LECTURE 2

Energy-momentum tensor of GWs

In general, we can have GW propagating around a dynamical background $\bar{g}_{\mu\nu}$ (instead of flat space with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$). The metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad (20)$$

satisfies Einstein equations. To have a clear distinction between the background and the GW we need a large spatial variation of the background compared to that of the GW:

$$\lambda \ll L_B,$$

or alternatively, that $h_{\mu\nu}$ is peaked at a large frequency f such that

$$f >> f_B .$$

In these regimes, one can define an energy-momentum tensor for GWs. We will use Noether's theorem for spacetime translation symmetry which leads to the following conserved energy-momentum tensor

$$t^{\mu\nu} = \left\langle - \frac{\partial \mathcal{L}^{F.P.}}{\partial_\mu h_{\alpha\beta}} \partial^\nu h_{\alpha\beta} + \eta^{\mu\nu} \mathcal{L}^{F.P.} \right\rangle, \qquad (21)$$

where the average $\langle\rangle$ is over a volume of size l with $\bar{\lambda} \ll l \ll L_B$ or alternatively over a timescale τ with $f^{-1} \ll \tau \ll f_B^{-1}$.

Using Eq. (2) in Lorentz gauge (Eq. 4) + $\bar{h}=0$ we find[3]

$$-\frac{\partial \mathcal{L}^{F.P.}}{\partial_\mu h_{\alpha\beta}} = \frac{1}{32\pi G} \partial^\mu h^{\alpha\beta}, \qquad \langle \mathcal{L}^{F.P.} \rangle \overset{\text{by parts for waves } (\partial_t \sim -\partial_x)}{\propto} \langle h_{\mu\nu} \Box h^{\mu\nu} \rangle \overset{e.o.m.}{=} 0,$$

so that

$$t^{\mu\nu} = \frac{1}{32\pi G} \langle \partial^\mu h^{\alpha\beta} \partial^\nu h_{\alpha\beta} \rangle . \qquad (22)$$

This result can also be obtained from the low-frequency part of Einstein equations. In fact, Eq. 22 is invariant under linearized diffeomorphisms, so for simplicity we can evaluate it in TT gauge. Thus, the GW energy density is

$$\boxed{t^{00} = \frac{1}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle} \qquad (23)$$

3) Inside $\langle\rangle$ we can integrate by parts both time and spatial derivatives since for a wave solution f, we have $\partial_t f \sim -\partial_x f$. Thus we can choose the appropriate one for a time or spatial average. The boundary terms are of order $\bar{\lambda}/L_B$ so they can be neglected.

Far away from sources $t^{\mu\nu}$ is conserved: $\partial_\mu t^{\mu\nu} = 0$, but near sources, the conservation of the total energy-momentum tensor is

$$\bar{\nabla}_\mu (T^{\mu\nu} + t^{\mu\nu}) = 0, \qquad (24)$$

where $\bar{\nabla}_\mu$ is the covariant derivative of the background metric $\bar{g}_{\mu\nu}$ and $T^{\mu\nu}$ describes the external sources.

Away from sources we define the energy flux of GW, $\frac{dE}{dt} = -\frac{dE_V}{dt}$, where E_V is the GW energy in a volume V

$$\frac{dE}{dt} = -\int_V d^3x \, \partial_t t^{00} = \int_V d^3x \, \partial_i t^{0i} = r^2 \int_{\partial V} d\Omega \, t^{0r} \underset{\partial_t \sim -\partial_r}{\overset{\text{large } r}{\simeq}} r^2 \int_{\partial V} d\Omega \, t^{00} \qquad (25)$$

combining this with Eq. 23 we find

$$\frac{dE}{dt\,dA} = \frac{1}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle . \qquad (26)$$

The energy flow through a surface in frequency domain (See Eq. 15) is given by

$$\frac{dE}{dA} = \frac{\pi}{2G} \int_0^\infty df \, f^2 \left(h_+^2(f) + h_\times^2(f) \right) \qquad (27)$$

where we got rid of the average by explicitly performing the integral. This allows us to find the energy spectrum:

$$\boxed{\frac{dE}{df} = \frac{\pi}{2G} r^2 f^2 \int d\Omega \, \left(h_+^2(f) + h_\times^2(f)\right)} \quad (28)$$

$\hookrightarrow$ GW sensitivity curves 1408.0740

Power Radiated in GW's

Given a system of dynamical matter, we want to compute the power radiated in GW's by this system. We work in the weak field ($|h_{\mu\nu}| \ll 1$) and low-velocity (small typical velocities of the matter sources) limit.

From Eq. 8 we can write the graviton solution as

$$\bar{h}_{\mu\nu} = -16\pi G \int d^4x' \, G_R(x-x') \, T_{\mu\nu}(x') \quad (29)$$

where the retarded Green's function solves $\Box_x G_R(x-x') = \delta^4(x-x')$ and is given by

$$G_R(x-x') = -\frac{1}{4\pi|\bar{x}-\bar{x}'|} \, \delta\left(t^{ret} - t'\right) \quad (30)$$

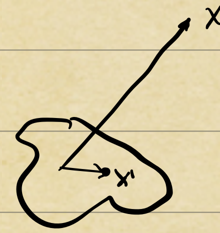
$$\text{with } t^{ret} = t - |\bar{x}-\bar{x}'|$$

$'$ are coordinates at the source

so that after projecting to TT gauge using Eqs. 16 and 17 we find

$$h_{ij}^{TT} = 4G \, \Lambda_{ij,kl}(\hat{x}) \int d^3x' \, \frac{T^{kl}(t^{ret}, \bar{x}')}{|\bar{x}-\bar{x}'|} \quad (31)$$

Assuming we are observing far from the source we can expand $|x - x'| \simeq |x| - x'\cdot\hat{x}$



so that

$$h_{ij}^{TT} \simeq \frac{4G}{r}\Lambda_{ij,kl}(\hat{x})\int d^3x'\, T^{kl}(t - r + x'\cdot\hat{x}\,, \vec{x}') \quad (32)$$

where defined $r \equiv |x|$. We can further expand the energy-momentum tensor assuming a slowly varying source. For a purely gravitational system, this is not an assumption, but a requirement of the weak field approximation. From the Virial theorem $v^2 \sim \frac{Gm}{r} \sim h \ll 1$. Then,

$$T_{ij}(t - r + x'\cdot\hat{x}, x') \simeq T_{ij}(t-r, x') + x'\cdot\hat{x}\;\dot{T}_{ij}(t-r, x') + \dots \quad (33)$$

where we will neglect the second and higher order terms which are suppresed by the slow motion of the source.

Now we use the conservation of $T_{\mu\nu}$ to rewrite T_{ab} in terms of T_{00} and also integrate by parts obtaining

$$\int_{x'} T^{ij} = \int_{x'} (\partial_k x^{(i}) T^{j)k} = -\int_{x'} x^{(i} \partial_k T^{j)k}$$

$$\overset{\partial_\mu T^{\mu\nu}=0}{=} \int_{x'} x^{(i} \dot{T}^{j)0} = \int_{x'} \frac{1}{2} \partial_k (x^i x^j) \dot{T}^{k0} = -\frac{1}{2} \int_{x'} x^i x^j \partial_k \dot{T}^{k0}$$

$$\overset{\partial_\mu T^{\mu\nu}=0}{=} \int_{x'} \frac{1}{2} x^i x^j \ddot{T}^{00} \qquad (34)$$

where $x^{(i} y^{j)} \equiv \frac{1}{2}(x^i y^j + x^j y^i)$ and $\int_{x'} \equiv \int d^3x'$. Here we can use $\partial_\mu T^{\mu\nu} = 0$ since we work in the linearized theory and ignore backreaction from GW's (the $t_{\mu\nu}$ contribution).

Puting all the above together we find

$$h^{TT}_{ij} \simeq \frac{2G}{r} \Lambda_{ij,kl}(\hat{x}) \int d^3x' \, x^i x^j \ddot{\rho}(t-r) \qquad (35)$$

where we used $T^{00} = \rho$ the energy density.

Rembering that the quadrupole moment is given by

$$Q^{ij} = \int d^3x \, \rho \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij}\right)$$

we get the famous quadrupole radiation formula

$$\boxed{h_{ij}^{TT} \simeq \frac{2G}{r} \Lambda_{ij,kl}(\hat{x}) \ddot{Q}_{ij}(t-r)} \qquad (36)$$

There are no monopole or dipole contributions since the mass and momentum conservation of the source imply

$$\dot{M} = 0 \qquad M \equiv \int d^3x \, T^{00}$$

$$\ddot{D}_i = \dot{P}_i = 0 \qquad D^i \equiv \int d^3x \, T^{00} x^i$$

$$P^i \equiv \int d^3x \, T^{0i}$$

The power radiated is given by

$$\frac{dP}{d\Omega} = \frac{r^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle = \frac{G}{8\pi} \Lambda_{ij,kl} \langle \dddot{Q}_{ij} \dddot{Q}_{kl} \rangle \qquad (37)$$

thus

$$P^{Quad} = \frac{G}{5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle \qquad (38)$$

Binary Systems

We consider two masses m_1, m_2 with relative coordinate $\bar{X}_0 = \bar{X}_1 - \bar{X}_2$ undergoing circular motion:

$$\bar{x}_0 = (R\cos(\omega_s t + \pi/2), R\sin(\omega_s t + \pi/2), 0) \quad (39)$$

so that their mass momenta is

$$M^{ij} = \int d^3x\, \rho\, x^i x^j = \mu\, x_0^i x_0^j, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad (40)$$

whose traceless part gives the quadrupole Q^{ij}. We now find that for a wave traveling in the direction $\hat{n} = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta)$,[4]

$$h_+ = \frac{G}{r}(\ddot{M}_{xx} - \ddot{M}_{yy})$$

$$= \frac{4G\mu\omega_s^2 R^2}{r}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{ret} + 2\phi) \quad (41)$$

$$h_\times = \frac{2G}{r}\ddot{M}_{xy}$$

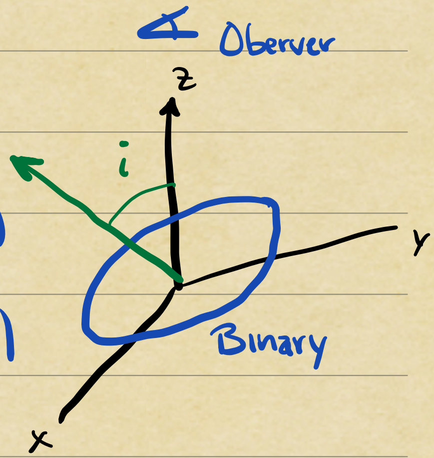
$$= \frac{4G\mu\omega_s^2 R^2}{r}\cos\theta\,\sin(2\omega_s t_{ret} + 2\phi) \quad (42)$$

We can now assume that the distance to the GW source, r, is almost constant and a coordinate system with ϕ fixed. Thus θ measures the inclination w.r.t. the line of sight and translating time so that $2\omega t_{ret} + 2\phi \to 2\omega t + \underbrace{2\omega r + 2\phi + 2\omega\Delta t}_{2\pi n}$ we find

4) This can be found by first computing this for a wave traveling in $\hat{z}$ direction and the applying a rotation to M_{ij}: of the form $M_{ij} = R_{ik}R_{jl}M_{kl}$.

$$h_+ = \frac{\overbrace{4GM\omega_s^2R^2}^{h}}{r}\left(\frac{1+\cos^2 i}{2}\right)\cos(2\omega t) \quad (43)$$

$$h_\times = \frac{4GM\omega_s^2R^2}{r}\cos i \ \sin(2\omega t) \quad (44)$$



Since the binary is held together by gravity we have

$$\frac{GM}{R^2} = \frac{v^2}{R} = \frac{(\omega_s R)^2}{R} \quad (45)$$

so that $R^3 = GM/\omega_s^2$ then we have

$$h = \frac{4}{r}\left(GM_c\right)^{5/3}\left(\pi f_{GW}\right)^{2/3} \quad (46)$$

where $f_{GW} = \omega_{GW}/2\pi$, the GW frequency is twice of that of the source, $\omega_{GW} = 2\omega_s$ and the chirp mass is

$$M_c \equiv \mu^{3/5}(m_1+m_2)^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \quad (47)$$

The power radiated is

$$\frac{dP}{d\Omega} = \frac{2}{\pi G}\left(\frac{G M_c \omega_{GW}}{2}\right)^{10/3} g(i) \quad (48)$$

Direct calc. + EFT's
+ Amplitudes

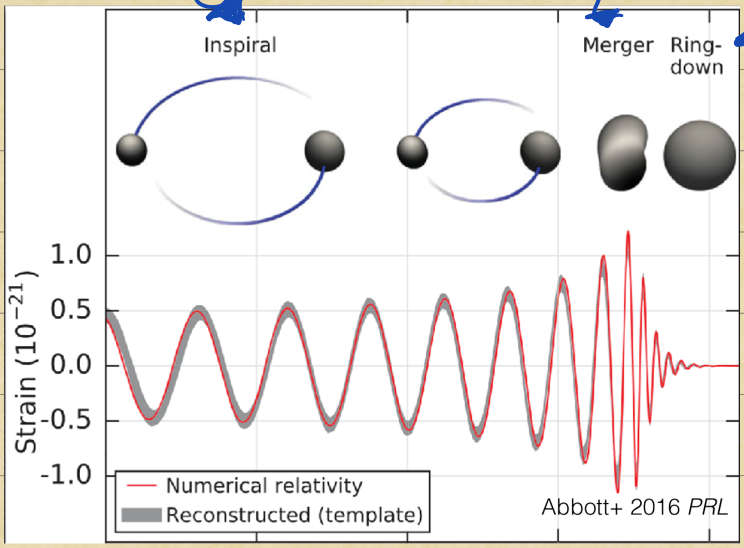
– Post-Newtonian $G, v \ll 1$

– Post-Minkowskian $G \ll 1$

Numerical GR

BH perturbation theory

leading order determines chirp mass



determines total mass

At higher PN orders can determine m_1/m_2, but there is a degeneracy with spins.

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