=60=1^2+3^2+3^2+4^2+5^2$ **Two triplets.**

Now we know a little about the icosahedral symmetry group. How can we apply this to the quarks?

Inspired by U(2)

$$Y_u = \begin{pmatrix} 0 & \lambda^6 & 0 \\ -\lambda^6 & \lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \qquad Y_d = \begin{pmatrix} 0 & \lambda^6 & 0 \\ -\lambda^6 & \lambda^5 & \lambda^5 \\ 0 & \lambda^5 & \lambda^3 \end{pmatrix}$$

$$\lambda = \sin\left(\theta_{12}^{CKM}\right) \approx .22$$

The above textures are known to result in successful predictions for the quark masses and mixing angles after electroweak symmetry breaking.

Based on: $2 \otimes 2 = 1 \oplus 3$

But the Icosahedral Group does not have a 2 dimensional irreducible representation.....

The Double Cover of I, I'

- To each element $g \in I$, associate another element $gR \ni R^2 = e$
- Therefore, $|I'| = 2|I| = 120$
- The characters(traces) of the new elements are related to the old by: $\chi(gR) = \pm\chi(g)$
- Furthermore, each conjugacy class gets a partner: $C_n^k R$
- One notable exception: $15C_2$
- We get 4 more irreps: $\underline{2^2 + 2^2} + 4^2 + 6^2 = 60$

$$I' \subseteq SU(2)$$

Two dimensional irreps!

Summarize all of this information with a character table.

I' Character Table

The traces (characters) of elements of a certain dimensional irreducible representation in a particular conjugacy class share the same trace.

$\mathcal{I}'$	**1**	**3**	**3**$'$	**4**	**5**	**2**	$2'$	$4'$	**6**
e	1	3	3	4	5	2	2	4	6
$12C_5$	1	ϕ	$1-\phi$	-1	0	ϕ	$1-\phi$	1	-1
$12C_5^2$	1	$1-\phi$	ϕ	-1	0	$\phi-1$	$-\phi$	-1	1
$20C_3$	1	0	0	1	-1	1	1	-1	0
$30C_2$	1	-1	-1	0	1	0	0	0	0
R	1	3	3	4	5	-2	-2	-4	-6
$12C_5R$	1	ϕ	$1-\phi$	-1	0	$-\phi$	$\phi-1$	-1	1
$12C_5^2R$	1	$1-\phi$	ϕ	-1	0	$1-\phi$	ϕ	1	-1
$20C_3R$	1	0	0	1	-1	-1	-1	1	0

Notable Kronecker Products of I'.

Use Character Table to easily obtain Kronecker Products (known)

$$3'\otimes 3' = 1 \oplus 3' \oplus 5 \quad 3 \otimes 3 = 1 \oplus 3 \oplus 5 \quad 3 \otimes 3' = 4 \oplus 5$$

$$2 \otimes 2 = 1 \oplus 3 \quad 2 \otimes 2' = 4 \quad 2' \otimes 2' = 1 \oplus 3'$$

$$4 \otimes 4 = 1 \oplus 3 \oplus 3' \oplus 4 \oplus 5$$

$$5 \otimes 5 = 1 \oplus 3 \oplus 3' \oplus 4 \oplus 4 \oplus 5 \oplus 5$$

These Kronecker Products will allow us to construct a simple Lagrangian (superpotential) that is invariant under the discrete symmetry that generates our observed quark masses and mixings.

All of this is abstract. We need actual representations.

Tensor Product Decomposition

$$2 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$2 \otimes 2 = 1 \oplus 3$$

$$1 = a_2 b_1 - a_1 b_2$$

$$3 = \begin{pmatrix} -ia_1 b_1 + ia_2 b_2 \\ a_1 b_1 + a_2 b_2 \\ ia_2 b_1 + ia_1 b_2 \end{pmatrix}$$

$$2' \otimes 2' = 1 \oplus 3'$$

$$1 = a_2 b_1 - a_1 b_2$$

$$3' = \begin{pmatrix} -ia_1 b_1 - ia_2 b_2 \\ -a_1 b_1 + a_2 b_2 \\ a_2 b_1 + a_1 b_2 \end{pmatrix}$$

$$2 \otimes 2' = 4$$

$$4 = \begin{pmatrix} -a_2 b_1 - a_1 b_2 \\ ia_2 b_1 - ia_1 b_2 \\ -ia_1 b_1 - ia_2 b_2 \\ -a_1 b_1 + a_2 b_2 \end{pmatrix}$$

Now we have the explicit representations… What do we do next?

How to build an I' Flavor Model

Assume 3rd generation quarks transform as singlets, and 1st and 2nd generation quarks as doublets.

Embed left- and right-handed quarks in **only** 2 dimensional irreps not 2' so as to not interfere with existing lepton sector directly:

$$3 \otimes 3' = 4 \oplus 5$$

$$2 \otimes 2 = 1 \oplus 3 \qquad 2 \otimes 2' = 4 \qquad 2' \otimes 2' = 1 \oplus 3'$$

Thus, we will have flavon fields that transform as 1's, 2's or 3's..

I' Quark Model

$\alpha = e^{2\pi i/9}$

Field	Q	Q_3	u^c	d^c	t^c	b^c	$H_{u,d}$	ρ	σ	χ	η	η_2	ψ	σ^0	χ^0	η^0	ψ^0
$\mathcal{I}'$	2	1	2	2	1	1	1	1	1	1	2	2	3	1	1	2	3
$\mathcal{Z}_9$	1	α	α^8	α^5	α^8	α^6	1	α^8	α^5	α^2	1	α	α^8	α^5	α^2	1	α^8

Include additional $\mathcal{Z}_9$ to distinguish similar irreps $\mathcal{I}'$ as well as to prevent proliferation of possible terms in flavon potential. Then for the up-type quark sector we'll have:

$$
\begin{aligned}
W_u &= y_{u1} Q_3 t^c H_u + \frac{y_{u2}}{M} Q_3 u^c \eta_1 H_u + \frac{y'_{u2}}{M^2} Q_3 u^c \eta_2 \rho H_u + \frac{y''_{u2}}{M^2} Q_3 u^c \eta_2 \psi H_u + \frac{y_{u3}}{M} Q t^c \eta_2 H_u \\
&+ \frac{y_{u4}}{M^2} Q u^c \sigma\sigma H_u + \frac{y'_{u4}}{M^2} Q u^c \rho\chi H_u + \frac{y''_{u4}}{M^2} Q u^c \eta\eta_2 H_u + \frac{y'''_{u4}}{M^2} Q u^c \chi\psi H_u,
\end{aligned}
$$

For the down-type sector we'll have:

$$
W_d = \frac{y_{d1}}{M} Q_3 b^c \chi H_d + \frac{y_{d2}}{M^2} Q_3 d^c \eta_2 \chi H_d + \frac{y_{d3}}{M^2} Q b^c \eta_2 \chi H_d + \frac{y_{d4}}{M^2} Q d^c \rho\sigma H_d + \frac{y'_{d4}}{M^2} Q d^c \chi\chi H_d + \frac{y''_{d4}}{M^2} Q d^c \sigma\psi H_d
$$

Breaking I'

Assume the following patterns for the vacuum expectation values of the flavon fields:

$$\langle\psi\rangle = \langle(\psi^1,\psi^2,\psi^3)^T\rangle = \frac{v_3}{2}(-i,1,0)^T, \quad \langle\eta_1\rangle = \langle(\eta_1^1,\eta_1^2)^T\rangle = (v_{21},0)^T,$$
$$\langle\eta_2\rangle = \langle(\eta_2^1,\eta_2^2)^T\rangle = (v_{22},0)^T, \quad \langle\rho\rangle = v_\rho, \quad \langle\chi\rangle = v_\chi, \quad \langle\sigma\rangle = v_\sigma,$$

With the above vevs we assume the following orders:

$$\lambda = \sin(\theta_{12}^{CKM}) \approx .22$$

$$\left|\frac{v_3}{M}\right| \sim \left|\frac{v_{21}}{M}\right| \sim \left|\frac{v_{22}}{M}\right| \sim \lambda^2, \qquad \left|\frac{v_\rho}{M}\right| \sim \left|\frac{v_\chi}{M}\right| \sim \left|\frac{v_\sigma}{M}\right| \sim \lambda^3,$$

Now we are ready to write down our mass matrices...

Quark Mass Matrices

$$\mathcal{M}_u = \begin{pmatrix} 0 & -y_{u4}\frac{v_\sigma^2}{M^2} - y'_{u4}\frac{v_\rho v_\chi}{M^2} & 0 \\ y_{u4}\frac{v_\sigma^2}{M^2} + y'_{u4}\frac{v_\rho v_\chi}{M^2} & y''_{u4}\frac{v_{21}v_{22}}{M^2} + y'''_{u4}\frac{v_3 v_\chi}{M^2} & y_{u3}\frac{v_{22}}{M} \\ 0 & y_{u2}\frac{v_{21}}{M} + y'_{u2}\frac{v_{22}v_\rho}{M^2} & y_{u1} \end{pmatrix} v_u \equiv \begin{pmatrix} 0 & -\tilde{y}_{u4}\lambda^6 & 0 \\ \tilde{y}_{u4}\lambda^6 & \tilde{y}''_{u4}\lambda^4 + \tilde{y}'''_{u4}\lambda^5 & \tilde{y}_{u3}\lambda^2 \\ 0 & \tilde{y}_{u2}\lambda^2 & \tilde{y}_{u1} \end{pmatrix} v_u,$$

$$\mathcal{M}_d = \begin{pmatrix} 0 & -y_{d4}\frac{v_\rho v_\sigma}{M^2} - y'_{d4}\frac{v_\chi^2}{M^2} & 0 \\ y_{d4}\frac{v_\rho v_\sigma}{M^2} + y'_{d4}\frac{v_\chi^2}{M^2} & y''_{d4}\frac{v_3 v_\sigma}{M^2} & y_{d3}\frac{v_{22}v_\chi}{M^2} \\ 0 & y_{d2}\frac{v_{22}v_\chi}{M^2} & y_{d1}\frac{v_\chi}{M} \end{pmatrix} v_d \equiv \begin{pmatrix} 0 & -\tilde{y}_{d4}\lambda^3 & 0 \\ \tilde{y}_{d4}\lambda^3 & \tilde{y}''_{d4}\lambda^2 & \tilde{y}_{d3}\lambda^2 \\ 0 & \tilde{y}_{d2}\lambda^2 & \tilde{y}_{d1} \end{pmatrix} \lambda^3 v_d.$$

These can be diagonalized to yield the quark masses and mixings....

Results (at leading order)

Masses:

$$m_u \simeq \frac{|\tilde{y}_{u1}||\tilde{y}_{u4}|^2}{|\tilde{y}_{u2}\tilde{y}_{u3} - \tilde{y}_{u1}\tilde{y}_{u4}|}\lambda^8 v_u, \qquad m_c \simeq \left|\frac{\tilde{y}_{u2}\tilde{y}_{u3}}{\tilde{y}_{u1}} - \tilde{y}''_{u4}\right|\lambda^4 v_u, \qquad m_t \simeq \left(|\tilde{y}_{u1}| + \frac{|\tilde{y}_{u2}|^2 + |\tilde{y}_{u3}|^2}{2|\tilde{y}_{u1}|}\lambda^4\right) v_u,$$

$$m_d \simeq \frac{|\tilde{y}_{d4}|^2}{|\tilde{y}''_{d4}|}\lambda^7 v_d, \qquad m_s \simeq |\tilde{y}''_{d4}|\lambda^5 v_d, \qquad m_b \simeq \left(|\tilde{y}_{d1}| + \frac{|\tilde{y}_{d2}|^2 + |\tilde{y}_{d3}|^2}{2|\tilde{y}_{d1}|}\lambda^4\right)\lambda^3 v_d.$$

Mixing Angles:

$$V_{ud} \sim V_{cs} \sim V_{tb} \sim 1, \quad V_{us} \sim -\frac{\tilde{y}_{d4}}{\tilde{y}''_{d4}}\lambda - \frac{\tilde{y}_{u1}\tilde{y}_{u4}}{\tilde{y}_{u2}\tilde{y}_{u3} - \tilde{y}_{u1}\tilde{y}''_{u4}}\lambda^2 \sim -V^*_{cd}, \quad V_{cb} \sim \left(\frac{\tilde{y}_{d3}}{\tilde{y}_{d1}} - \frac{\tilde{y}_{u3}}{\tilde{y}_{u1}}\right)\lambda^2 \sim -V^*_{ts},$$

$$V_{ub} \sim \frac{\tilde{y}_{u4}\tilde{y}_{u1}}{\tilde{y}_{u2}\tilde{y}_{u3} - \tilde{y}_{u1}\tilde{y}''_{u4}}\left(\frac{\tilde{y}_{u3}}{\tilde{y}_{u1}} - \frac{\tilde{y}_{d3}}{\tilde{y}_{d1}}\right)\lambda^4, \qquad V_{td} \sim \frac{\tilde{y}^*_{d4}}{\tilde{y}''^*_{d4}}\left(-\frac{\tilde{y}^*_{d3}}{\tilde{y}^*_{d1}} + \frac{\tilde{y}^*_{u3}}{\tilde{y}^*_{u1}}\right)\lambda^3.$$

These match! (provided couplings are O(1))

The Flavon Potential

Recall the earlier charge assignments for the flavon fields and these 'driving' fields.

Field	ρ	σ	χ	η	η_2	ψ	σ^0	χ^0	η^0	ψ^0
$\mathcal{I}'$	1	1	1	2	2	3	1	1	2	3
$\mathcal{Z}_9$	α^8	α^5	α^2	1	α	α^8	α^5	α^2	1	α^8

Recall that a global $U(1)_R$ symmetry is present in N=1 SUSY (before supersymmetric breaking terms are added) such that the total R-charge of any term in the superpotential is +2. Flavon fields necessarily have R-charge 0. Introduce 'driving fields' of R-charge 2, which couple linearly to flavons.

$$\begin{aligned} W_{\text{fl}} \;&=\; M_\eta \eta_1 \eta^0 + g_1 \eta^0 \eta_2 \rho + g_2 \sigma^0 \sigma \rho + g_3 \sigma^0 \chi\chi + g_4 \chi^0 \rho\rho + g_5 \chi^0 \sigma\chi + g_6 \chi^0 \psi\psi + g_7 \eta^0 \eta_2 \psi + \\ &+\; g_8 \eta_1 \eta_2 \psi^0 + g_9 \chi \psi \psi^0 \end{aligned}$$

Flavon Potential(II)

Assume that the driving fields develop positive supersymmetric breaking mass-square terms so that they have a zero vev. Thus, we need only to calculate when the following F-terms vanish:

$$\frac{\partial W_{\text{fl}}}{\partial \sigma^0} = g_2 \sigma \rho + g_3 \chi \chi, \qquad \frac{\partial W_{\text{fl}}}{\partial \chi^0} = g_4 \rho \rho + g_5 \sigma \chi + g_6 (\psi^1 \psi^1 + \psi^2 \psi^2 + \psi^3 \psi^3),$$

$$\begin{aligned}
\frac{\partial W_{\text{fl}}}{\partial \eta^{0\,1}} &= M_\eta \eta_1^2 + g_7 \eta_2^1 (-i\psi^1 + \psi^2) + g_7 \eta_2^2 i \psi^3 + g_1 \rho \eta_2^2, \\
\frac{\partial W_{\text{fl}}}{\partial \eta^{0\,2}} &= -M_\eta \eta_1^1 + g_7 \eta_2^2 (i\psi^1 + \psi^2) + g_7 \eta_2^1 i \psi^3 - g_1 \rho \eta_2^1, \\
\frac{\partial W_{\text{fl}}}{\partial \psi^{0\,1}} &= g_8 (-i \eta_1^1 \eta_2^1 + i \eta_1^2 \eta_2^2) + g_9 \chi \psi^1, \quad \frac{\partial W_{\text{fl}}}{\partial \psi^{0\,2}} = g_8 (\eta_1^1 \eta_2^1 + \eta_1^2 \eta_2^2) + g_9 \chi \psi^2, \\
\frac{\partial W_{\text{fl}}}{\partial \psi^{0\,3}} &= = g_8 (i \eta_1^2 \eta_2^1 + i \eta_1^1 \eta_2^2) + g_9 \chi \psi^3.
\end{aligned}$$

Flavon Potential (III)

The preceding equations have a solution when:

$$v_\sigma = -\frac{g_4}{g_5}\left(\frac{g_3 g_5}{g_4 g_2}\right)^{1/3} v_\rho, \quad v_\chi = v_\rho \left(\frac{g_2 g_4}{g_3 g_5}\right)^{1/3}, \quad v_{21} = -\frac{g_1 v_{22} v_\rho}{M_\eta},$$

$$v_3 = \frac{2 g_1 v_{22}^2}{M_\eta}\frac{g_8}{g_9}\left(\frac{g_3 g_5}{g_4 g_2}\right)^{1/3},$$

Presumably, the two flat directions will be lifted by SUSY breaking terms. We have shown there exists a region of parameter space in which the flavon vevs do not vanish.

Conclusion

- The absence of explanation in the Standard Model for the observed fermionic masses and mixings leads us to look beyond the Standard Model for an answer. Perhaps this problem will be solved with discrete symmetries.
- As our work has shown, icosahedral symmetry is a viable symmetry to use in exploring solutions to this problem: arXiv:1011.4928 [hep-ph] (quarks), 0812.1057[hep-ph] (leptons).
- Still a lot of work to be done with icosahedral symmetry (e.g. lepton sector flavon dynamics, alternative models with fields embedded in different representations, GUT embeddings, etc..)

Stay Tuned!