CALCULATING HARD SCATTERING

LEARNING OBJECTIVES

By the end of this lecture, you will be able to:

- Describe the different ingredients that feed into a fixed order calculation at leading order
- Outline the difficulties with calculations beyond leading order and common methods of dealing with them
- Explain how weighted events are unweighted in order to simulate realistic collider outcomes

CALCULATIONS AT FIXED ORDER IN PERTURBATION THEORY

- Standard Model has gauge group $SU(3) \times SU(2)_L \times U(1)$
- Cannot compute anything exactly, so we expand in some (small) coupling
- Strong sector of $SU(3)$ has largest coupling/biggest effect, so usually compute corrections in this first - Quantum Chromodynamics
- Allowed to do this because of asymptotic freedom - sign of β function means that strong coupling decreases with energy

FIXED ORDER CALCULATIONS

- At a hadron collider, cross section given by collinear factorisation formula

$$\sigma_{2\to n} = \sum_{a,b} \int_0^1 \mathrm{d}x_a \mathrm{d}x_b \, f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F) \, \hat{\sigma}_{ab\to n}(\mu_F, \mu_R)$$

$$= \sum_{a,b} \int_0^1 \mathrm{d}x_a \mathrm{d}x_b \, f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F) \, \frac{1}{2\hat{s}} \int \mathrm{d}\Phi_n \, |\mathcal{M}_{ab\to n}|^2 (\Phi_n; \mu_F, \mu_R)$$

Initial state phase space

Parton distribution functions

Flux factor

Final state phase space

Squared amplitude

LEADING ORDER EVENT GENERATION

1. Generate phase space point
2. Evaluate matrix element $|\mathscr{M}|^2$ on phase space point and convolve with parton luminosity
3. Store event, specifying four momenta of all particles and weight = PDFs $\otimes$ matrix element $\times$ phase space Jacobian
4. Repeat N times to get a set of events
5. Analyse results - compute expected values and errors on set of events, apply cuts, bin in histogram

PHASE SPACE GENERATION

- Phase space integral to be performed is over

$$d\Phi_n = \left[\prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} (2\pi)\delta(p_i^2 - m_i^2)\,\Theta(E_i) \right] (2\pi)^4 \delta^4 \left(x_a P_{h_1} + x_b P_{h_2} - \sum_{i=1}^{n} p_i \right)$$

- Lorentz invariant volume
- On-shellness condition
- Positive energy
- Total four-momentum conservation

- Can rewrite in terms of nice variables like angles, invariant masses etc. and then importance sample

2 BODY PHASE SPACE

- Consider decay of one to two particles (e.g. $K \to \pi\pi$ in K rest frame)

$$\mathrm{d}\Phi_2 = \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\,(2\pi)\,\delta(p_1^2 - m_1^2)\,\Theta(E_1)\,\frac{\mathrm{d}^4 p_2}{(2\pi)^4}\,(2\pi)\,\delta(p_2^2 - m_2^2)\Theta(E_2)\,\delta^4\left(p_K^\mu - p_{\pi,1}^\mu - p_{\pi,2}^\mu\right)$$

$$= \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\,(2\pi)\,\delta(p_1^2 - m_1^2)\,\Theta(E_1)\,(2\pi)\,\delta\left[(p_K - p_1)^2 - m_2^2\right]\,\Theta(M_K - E_1)$$

$$= \frac{|\vec{p}_1|^2\,\mathrm{d}E_1\,\mathrm{d}|\vec{p}_1|\,\mathrm{d}^2\Omega_1}{(2\pi)^4}\,(2\pi)\,\delta\left(E_1^2 - |\vec{p}_1|^2 - m_1^2\right)\,\Theta(E_1)$$

$$\times\,(2\pi)\,\delta\left[(M_K - E_1)^2 - |\vec{p}_1^{\,2}| - m_2^2\right]\,\Theta(M_K - E_1)$$

$$= \frac{\lambda^{1/2}(M_K^2, m_1^2, m_2^2)}{8M_K^2}\,\frac{\mathrm{d}^2\Omega_1}{(2\pi)^2}$$

2 BODY PHASE SPACE

- Källén function given by

$$\lambda(a,b,c) = (a-b-c)^2 - 4bc$$

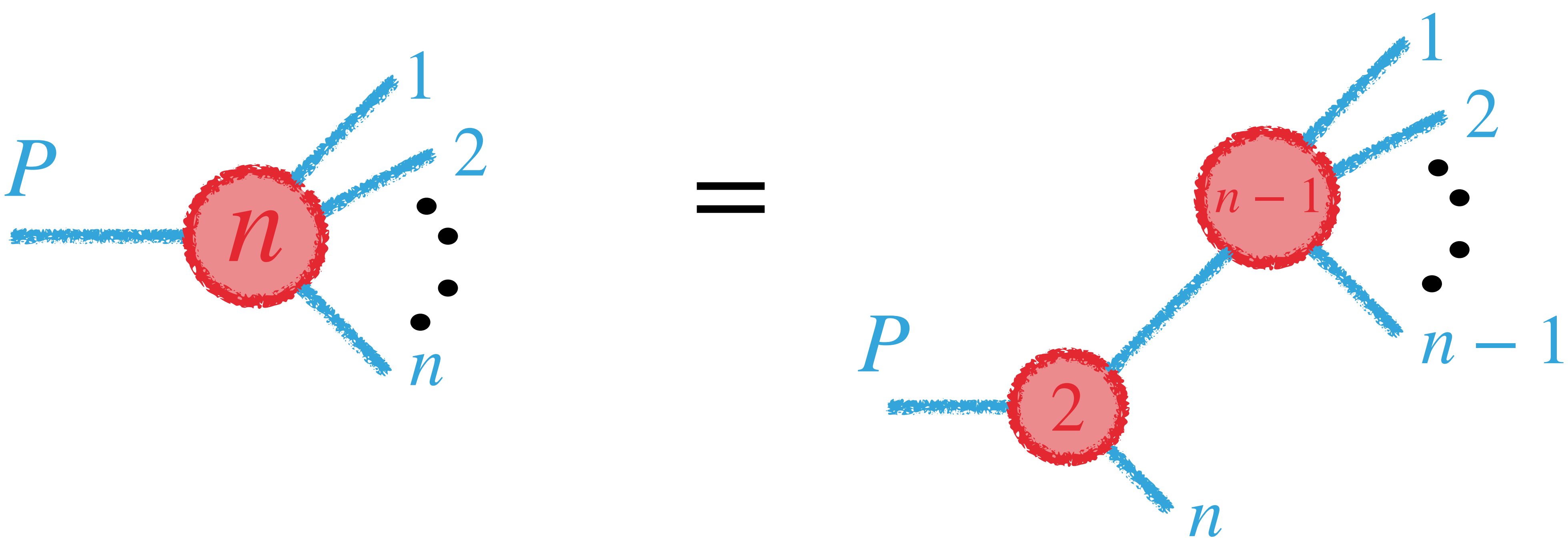
- Exercise: show that, when both final state particles are massless,

$$\int \mathrm{d}\Phi_2 = \frac{1}{8\pi}$$

N-BODY PHASE SPACE

- Can construct N-body phase space recursively from 2-body case, by introducing massive intermediate states $p_{12\ldots(n-1)} = \sum_{i=1}^{n-1} p_i$

$$\mathrm{d}\Phi_n(P; p_1, \ldots, p_n) = \mathrm{d}m^2_{12\ldots(n-1)}\, \mathrm{d}\Phi_2(P; p_{12\ldots(n-1)}, p_n)\, \mathrm{d}\Phi_{n-1}\left(p_{12\ldots(n-1)}; p_{1\ldots}, p_{n-1}\right)$$



MATRIX ELEMENT EVALUATION

- Can do by hand at tree level for ≤ 3 final state particles
- Beyond this, number of diagrams grows factorially

n gluons	Number of diagrams
0	1
1	2
2	8
3	48
4	384

$$e^+e^- \to q\bar{q} + ng$$

MATRIX ELEMENT EVALUATION

- Textbook methods are insufficient for most realistic problems (squaring, completeness relations, traces)
- Many helicity combinations vanish - wasted computation
- Parke and Taylor (1985) computed $gg \to gggg$: 220 diagrams, 100 pages calculation, 14 pages of result
- Within a year later they realised

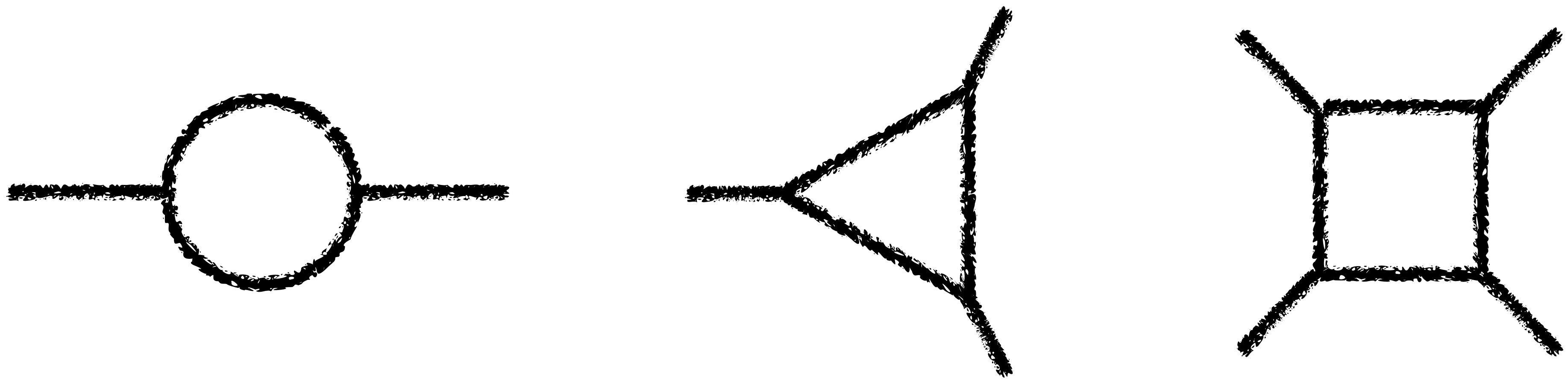
$$\mathcal{M}_6 = \frac{\langle 12 \rangle^3}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

MATRIX ELEMENT EVALUATION

- Spinor-helicity formalism trades 4-component Dirac spinors for 2-component left- and right-handed Weyl
- Evaluate spinor brackets numerically at amplitude level, then square a complex number
- Angular momentum conservation means some terms vanish immediately - no waste!
- Can be sped up further with recursion relations, colour sampling

MATRIX ELEMENT EVALUATION

- Loop amplitudes are harder. Rely on reduction of loop integrals to a basis of functions at one-loop:



- Methods to perform integral reduction: Passarino-Veltmann/Ossau-Papadopoulos-Pittau
- Two-loop much harder, basis of functions unknown

VIRTUAL CORRECTIONS

- At one-loop, all integrals can be expressed in terms of logarithms and dilogarithms

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t} \log(1-t)$$

- Beyond one-loop, more general functions appear

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

Multiple polylogarithms

$$G(a; z) = \log\left(1 - \frac{z}{a}\right) \qquad G(0,1; z) = -\mathrm{Li}_2(z)$$

VIRTUAL CORRECTIONS

- At one-loop, all integrals can be expressed in terms of logarithms and dilogarithms

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t} \log(1-t)$$

- Beyond one-loop, more general functions appear

$$\mathcal{L}_n(z,q) = \sum_{k=-\infty}^{\infty} \mathrm{Li}_n\left(zq^k\right), \quad q = e^{2\pi i \tau}$$

Elliptic polylogarithms

INTEGRATION-BY-PARTS IDENTITIES

- At 2-loops, Feynman diagram evaluation leaves 1000s of integrals to be evaluated
- Not all are independent! In dimensional regularisation,

$$0 = \int \mathrm{d}^D k_i \frac{\partial}{\partial k_i^\mu} \frac{k^\mu}{D_1^{a_1} \dots D_n^{a_n}}$$

- Integration-by-parts identities - relate integrals with different propagator powers and solve recursively

INTEGRATION-BY-PARTS IDENTITIES

- Reduces every integral into a linear combination of basis vectors (`master' integrals)
- $N_{master} \ll N_{total}$!
- In practice, use recursion relations to generate linear relations among integrals, and truncate tower of relations - system of finite size to solve
- Automated using Laporta's algorithm

INTEGRATION-BY-PARTS EXAMPLE – THE BUBBLE

Integral vanishes unless $n_1, n_2 > 0$

$$\mathrm{Bub}(n_1, n_2) = \int \frac{\mathrm{d}^D k}{[k^2]^{n_1}[(k+p)^2]^{n_2}}$$

$$0 = \int \mathrm{d}^D k \frac{\partial}{\partial k^\mu} k^\mu (\ldots) \qquad 0 = \int \mathrm{d}^D k \frac{\partial}{\partial k^\mu} p^\mu (\ldots)$$

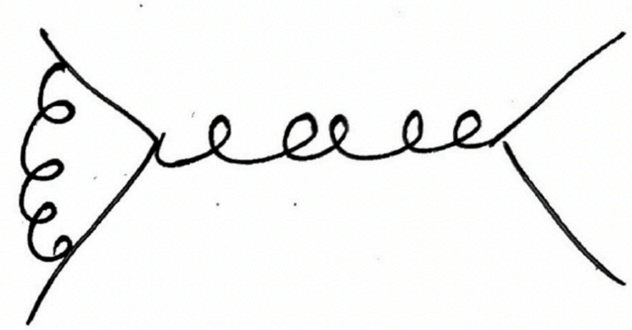
$$\begin{aligned}\mathrm{Bub}(n_1, n_2) &= \frac{n_1 + n_2 - 1 - D}{p^2(n_2 - 1)} \mathrm{Bub}(n_1, n_2 - 1) + \frac{1}{p^2} \mathrm{Bub}(n_1 - 1, n_2) \\ &= \frac{1}{p^2} \mathrm{Bub}(n_1, n_2 - 1) + \frac{n_1 + n_2 - 1 - D}{p^2(n_2 - 1)} \mathrm{Bub}(n_1 - 1, n_2)\end{aligned}$$

NEXT-TO-LEADING ORDER COMPUTATIONS

- Loop amplitudes themselves are divergent in the UV and IR. Can cure UV by renormalisation, but IR remain for $k \to 0$

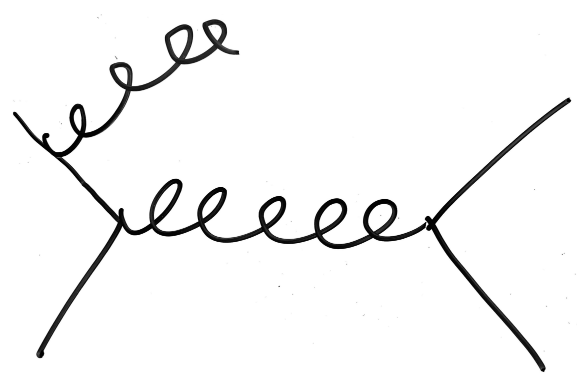
- KLN theorem (with caveats) - IR divergences cancel when combining real and virtual corrections

VIRTUAL →



$$\frac{A}{\epsilon} + B$$

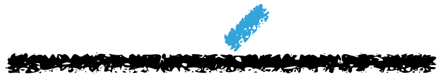
REAL →



$$-\frac{A}{\epsilon} + C$$

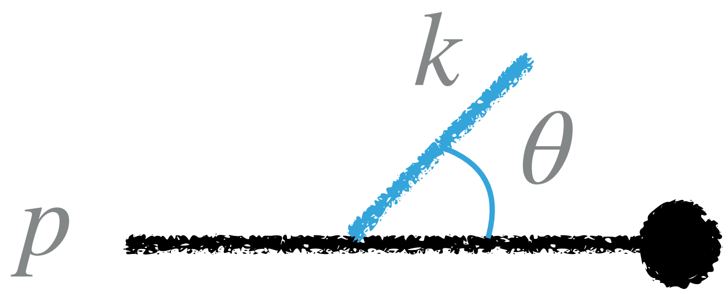
NEXT-TO-LEADING ORDER COMPUTATIONS

- Loop amplitudes are divergent - manifested analytically by poles in dimensional regulator $d = 4 - 2\epsilon$
- Real amplitudes diverge when integrated over emission phase space in limits when gluon becomes soft (low energy) or collinear.
- Cannot physically distinguish one emission here - could be an infinite number!



NEXT-TO-LEADING ORDER COMPUTATIONS

- Consider gluon emission off a hard incoming quark line



- Intermediate propagator goes as

$$\sim \frac{1}{(p-k)^2} \sim \frac{1}{2p \cdot k} \sim \frac{1}{E_k(1-\cos\theta)}$$

- Diverges for $E_k \to 0$ (soft) or $\theta \to 0$ (collinear)!

DEALING WITH IR SINGULARITIES

- Problem: how do we combine the divergences from virtual and real contributions to get something finite in four dimensions?
- Virtual singularities live in LO phase space (same number of final state particles). Real singularities have an extra emission which must be integrated over
- Need to combine the two in a way that can be integrated numerically, i.e. using a Monte Carlo

DEALING WITH IR SINGULARITIES

- Consider a toy model: UV and IR finite Born term with n particles in final state

$$\mathscr{B}_n = \sum |\mathscr{M}_n^{(0)}|^2$$

- UV renormalised virtual correction takes the form

$$\mathscr{V}_n = \frac{V_n}{\epsilon} = \sum |\mathscr{M}_n^{(0)}\mathscr{M}_n^{(1)*}|$$

- Real emission depends only on $x \in [0,1]$ and diverges for $x \to 0$:

$$\mathscr{R}_n(x) = \frac{R_n(x)}{x}$$

DEALING WITH IR SINGULARITIES

- Obtain NLO cross section by integrating over x and combining Born, real and virtual contributions:

$$\sigma^{\mathrm{NLO}} = \left[\mathscr{B}_n + \mathscr{V}_n\right] \mathscr{O}_n + \int_0^1 \mathrm{d}x\, \mathscr{R}_n(x)\mathscr{O}_{n+1}(x)$$

- Here $\mathscr{O}_n$ is an infrared-safe observable, which makes sure that the LO process is free of singularities. Require that

$$\lim_{x\to 0} \mathscr{O}_{n+1}(x) = \mathscr{O}_n$$

Definition of IR safety

DEALING WITH IR SINGULARITIES

- Obtain NLO cross section by integrating over x and combining Born, real and virtual contributions:

$$\sigma^{\mathrm{NLO}} = \left[\mathscr{B}_n + \frac{V_n}{\epsilon}\right] \mathcal{O}_n + \int_0^1 \frac{\mathrm{d}x}{x} R_n(x) \mathcal{O}_{n+1}(x)$$

- Here $\mathcal{O}_n$ is an infrared-safe observable, which makes sure that the LO process is free of singularities. Require that

$$\lim_{x\to 0} \mathcal{O}_{n+1}(x) = \mathcal{O}_n$$

Definition of IR safety

PHASE SPACE SLICING

- Let's regulate the divergent real integral:

$$\sigma^{(1)} = \frac{V_n}{\epsilon}\mathcal{O}_n + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x)$$

and split the range with a parameter δ

$$\sigma^{(1)} = \frac{V_n}{\epsilon}\mathcal{O}_n + \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x) + \int_\delta^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x)$$

- So far, just a rewriting.

PHASE SPACE SLICING

- Now choose $\delta \ll 1$; in the first integral, we can approximate

$$R_n(x)\mathcal{O}_{n+1}(x) \approx R_n(0)\mathcal{O}_n$$

to get

$$\sigma^{(1)} = \frac{V_n}{\epsilon}\mathcal{O}_n + R_n(0)\mathcal{O}_n \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} + \int_\delta^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x)$$

$$= \left(\frac{V_n}{\epsilon} + R_n(0)\int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}}\right)\mathcal{O}_n + \int_\delta^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x)$$

PHASE SPACE SLICING

- Second integral is not divergent for finite δ, so can set $\epsilon = 0$. Performing first integral, we find

$$\sigma^{(1)} = \left(\frac{V_n}{\epsilon} - R_n(0)\frac{\delta^{-\epsilon}}{\epsilon}\right)\mathcal{O}_n + \int_\delta^1 \frac{\mathrm{d}x}{x} R_n(x)\mathcal{O}_{n+1}(x)$$

$$= \left[\frac{\big(V_n - R_n(0)\big)}{\epsilon} + R_n(0)\,\log\delta\right]\mathcal{O}_n + \int_\delta^1 \frac{\mathrm{d}x}{x} R_n(x)\mathcal{O}_{n+1}(x)$$

- To cancel real and virtual divergences, require that $R_n(0) = V_n$.

PHASE SPACE SLICING

- Final result is

$$\sigma^{(1)} = V_n \mathcal{O}_n \log \delta + \int_\delta^1 \frac{\mathrm{d}x}{x} R_n(x) \mathcal{O}_{n+1}(x)$$

- No dependence on ϵ, so can be integrated numerically in four dimensions!
- Result depends on approximated real matrix element at small δ - receives corrections $\mathcal{O}(\delta)$, meaning we must choose δ small to minimise power corrections
- Numerically challenging, as $R_n(x)$ is divergent at small x

PHASE SPACE SLICING

- Also have to construct $R_n(0)\mathcal{O}_n$ term to use in the approximation, i.e. take the unresolved limit of real ME.
- Non-trivial even at NLO, as one has to avoid overlapping divergences. At NNLO, much harder (two emissions).
- Effective field theory approaches (SCET) can provide the below-cut contribution via nifty factorisation theorems and can disentangle soft/collinear divergences

$$\frac{\mathrm{d}\sigma(x < \delta)}{\mathrm{d}x} = H \otimes B \otimes B \otimes S \otimes J + \mathcal{O}(\delta)$$

LOCAL SUBTRACTION METHODS

- Recall the integral we had before introducing δ :

$$\sigma^{(1)} = \frac{V_n}{\epsilon}\mathcal{O}_n + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x)$$

- Can add zero in a clever way:

$$\sigma^{(1)} = \frac{V_n}{\epsilon}\mathcal{O}_n + R_n(0)\mathcal{O}_n\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R_n(x)\mathcal{O}_{n+1}(x) - R_n(0)\mathcal{O}_n\int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}$$

LOCAL SUBTRACTION METHODS

- Integrate the first term, rearrange the second:

$$\sigma^{(1)} = \left(\frac{V_n}{\epsilon} - \frac{R_n(0)}{\epsilon}\right)\mathcal{O}_n + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}}\left[R_n(x)\mathcal{O}_{n+1}(x) - R_n(0)\mathcal{O}_n\right]$$

- Inspect second term as $x \to 0$:

$$\lim_{x\to 0}\left[R_n(x)\mathcal{O}_{n+1}(x) - R_n(0)\mathcal{O}_n\right] = R_n(0)\mathcal{O}_{n+1}(0) + x\frac{\mathrm{d}}{\mathrm{d}x}R_n(x)\mathcal{O}_{n+1}(x)\bigg|_{x=0} - R_n(0)\mathcal{O}_n$$

- For IR-safe observables (recall def.), this is $\mathcal{O}(x)$ and so finite.

LOCAL SUBTRACTION METHODS

- Note that we did not have to introduce any additional parameters.
- Large cancellations still exist between subtraction term and real matrix element, but these can happen at the integrand level if the subtraction term is chosen correctly - numerically stable
- Difficult in general to find subtraction terms which capture all the divergences of the matrix element, and which can be integrated analytically.

LOCAL SUBTRACTION METHODS

- At NLO two main methods exist: Catani-Seymour dipole subtraction (CS), and Frixione-Kunszt-Signer subtraction (FKS) - can be fully automated
- At NNLO things are much harder. Many options (STRIPPER and antenna subtraction most developed)
- At N3LO nothing exists. Only slicing methods possible (or different techniques altogether, such as projection-to-Born).

BACK TO EVENT GENERATION…

- We now know how to construct the weights we need for the phase space points we generate (up to NLO).
- For given points in phase space $\overrightarrow{v}_i$, we associate weights w_i calculated from ME, PDFs, cuts etc.
- Potential problem - many points may have a small weight and contribute essentially nothing to cross section.
- In Nature, events don't have a weight - more events where cross section is large, fewer where it is small

UNWEIGHTING

- Solution - recalling hit-or-miss approach, do not keep all events, but keep event $\vec{v}_i$ with probability

$$P_i = \frac{w_i}{w_{\max}}$$

- Before - same number of events in areas of phase space with very different probabilities, events must carry weight
- After - number of events is proportional to probability of areas of phase space, all events have same weight
- Events distributed as in Nature!

DEFINITION OF A MONTE CARLO EVENT GENERATOR

- An MC event generator produces simulated events with the same probability as they occur in Nature - it mimics a collider.
- Performs a large number of difficult integrals, then unweights to give four momenta of detected particles
- It is common to refer to codes which use MC techniques but do not provide fully exclusive information on the final state as "Monte Carlos" - these normally predict cross sections or kinematic distributions, but cannot be unweighted

SUMMARY

- Hard scattering calculations depend on many ingredients: phase space integrals, matrix elements, parton distribution functions
- Beyond leading order, need methods to combine real and virtual divergences which allow integration in four dimensions
- When all ingredients are put together, can calculate the weight of each event. Need an unweighting procedure to general events with the same frequency as in Nature.