Gravitational Waves and Cosmological Phenomenology

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Conventions: $\hbar = c = 1$ $\quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$

LECTURE 1

Gravitational waves for particle physicists

Consider a massless spin-2 field described by a symmetric rank 2 tensor[1], $h_{\mu\nu}$, corresponding to a (for now) linearized graviton. To propagate the correct number of degrees of freedom, just as for a photon, we need to impose additional requirements. We assume that the Lagrangian describing this field is invariant under gauge transformations (linearized diffeomorphisms) of $h_{\mu\nu}$:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu , \qquad (1)$$

and find the unique answer at quadratic order

$$\mathcal{L}^{F.P.} = -\frac{1}{2\kappa}\left(\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho{}_\nu\right) \qquad (2)$$

with $|h_{\mu\nu}| \ll 1$ and $h = \eta^{\mu\nu}h_{\mu\nu}$. This is the Fierz-Pauli

1) Particles are described by irreducible representations of the Lorentz group. The simplest tensor which includes a massless spin 2 irrep is a symmetric rank 2 tensor.

Lagrangian. The overall multiplicative constant, K, can be fixed to recover Newtonian mechanics as described below.

The corresponding equations of motion are given by

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = 0 \qquad (3)$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$. We can now use the freedom from the gauge invariance to simplify this equation. Working in Lorentz (de Donder) gauge

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0,} \qquad (4)$$

the e.o.m is now

$$\boxed{\Box \bar{h}_{\mu\nu} = 0.} \qquad (5)$$

We still have residual gauge freedom that allows us to shift $h_{\mu\nu}$ as in Eq. (1) with $\Box \xi_\mu = 0$.

Let's count the degrees of freedom

	$h_{\mu\nu}$	10
$+$	$\partial^\nu \bar{h}_{\mu\nu} = 0$	-4
$+$	Eq. (1) w/ $\Box \xi_\mu = 0$	-4
	$h^{TT}_{\mu\nu}$	2 $\leftarrow$ physical d.o.f. of massless spin 2

Now, we can choose ξ_μ s.t. $\bar{h}=0$ and $h_{0i}=0$.

With this, the Lorentz condition implies

$$\partial h_{00}=0,$$

so that h_{00} describes the static contribution to the gravitational field, i.e., the Newtonian potential. More precisely $h_{00}=-2\phi$ which can be inferred from the geodesic equation in the weak field limit of General Relativity,

$$\frac{d^2x^\mu}{dt^2}=-2\frac{\partial h_{00}}{\partial x_\mu}=\frac{\partial\phi}{\partial x_\mu} \quad (6)$$

Let's pause and use this identification to fix κ above by recovering Newtonian mechanics. To do so, lets add a coupling of our graviton to an external source $T^{\mu\nu}$

$$\mathcal{L}^{ext}=\frac{1}{2}h_{\mu\nu}T^{\mu\nu} \quad (7)$$

The e.o.m in de Donder gauge is

$$\Box\bar{h}_{\mu\nu}=\Box h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\Box h=-\frac{\kappa}{2}T_{\mu\nu} \quad (8)$$

Note that for consistency, Eq. (4) implies $\partial^\mu T_{\mu\nu}=0$. So $T^{\mu\nu}$ is identified with a conserved stress-energy tensor.

It is useful to take the trace of Eq. (8)

$$\Box h - \frac{1}{2}\cdot 4 h = -\Box h = -\frac{\kappa}{2} T$$

which allows us to write the trace reversed e.o.m.

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T\right). \quad (9)$$

Looking at the 00 component and taking the Newtonian limit, where the source and field are slowly varying, we find

$$\Box h_{00} = -\frac{\kappa}{2}\left(T_{00} - \frac{1}{2} T_{00}\right) \sim -\frac{1}{4}\kappa\rho$$

v.s.

$$\nabla^2 \phi = 4\pi G\rho$$ Newtonian mechanics

Thus, taking $h_{00} = -2\phi$ we find

$$\kappa = 32\pi G. \quad (10)$$

With this, Eq. (2) is the linearized limit of GR. In GR a metric $g_{\mu\nu}$ describes the spacetime in which we live in. In the linearized limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, so $h_{\mu\nu}$ describes the deviations from flat space.

Propagation of gravitational waves

In vacuum, in the TT gauge[2] we can choose $h_{00}=0$ such that

$$h_0^{\mu}=0, \quad h_i^{\ i}=0 \quad \partial^i h_{ij}=0 \qquad (11)$$

this allows us to analyze the propagation of gravitational waves in vacuum.
The eom

$$\boxed{\Box h_{ij}^{TT}=0} \qquad (12)$$

has plane wave solutions of the form

$$h_{ij}=e_{ij}\, e^{ik\cdot x}$$

where the polarization vectors e_{ij} are transverse to the direction of propagation, $e_{ij}k^i=0$.
For $\bar{k}=k\hat{e}_z$

h_+ and h_- are the physical modes

$$e_{ij}=\begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (13)$$

[2] Sometimes Eq. 11 is used to define the TT gauge. Then this gauge is not valid in the prescence of a source ($T^{\mu\nu}$) since we are not allowed to set $h_{00}=0$.

In the presence of sources, $h_{\mu\nu}$ has to be decomposed in scalar, vector, and tensor components, and one has to construct gauge invariant objects. This case won't be treated here.

A general solution of Eq. (7) can be written as

$$h_{ij}^{TT} = \int \frac{d^3\bar{k}}{(2\pi)^3} \left(A_{ij}(\bar{k})\, e^{ik\cdot x} + A_{ij}^{*}(\bar{k})\, e^{-ik\cdot x} \right) \quad (14)$$

In frequency domain this reads

$$h_{ij}^{TT} = \int df\, f^2 \int d^2\hat{n} \left(A_{ij}(\hat{n}, f)\, e^{-2\pi i f (t - \hat{n}\cdot\bar{x})} + \text{c.c.} \right) \quad (15)$$

where $k^{\mu} = (\omega, \bar{k}) = (2\pi f, |k|\hat{n})$ and $|k| = \omega$

For stochastic GW A_{ij} is a full 3D matrix, but for GW from a localized astrophysical source A_{ij} is 2d transverse to $\hat{n}$.

It is useful to note that we can go from a generic $h_{\mu\nu}$ to TT gauge by using a projector

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}, \quad P_{ij} \equiv \delta_{ij} - n_i n_j, \quad (16)$$

which projects the transverse, traceless modes of a wave propagating in the $\hat{n}$ direction such that

$$h^{TT}_{ij} = \Lambda_{ij,kl} h_{kl} . \qquad (17)$$

Effect of GW on a test mass

In GR, $h_{\mu\nu}$ describes changes in the spacetime that can be produced by matter, and at the same time $h_{\mu\nu}$ changes how matter travels in this spacetime as given by the geodesic eq. The gauge invariance corresponds to the freedom of choosing coordinates without changing the physics. To analyze the effect of GW on a test mass we will work with coordinates suited for the detector frame which we denote $x^\mu = (t, \bar{x})$. At leading order we get Newtonian contributions so that the trajectory of a test mass is given by

$$\frac{d^2x^i}{dt^2} = \underbrace{-a^i}_{\text{gravitational acceleration}} \underbrace{- 2(\Omega \times v)^i}_{\text{Coriolis force}} + \underbrace{f^i}_{\text{External force: suspension mechanism}} \qquad (18)$$

In principle, this effects are much larger than that of GW which appears at higher order, but at certain frequencies it can become relevant. Assuming f^i compensates for a_i and all other Newtonian contributions are slowly varying, we can work on a free-fall frame.

Consider two test particles with trajectories x^μ and $x^\mu + \xi^\mu$ where $|\xi|$

is smaller than the scale of variation of the gravitational field. In the detector frame, one has

$$\boxed{\frac{d^2\xi_i}{dt^2} = \frac{1}{2}\ddot{h}^{TT}_{ij}\,\xi^j} \qquad (19)$$

which is the geodesic deviation equation.

This tells us that the nearby test particles feel a gravitational tidal force produced by the passing of the GW.

In a detector of size L this holds as long as $L \ll \lambda\!\!\!^{-}$, where $\lambda\!\!\!^{-}$ is the reduced wavelength of the GW.

For a ring of masses in the $z=0$ plane, we can understand the effect of the polarizations in Eq. 13 as follows. h_+ causes a shear in the x, y directions, while $h_\times$ causes it at 45° degrees (hence the subscripts). See Fig. below.