LECTURE 3

Propagation of GW in our Universe

The propagation of GW in curved spacetimes is described by Eq. 4 and 5 by taking $\partial_\mu \to \bar{\nabla}_\mu$.

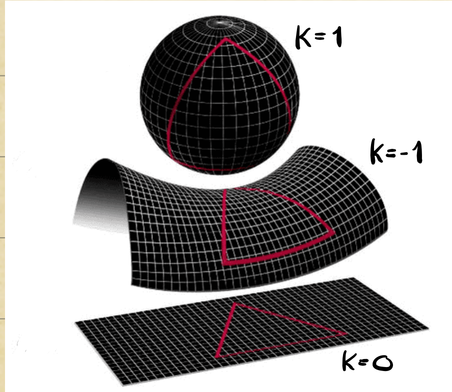
$$\bar{\nabla}^\nu \bar{h}_{\mu\nu} = 0 , \qquad \bar{\nabla}_\rho \bar{\nabla}^\rho h_{\mu\nu} = 0 . \qquad (49)$$

This holds for $\lambda \ll L_B$ and can be derived from the high frequency part of Einstein equations.

At large scales, our Universe is spatially homogeneous and isotropic. The most general metric describing this is the FRW metric:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right) \qquad (50)$$

where $a(t)$ is the scale factor describing the expansion of the Universe and $k = \{+1, 0, -1\}$ correspond to spherical, Euclidean, and hyperbolic spatial geometries respectively.



The e.o.m. (22) with metric given by (23) in TT gauge is:

$$\boxed{\ddot{h}_{ij} + 3H \dot{h}_{ij} - \frac{1}{a^2} \nabla^2 h_{ij} = 0 ,} \qquad (51)$$

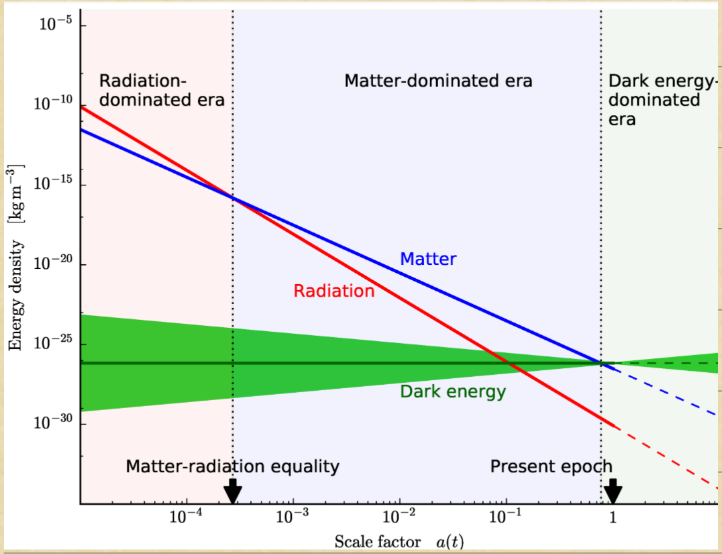
where $\boxed{H \equiv \frac{\dot{a}}{a}}$ is the Hubble rate and ∇^2 is the 3d spatial Laplacian.

Einstein's equations for the background metric in Eq. (23) reduce to the Friedmann's eq.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad (52)$$

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (53)$$

where we assumed that we have the stress-energy tensor of a perfect fluid with energy density ρ and pressure p. For a fluid with equation of state $p = \omega\rho$ then



$$\rho \propto a^{-3(1+\omega)} \qquad (54)$$

where

$$\omega = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{radiation} \\ -1 & \text{cosmological constant} \end{cases}$$

Going back to the GW eq. (), we can

solve it by goin to Fourier space and conformal time, η defined as

$$a\,d\eta = dt. \tag{55}$$

The metric is now conformally flat

$$\bar{g}_{\mu\nu} = a^2(\eta)\,\eta_{\mu\nu}. \tag{56}$$

In these coordinates we can go to Fourier space and change variables to $H_{ij} = a\,h_{ij}$ so Eq. 24 reads

$$H''_{ij}(\eta, k) + \left(k^2 - \frac{a''}{a}\right) H_{ij}(\eta, \bar{k}) = 0 \tag{57}$$

where $' \equiv \frac{d}{d\eta}$ and we define $\mathcal{H} = a'/a$.

Assuming $a(\eta) \propto \eta^p$, where the values

$$p = \begin{cases} -1 & \text{de Sitter} \\ 0 & \text{flat space} \\ 1 & \text{matter domination} \\ 2 & \text{radiation domination} \end{cases}$$

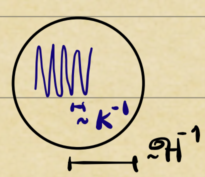
correspond to cosmologically relevant evolutions. Note $\mathcal{H} = \frac{p}{\eta}$.

The solution is

$$h^p_{ij}(\eta, \bar{k}) = \frac{\eta}{a(\eta)} \sum_\alpha e^\alpha_{ij} \left(A_\alpha(\bar{k})\, j_{p-1}(k\eta) + B_\alpha(\bar{k})\, y_{p-1}(k\eta) \right). \tag{58}$$

We can now analyze two limits:

- Subhorizon modes $k \gg \mathcal{H}$



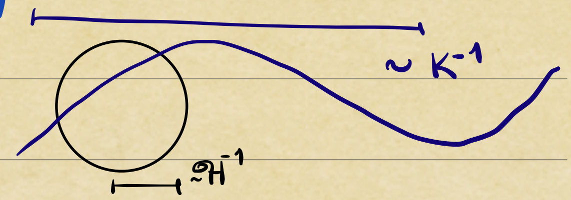
$$h^{p}_{ij}(\eta,\bar{k}) = \frac{1}{a(\eta)} \sum_{\alpha} e^{\alpha}_{ij} \left(A_{\alpha}(\bar{k})\, e^{ik\eta} + B_{\alpha}(\bar{k})\, e^{-ik\eta} \right) \quad (59)$$

We have plane waves since we are not probing the curvature of the spacetime.

$$\rho_{GW} \sim \frac{1}{16\pi G} |\dot{h}|^2 \sim \frac{1}{a^4} \longleftarrow \text{behave like radiation } \checkmark \quad (60)$$

$(\dot{h} = \frac{1}{a} h')$

- Superhorizon modes $k \ll \mathcal{H}$



$$h^{p}_{ij}(\eta,\bar{k}) = \frac{1}{a(\eta)} \sum_{\alpha} e^{\alpha}_{ij} \left(A_{\alpha}(\bar{k}) + B_{\alpha}(\bar{k}) \int^{\eta} \frac{d\eta'}{a^2(\eta')} \right) \quad (61)$$

We have a constant and a rapidly decaying (B term) contribution

The amplitude of the GW is similar to that of flat space but now we consider physical distances and the fact that frequencies redshift:

$$h = \frac{4}{a(t_0) r} (G M_c)^{5/3} (\pi f_{GW}(z_s))^{2/3} \quad (62)$$

where the redshif at the source is

$$1 + z_s = \frac{a(t_0)}{a(t_s)} . \quad (63)$$

This is introduced since in expanding universes, there is a time dilation of the time measured by the observer vs. the time measured at the source

$$\Delta t_{obs} = \frac{a(t_{obs})}{a(t_{emis})} \Delta t_{emis} \equiv (1+z)\, \Delta t_{emis} \qquad (64)$$

thus

$$f_{GW}(z_s) = (1+z_s)\, f_{GW}(z_0). \qquad (65)$$

so we can write

$$h = \frac{4}{d_L(z)} \left(G \mathcal{M}_c(z)\right)^{5/3} \left(\pi f_{GW}(z_s)\right)^{2/3} \qquad (66)$$

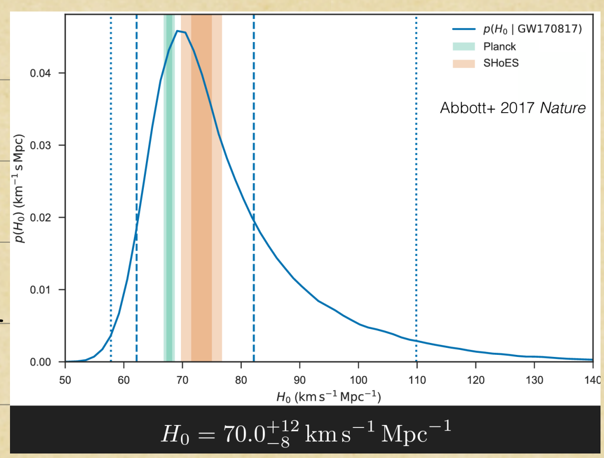
where $d_L = (1+z) a_0 r$ is the luminosity distance defined from the energy flux as $\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2}$ with $\mathcal{L}$ the absolute luminosity of the source, i.e. the power it radiates. We also defined the (redshifted) chirp mass $\mathcal{M}_c = (1+z) M_c$.

At low redshifts :

$$d_L H_0 = z + \mathcal{O}(z^2) \qquad (67)$$

inferred from GW

Obtained from E&M counterpart or host galaxy



So that we can measure the Hubble parameter today (H_0).

More generally

$$H_0 d_L = (1+z) \int_0^z \frac{dz'}{\sqrt{\Omega_r (1+z')^4 + \Omega_m (1+z')^3 + \Omega_k (1+z')^2 + \Omega_\Lambda}} \qquad (68)$$

for a Universe with radiation, matter, curvature, and cosmological constant where their normalized energy densities are $\Omega_i = \frac{\rho_i}{\rho_c}$, $\rho_c = \frac{8\pi G}{3H_0^2}$.

Including modified gravity effects

Given the existing cosmological tensions and unknown nature of dark energy, we can propose solutions by modifying General Relativity. Here, we will consider the possibility of new degrees of freedom and focus on the scalar case which could describe dark energy.

Additional dof could spoil observations within the Solar System where we know GR holds, thus these new dof should either be extremely weakly coupled or screened. The latter means that their effect is only observed at large distances, beyond our Solar System. Thus we focus on their propagation.

We can parametrize deviations from GR with an additional scalar degree of freedom with the following parameters

α_T speed of gravitational gaves

$$c_T^2 = 1 + \alpha_T$$

α_M running of effective Planck Mass

$$M_{pl}^2 = \frac{1}{8\pi G} \qquad \alpha_M \equiv \frac{1}{H}\frac{d \ln M_{pl}^2}{dt}$$

α_B "braiding" Mixing of scalar and metric Kinetic terms

α_K describes scalar dof kinetic term

α_H "disformal symmetries" Coupling between scalar and metric

The modification to Eq. describing the GW propagation is

$$\underbrace{h_{ij}'' + 2(1+\alpha_M)\mathcal{H} h_{ij}'}_{\text{change in friction}} + \underbrace{((1+\alpha_T) k^2}_{\text{propagation speed}} + \underbrace{a^2 m_g^2}_{\text{massive graviton}})h_{ij} = \underbrace{a^2 \Gamma \gamma_{ij}}_{\text{source}} \quad (69)$$

$+ (h_{00}, h_{i0})$ eom. $(\alpha_B, \alpha_H, \alpha_K)$

extra polarizations

Usually these are tested by taking $\alpha = \Omega_\Lambda$ or $\alpha = a(t)$.

Gama Ray Burst

- GW170817 & GRB170817

LIGO — Fermi, Integral — Baker 2017

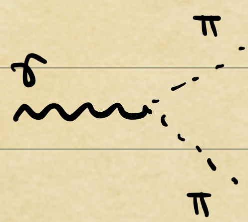
Observed at $\Delta t \simeq 1.7\,s$ $\longrightarrow$ $|\alpha_T| \lesssim 10^{-13}$

taking into account possible delay between GW and GRB

- For $\alpha_H \neq 0$

GW decay into π

(Creminelli et al 2018)



Since we observe GW

$\rightarrow \alpha_H \simeq 0$

- For $\alpha_M \neq 0$

This is just a change in the friction term so it only changes the amplitude as follows

$$h = h_{GR}\, e^{-\frac{1}{2}\int_0^z \frac{\alpha_M}{1+z}dz} \quad \leftarrow \text{changes effective luminosity distance}$$

From Eq. 66 $\longrightarrow$ $h_{GR} = \dfrac{1}{d_L}\left(G\mathcal{M}\right)^{5/3}\left(\pi f_{GW}\right)^{2/3}$

Baker 2020

Lagos 2019