Theory of EM fields

CAS course on “RF for Accelerators“,
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Thomas Flisgen
Ferdinand-Braun-Institut
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> ***„There is nothing more <u>practical</u> than a good <u>theory</u>!”***,
> Kurt Lewin (1890 - 1947), German-American psychologist

- When dealing with electromagnetic phenomena, we have a very good theory in form of Maxwell’s equations

- As it is a good theory, it is not just relevant for theory enthusiasts, but indeed it is very practical and useful to gain insight in electromagnetic phenomena relevant for accelerators

- In this talk: Recapitulation of Maxwell’s equations, conservation principles, classes of fields, some selected solutions and their properties

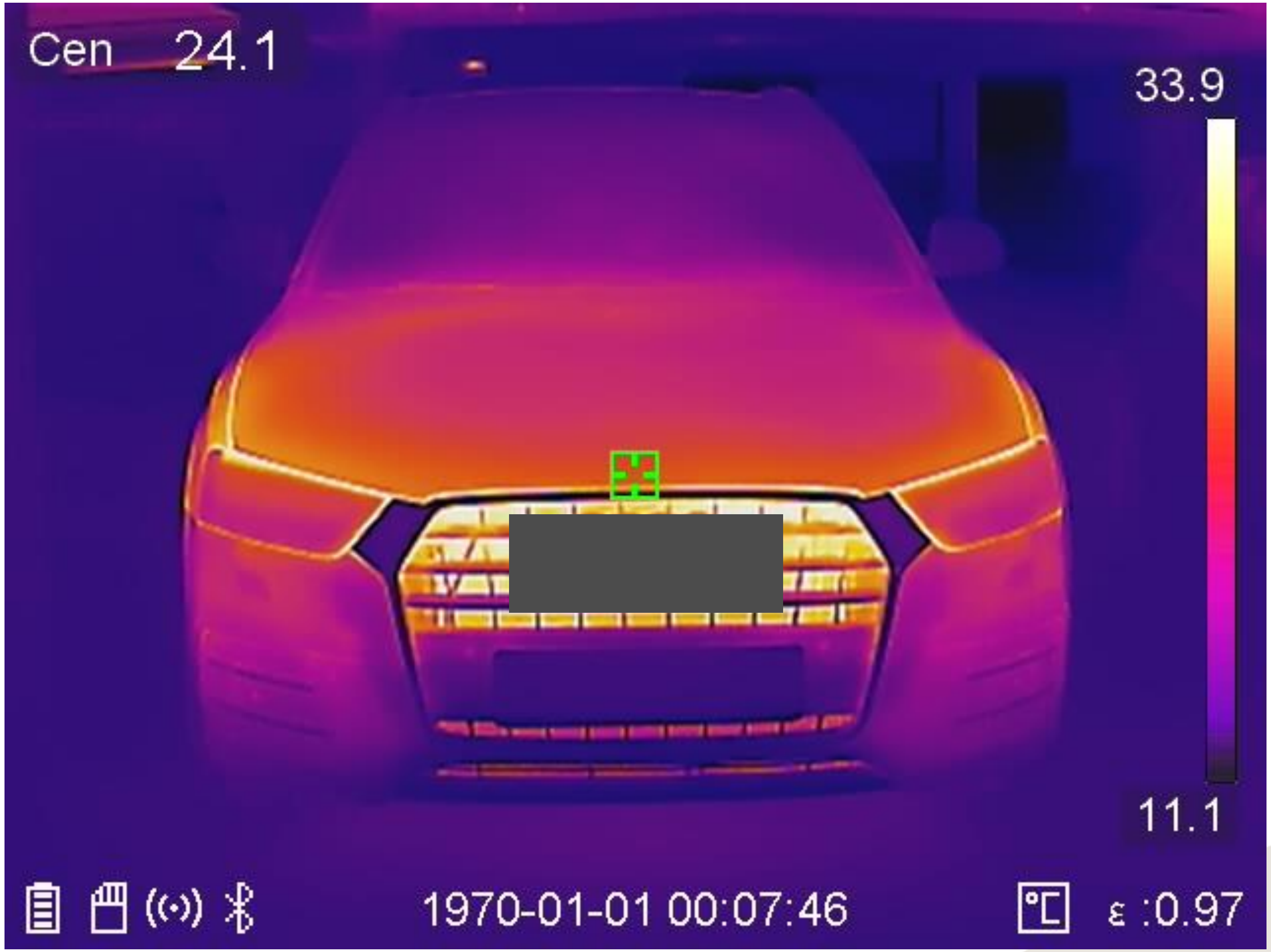
Leibniz Ferdinand Braun Institut

Fields

Fields describe states in space:

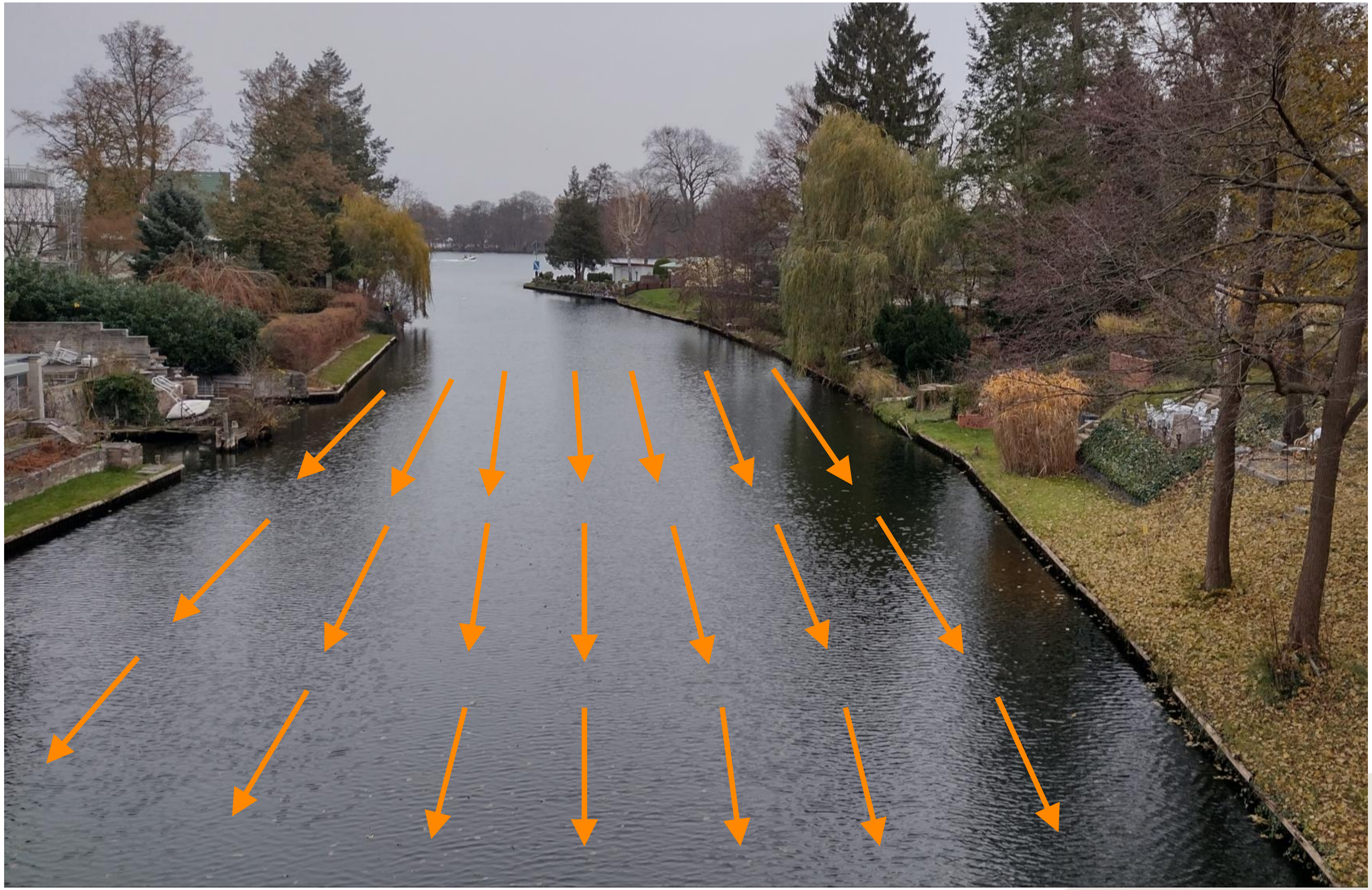
Example Scalar Field

Temperature distribution $T(\mathbf{r}, t)$ of car (infrared image)

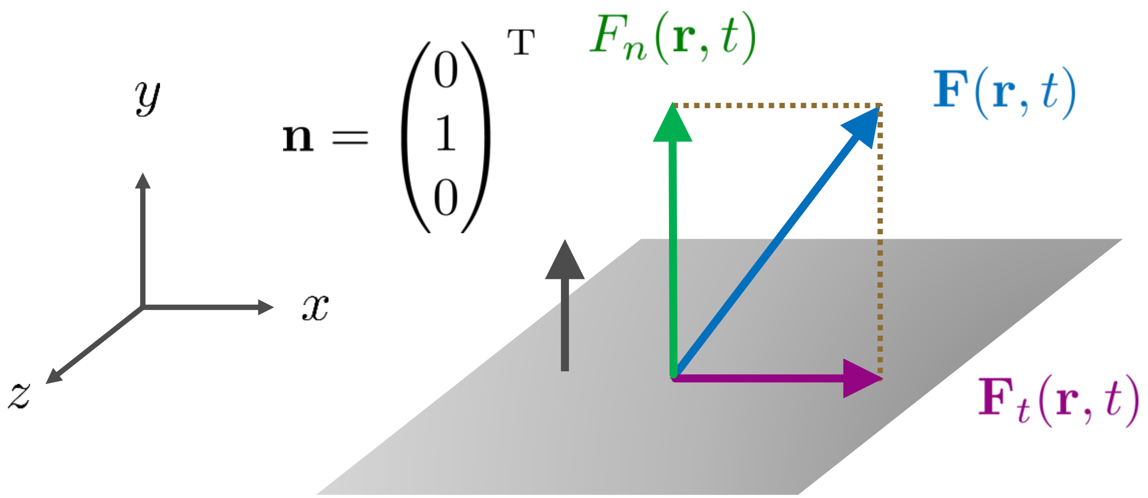


Example Vector Field

Velocity field $\mathbf{v}(\mathbf{r}, t)$ (sketch) of Müggelspree in autumn



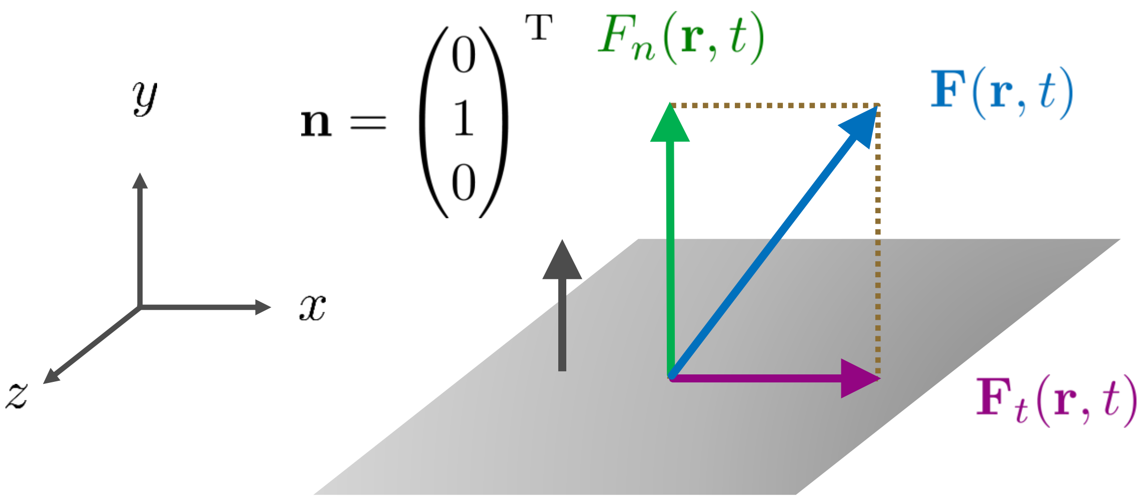
Decomposition of Fields on Plane into Normal and Tangential Components



$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}_n(\mathbf{r}, t) + \mathbf{F}_t(\mathbf{r}, t)$$

Fields on a plane can be split into:

Evaluation of Normal Component



$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{\mathrm{T}}$$

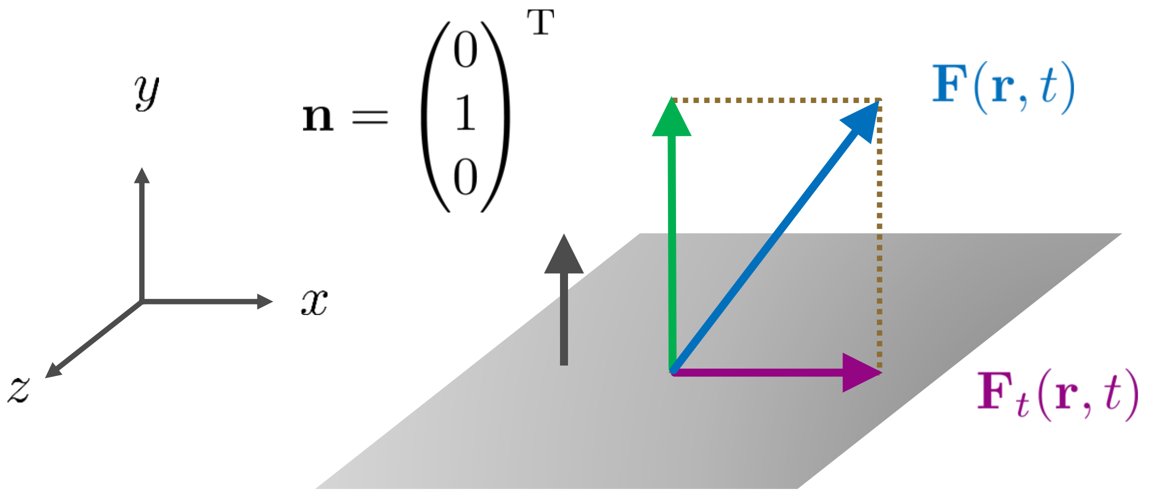
$$\mathbf{F}(\mathbf{r},t) = \mathbf{F}_n(\mathbf{r},t) + \mathbf{F}_t(\mathbf{r},t)$$

Normal component is obtained
by following dot product:

$$\mathbf{n} \cdot \mathbf{F}(\mathbf{r},t) = F_n(\mathbf{r},t)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} F_x(\mathbf{r},t) \\ F_y(\mathbf{r},t) \\ F_z(\mathbf{r},t) \end{pmatrix} = F_y(\mathbf{r},t) = F_n(\mathbf{r},t)$$

Evaluation of Tangential Component



Tangential component is obtained by following cross products:

$$-\mathbf{n} \times \mathbf{n} \times \mathbf{F}(\mathbf{r}, t) = \begin{pmatrix} F_x(\mathbf{r}, t) \\ 0 \\ F_z(\mathbf{r}, t) \end{pmatrix} = \mathbf{F}_t(\mathbf{r}, t)$$

$$= -\mathbf{n} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{\mathrm{T}} \times \begin{pmatrix} F_x(\mathbf{r}, t) \\ F_y(\mathbf{r}, t) \\ F_z(\mathbf{r}, t) \end{pmatrix} = -\mathbf{n} \times \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ F_x(\mathbf{r}, t) & F_y(\mathbf{r}, t) & F_z(\mathbf{r}, t) \end{vmatrix} = -\mathbf{n} \times \begin{pmatrix} F_z(\mathbf{r}, t) \\ 0 \\ -F_x(\mathbf{r}, t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}^{\mathrm{T}} \times \begin{pmatrix} F_z(\mathbf{r}, t) \\ 0 \\ -F_x(\mathbf{r}, t) \end{pmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & -1 & 0 \\ F_z(\mathbf{r}, t) & 0 & -F_x(\mathbf{r}, t) \end{vmatrix} = \begin{pmatrix} F_x(\mathbf{r}, t) \\ 0 \\ F_z(\mathbf{r}, t) \end{pmatrix}$$

Maxwell's Equations in Integral Representation

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V$$

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = 0$$

$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint_{\boldsymbol{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}$$

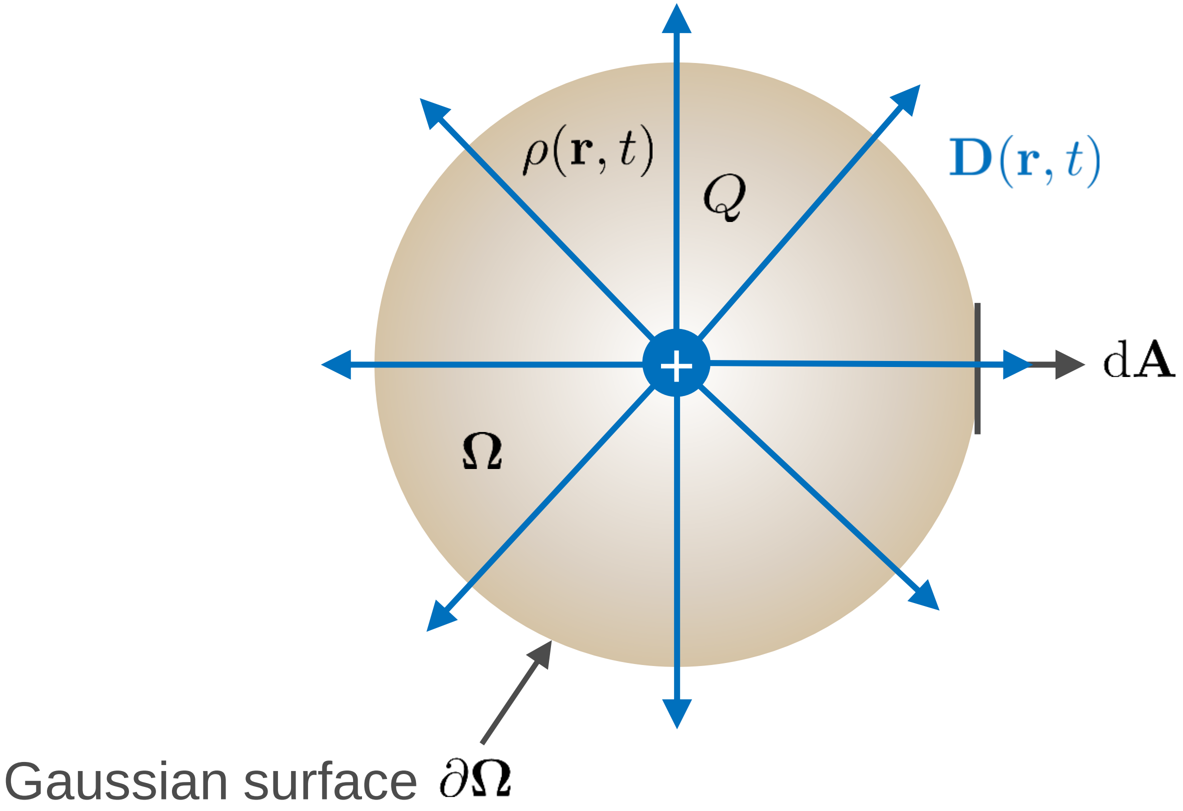
$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\boldsymbol{\Gamma}} \left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}$$

James Clerk Maxwell (1831-1879)

Figure: https://upload.wikimedia.org/wikipedia/commons/thumb/1/1e/James_Clerk_Maxwell_big.jpg/390px-James_Clerk_Maxwell_big.jpg

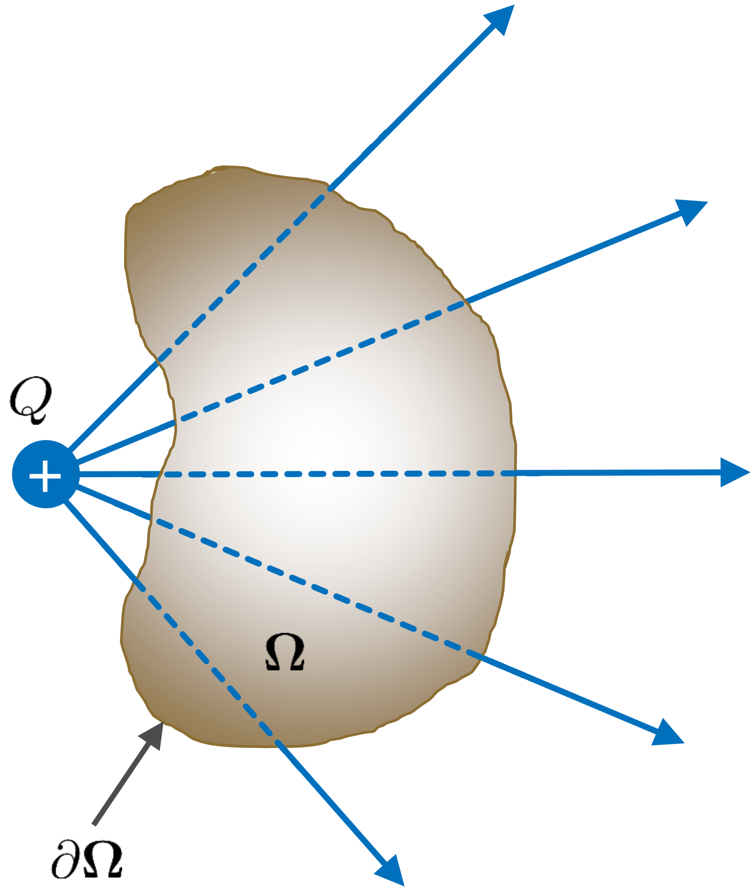
Gauss' Law (for Electricity) in Integral Form

Electric charges Q or electric charge densities $\rho(\mathbf{r},t)$ generate electric flux densities $\mathbf{D}(\mathbf{r},t)$!



$$\underbrace{\oiint_{\partial\mathbf{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = Q = \underbrace{\iiint_{\mathbf{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V}_{\text{total electric charge enclosed in Gaussian surface}}$$

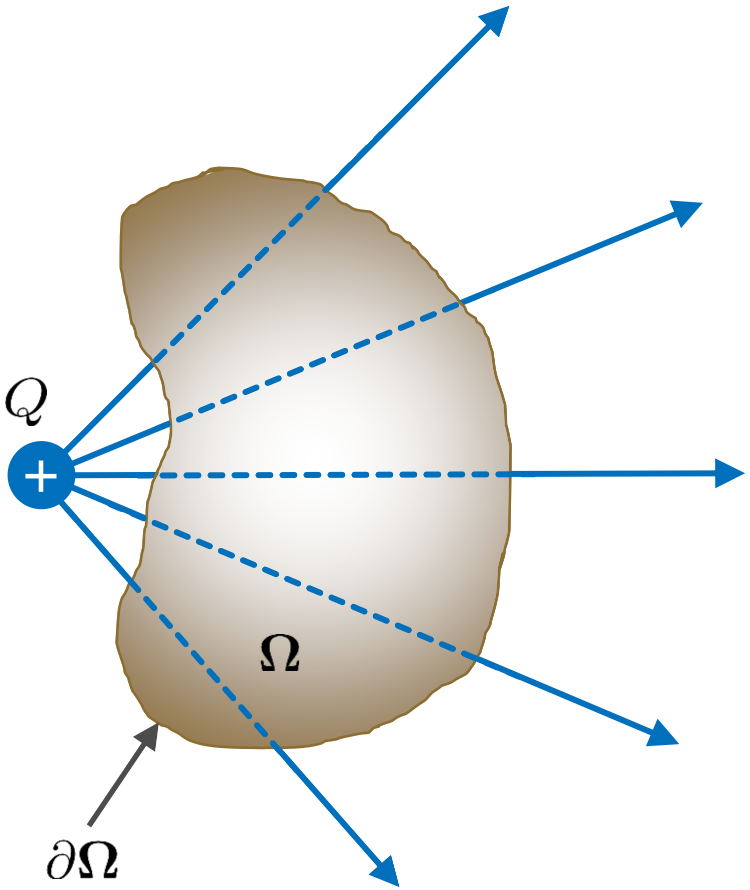
Quick Quiz – Value of Net Flux through Surface? (I / II)



$$\underbrace{\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = ???$$

total electric flux through Gaussian surface

Quick Quiz – Value of Net Flux through Surface? (II / II)

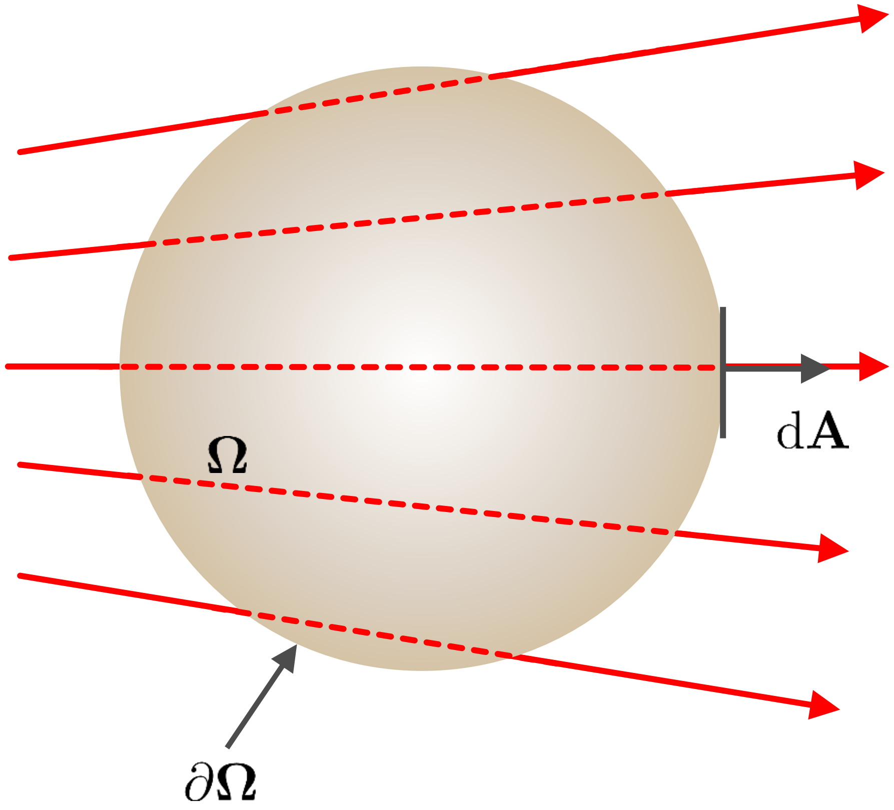


$$\underbrace{\oiint_{\partial\Omega} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = 0 = \iiint_{\Omega} \underbrace{\rho(\mathbf{r},t)}_{=0!}\, \mathrm{d}V$$

- Total electric flux through the Gaussian surface equals zero since no charges are contained in the volume!
- In other words: total amount of flux flowing into the Gaussian surface is equal to total amount of flux flowing out of the surface
- Absence of charges in the volume does not mean that the electric displacement fields are zero in the volume

Gauss' Law for Magnetism in Integral Form

Magnetic flux densities $\mathbf{B}(\mathbf{r},t)$ do not have sources, i.e. they are closed field lines!

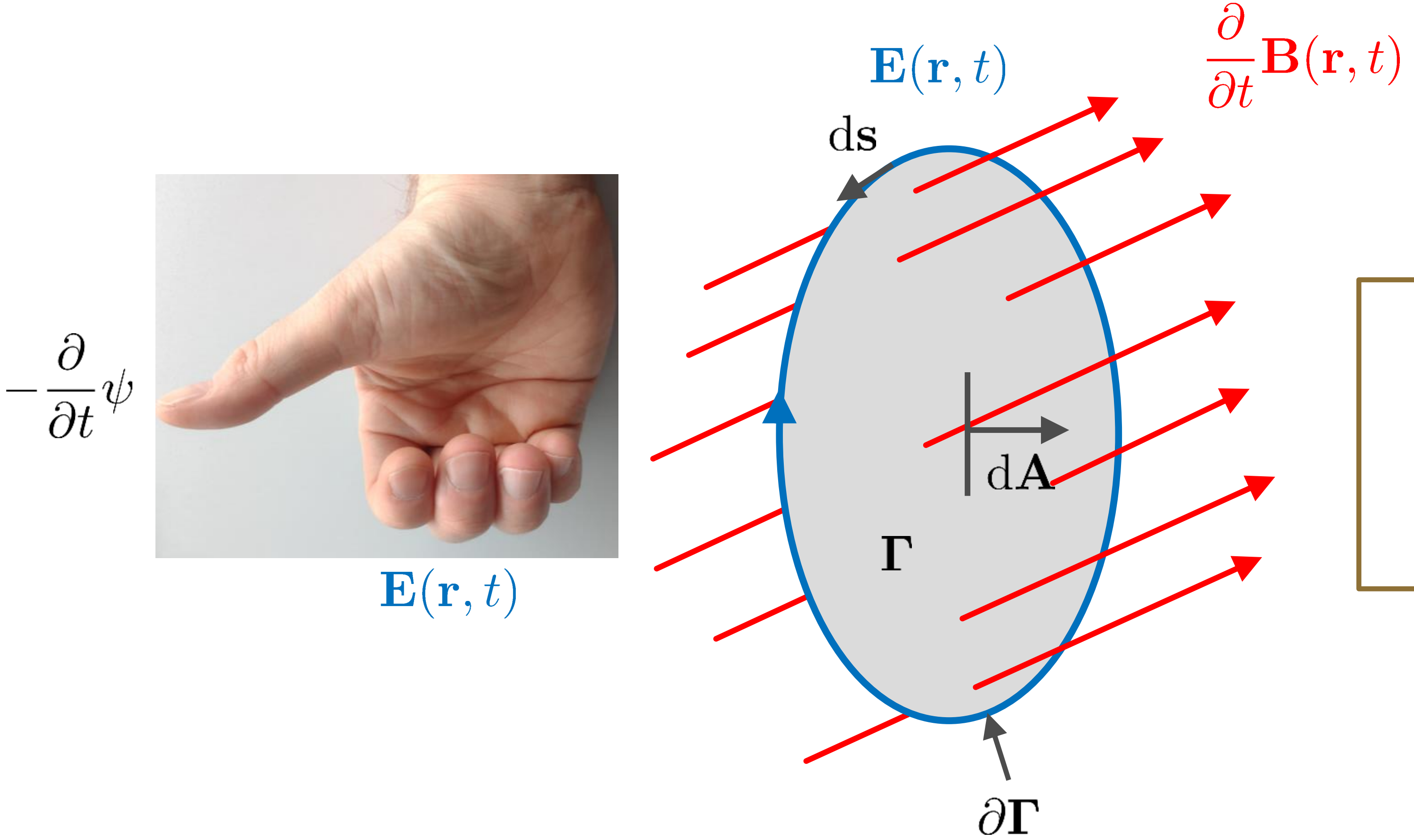


$$\underbrace{\oiint_{\partial\Omega} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total magnetic flux through Gaussian surface}} = 0$$

total magnetic flux through
Gaussian surface

Faraday's Law of Induction

Time-dependent magnetic flux densities $\mathbf{B}(\mathbf{r},t)$ generate curled electric field strength $\mathbf{E}(\mathbf{r},t)$!



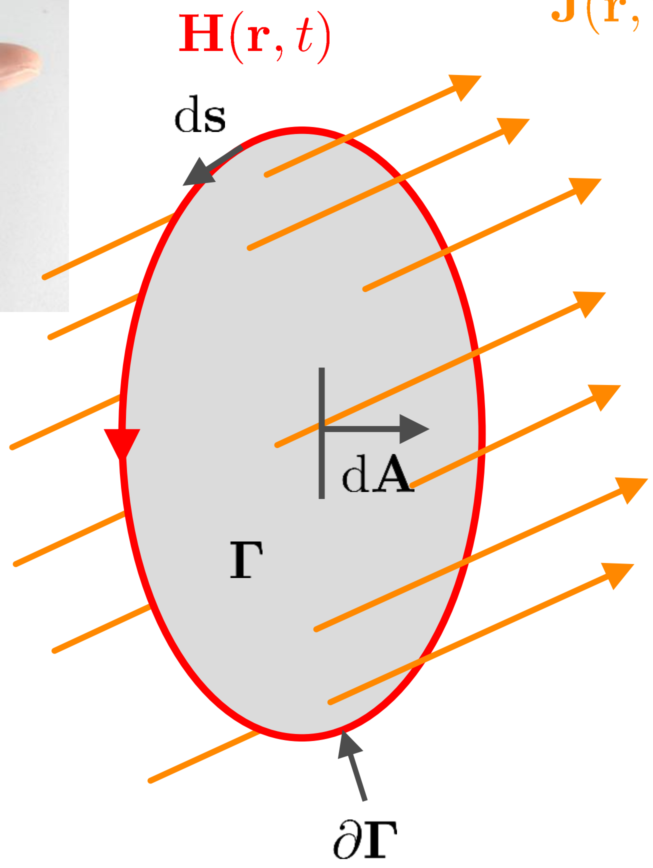
$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\underbrace{\iint_{\boldsymbol{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\frac{\partial}{\partial t}\psi}$$

Ampère's Law with Maxwell's Extension

Electric current densities $\mathbf{J}(\mathbf{r},t)$ and electric displacement currents densities $\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)$ generate curled magnetic field strengths $\mathbf{H}(\mathbf{r},t)$!

$\mathbf{H}(\mathbf{r},t)$

$\mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)$



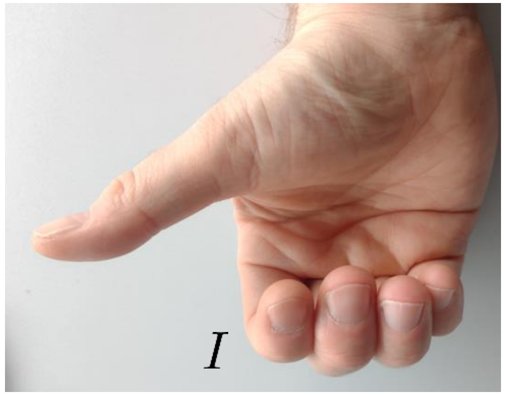
$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\boldsymbol{\Gamma}} \left(\mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}$$

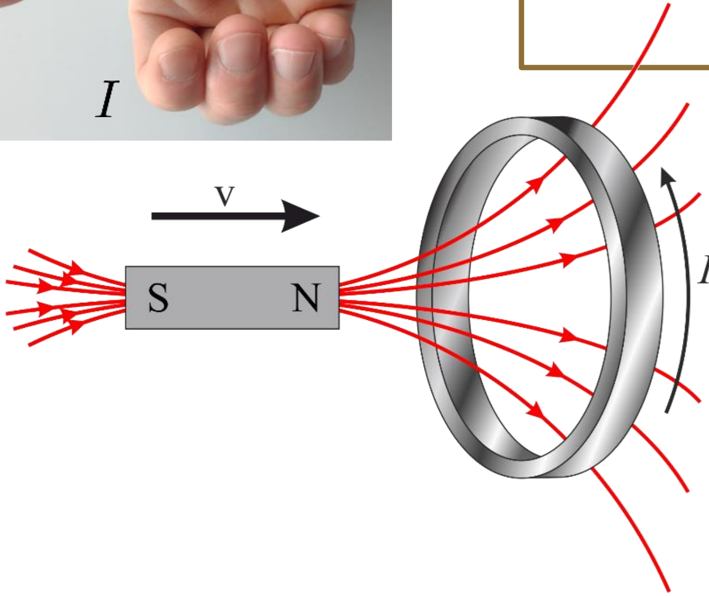
Faraday‘s Law of Induction – The Minus Sign - Lenz Law

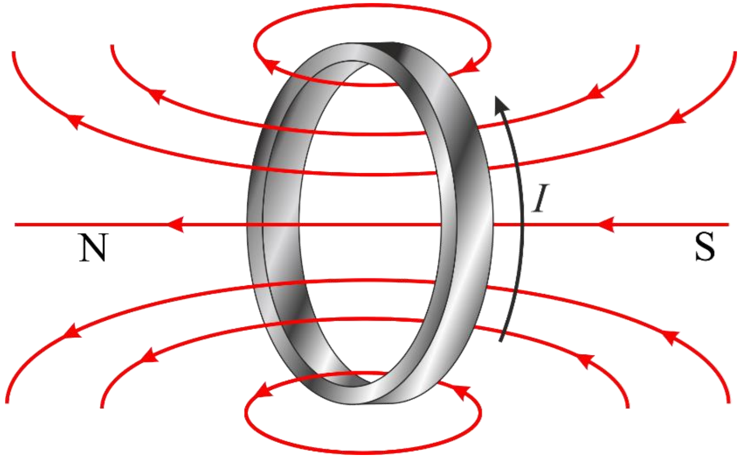
The direction of the induced electric field strength tends to produce a current that creates a magnetic flux to oppose the change in magnetic flux through the area enclosed by the current loop!

$$\oint_{\partial\Gamma} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\underbrace{\iint_{\Gamma} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\frac{\partial}{\partial t}\psi}$$

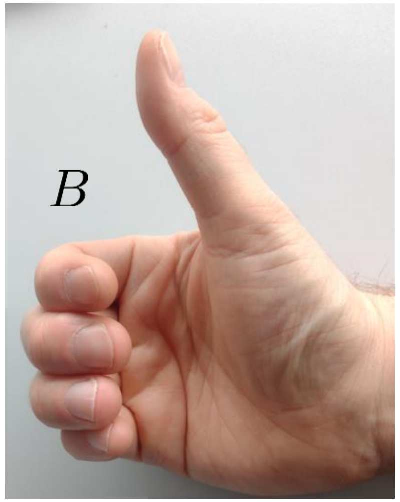
$-\frac{\partial}{\partial t}\psi$







I

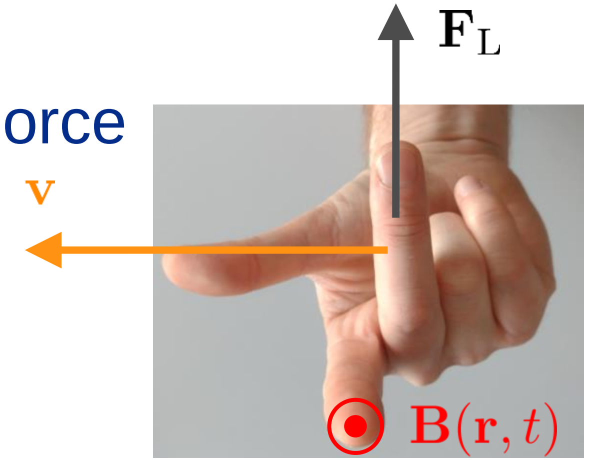


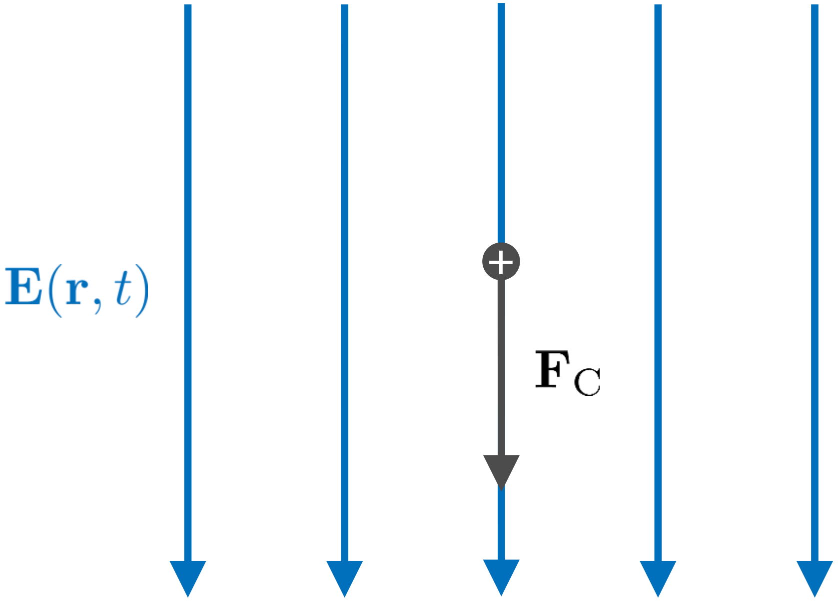
The minus sign in the induction law is also required for Maxwell‘s equations to be energy conserving!

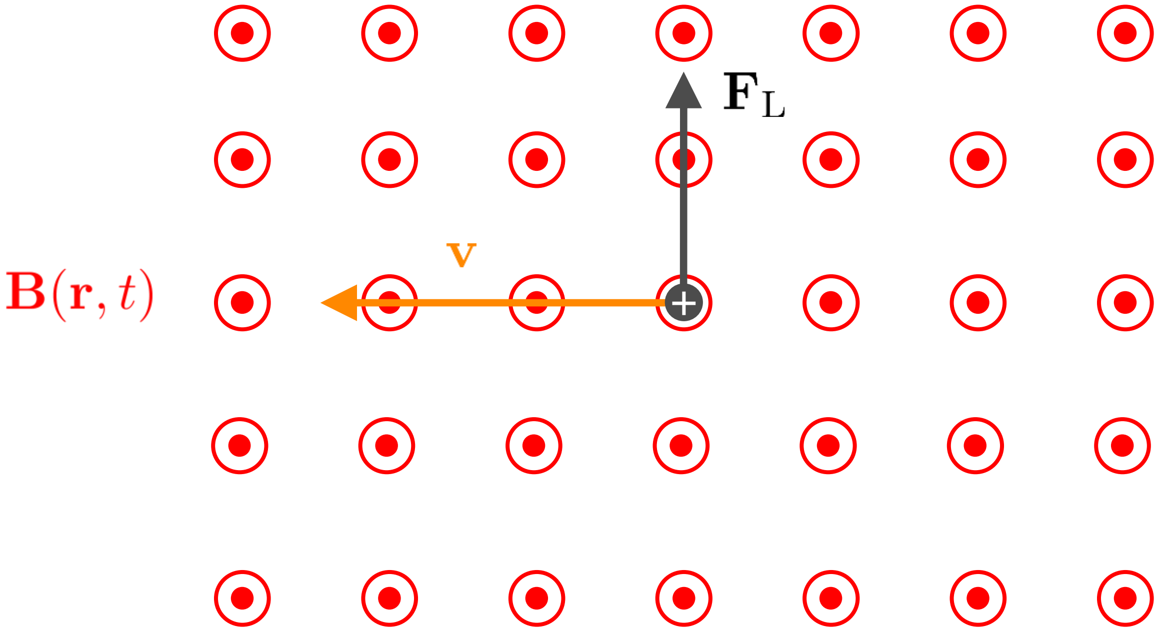
Figure adapted from D. Halliday, R. Resnick, J. Walker, Fundamentals of Physics, John Wiley & Sons Inc., 2014

Forces Acting on Charged Particles – Coulomb and Lorentz Force

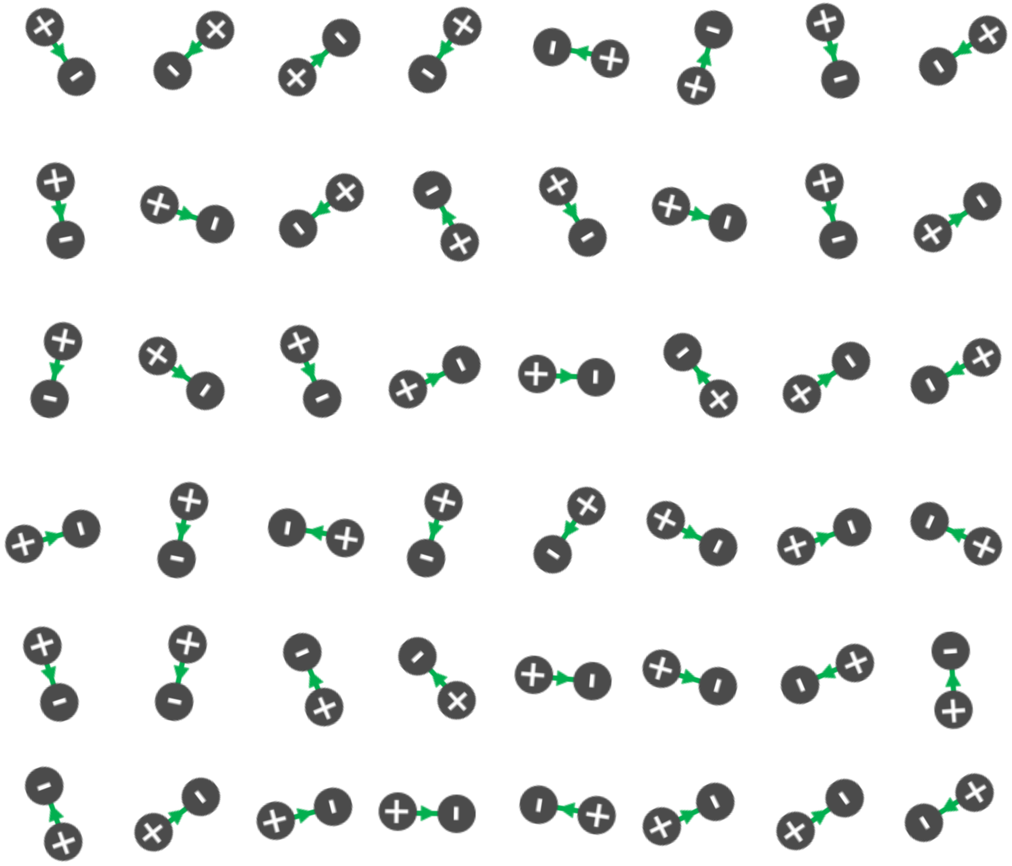
$$\mathbf{F} = q\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right] = \underbrace{q\,\mathbf{E}(\mathbf{r},t)}_{\mathbf{F}_\mathrm{C}} + \underbrace{q\,\mathbf{v}\times\mathbf{B}(\mathbf{r},t)}_{\mathbf{F}_\mathrm{L}}$$





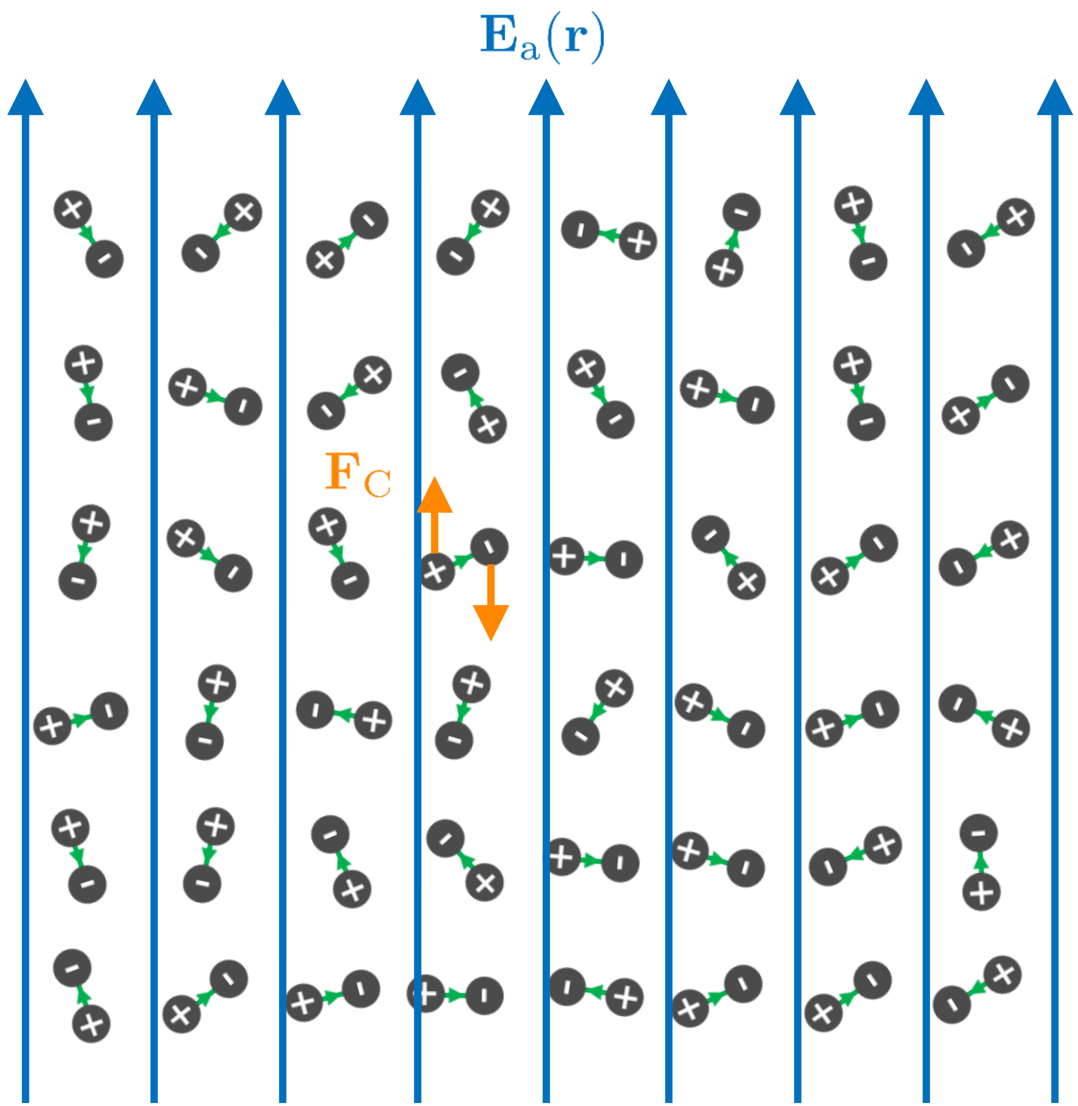


Electric Fields in Dielectric Materials



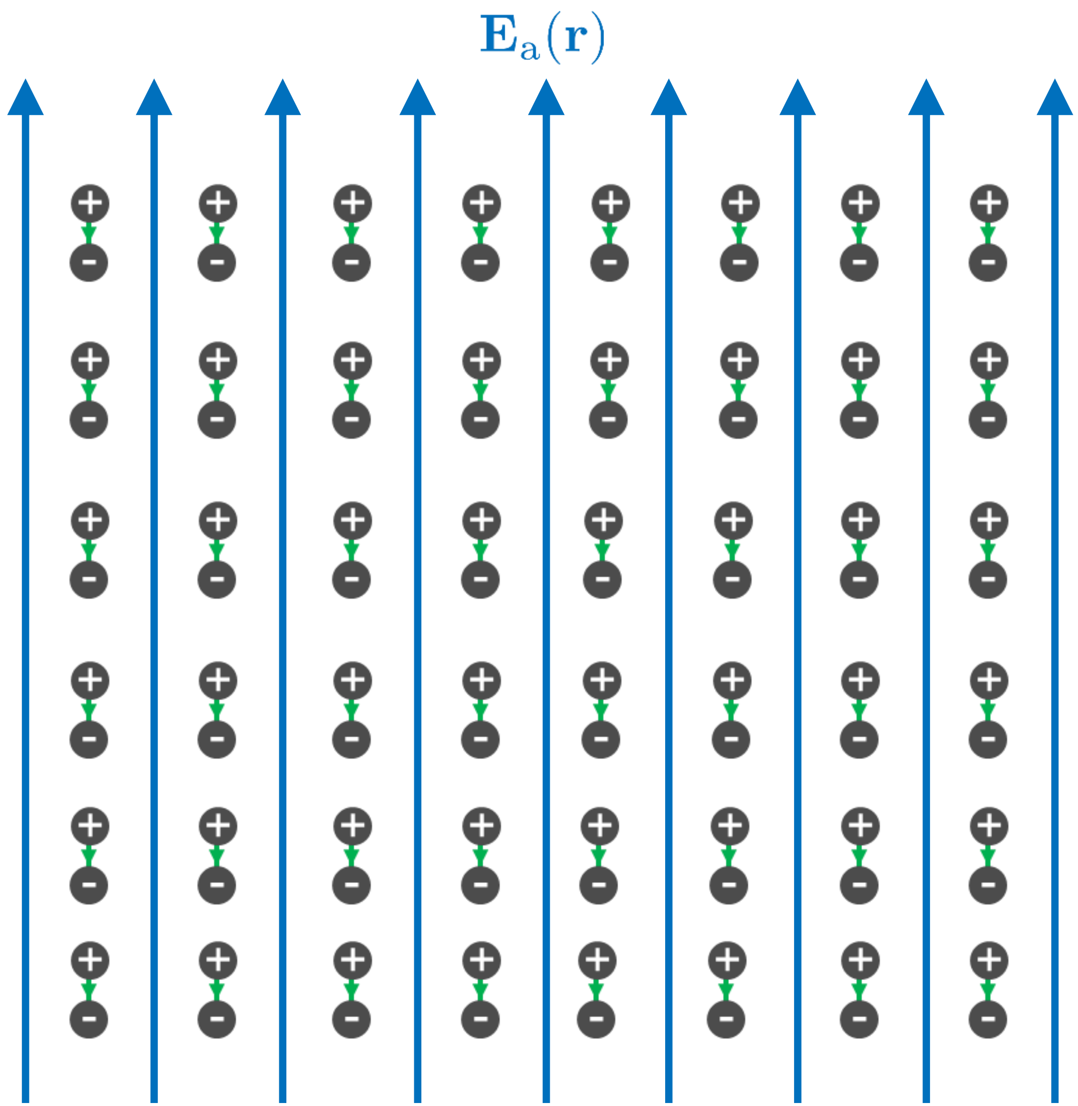
- No free charges in ideal dielectric materials, but bound charges only able to move at a small distances

Electric Fields in Dielectric Materials



- No free charges in ideal dielectric materials, but bound charges only able to move at a small distances
- Materials can be polarized by applied electric fields $\mathbf{E}_\mathrm{a}(\mathbf{r})$

Electric Fields in Dielectric Materials



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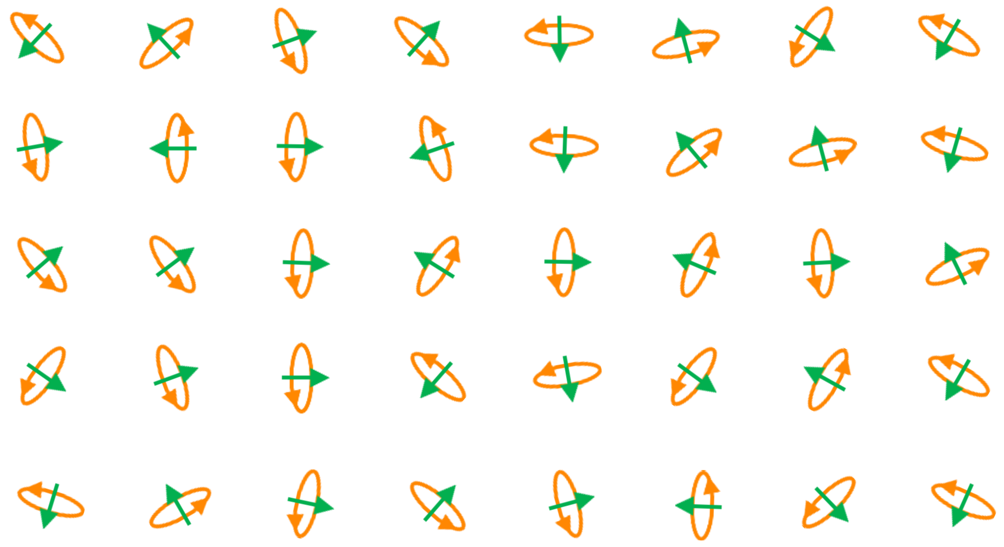
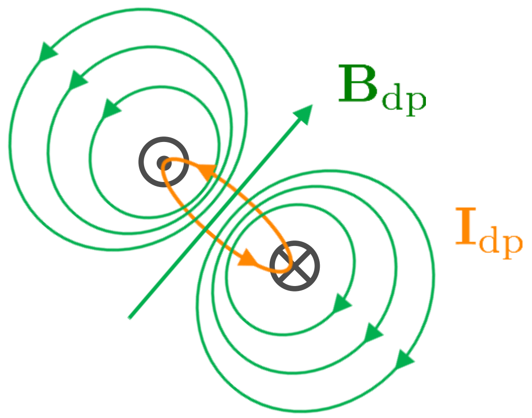
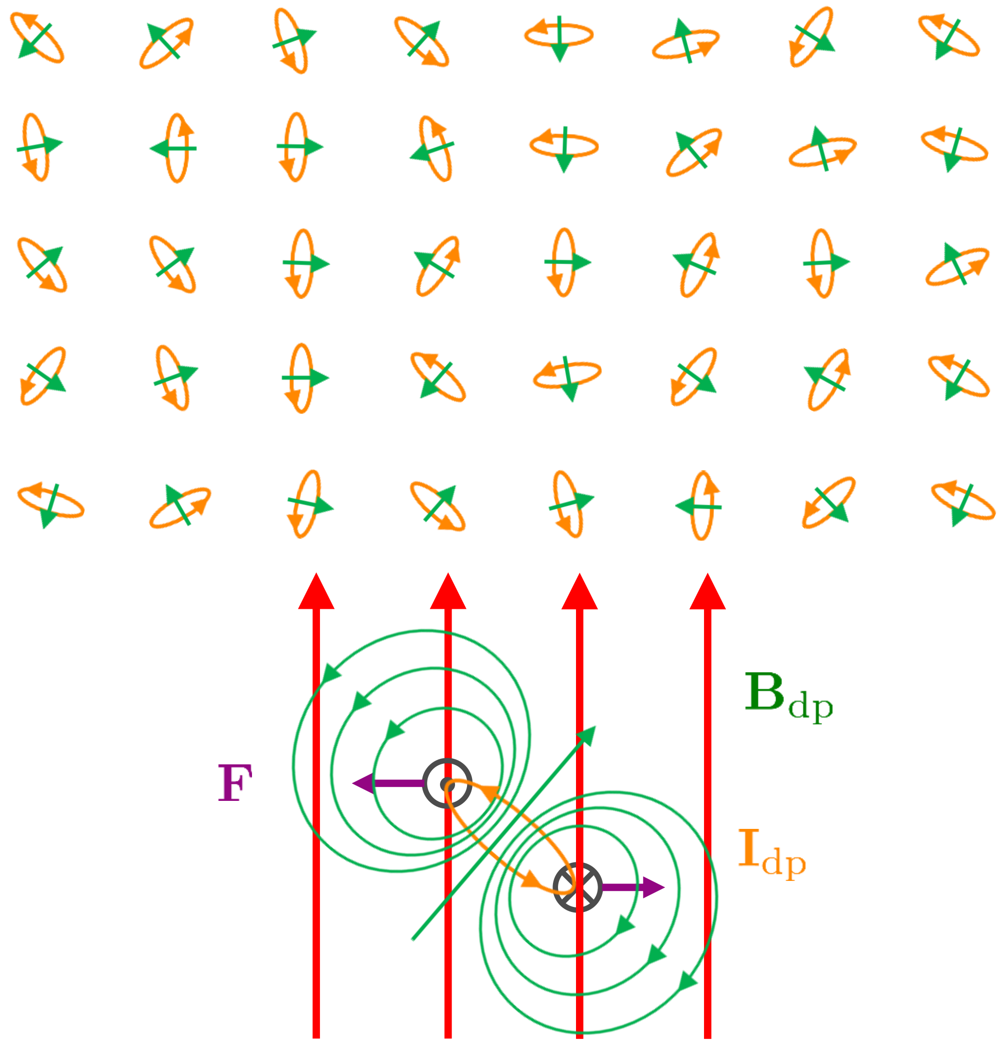
Magnetic Fields in Matter





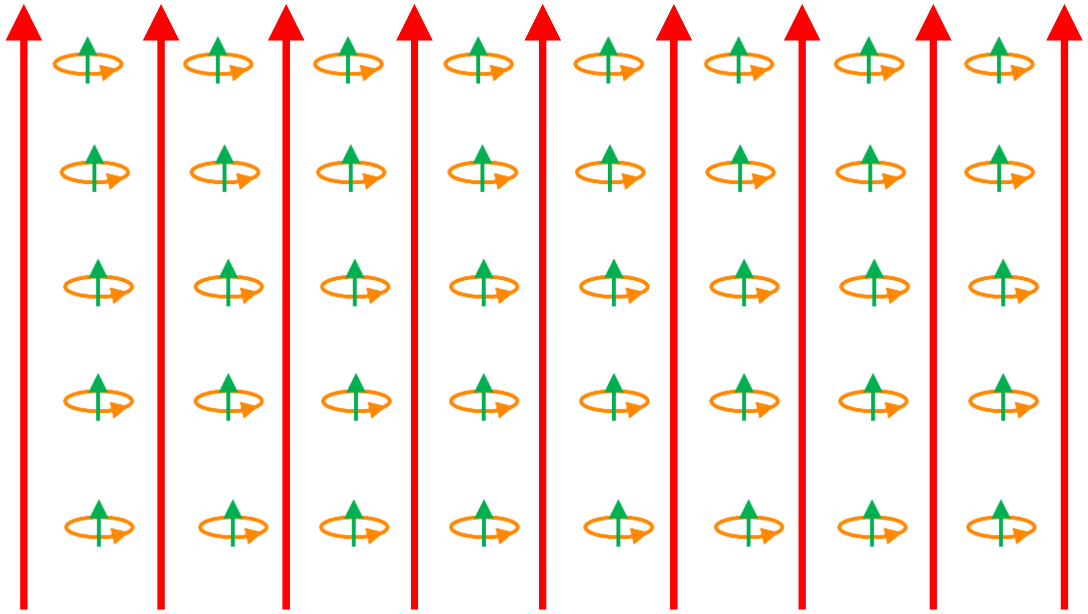
- Magnetic dipoles are present in matter e.g. due to rotation of electrons around the nucleus

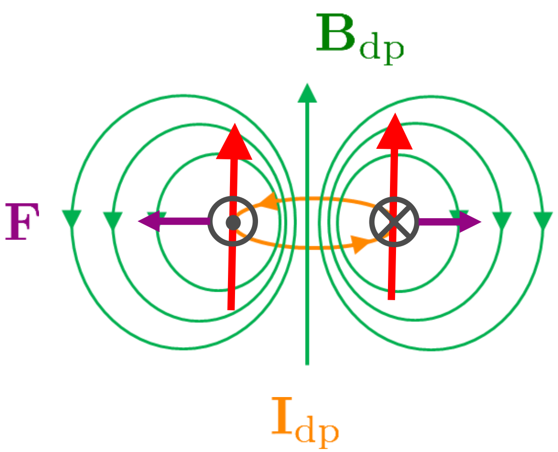
Magnetic Fields in Matter



- Magnetic dipoles are present in matter e.g. due to rotation of electrons around the nucleus
- Materials can be magnetized by applied $\mathbf{B}_{\mathrm{a}}(\mathbf{r})$

Magnetic Fields in Matter

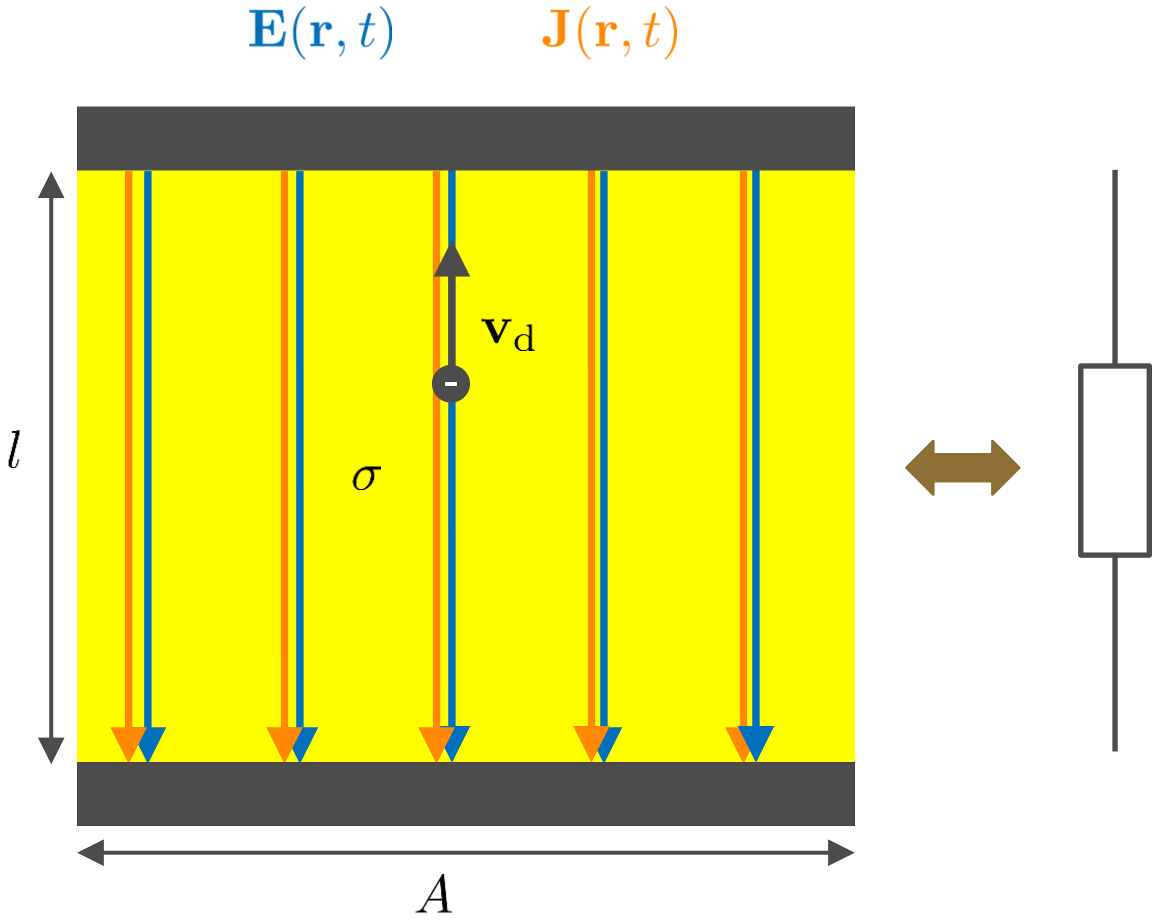




- Magnetic dipoles are present in matter e.g. due to rotation of electrons around the nucleus

- Materials can be magnetized by applied $\mathbf{B}_{\mathrm{a}}(\mathbf{r})$

Conducting Materials in Electric Fields - Ohm's Law



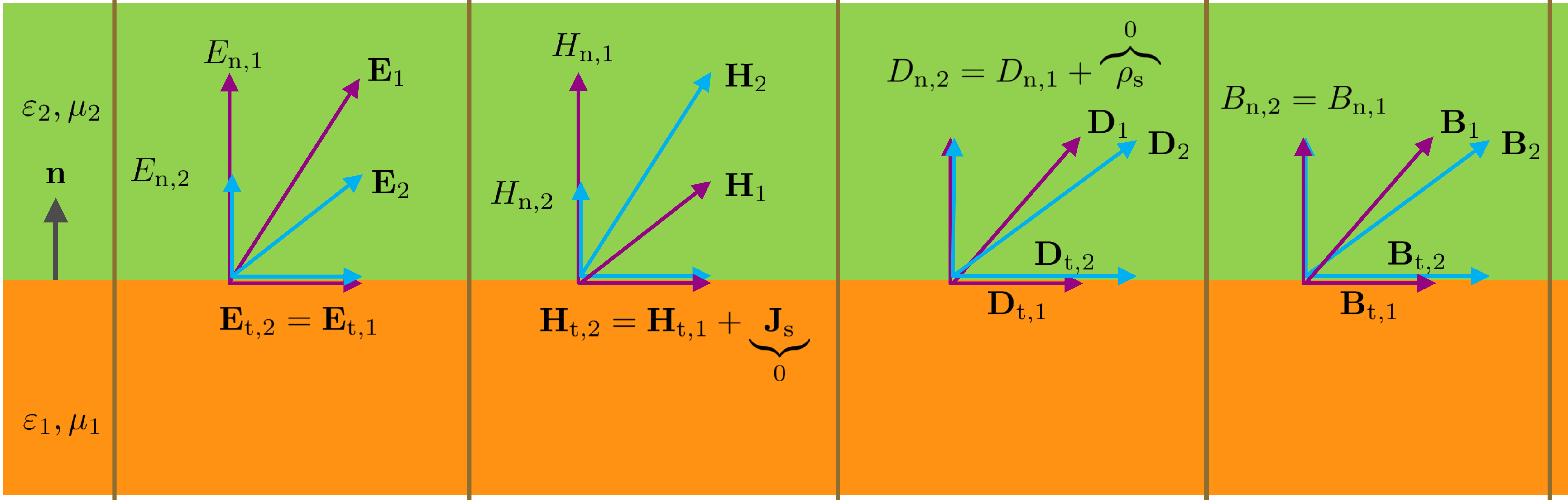
$$R = \frac{l}{\sigma A}$$

- Conducting materials in electric fields result in electric currents

$$\mathbf{J}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}_\mathrm{d}(\mathbf{r},t),$$

Continuity Constraints on Interface between two Materials

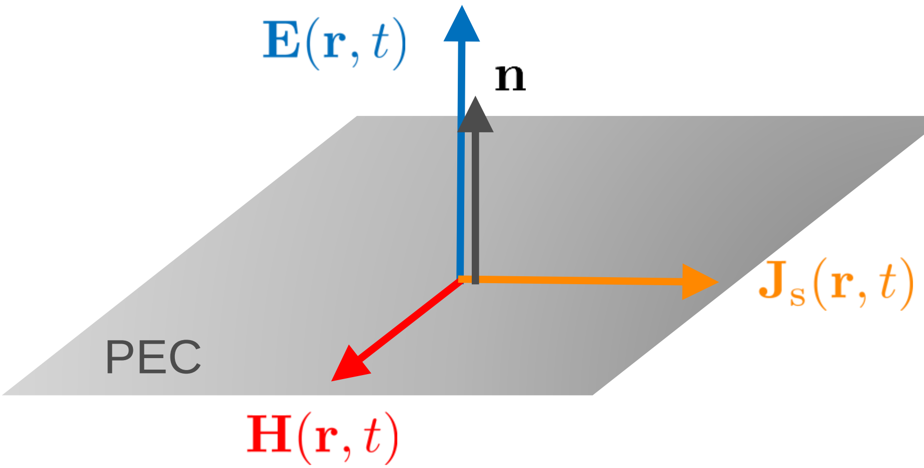
$\mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_1] = \mathbf{0}$	$\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_1] = \mathbf{J}_\mathrm{s}$	$\mathbf{n} \cdot [\mathbf{D}_2 - \mathbf{D}_1] = \rho_\mathrm{s}$	$\mathbf{n} \cdot [\mathbf{B}_2 - \mathbf{B}_1] = 0$



Fields on a Perfect Electric Conductor (PEC)

- Accurate approximation for metal surfaces with high conductivity, requiring magnetic fields to be tangential and electric fields to be normal

- Common boundary condition in calculations (equivalent to short circuit)



$\mathbf{n} \cdot \mathbf{E}(\mathbf{r}, t) \neq 0$ normal component of electric field unequal to zero

$\mathbf{n} \times \mathbf{E}(\mathbf{r}, t) = \mathbf{0}$ tangential electric field is zero

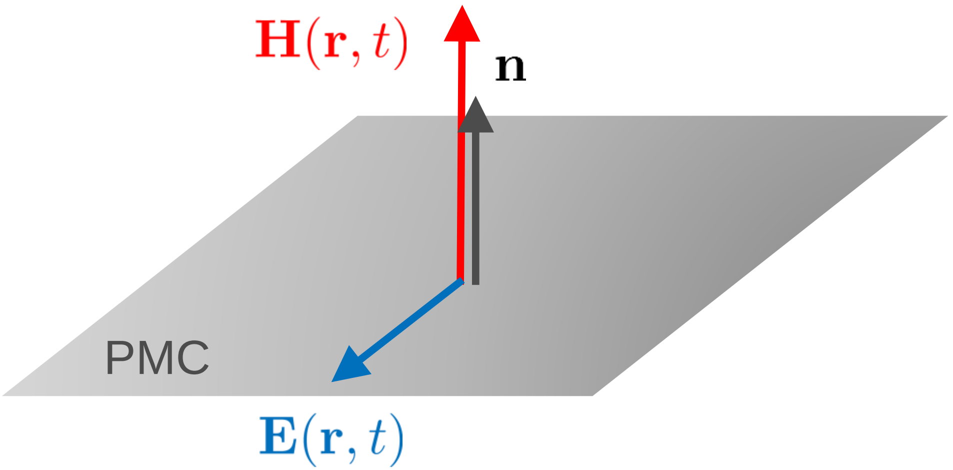
$\mathbf{n} \cdot \mathbf{H}(\mathbf{r}, t) = 0$ normal component of magnetic field is equal to zero

$\mathbf{n} \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}_\mathrm{s}(\mathbf{r}, t)$ tangential component of magnetic field is unequal to zero, i.e. equal to surface current density

Fields on a Perfect Magnetic Conductor (PMC)

- Hypothetical material requiring magnetic fields to be normal and electric fields to be tangential (magnetic analogue of PEC)

- Common boundary condition in calculations (equivalent to open circuit)



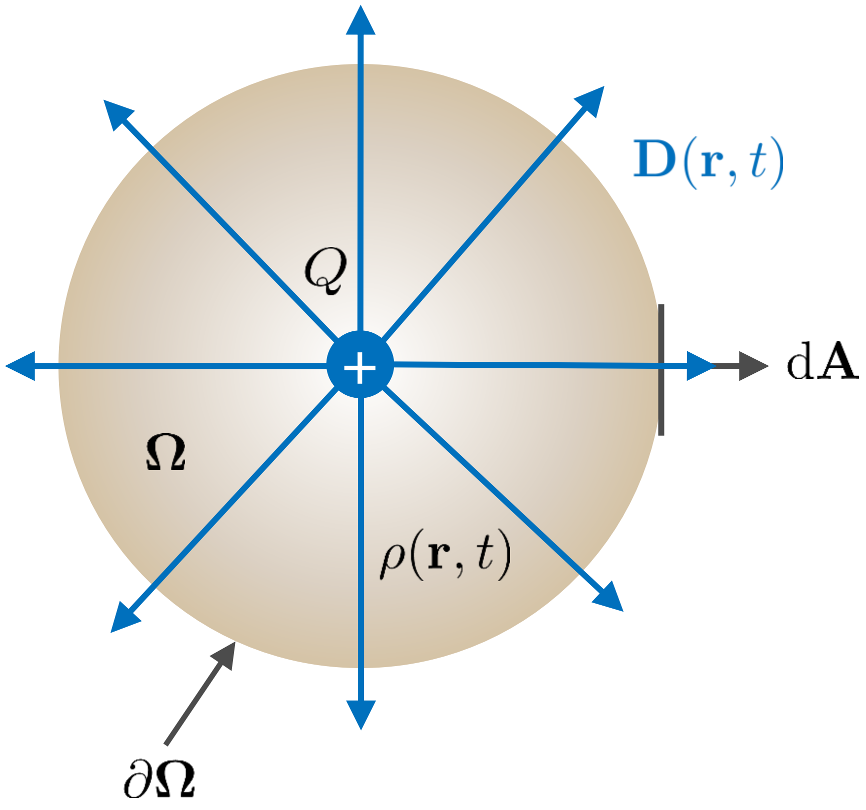
$\mathbf{n} \cdot \mathbf{E}(\mathbf{r}, t) = 0$ normal component of electric field equals zero

$\mathbf{n} \times \mathbf{E}(\mathbf{r}, t) \neq \mathbf{0}$ tangential electric field unequal to zero

$\mathbf{n} \cdot \mathbf{H}(\mathbf{r}, t) \neq 0$ normal component of magnetic field is unequal to zero

$\mathbf{n} \times \mathbf{H}(\mathbf{r}, t) = \mathbf{0}$ tangential component of magnetic field equals zero

Gauss' Law (for Electricity) in Integral Form for (infinitely) small Volumes



$$\oint\!\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V \qquad \Big|\cdot\frac{1}{V}$$

$$\frac{1}{V}\oint\!\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \frac{1}{V}\iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V \qquad \Big|\lim_{V\to 0}$$

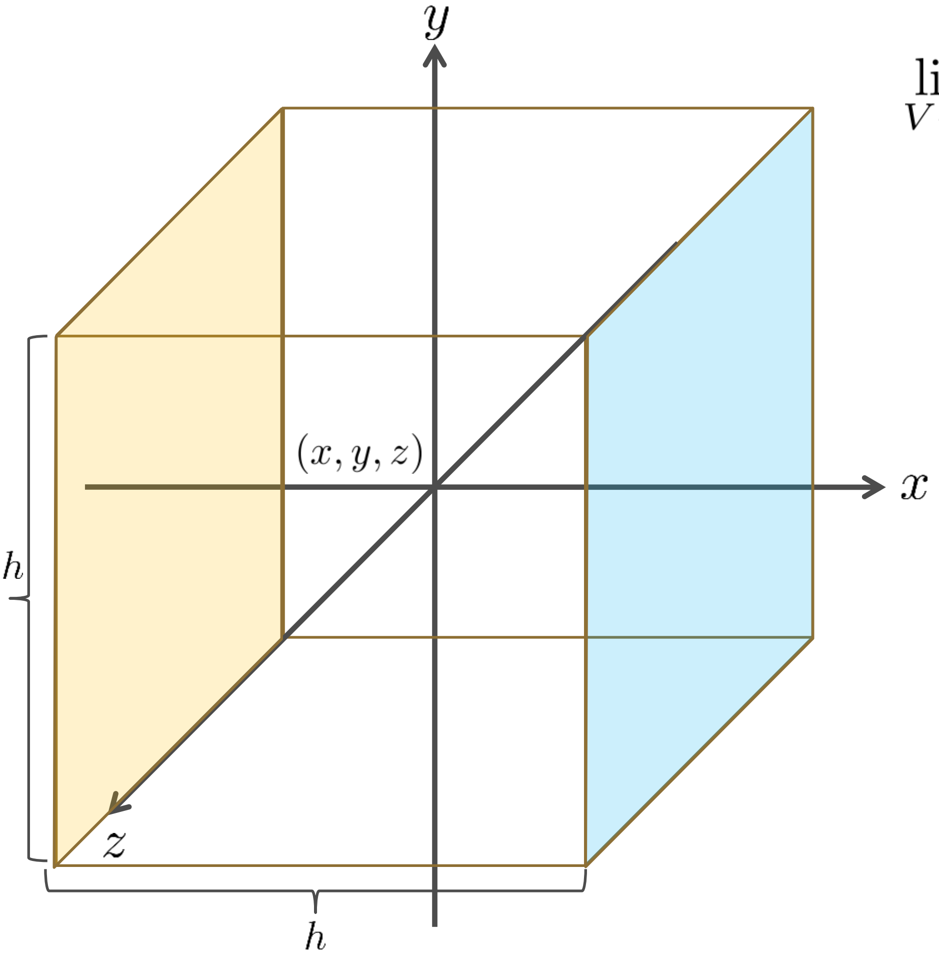
$$\underbrace{\lim_{V\to 0}\frac{1}{V}\oint\!\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\nabla\cdot\mathbf{D}(\mathbf{r},t)=\mathrm{div}\,\mathbf{D}(\mathbf{r},t)} = \underbrace{\lim_{V\to 0}\frac{1}{V}\iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V}_{\rho(\mathbf{r},t)}$$

normalized electric flux through infinitely small Gaussian surface

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

Definition of Divergence in a Cartesian System - Integral Decomposition

$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{V\to 0} \frac{1}{V} \oiint_{\partial\mathbf{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

Definition of Divergence in a Cartesian System - Integral Decomposition

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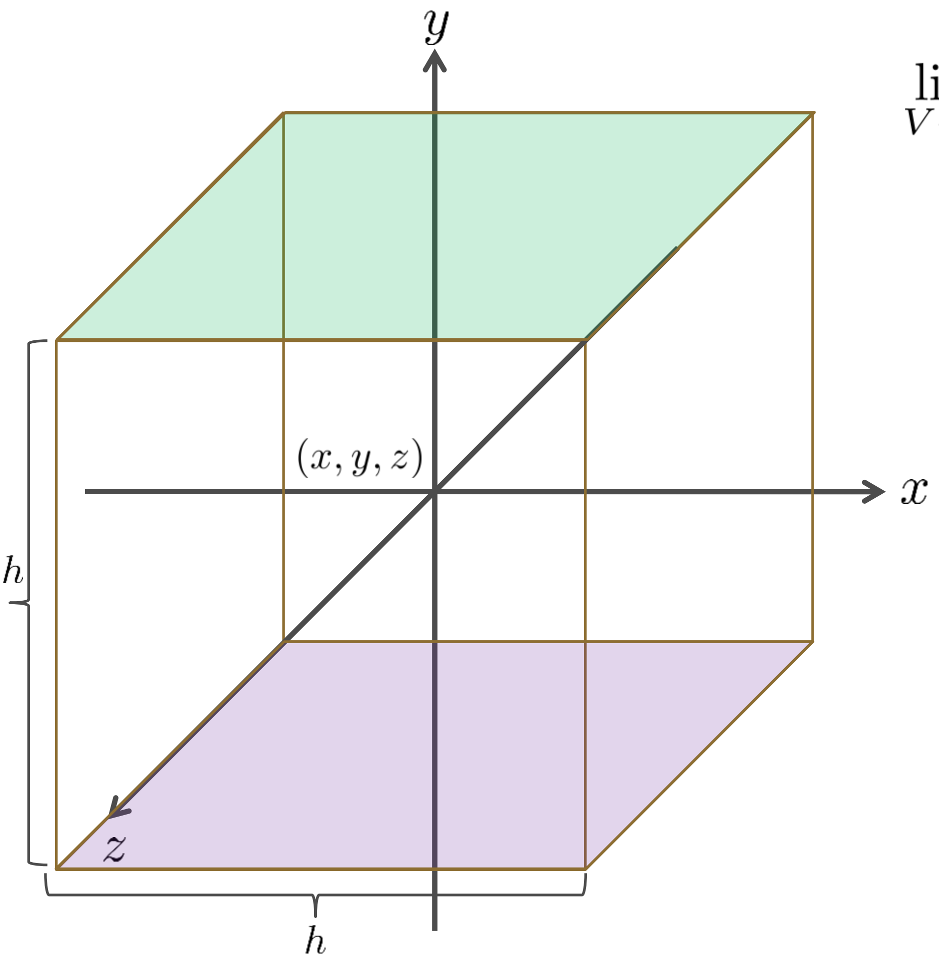


$$\lim_{V \to 0} \frac{1}{V} \oiint_{\partial \mathbf{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

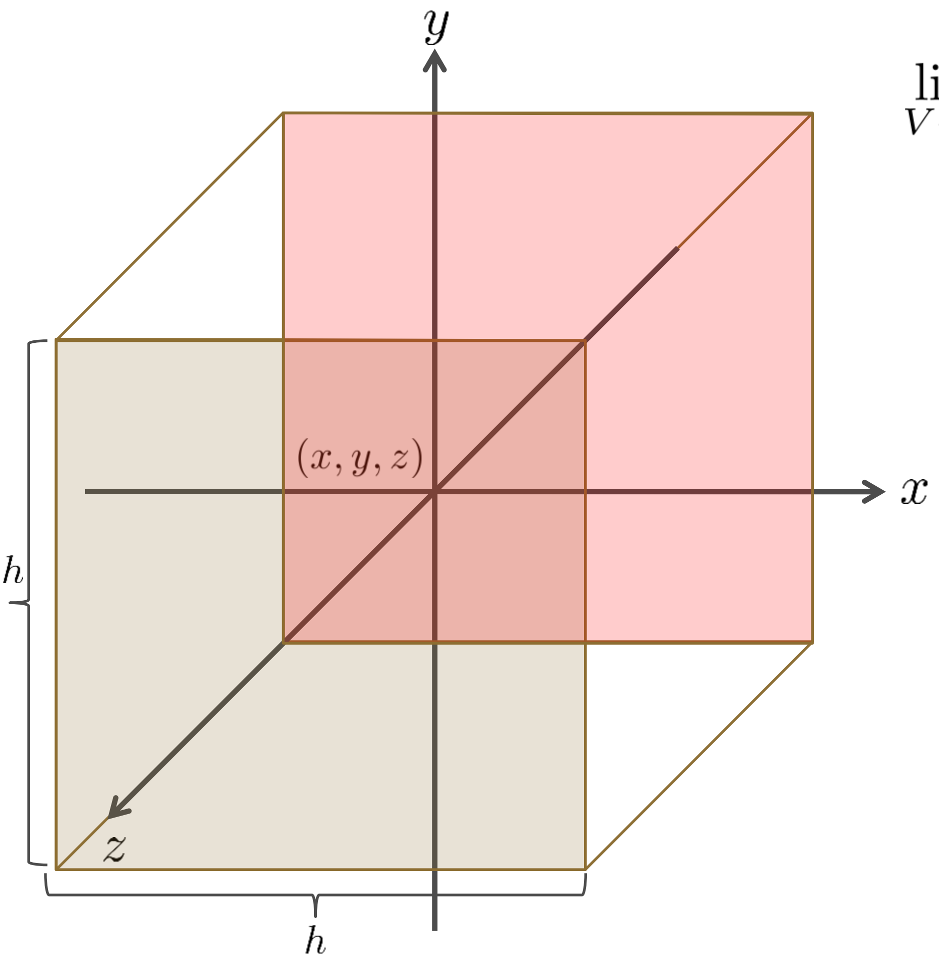
$$= \lim_{h \to 0} \frac{1}{h^3} \Bigg\{ \iint_{A_x^+} \mathbf{D}\left(x + \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_x^-} \mathbf{D}\left(x - \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A}$$

$$+ \iint_{A_y^+} \mathbf{D}\left(x, y + \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_y^-} \mathbf{D}\left(x, y - \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A}$$

$$+ \iint_{A_z^+} \mathbf{D}\left(x, y, z + \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_z^-} \mathbf{D}\left(x, y, z - \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} \Bigg\}$$

Definition of Divergence in a Cartesian System - Integral Decomposition

$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{V \to 0} \frac{1}{V} \oiint_{\partial\mathbf{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

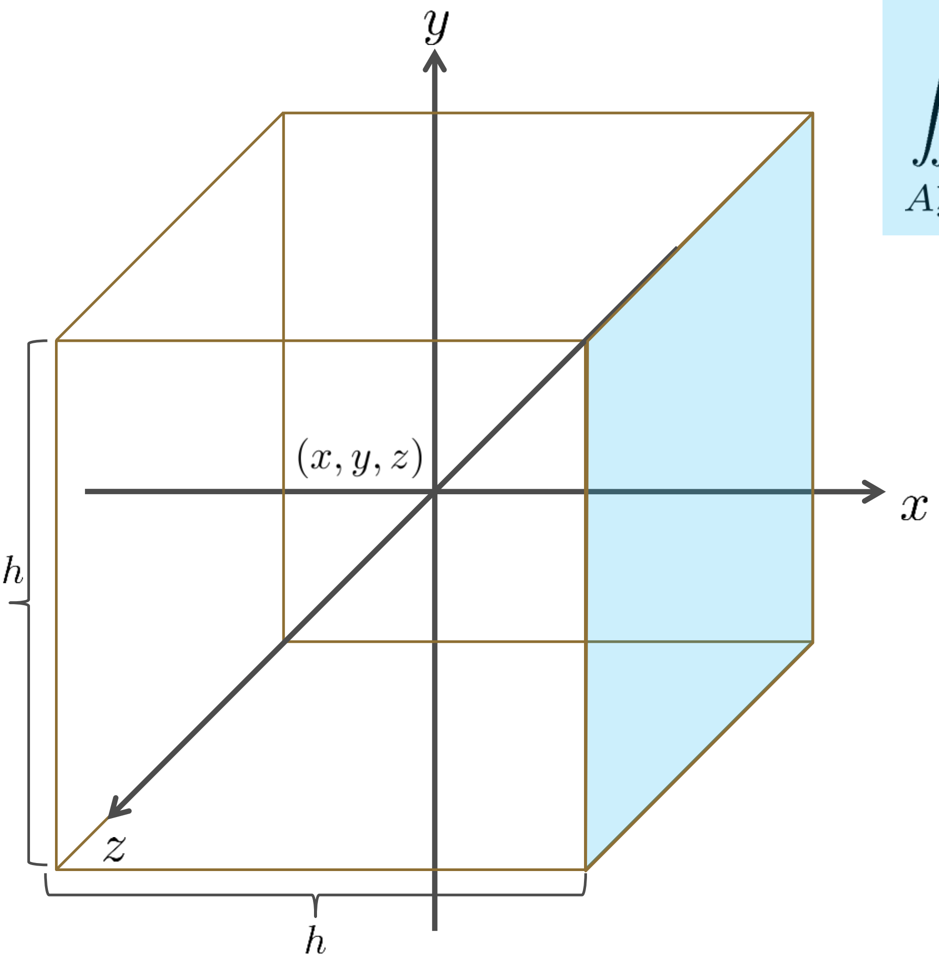
$$= \lim_{h \to 0} \frac{1}{h^3} \Bigg\{ \iint_{A_x^+} \mathbf{D}\left(x + \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_x^-} \mathbf{D}\left(x - \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A}$$

$$+ \iint_{A_y^+} \mathbf{D}\left(x, y + \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_y^-} \mathbf{D}\left(x, y - \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A}$$

$$+ \iint_{A_z^+} \mathbf{D}\left(x, y, z + \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_z^-} \mathbf{D}\left(x, y, z - \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} \Bigg\}$$

Integral Evaluation using Midpoint Rule (I / II)

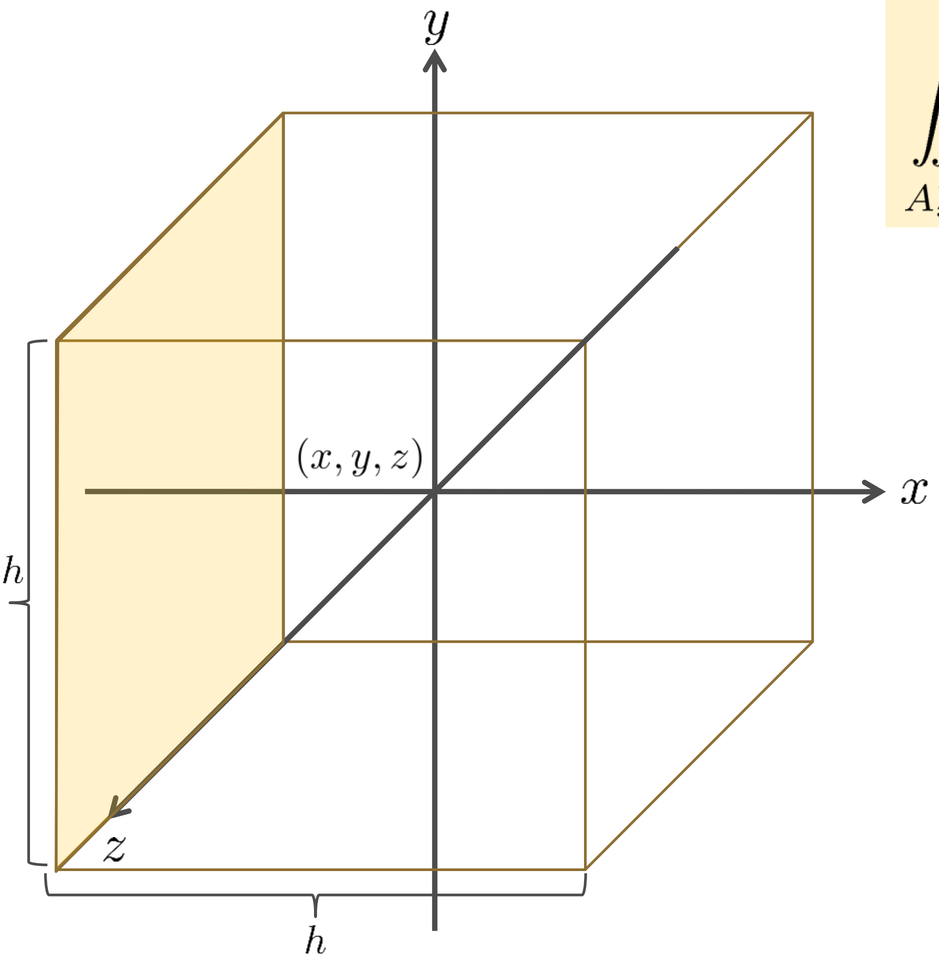
$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\iint\limits_{A_x^+} \mathbf{D}\left(x + \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A} =$$

Integral Evaluation using Midpoint Rule (II / II)

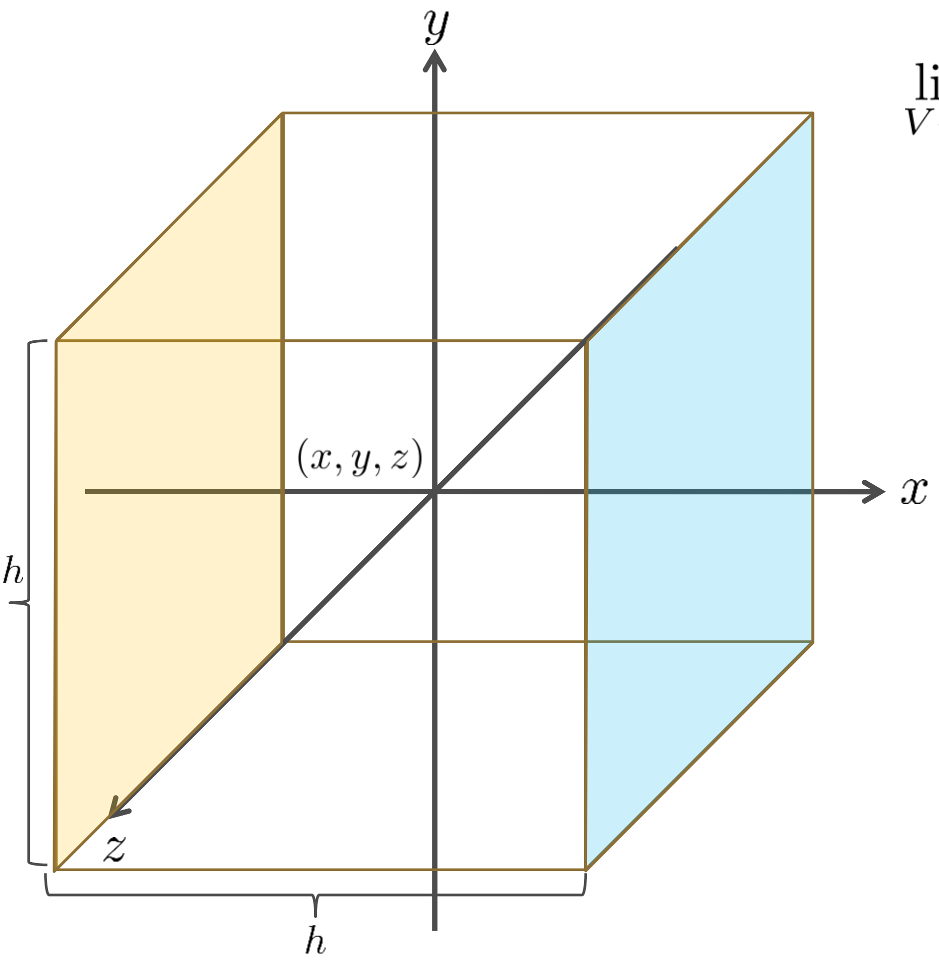
$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\iint_{A_x^-} \mathbf{D}\left(x - \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A} =$$

Definition of Divergence in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{V \to 0} \frac{1}{V} \oiint_{\partial \boldsymbol{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

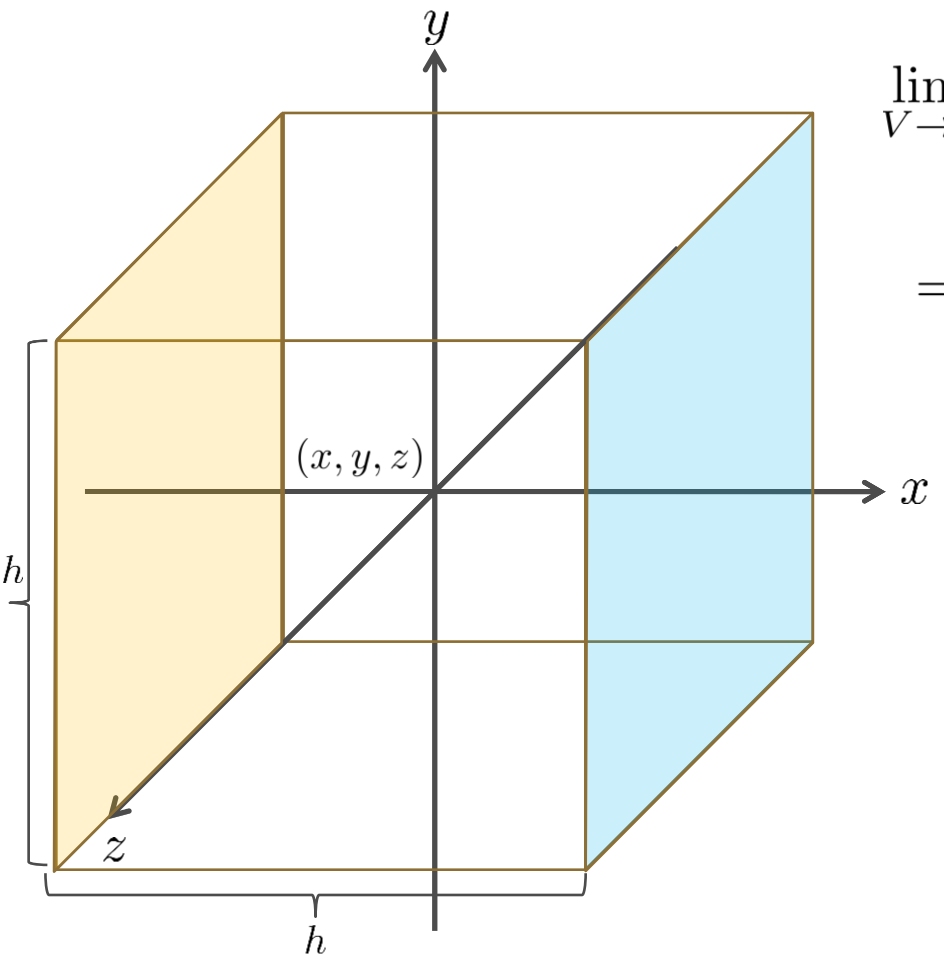
$$= \lim_{h \to 0} \frac{1}{h^3} \Bigg\{ \underbrace{\iint_{A_x^+} \mathbf{D}\left(x + \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A}}_{D_x\left(x+\frac{h}{2},y,z,t\right)h^2+\mathcal{O}(h^4)} + \underbrace{\iint_{A_x^-} \mathbf{D}\left(x - \frac{h}{2}, y, z, t\right) \cdot \mathrm{d}\mathbf{A}}_{-D_x\left(x-\frac{h}{2},y,z,t\right)h^2+\mathcal{O}(h^4)}$$

$$+ \iint_{A_y^+} \mathbf{D}\left(x, y + \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_y^-} \mathbf{D}\left(x, y - \frac{h}{2}, z, t\right) \cdot \mathrm{d}\mathbf{A}$$

$$+ \iint_{A_z^+} \mathbf{D}\left(x, y, z + \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} + \iint_{A_z^-} \mathbf{D}\left(x, y, z - \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{A} \Bigg\}$$

Definition of Divergence Operator in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



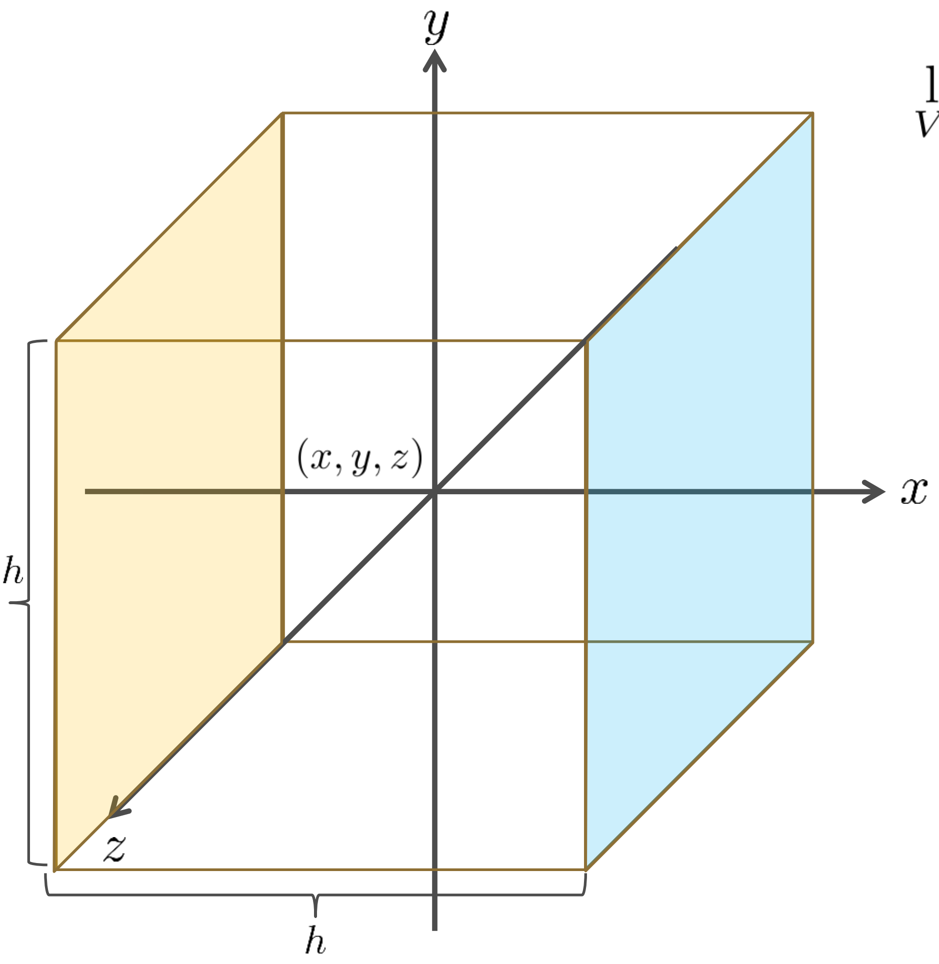
$$\lim_{V\to 0} \frac{1}{V} \oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r},t) = \operatorname{div} \mathbf{D}(\mathbf{r},t)$$

$$= \lim_{h\to 0} \frac{1}{h^{\cancel{3}}} \Bigg\{ \left[D_x\left(x + \frac{h}{2}, y, z, t\right) \cancel{h^2} - D_x\left(x - \frac{h}{2}, y, z, t\right) \cancel{h^2}\right] + \left[D_y\left(x, y + \frac{h}{2}, z, t\right) \cancel{h^2} - D_y\left(x, y - \frac{h}{2}, z, t\right) \cancel{h^2}\right] + \left[D_z\left(x, y, z + \frac{h}{2}, t\right) \cancel{h^2} - D_z\left(x, y, z - \frac{h}{2}, t\right) \cancel{h^2}\right] + \mathcal{O}(h^{\cancel{4}}) \Bigg\}$$

Leibniz Ferdinand Braun Institut

Definition of Divergence Operator in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$

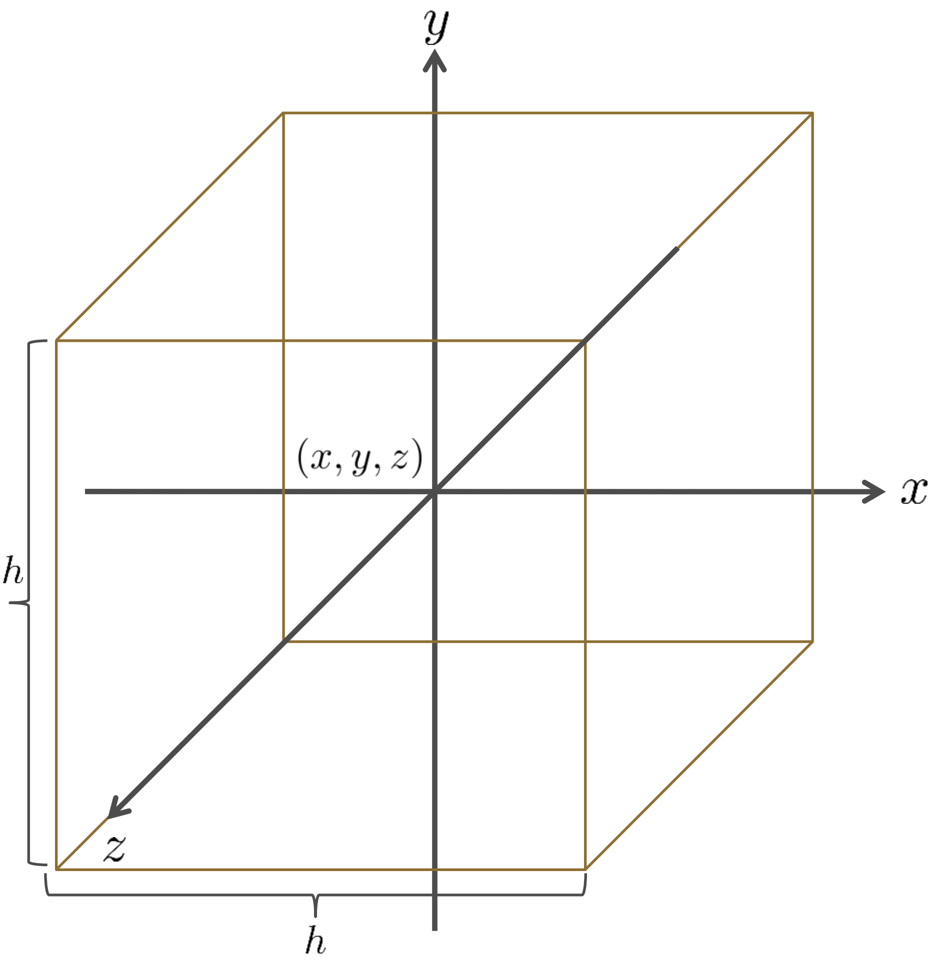


$$\lim_{V \to 0} \frac{1}{V} \oiint_{\partial \boldsymbol{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

$$= \left\{ \underbrace{\lim_{h \to 0} \frac{D_x\left(x + \frac{h}{2}, y, z, t\right) - D_x\left(x - \frac{h}{2}, y, z, t\right)}{h}}_{\frac{\partial}{\partial x} D_x(x,y,z,t)} + \underbrace{\lim_{h \to 0} \frac{D_y\left(x, y + \frac{h}{2}, z, t\right) - D_y\left(x, y - \frac{h}{2}, z, t\right)}{h}}_{\frac{\partial}{\partial y} D_y(x,y,z,t)} + \underbrace{\lim_{h \to 0} \frac{D_z\left(x, y, z + \frac{h}{2}, t\right) - D_z\left(x, y, z - \frac{h}{2}, t\right)}{h}}_{\frac{\partial}{\partial z} D_z(x,y,z,t)} + \underbrace{\lim_{h \to 0} \mathcal{O}(h)}_{0} \right\}$$

Definition of Divergence Operator in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{V \to 0} \frac{1}{V} \oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} = \nabla \cdot \mathbf{D}(\mathbf{r}, t) = \operatorname{div} \mathbf{D}(\mathbf{r}, t)$$

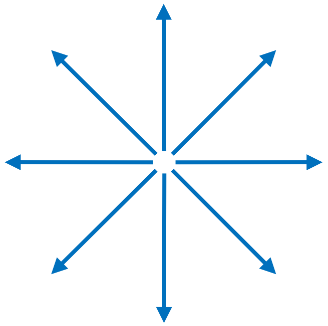
$$= \left\{ \frac{\partial}{\partial x} D_x\left(x, y, z, t\right) + \frac{\partial}{\partial y} D_y\left(x, y, z, t\right) + \frac{\partial}{\partial z} D_z\left(x, y, z, t\right) \right\}$$

Remarks on Vector Operator Divergence

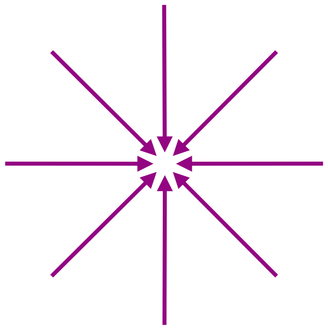
$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=\operatorname{div}\mathbf{D}(\mathbf{r},t)=s(\mathbf{r},t)=\underbrace{\frac{\partial}{\partial x}D_x(x,y,z,t)+\frac{\partial}{\partial y}D_y(x,y,z,t)+\frac{\partial}{\partial z}D_z(x,y,z,t)}_{\text{for Cartesian coordinate system}},\qquad \underbrace{\nabla=\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]}_{\text{del or nabla}}$$

- Divergence acts on a vector field and gives back a scalar field
- Divergence indicates the source strength of the field per unit volume (how much vectors diverge in a small neighbourhood around the point)
- Divergence of some characteristic field distributions:

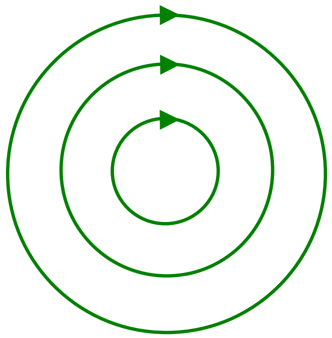
$\nabla\cdot\mathbf{F}_{\text{source}}(\mathbf{r})>0$



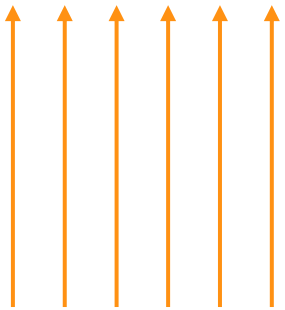
$\nabla\cdot\mathbf{F}_{\text{sink}}(\mathbf{r})<0$



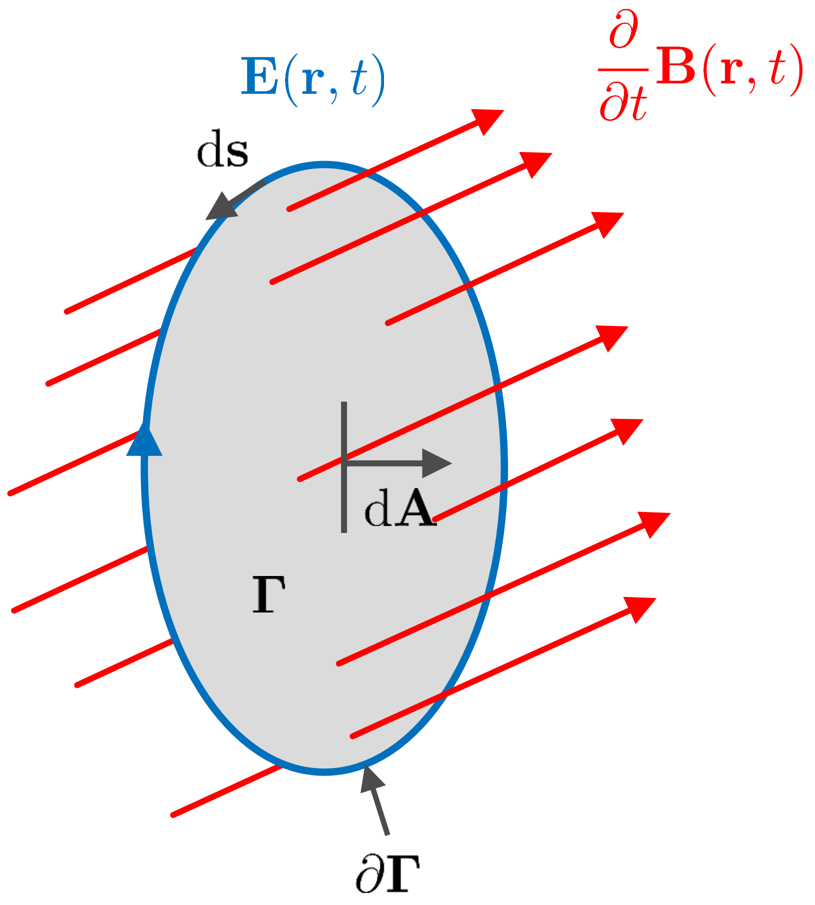
$\nabla\cdot\mathbf{F}_{\text{curl}}(\mathbf{r})=0$



$\nabla\cdot\mathbf{F}_{\text{homo}}(\mathbf{r})=0$



Faraday's Law of Induction in Integral Form for (infinitely) small Area



$$\oint_{\partial\Gamma} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint_{\Gamma} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} \qquad \Big|\cdot\frac{1}{A}$$

$$\frac{1}{A}\oint_{\partial\Gamma} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\frac{1}{A}\iint_{\Gamma} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} \qquad \Big|\lim_{A\to 0}$$

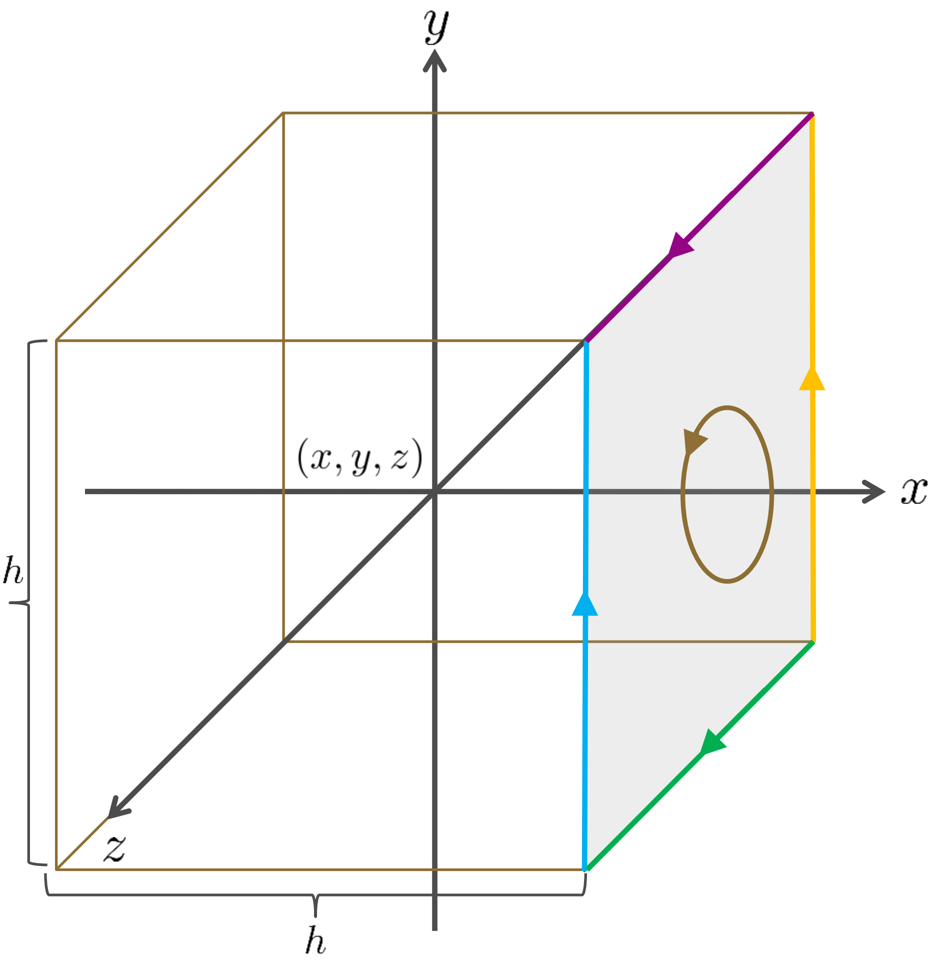
$$\underbrace{\lim_{A\to 0}\frac{1}{A}\oint_{\partial\Gamma} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s}}_{\mathbf{n}\cdot[\nabla\times\mathbf{E}(\mathbf{r},t)=\mathbf{n}\cdot[\mathrm{curl}\,\mathbf{E}(\mathbf{r},t)]} = -\underbrace{\lim_{A\to 0}\frac{1}{A}\iint_{\Gamma} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\mathbf{n}\cdot[\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)]}$$

normalized closed line integral on boundary of infinitely small area

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$

Definition of (x-Component) of Curl in a Cartesian System - Integral Decomposition

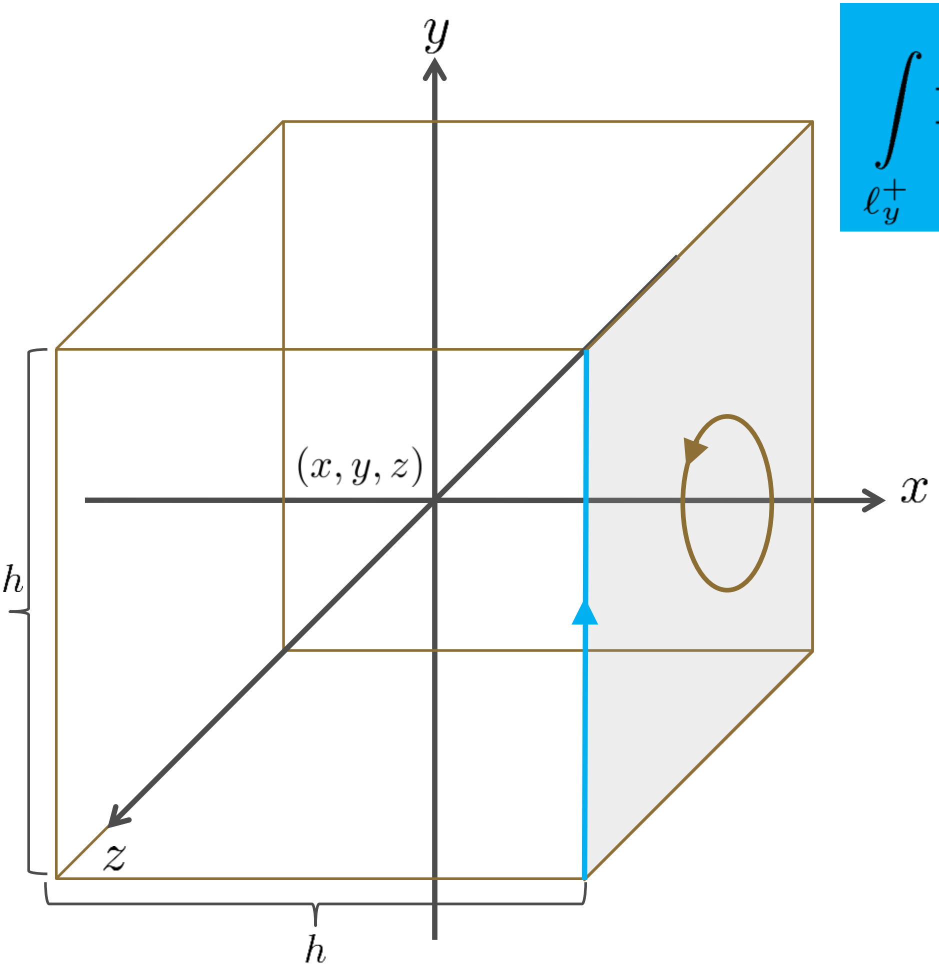
$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{A_x \to 0} \frac{1}{A_x} \oint_{\partial A_x^+} \mathbf{E}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{s} = \mathbf{n}_x \cdot \left[\nabla \times \mathbf{E}(\mathbf{r},t)\right] = \mathbf{n}_x \cdot \left[\mathrm{curl}\,\mathbf{E}(\mathbf{r},t)\right]$$

Integral Evaluation using Midpoint Rule (I / II)

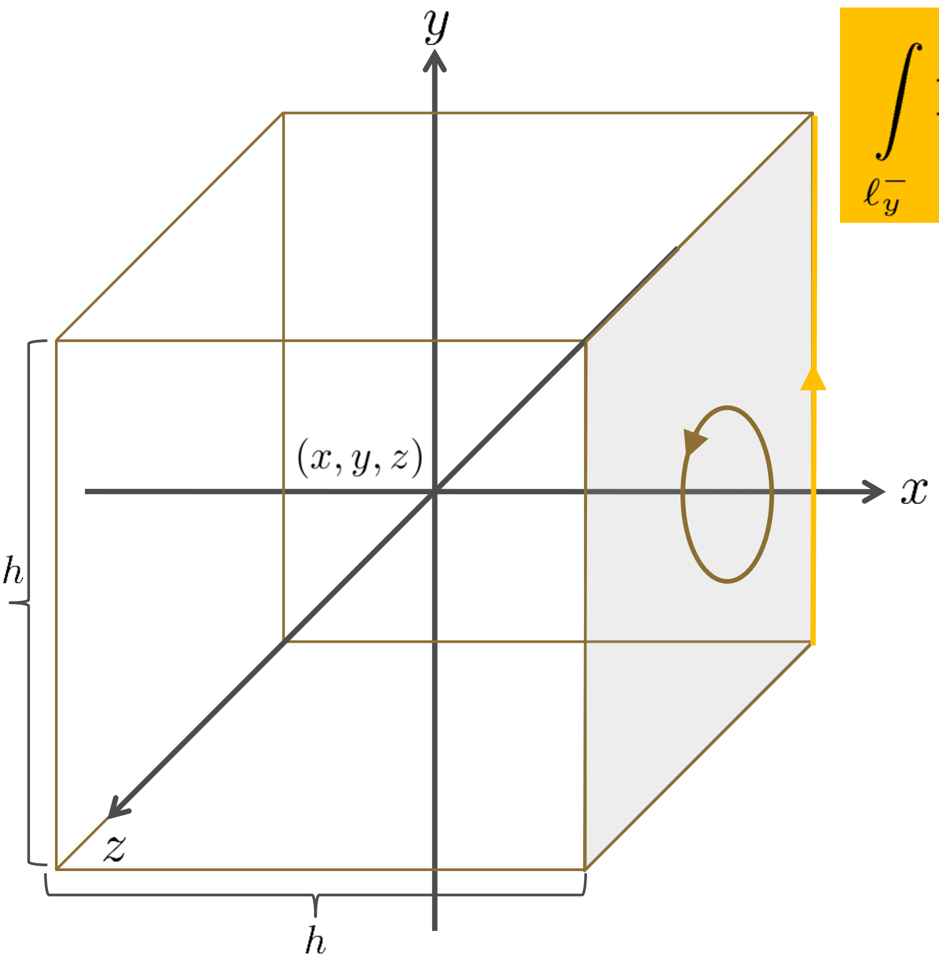
$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\int_{\ell_y^+} \mathbf{E}\left(x + \frac{h}{2}, y, z + \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{s} =$$

Integral Evaluation using Midpoint Rule (II / II)

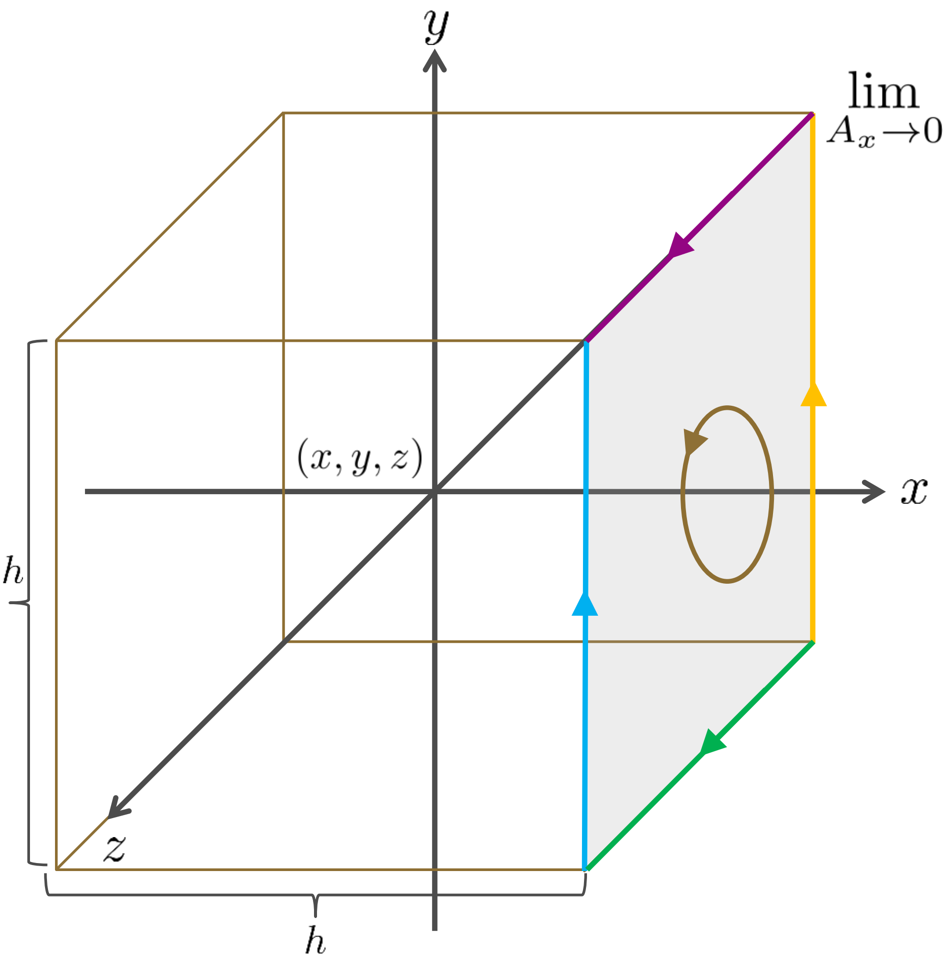
$$\mathbf{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\int_{\ell_y^-} \mathbf{E}\left(x + \frac{h}{2}, y, z - \frac{h}{2}, t\right) \cdot \mathrm{d}\mathbf{s} =$$

Definition of (x-Component) of Curl in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$

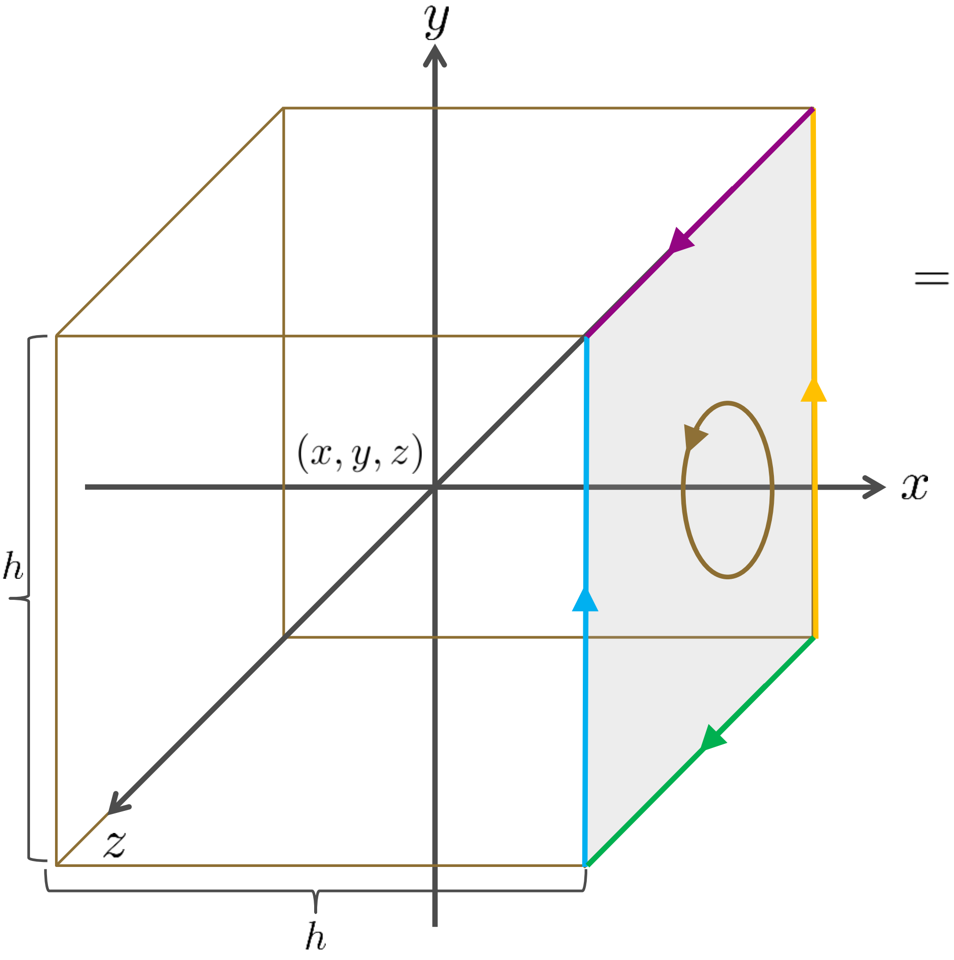


$$\lim_{A_x \to 0} \frac{1}{A_x} \oint_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{s} = \mathbf{n}_x \cdot \left[\nabla \times \mathbf{E}(\mathbf{r},t)\right] = \mathbf{n}_x \cdot \left[\mathrm{curl}\, \mathbf{E}(\mathbf{r},t)\right]$$

$$= \lim_{h \to 0} \frac{1}{h^2} \left\{ \underbrace{\int_{\ell_y^+} \mathbf{E}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{s}}_{-E_y\left(x+\frac{h}{2},y,z+\frac{h}{2},t\right) h + \mathcal{O}(h^3)} + \underbrace{\int_{\ell_y^-} \mathbf{E}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{s}}_{E_y\left(x+\frac{h}{2},y,z-\frac{h}{2},t\right) h + \mathcal{O}(h^3)} \right\}$$

Definition of (x-Component) of Curl in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{A_x \to 0} \frac{1}{A_x} \oint_{\partial \boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{s} = \mathbf{n}_x \cdot \left[\nabla \times \mathbf{E}(\mathbf{r}, t)\right] = \mathbf{n}_x \cdot \left[\operatorname{curl} \mathbf{E}(\mathbf{r}, t)\right]$$

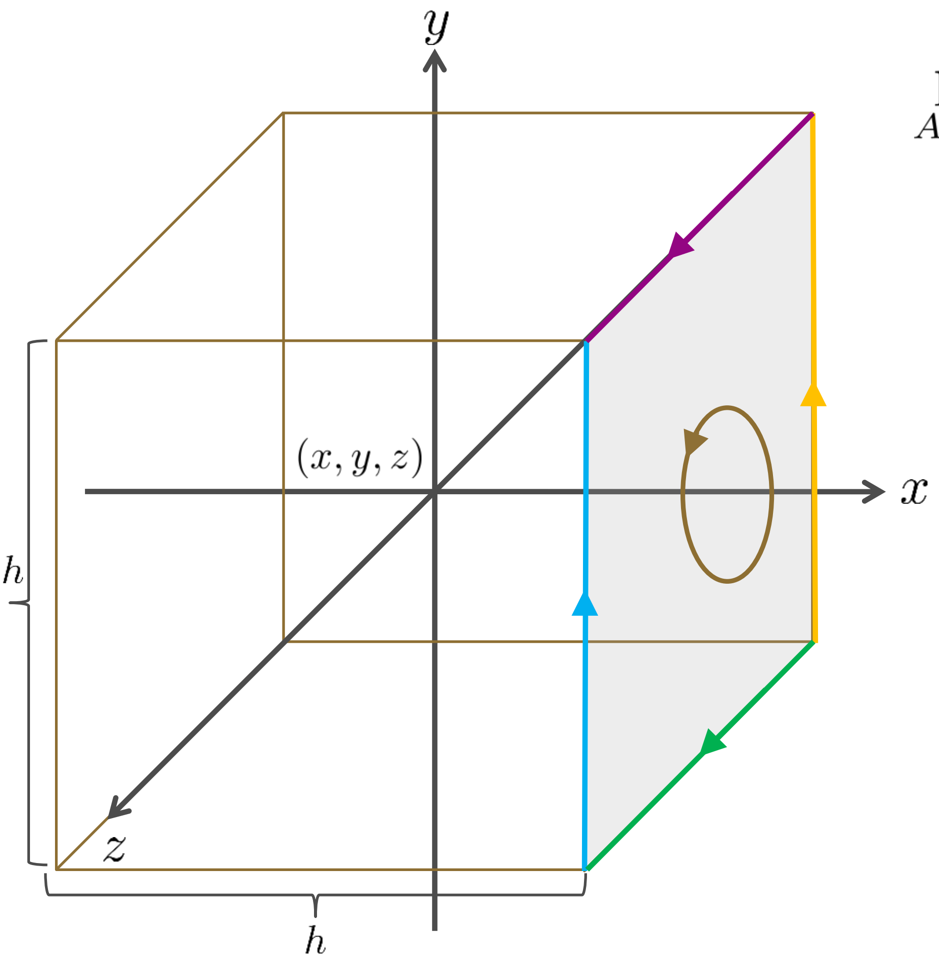
$$= \lim_{h \to 0} \frac{1}{h^{\cancel{2}}} \Bigg\{ - E_y\left(x + \frac{h}{2}, y, z + \frac{h}{2}, t\right) \cancel{h} + E_y\left(x + \frac{h}{2}, y, z - \frac{h}{2}, t\right) \cancel{h}$$

$$E_z\left(x + \frac{h}{2}, y + \frac{h}{2}, z, t\right) \cancel{h} - E_z\left(x + \frac{h}{2}, y - \frac{h}{2}, z, t\right) \cancel{h}$$

$$+ \mathcal{O}(h^{\cancel{3}}) \Bigg\}$$

Definition of (x-Component) of Curl in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{A_x \to 0} \frac{1}{A_x} \oint_{\partial \boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{s} = \mathbf{n}_x \cdot \left[\nabla \times \mathbf{E}(\mathbf{r}, t)\right] = \mathbf{n}_x \cdot \left[\operatorname{curl} \mathbf{E}(\mathbf{r}, t)\right]$$

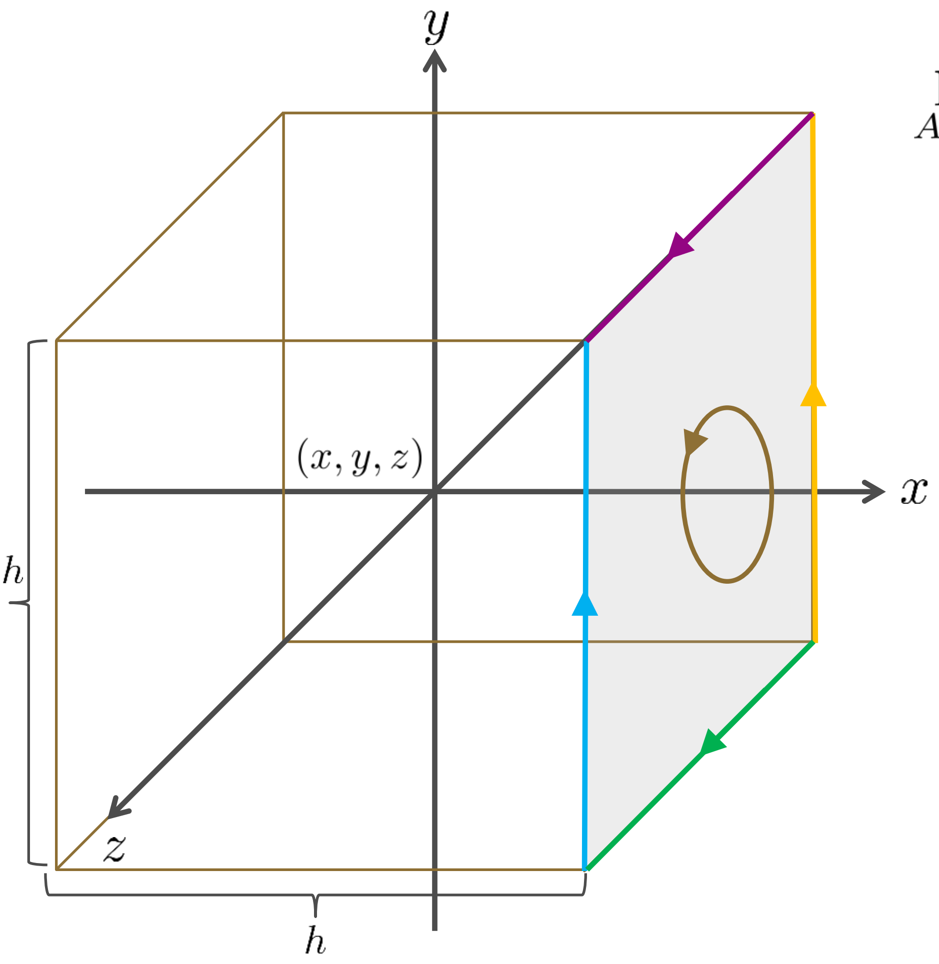
$$= \Bigg\{ - \underbrace{\lim_{h \to 0} \frac{E_y\left(x + \frac{h}{2}, y, z + \frac{h}{2}, t\right) - E_y\left(x + \frac{h}{2}, y, z - \frac{h}{2}, t\right)}{h}}_{\frac{\partial}{\partial z} E_y(x,y,z,t)}$$

$$+ \underbrace{\lim_{h \to 0} \frac{E_z\left(x + \frac{h}{2}, y + \frac{h}{2}, z, t\right) - E_z\left(x + \frac{h}{2}, y - \frac{h}{2}, z, t\right)}{h}}_{\frac{\partial}{\partial y} E_z(x,y,z,t)}$$

$$+ \underbrace{\lim_{h \to 0} \mathcal{O}(h)}_{0} \Bigg\}$$

Definition of (x-Component) of Curl in a Cartesian System

$$\boldsymbol{\Omega} = \left[x - \frac{h}{2}, x + \frac{h}{2}\right] \times \left[y - \frac{h}{2}, y + \frac{h}{2}\right] \times \left[z - \frac{h}{2}, z + \frac{h}{2}\right]$$



$$\lim_{A_x \to 0} \frac{1}{A_x} \oint_{\partial \boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{s} = \mathbf{n}_x \cdot [\nabla \times \mathbf{E}(\mathbf{r}, t)] = \mathbf{n}_x \cdot [\mathrm{curl}\, \mathbf{E}(\mathbf{r}, t)]$$

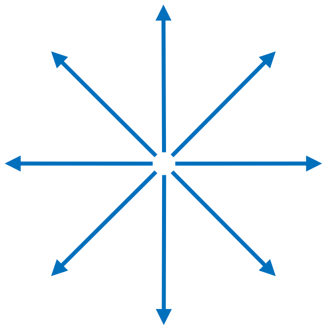
$$= \left\{ -\frac{\partial}{\partial z} E_y(x, y, z, t) + \frac{\partial}{\partial y} E_z(x, y, z, t) \right\}$$

Remarks on Vector Operator Curl

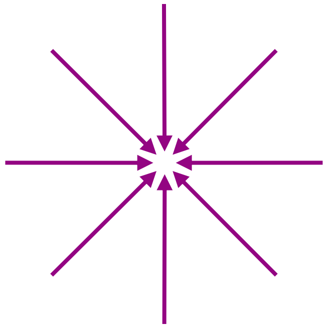
$$\nabla \times \mathbf{F}(\mathbf{r},t) = \operatorname{curl}\mathbf{F}(\mathbf{r},t) = \mathbf{C}(\mathbf{r},t) = \underbrace{\begin{pmatrix} \frac{\partial}{\partial y}F_z(x,y,z,t) - \frac{\partial}{\partial z}F_y(x,y,z,t) \\ \frac{\partial}{\partial z}F_x(x,y,z,t) - \frac{\partial}{\partial x}F_z(x,y,z,t) \\ \frac{\partial}{\partial x}F_y(x,y,z,t) - \frac{\partial}{\partial y}F_x(x,y,z,t) \end{pmatrix}}_{\text{for Cartesian coordinate system}}, \quad \underbrace{\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]}_{\text{del or nabla}}$$

- Acts on a vector field and gives back a vector field!
- Measures the rotation (direction and magnitude) of a vector field in a point
- Curl of some characteristic field distributions:

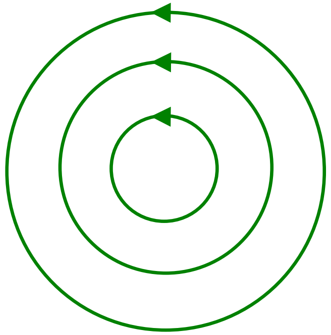
$\nabla \times \mathbf{F}_{\text{source}}(\mathbf{r}) = \mathbf{0}$



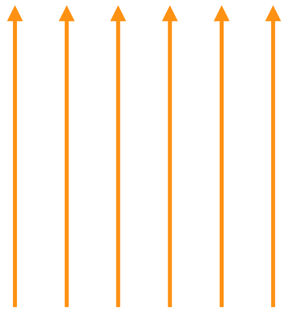
$\nabla \times \mathbf{F}_{\text{sink}}(\mathbf{r}) = \mathbf{0}$



$\nabla \times \mathbf{F}_{\text{curl}}(\mathbf{r}) \neq \mathbf{0}$

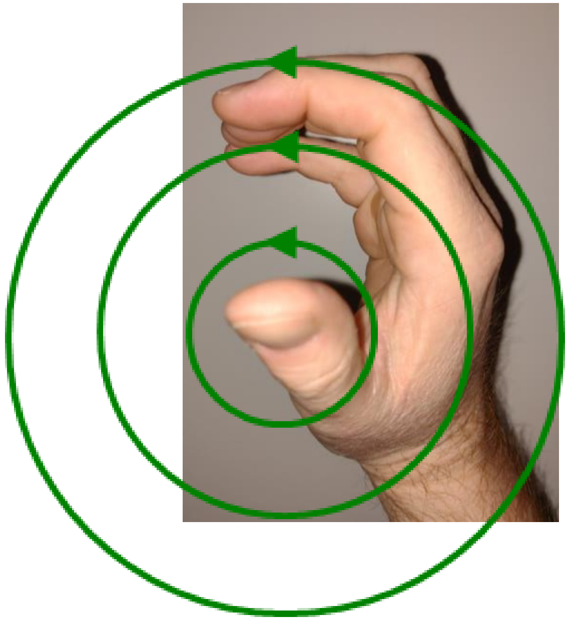


$\nabla \times \mathbf{F}_{\text{homo}}(\mathbf{r}) = \mathbf{0}$

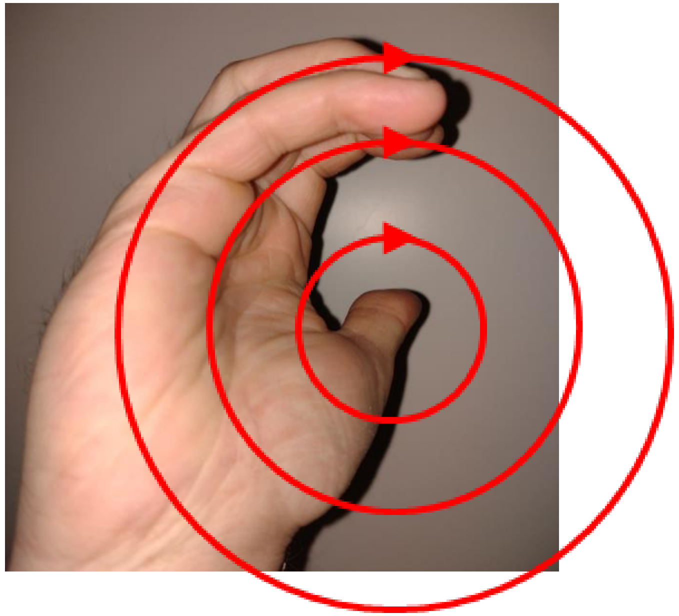


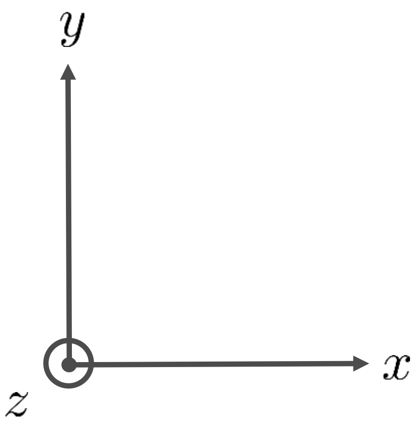
Directions of Fields Resulting from Curl

$$\nabla \times \mathbf{F}_{\text{curl}}(\mathbf{r}) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, a > 0$$

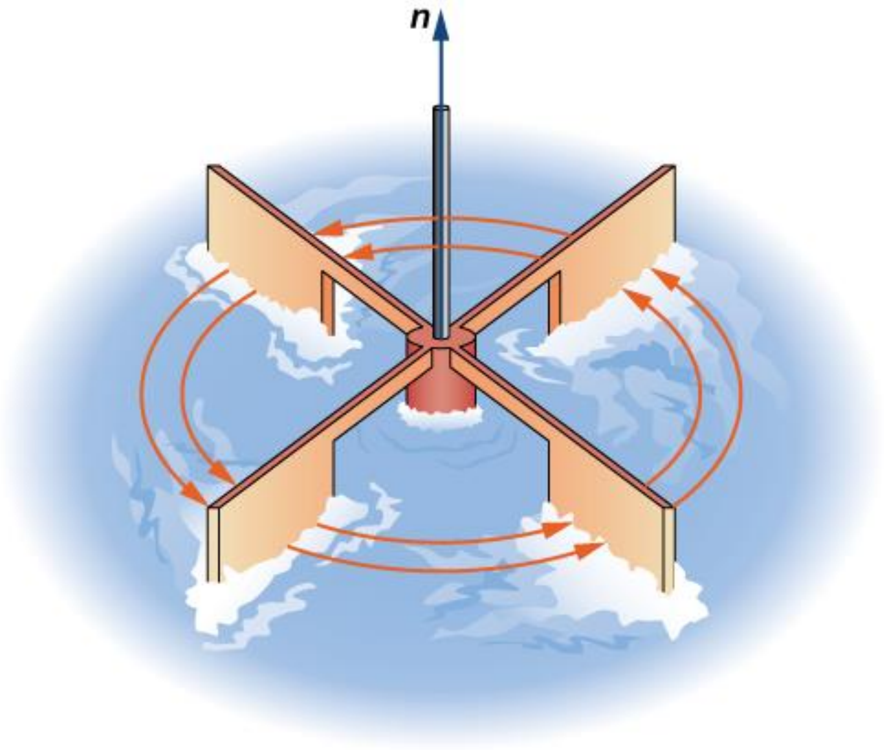


$$\nabla \times \mathbf{F}_{\text{curl}}(\mathbf{r}) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, a < 0$$





Imagine Curl Operator as a Paddle Wheel



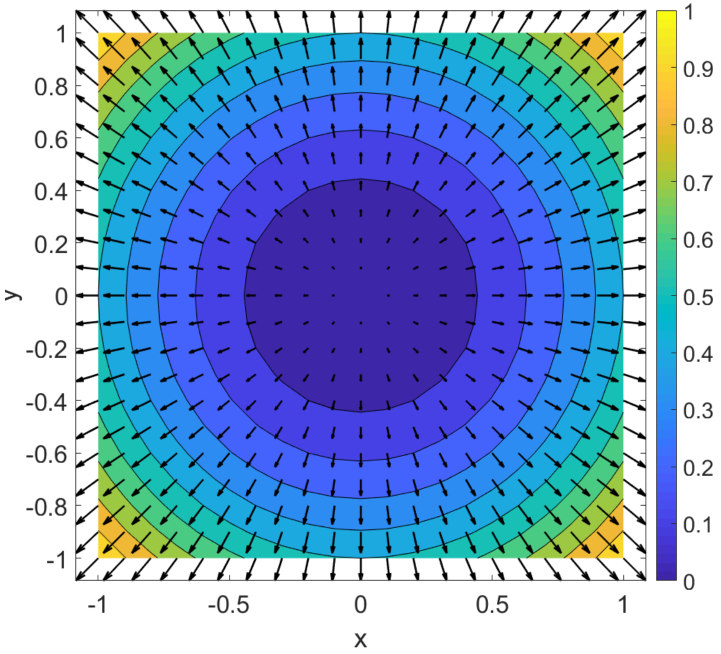
Source: https://query.libretexts.org/Kiswahili/Kitabu%3A_Calculus_%28OpenStax%29/16%3A_Vector_Calculus/16.05%3A_Tofauti_na_Curl

Gradient Operator

$$\nabla \phi(\mathbf{r}, t) = \operatorname{grad} \phi(\mathbf{r}, t) = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} \phi(x, y, z, t) \\ \frac{\partial}{\partial y} \phi(x, y, z, t) \\ \frac{\partial}{\partial z} \phi(x, y, z, t) \end{pmatrix}}_{\text{for Cartesian coordinate system}}$$

$$\phi(x, y, z) = \frac{1}{2}\left(x^2 + y^2\right)$$

$$\nabla \phi(x, y, z) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$



- Application to scalar fields, result is a vector field
- Direction points in the direction of largest increase from the scalar field at the point
- Magnitude is the slope towards the maximum change at the point

Laplace Operator

For scalar fields:

$$\Delta\phi(\mathbf{r},t) = \nabla\cdot[\nabla\phi(\mathbf{r},t)] = \operatorname{div}\operatorname{grad}\phi(\mathbf{r},t) = \underbrace{\frac{\partial^2}{\partial x^2}\phi(x,y,z,t) + \frac{\partial^2}{\partial y^2}\phi(x,y,z,t) + \frac{\partial^2}{\partial z^2}\phi(x,y,z,t)}_{\text{for Cartesian coordinate system}}$$

For vector fields:

$$\Delta\mathbf{F}(\mathbf{r},t) = \nabla\left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right] - \nabla\times\nabla\times\mathbf{F}(\mathbf{r},t) = \operatorname{grad}\operatorname{div}\mathbf{F}(\mathbf{r},t) - \operatorname{curl}\operatorname{curl}\mathbf{F}(\mathbf{r},t)$$

$$= \underbrace{\begin{pmatrix} \frac{\partial^2}{\partial x^2}F_x(x,y,z,t) + \frac{\partial^2}{\partial y^2}F_x(x,y,z,t) + \frac{\partial^2}{\partial z^2}F_x(x,y,z,t) \\ \frac{\partial^2}{\partial x^2}F_y(x,y,z,t) + \frac{\partial^2}{\partial y^2}F_y(x,y,z,t) + \frac{\partial^2}{\partial z^2}F_y(x,y,z,t) \\ \frac{\partial^2}{\partial x^2}F_z(x,y,z,t) + \frac{\partial^2}{\partial y^2}F_z(x,y,z,t) + \frac{\partial^2}{\partial z^2}F_z(x,y,z,t) \end{pmatrix}}_{\text{for Cartesian coordinate system}}$$

Important Properties of Differential Operators

Differential operators are linear:

$$\nabla\left[k\phi(\mathbf{r})\right] = k\nabla\phi(\mathbf{r})$$

$$\nabla\cdot\left[k\mathbf{F}(\mathbf{r})\right] = k\nabla\cdot\mathbf{F}(\mathbf{r})$$

$$\nabla\times\left[k\mathbf{F}(\mathbf{r})\right] = k\nabla\times\mathbf{F}(\mathbf{r})$$

$$\Delta\left[k\phi(\mathbf{r})\right] = k\Delta\phi(\mathbf{r})$$

$$\nabla\left[\phi(\mathbf{r})+\psi(\mathbf{r})\right] = \nabla\phi(\mathbf{r})+\nabla\psi(\mathbf{r})$$

$$\nabla\cdot\left[\mathbf{F}(\mathbf{r})+\mathbf{G}(\mathbf{r})\right] = \nabla\cdot\mathbf{F}(\mathbf{r})+\nabla\cdot\mathbf{G}(\mathbf{r})$$

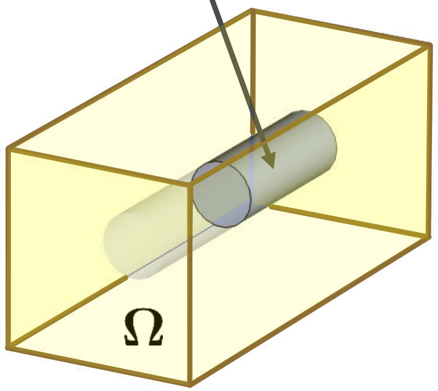
$$\nabla\times\left[\mathbf{F}(\mathbf{r})+\mathbf{G}(\mathbf{r})\right] = \nabla\times\mathbf{F}(\mathbf{r})+\nabla\times\mathbf{G}(\mathbf{r})$$

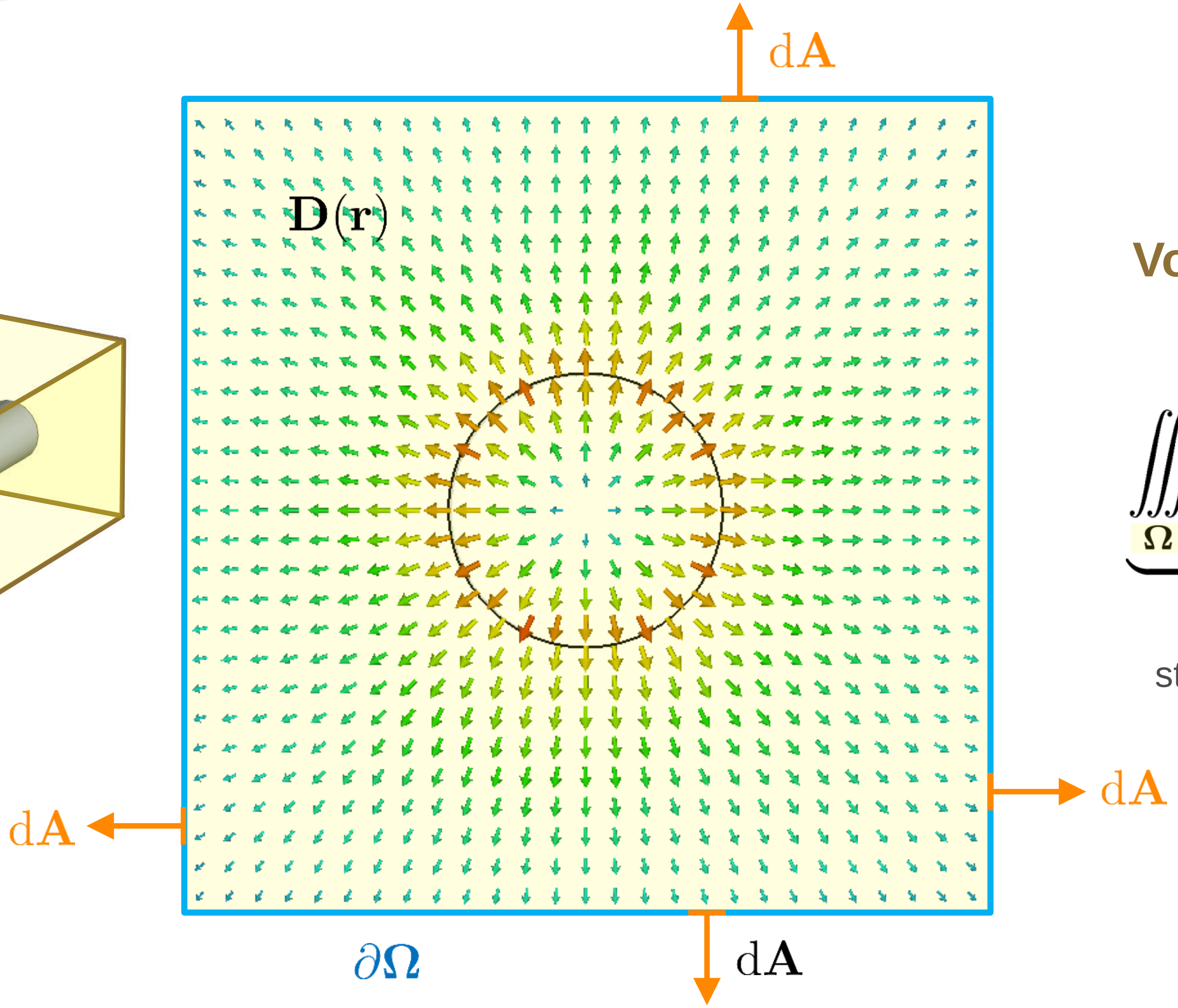
$$\Delta\left[\phi(\mathbf{r})+\psi(\mathbf{r})\right] = \Delta\phi(\mathbf{r})+\Delta\psi(\mathbf{r})$$

Maxwell's Equations in Integral and Differential Representation

Integral Form		Differential Form
$\oint\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V$	$\Big\vert \lim_{V\to 0} \frac{1}{V}\ldots \to$	$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$
$\oint\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = 0$	$\Big\vert \lim_{V\to 0} \frac{1}{V}\ldots \to$	$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$
$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint_{\boldsymbol{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}$	$\Big\vert \lim_{A\to 0} \frac{1}{A}\ldots \to$	$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$
$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\boldsymbol{\Gamma}} \left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}$	$\Big\vert \lim_{A\to 0} \frac{1}{A}\ldots \to$	$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)$

Divergence Theorem – Connection between a Volume and Surface Integral





Volume Integral | **Surface Integral**

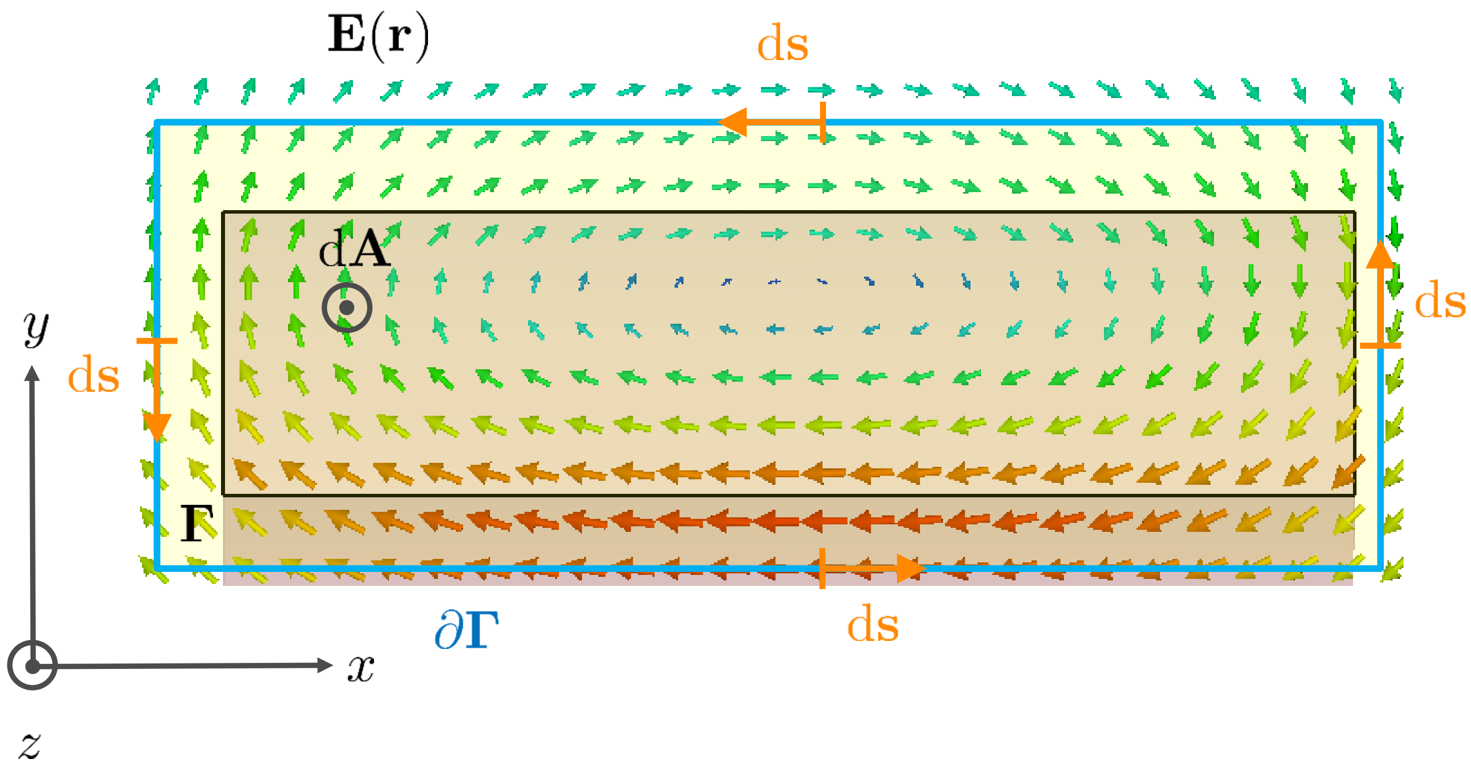
source strength per unit volume

$$\underbrace{\iiint_{\Omega} \overbrace{\nabla \cdot \mathbf{D}(\mathbf{r})} \, \mathrm{d}V}_{\text{total net source strength in volume}} = \underbrace{\oint\!\!\!\oint_{\partial\Omega} \mathbf{D}(\mathbf{r}) \cdot \mathrm{d}\mathbf{A}}_{\text{total net flux out of volume}}$$

total net source strength in volume

total net flux out of volume

Kelvin–Stokes' Theorem



Surface Integral **Path Integral**

$$\iint_{\mathbf{\Gamma}} \overbrace{\nabla \times \mathbf{E}(\mathbf{r})}^{\text{curl per unit area}} \cdot \mathrm{d}\mathbf{A} = \oint_{\partial\mathbf{\Gamma}} \mathbf{E}(\mathbf{r}) \cdot \mathrm{d}\mathbf{s}$$

total net curl of area

closed path integral on boundary of area

Conservation of Charges (in a Point)

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t) \qquad \Big|\, \nabla \cdot$$

$$\underbrace{\nabla \cdot [\nabla \times \mathbf{H}(\mathbf{r},t)]}_{0} = \nabla \cdot \left[\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right] + \nabla \cdot \mathbf{J}(\mathbf{r},t)$$

| exchange of derivates (Schwarz's theorem)

$$0 = \frac{\partial}{\partial t}[\nabla \cdot \mathbf{D}(\mathbf{r},t)] + \nabla \cdot \mathbf{J}(\mathbf{r},t) \qquad \Big|\, \nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

exploiting Gauss' law of electricity

Conservation of charges in a point:

$$0 = \frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot \mathbf{J}(\mathbf{r},t)$$

Conservation of Charges (in a Volume)

$$0 = \frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot \mathbf{J}(\mathbf{r},t) \qquad \Big| \iiint_{\boldsymbol{\Omega}} \ldots \,\mathrm{d}V$$

$$\iiint_{\boldsymbol{\Omega}} 0 \,\mathrm{d}V = \iiint_{\boldsymbol{\Omega}} \frac{\partial}{\partial t}\rho(\mathbf{r},t)\,\mathrm{d}V + \iiint_{\boldsymbol{\Omega}} \nabla \cdot \mathbf{J}(\mathbf{r},t)\,\mathrm{d}V$$

exchange of integration and derivation

$$\iiint_{\boldsymbol{\Omega}} 0 \,\mathrm{d}V = \frac{\partial}{\partial t}\iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V + \iiint_{\boldsymbol{\Omega}} \nabla \cdot \mathbf{J}(\mathbf{r},t)\,\mathrm{d}V$$

replace volume integral over charge density

$$0 = \frac{\partial}{\partial t}Q_{\mathrm{tot}}(t) + \iiint_{\boldsymbol{\Omega}} \nabla \cdot \mathbf{J}(\mathbf{r},t)\,\mathrm{d}V$$

apply divergence theorem

$$0 = \frac{\partial}{\partial t}Q_{\mathrm{tot}}(t) + \oiint_{\partial\boldsymbol{\Omega}} \mathbf{J}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{A}$$

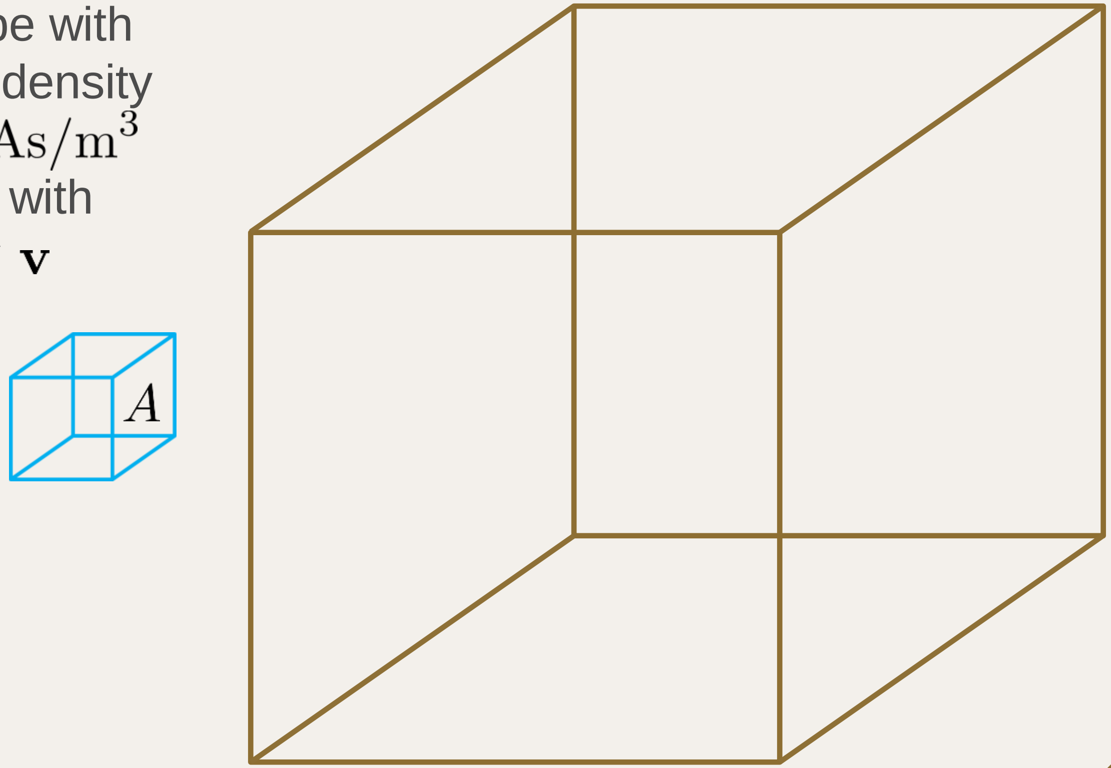
replace surface integral over current densities

Conservation of charges in volume:

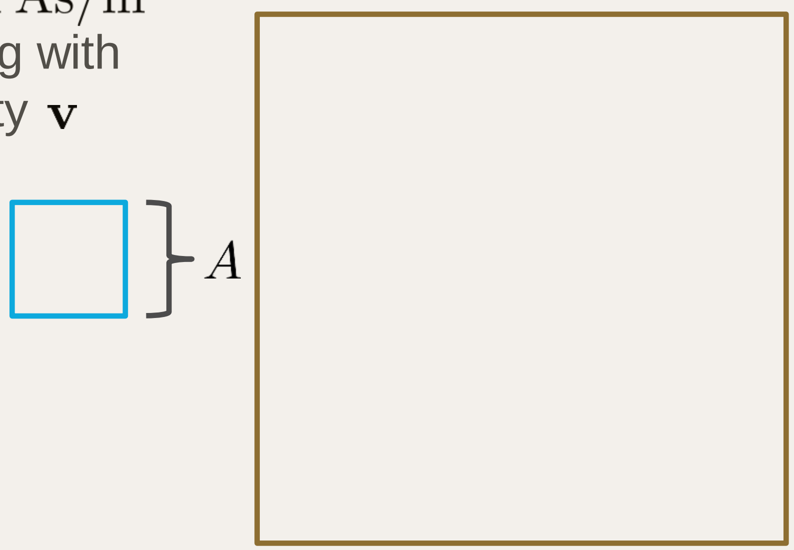
$$0 = \frac{\partial}{\partial t}Q_{\mathrm{tot}}(t) + I_{\mathrm{tot,out}}(t)$$

Example: Charged Cube and Test Volume

3D Sketch



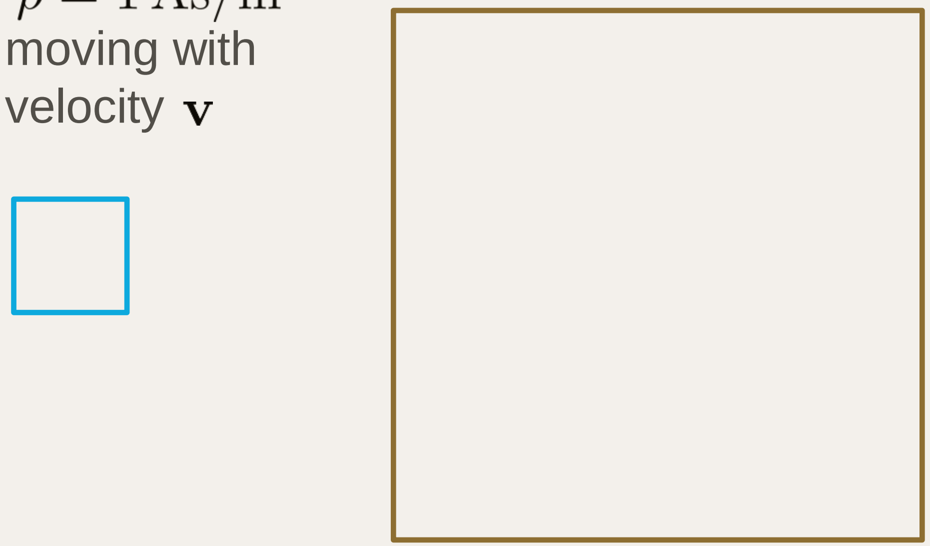
Cutaway View



Case 1: Charged Cube is Moving Outside the Test Volume

unit cube with charge density $\rho = 1\,\mathrm{As/m^3}$ moving with velocity $\mathbf{v}$

test volume $\boldsymbol{\Omega}$



$$0 = \underbrace{\frac{\partial}{\partial t} Q_{\mathrm{tot}}(t)} + \underbrace{I_{\mathrm{tot,out}}(t)}$$

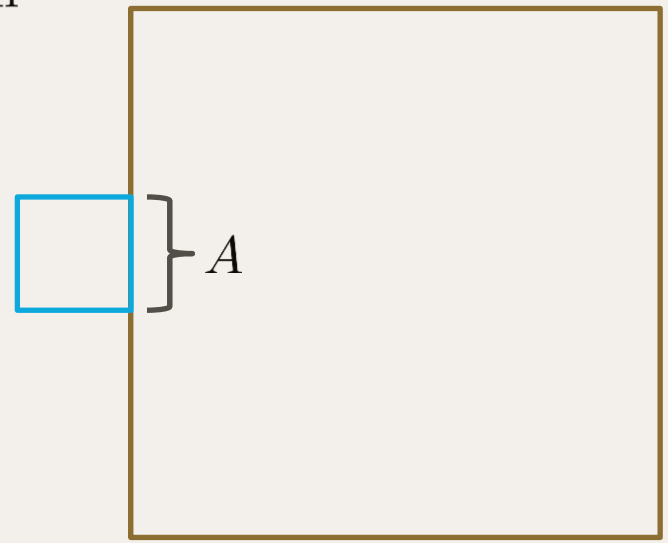
Case 2: Charged Cube is Moving Into the Test Volume

$$
\begin{aligned}
Q_{\mathrm{tot}}(t) &= \rho V \\
&= \rho A d \\
&= \rho A v t
\end{aligned}
$$

$$
\begin{aligned}
I_{\mathrm{tot,out}}(t) &= -AJ \\
&= -A\rho v
\end{aligned}
$$

unit cube with charge density $\rho = 1\,\mathrm{As/m}^3$ moving with velocity $\mathbf{v}$

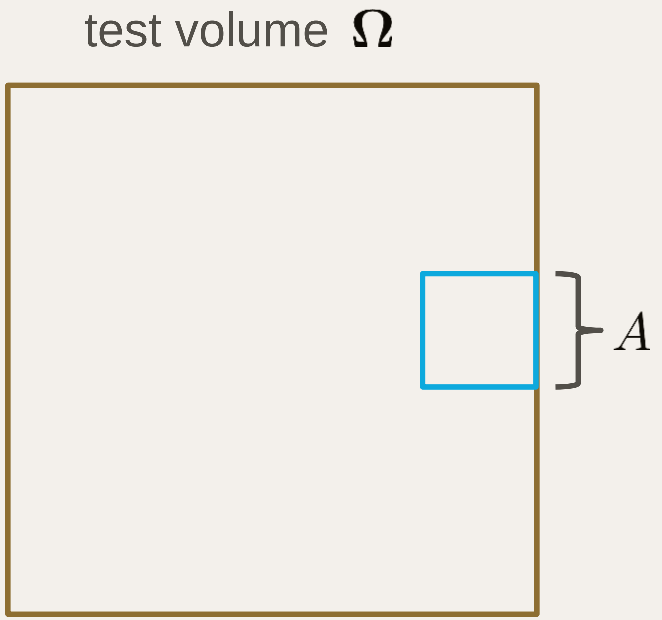
test volume $\boldsymbol{\Omega}$



$$
0 = \underbrace{\frac{\partial}{\partial t} Q_{\mathrm{tot}}(t)}_{} + \underbrace{I_{\mathrm{tot,out}}(t)}_{}
$$

Case 3: Charged Cube is Moving out of the Test Volume

unit cube with charge density $\rho = 1\,\mathrm{As/m^3}$ moving with velocity $\mathbf{v}$



$$0 = \underbrace{\frac{\partial}{\partial t} Q_{\mathrm{tot}}(t)} + \underbrace{I_{\mathrm{tot,out}}(t)}$$

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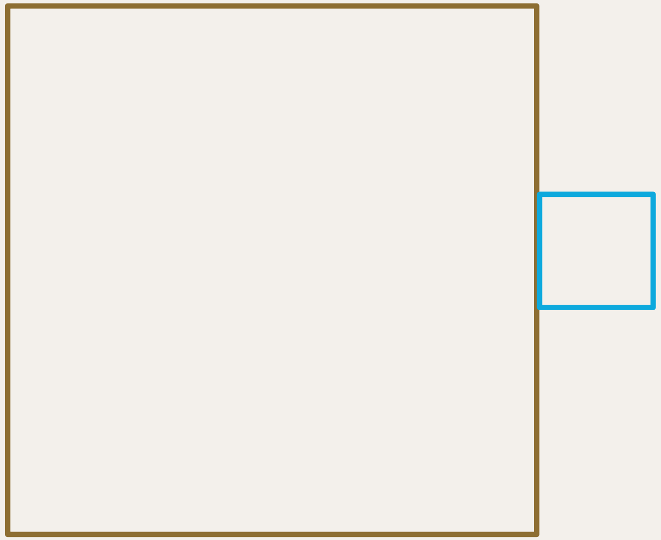
Case 4: Charged Cube is Moving outside the Test Volume

unit cube with charge density $\rho = 1\,\mathrm{As/m^3}$ moving with velocity $\mathbf{v}$

test volume $\boldsymbol{\Omega}$



$$0 = \underbrace{\frac{\partial}{\partial t} Q_{\mathrm{tot}}(t)} + \underbrace{I_{\mathrm{tot,out}}(t)}$$

Results from Conservation of Charges

Conservation of charges in a point:

$$0 = \frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot \mathbf{J}(\mathbf{r},t)$$

Conservation of charges in volume:

$$0 = \frac{\partial}{\partial t}Q_{\text{tot}}(t) + I_{\text{tot,out}}(t)$$

- If charge in a volume changes, exactly this amount of charge has to be transported through the surface of the volume, leading to a current
- Charges are conserved, they neither can be created nor destroyed, but result from separation requiring flow of charges, (again leading to currents)
- „Switching on and off“ charges (like a light bulb) violates Maxwell’s equations

Conservation of Energy or Poynting* Theorem (in a Point)

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \qquad \Big| \, \mathbf{H}(\mathbf{r},t)\cdot$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t) \qquad \Big| \, \mathbf{E}(\mathbf{r},t)\cdot$$

$$\mathbf{H}(\mathbf{r},t)\cdot\nabla \times \mathbf{E}(\mathbf{r},t) = -\mathbf{H}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t)\cdot\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{E}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t)$$

$-$

$$\mathbf{H}(\mathbf{r},t)\cdot\nabla \times \mathbf{E}(\mathbf{r},t) - \mathbf{E}(\mathbf{r},t)\cdot\nabla \times \mathbf{H}(\mathbf{r},t) = -\mathbf{H}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) - \mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) - \mathbf{E}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t)$$

Conservation of energy in a point:

$$\nabla\cdot\underbrace{\left[\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\right]}_{\mathbf{S}(\mathbf{r},t)} = \underbrace{-\mathbf{H}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)}_{\frac{\partial}{\partial t}w_{\mathrm{magn}}(\mathbf{r},t)} \underbrace{-\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)}_{\frac{\partial}{\partial t}w_{\mathrm{elec}}(\mathbf{r},t)} \underbrace{-\mathbf{E}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t)}_{p_{\mathrm{diss}}(\mathbf{r},t)}$$

energy flow per unit area (Poynting* vector)

change of energy stored in magnetic fields per unit volume

change of energy stored in electric fields per unit volume

power dissipated per unit volume

*John Henry Poynting, engl. physicist, 1852 – 1914

Conservation of Energy or Poynting* Theorem (in a Volume)

$$\nabla \cdot \mathbf{S}(\mathbf{r},t) = -\frac{\partial}{\partial t} w_{\text{magn}}(\mathbf{r},t) - \frac{\partial}{\partial t} w_{\text{elec}}(\mathbf{r},t) - p_{\text{diss}}(\mathbf{r},t) \quad \Big| \iiint_{\mathbf{\Omega}} \ldots \, \mathrm{d}V$$

$$\iiint_{\mathbf{\Omega}} \nabla \cdot \mathbf{S}(\mathbf{r},t)\,\mathrm{d}V = -\iiint_{\mathbf{\Omega}} \frac{\partial}{\partial t} w_{\text{magn}}(\mathbf{r},t)\,\mathrm{d}V - \iiint_{\mathbf{\Omega}} \frac{\partial}{\partial t} w_{\text{elec}}(\mathbf{r},t)\,\mathrm{d}V - \iiint_{\mathbf{\Omega}} p_{\text{diss}}(\mathbf{r},t)\,\mathrm{d}V$$

exchange of integration and derivation

$$\iiint_{\mathbf{\Omega}} \nabla \cdot \mathbf{S}(\mathbf{r},t)\,\mathrm{d}V = -\frac{\partial}{\partial t} \underbrace{\iiint_{\mathbf{\Omega}} w_{\text{magn}}(\mathbf{r},t)\,\mathrm{d}V}_{W_{\text{tot,magn}}(t)} - \frac{\partial}{\partial t} \underbrace{\iiint_{\mathbf{\Omega}} w_{\text{elec}}(\mathbf{r},t)\,\mathrm{d}V}_{W_{\text{tot,elec}}(t)} - \underbrace{\iiint_{\mathbf{\Omega}} p_{\text{diss}}(\mathbf{r},t)\,\mathrm{d}V}_{P_{\text{tot,diss}}(t)}$$

apply divergence theorem

Conservation of energy in volume:

$$\underbrace{\oiint_{\partial\mathbf{\Omega}} \mathbf{S}(\mathbf{r},t)\,\mathrm{d}\mathbf{A}}_{P_{\text{tot,out}}(t)} = -\underbrace{\frac{\partial}{\partial t} W_{\text{tot,magn}}(t)}_{} - \underbrace{\frac{\partial}{\partial t} W_{\text{tot,elec}}(t)}_{} - \underbrace{P_{\text{tot,diss}}(t)}_{}$$

total energy flow out of volume

change of energy stored in magnetic fields in volume

change of energy stored in electric fields in volume

power dissipated in volume

*John Henry Poynting, engl. physicist, 1852 – 1914

Interpretation of Equations

Conservation of energy in a point:

$$\nabla \cdot \mathbf{S}(\mathbf{r},t) + p_{\mathrm{diss}}(\mathbf{r},t) = -\frac{\partial}{\partial t} w_{\mathrm{magn}}(\mathbf{r},t) - \frac{\partial}{\partial t} w_{\mathrm{elec}}(\mathbf{r},t)$$

Conservation of energy in volume:

$$P_{\mathrm{tot,out}}(t) + P_{\mathrm{tot,diss}}(t) = -\frac{\partial}{\partial t} W_{\mathrm{tot,magn}}(t) - \frac{\partial}{\partial t} W_{\mathrm{tot,elec}}(t)$$

- Equations balancing change of energy per time (i.e. power)
- Sum of total power propagating out of the volume and power dissipating in the volume is equal to loss of energy stored in fields, i.e. energy is conserved
- If no power propagates out of the volume and no power is dissipated in the volume, total energy stored in fields is constant (derivative w.r.t. time is zero), but field energy may be converted from electric to magnetic fields and vice versa

Classification of Electromagnetic Fields*

Source Fields		Curl Fields		
Electrostatics	Stationary Currents	Magnetostatics	Quasi-stationary	General Propagation of Waves
$\nabla \cdot \mathbf{D} = \rho$ $\nabla \times \mathbf{E} = \mathbf{0}$	$\nabla \times \mathbf{E} = \mathbf{0}$ $\nabla \cdot \mathbf{J} = 0$ $\mathbf{J} = \sigma\,\mathbf{E}$	$\nabla \times \mathbf{H} = \mathbf{J}$ $\nabla \cdot \mathbf{B} = 0$	$\nabla \times \mathbf{H} = \sigma\,\mathbf{E}$ $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ $\nabla \cdot \mathbf{D} = 0$ $\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{D} = \rho$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ $\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}$

*Klaus W. Kark, Antennen und Strahlungsfelder - Elektromagnetische Wellen auf Leitungen, im Freiraum und ihre Abstrahlung, 3., erweiterte Auflage, Vieweg + Teubner, 2010

Electrostatics

Electrostatics – Simplifications of Maxwell's Equations

$$\nabla \times \mathbf{E}(\mathbf{r}) = \underbrace{-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r})}_{\mathbf{0}} \quad ✔$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

Electric field strength is curl-free, so it can be expressed gradient of a scalar potential

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

With this approach, we can ensure that induction law is fulfilled (and search for just one rather than for three function required):

$$\nabla \times \mathbf{E}(\mathbf{r}) = \nabla \times [-\nabla\phi(\mathbf{r})] = \mathbf{0}$$

Electrostatics – Derivation of Poisson‘s equation

Starting with Gauss’ law for electric fields:

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

Employing the material equation $\varepsilon \mathbf{E}(\mathbf{r}) = \mathbf{D}(\mathbf{r})$ (assuming homogeneous material):

$$\nabla \cdot \left[\varepsilon \mathbf{E}(\mathbf{r})\right] = \rho(\mathbf{r}) \quad \leftrightarrow \quad \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon}$$

Expressing electric fields by scalar potential ($\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$) delivers Poisson equation for electric potential:

$$\nabla \cdot \left[-\nabla\phi(\mathbf{r})\right] = -\Delta\phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon}$$

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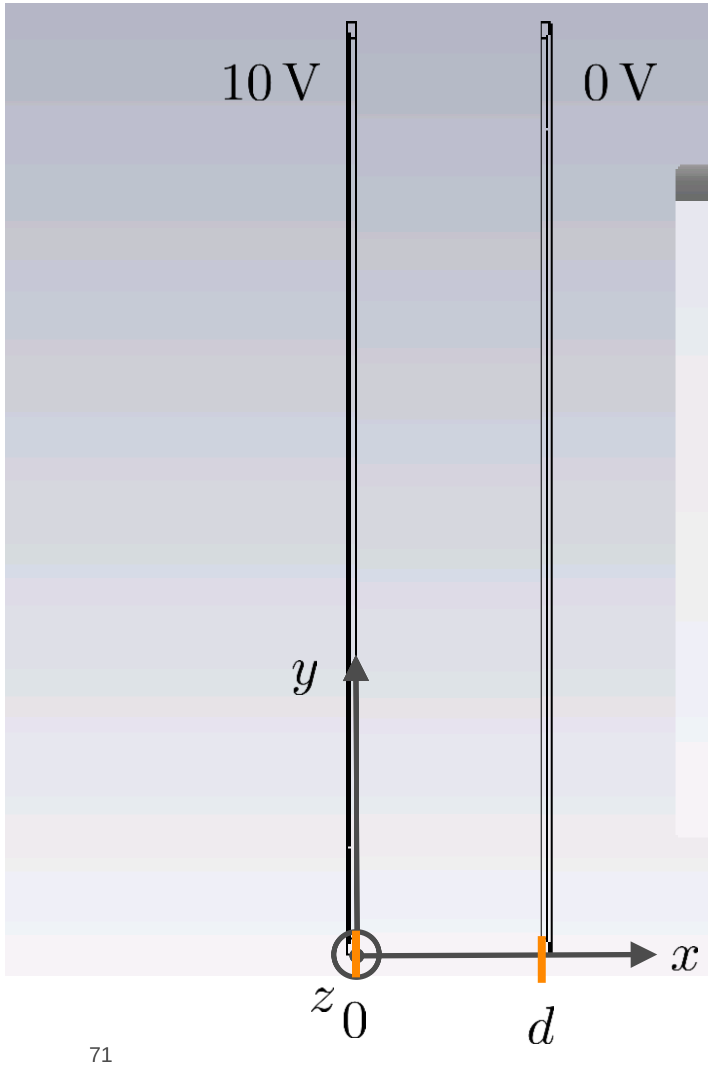
Electrostatics – A simple example: Capacitor

Depicted with CST Studio Suite®



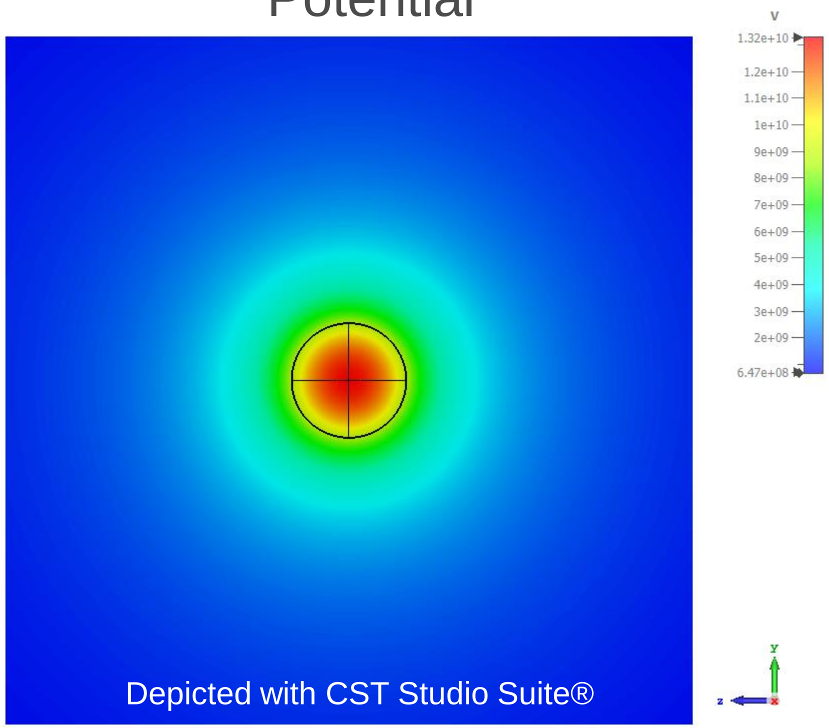
A capacitor is free of charges between its plates:

$$\Delta\phi(x, y, z) = 0$$

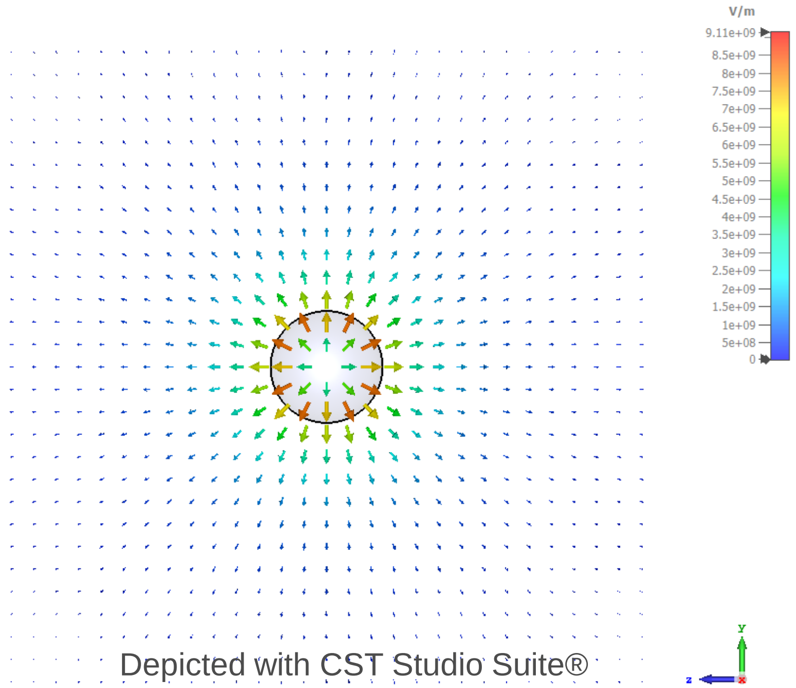
Electrostatics – Solution Poisson's equation for a Spherical Charge

$$-\Delta\phi(\mathbf{r}) = \frac{\rho(r)}{\varepsilon} = \begin{cases} \frac{3Q}{4\pi\varepsilon R^3}, & r \leq R \\ 0, & r > R \end{cases}$$

Potential



Electric Field



$$\phi(r) = \begin{cases} -\frac{Q}{8\pi\varepsilon R^3} r^2 + \frac{3Q}{8\pi\varepsilon R}, & r \leq R \\ \frac{Q}{4\pi\varepsilon r}, & r > R \end{cases}$$

$$E_r(r) = \begin{cases} \frac{Q}{4\pi\varepsilon R^3} r, & r \leq R \\ \frac{Q}{4\pi\varepsilon r^2}, & r > R \end{cases}$$

Magnetostatics

Magnetostatics – Maxwell Simplifications

$$\nabla \times \mathbf{H}(\mathbf{r}) = \underbrace{\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)}_{\mathbf{0}} + \mathbf{J}(\mathbf{r},t)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \checkmark$$

The magnetic flux density is divergence-free so that it can be expressed as curl of a (Coulomb gauged) vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

With this approach, we can ensure that Gauss' law for magnetism holds:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \nabla \cdot (\nabla \times \mathbf{A}(\mathbf{r})) = 0$$

Magnetostatics – Derivation of Poisson‘s equation

Starting with Ampère‘s law for the static case:

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r})$$

Employing the material equation $\mu \mathbf{H}(\mathbf{r}) = \mathbf{B}(\mathbf{r})$ (assuming homogeneous material):

$$\nabla \times \left[\frac{1}{\mu} \mathbf{B}(\mathbf{r}) \right] = \mathbf{J}(\mathbf{r}) \quad \leftrightarrow \quad \nabla \times \mathbf{B}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})$$

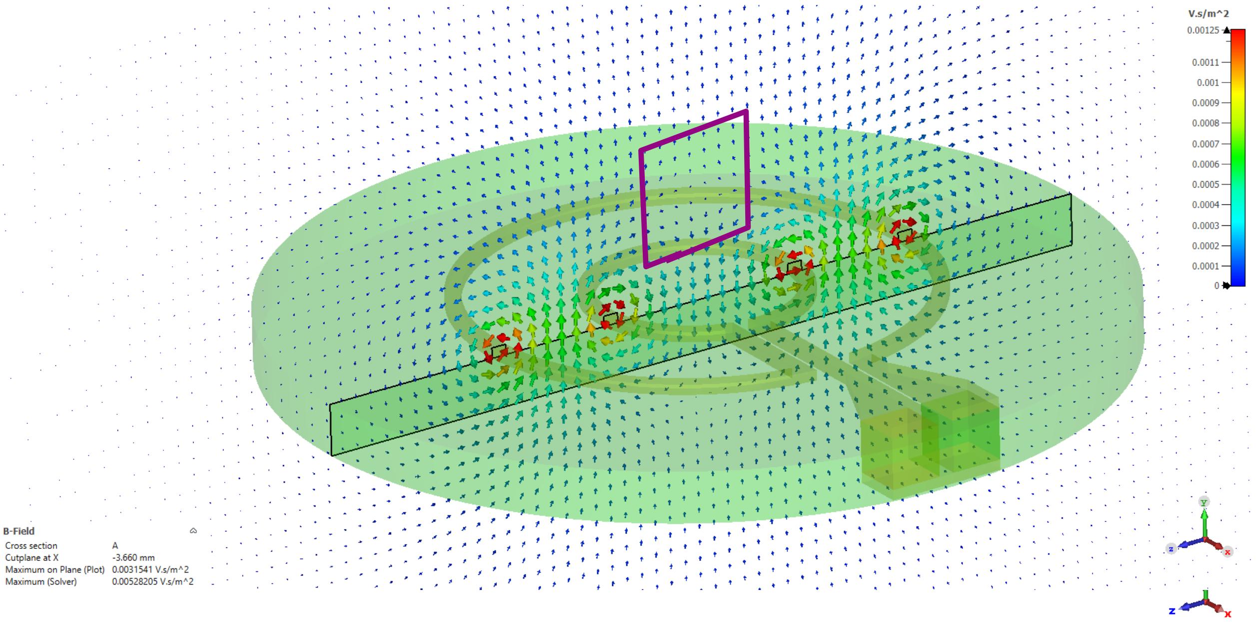
Expressing magnetic flux density by vector potential ($\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$):

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \big(\underbrace{\nabla \cdot \mathbf{A}(\mathbf{r})}_{0} \big) - \Delta \mathbf{A}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})$$

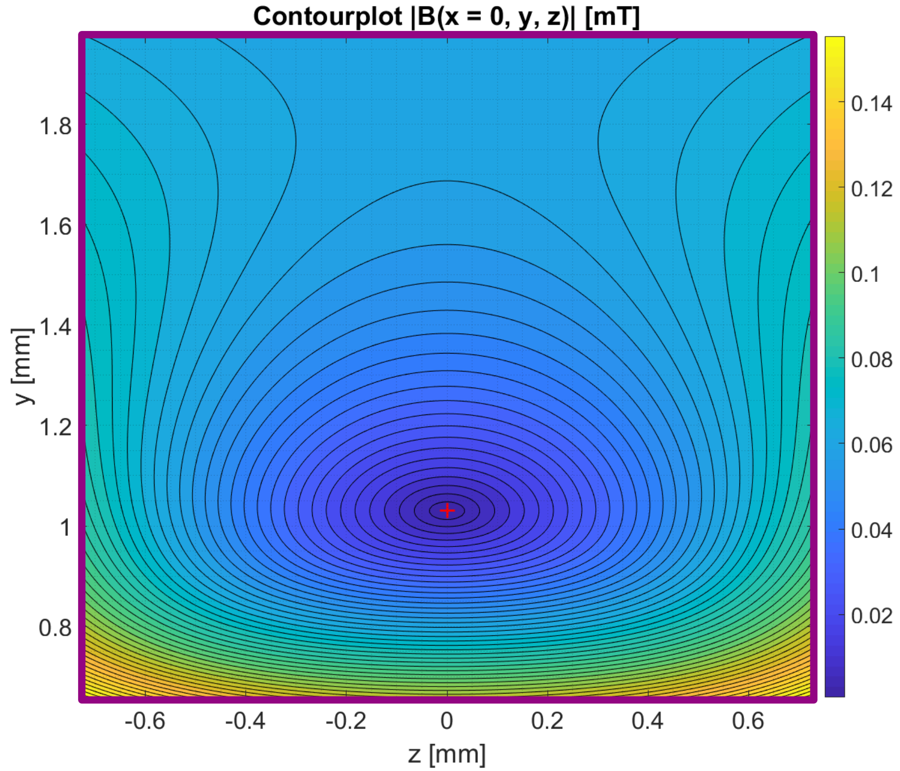
Gives Poisson equation for magnetic vector potential

$$-\Delta \mathbf{A}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})$$

Magnetostatics – Example: Double Loop Structure*



Simulated with CST Studio Suite®



Evaluated with analytical solution

*B. Jian and W. van Wijngaarden, "Double-loop microtrap for ultracold atoms," J. Opt. Soc. Am. B 30, 238-243 (2013)

Electromagnetic Waves

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Wave Equation arising from Maxwell‘s Equations

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \quad \Big| \nabla \times$$ taking the curl on both sides

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = \nabla \times \left(-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\right)$$ | exchanging curl and time derivative

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{B}(\mathbf{r},t)\right)$$ | applying the material law

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\left(\nabla \times \mu\mathbf{H}(\mathbf{r},t)\right)$$ | assuming constant permeability

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu\frac{\partial}{\partial t}\nabla \times \mathbf{H}(\mathbf{r},t)$$ | using Ampère’s law

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)$$ | deriving expression in brackets

$$= -\mu\frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) - \mu\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

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Wave Equation arising from Maxwell‘s Equations (cont.)

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) - \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

applying the material law

$$= -\varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) - \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) + \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

curl-curl equation

$$\nabla\Big(\underbrace{\nabla \cdot \mathbf{E}(\mathbf{r},t)}_{\frac{\rho(\mathbf{r},t)}{\varepsilon}}\Big) - \Delta\mathbf{E}(\mathbf{r},t) + \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

for charge-free case

$$\Delta\mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

wave equation (with excitation)

Properties of Solution of Wave Equation

Wave equation with excitation

$$\Delta \mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

far from the sources, excitation vanishes

$$\Delta \mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mathbf{0}$$ with speed of light $c = \frac{1}{\sqrt{\varepsilon\mu}}$

Assume that E has only one component (e.g. *x*), and the axis of propagation is z:

$$\frac{\partial^2}{\partial z^2}E_x(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_x(z,t) = 0$$

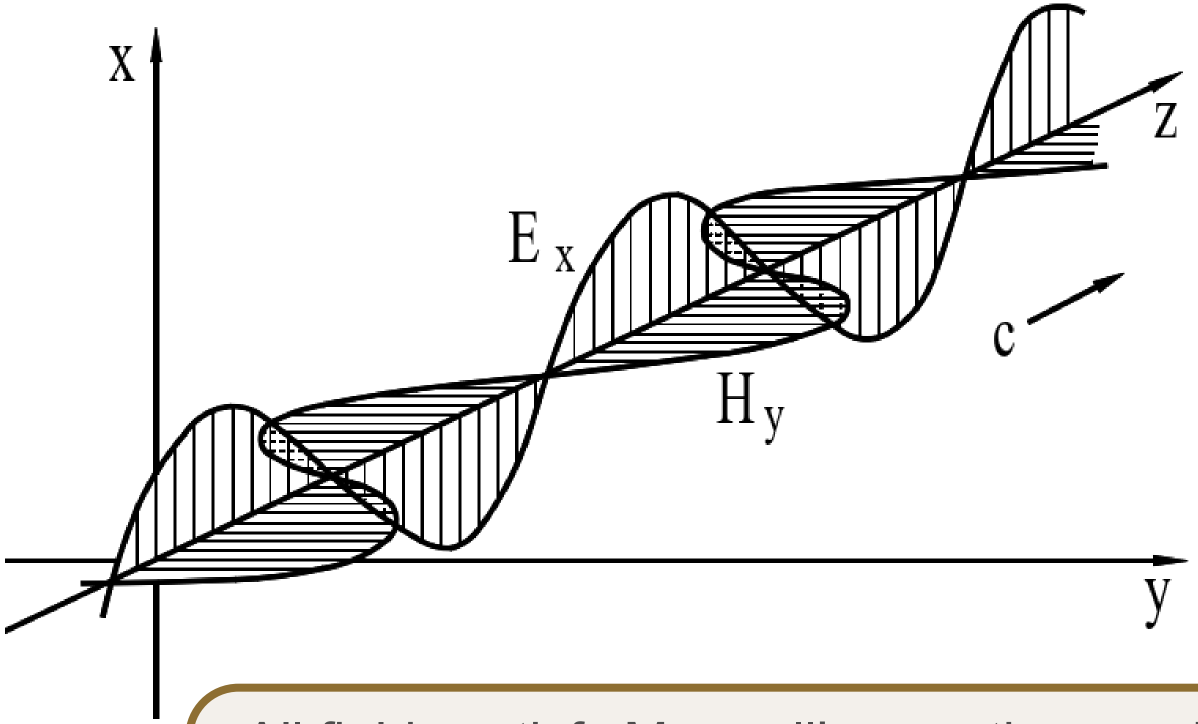
the general solution reads

$$E_x(z,t) = f(z-ct) + g(z+ct)$$

something traveling in +*z* direction

something traveling in -*z* direction with speed *c*

Waves in Free Space – Plane Wave propagating in +z-Direction



$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}_x E_0 \cos(\omega t - kz)$$
$$\mathbf{H}(\mathbf{r},t) = \mathbf{e}_y H_0 \cos(\omega t - kz)$$
$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) = \mathbf{e}_z S_0 \cos^2(\omega t - kz),$$

where

$$k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}$$ wave number

$$H_0 = \frac{E_0}{\mu_0 c_0} = \frac{E_0}{Z_0}$$ amplitude of magnetic field

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega$$ free space impedance

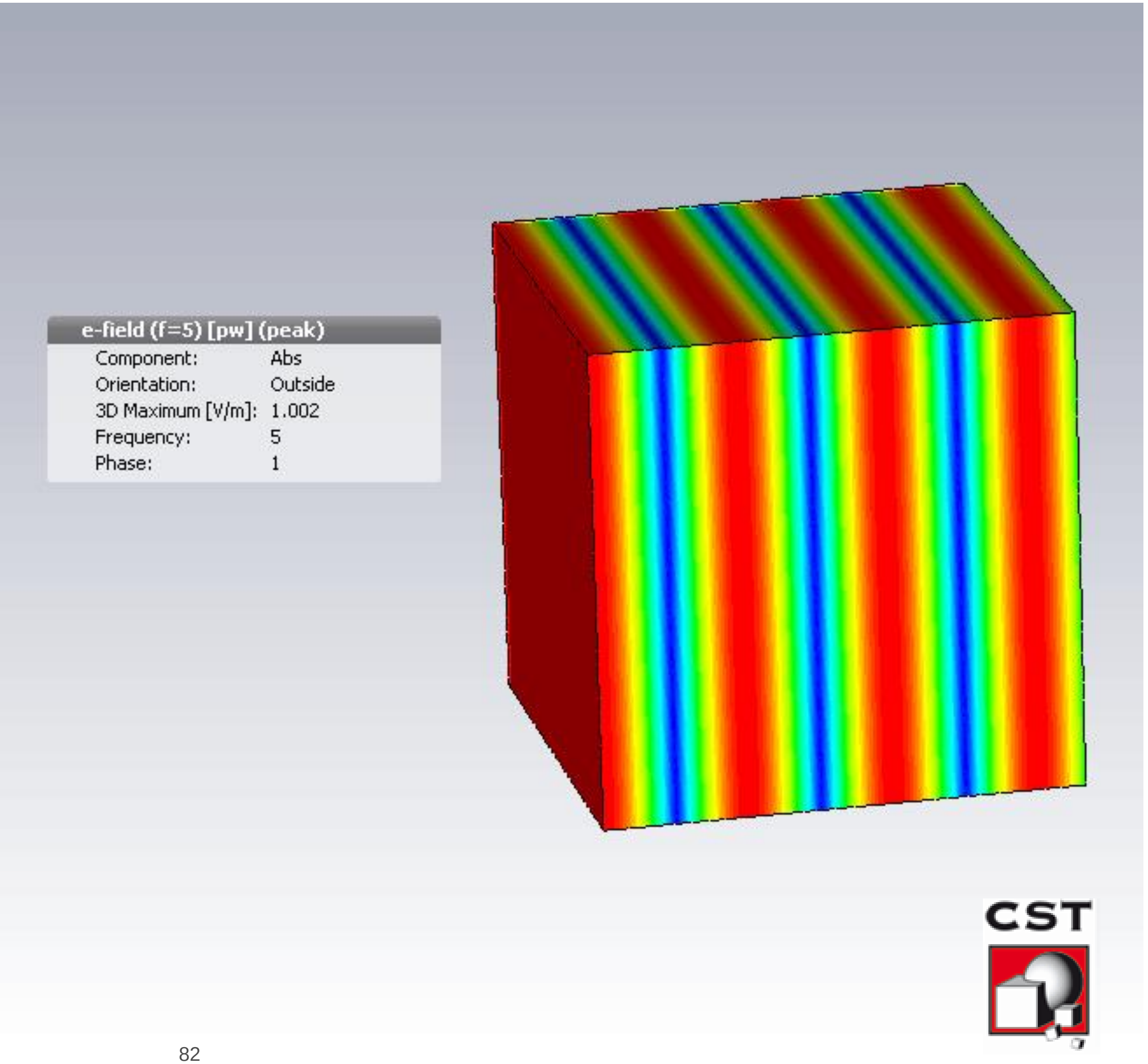
$$S_0 = \frac{E_0^2}{Z_0}$$ amplitude of power density

All fields satisfy Maxwell's equations and material equations in particular they satisfy:

$$\Delta\mathbf{E}(\mathbf{r},t) - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mathbf{0}$$

$$\Delta\mathbf{H}(\mathbf{r},t) - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{r},t) = \mathbf{0}$$

Waves in Free Space – Plane Wave propagating in +z-Direction



$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}_x E_0 \cos(\omega t - kz)$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{e}_y H_0 \cos(\omega t - kz)$$

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) = \mathbf{e}_z S_0 \cos^2(\omega t - kz),$$

where

$$k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}$$ wave number

$$H_0 = \frac{E_0}{\mu_0 c_0} = \frac{E_0}{Z_0}$$ amplitude of magnetic field

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega$$ free space impedance

$$S_0 = \frac{E_0^2}{Z_0}$$ amplitude of power density

Influence on Conducting Matter on Waves (I / II)

In conducting matter, Ohmic electric current densities will flow:

$$\mathbf{J}(\mathbf{r}, t) = \sigma \mathbf{E}(\mathbf{r}, t)$$

Replacing the electric current density in the wave equation with the upper relation gives

$$\Delta \mathbf{E}(\mathbf{r}, t) - \varepsilon\mu \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = \mu\sigma \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t)$$

Transforming this equation into frequency domain delivers

$$\Delta \underline{\mathbf{E}}(\mathbf{r}) + \varepsilon\mu\omega^2 \underline{\mathbf{E}}(\mathbf{r}) = j\omega\mu\sigma \underline{\mathbf{E}}(\mathbf{r})$$

with complex phasors. Now, consider a plane wave propagation in +z-direction:

$$\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{e}_x E_0 \mathrm{e}^{-j\underline{k}z}$$

Plugging this into the frequency-domain representation of the wave equation gives

$$\underline{k}^2 = \varepsilon\mu\omega^2 - j\omega\mu\sigma$$

Influence on Conducting Matter on Waves (II / II)

The wave number is complex-valued

$$\underline{k} = k' - jk''$$

with the following real and imaginary parts

$$k' = \frac{\mu\sigma\omega}{2\sqrt{-\frac{1}{2}\varepsilon\mu\omega^2 + \frac{1}{2}\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4}}}$$

$$k'' = \sqrt{-\frac{1}{2}\varepsilon\mu\omega^2 + \frac{1}{2}\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4}}$$

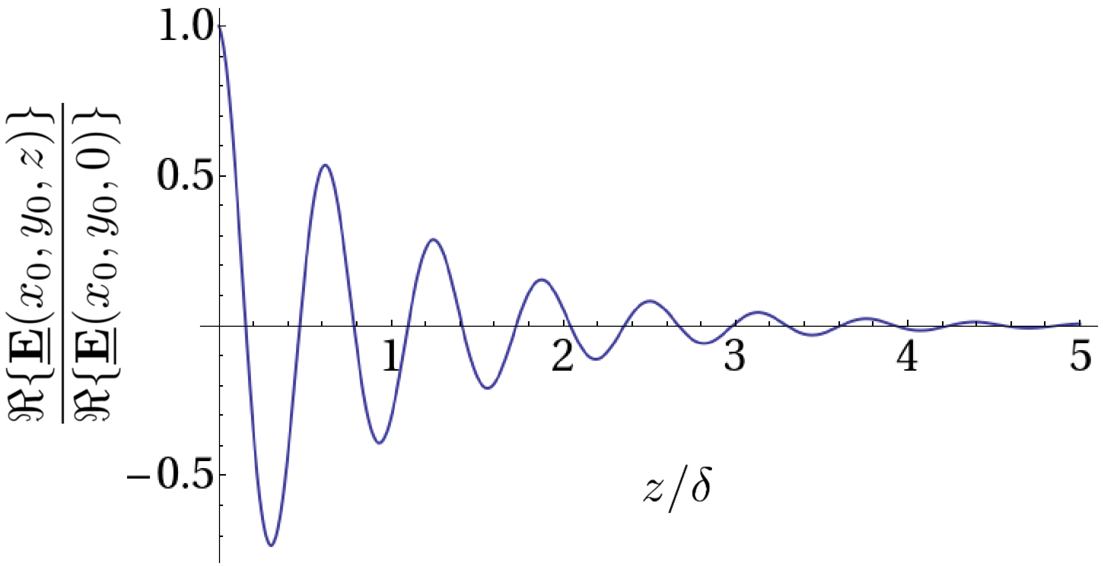
The real part describes the propagation of the wave while the imaginary part describes the exponential decay of the field strength in the conductor

$$\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{e}_x E_0 \mathrm{e}^{-j\underline{k}z} = \mathbf{e}_x E_0 \mathrm{e}^{-jk'z}\,\mathrm{e}^{-k''z}$$

The distance which is required for the fields to drop by a factor of exp(-1) is called <u>skin depth</u>:

$$\delta = \frac{1}{k''} = \frac{\sqrt{2}}{\sqrt{-\varepsilon\mu\omega^2 + \sqrt{\mu^2\omega^2\left(\sigma^2 + \varepsilon^2\omega^2\right)}}} \approx \sqrt{\frac{2}{\mu\omega\sigma}}$$

Skin Depth / Amplitude Decay in Conducting Matter



If frequency and/or the conductivity are “large”

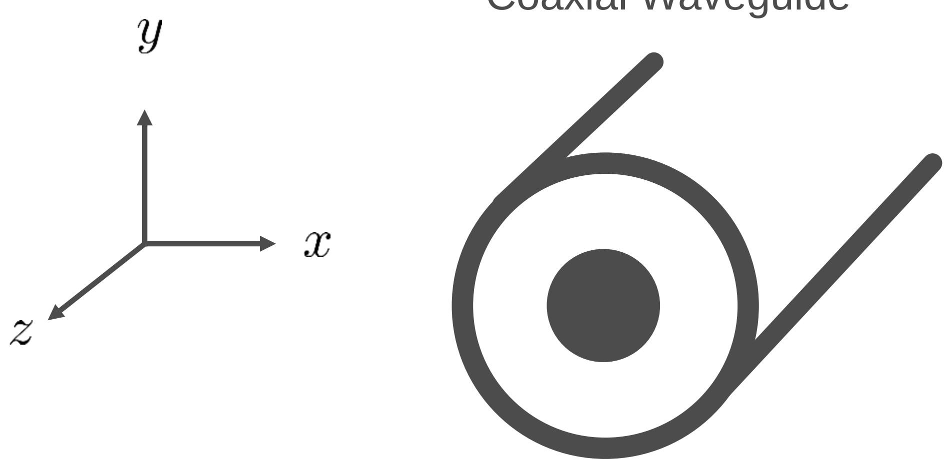
$$\delta \approx \sqrt{\frac{2}{\mu\omega\sigma}} \to 0$$

the fields do (almost) not penetrate into the metal. For copper at 1 GHz: $\delta \approx 2\,\mu\text{m}$.

Thus, power loss due to Ohmic currents is relevant only within a layer of thickness δ .

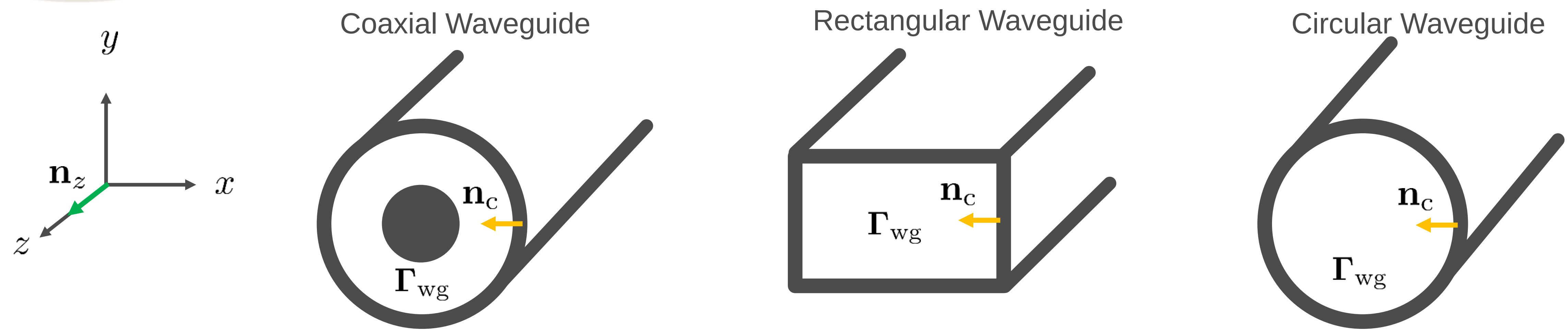
Fields in Waveguides

Coaxial Waveguide



Transverse Electric Magnetic Mode (TEM) $E_z = H_z = 0$, $f_{co} = 0$ GHz	✓
Transverse Electric Mode (TE) $E_z = 0$, $f_{co} > 0$ GHz	✓
Transverse Magnetic Mode (TM) $H_z = 0$, $f_{co} > 0$ GHz	✓

Calculation of Waveguide Modes and their Properties



$$\Delta_\mathrm{t} \begin{Bmatrix} \phi(\mathbf{r}_\mathrm{t}) \\ \psi(\mathbf{r}_\mathrm{t}) \end{Bmatrix} + k_\mathrm{t}^2 \begin{Bmatrix} \phi(\mathbf{r}_\mathrm{t}) \\ \psi(\mathbf{r}_\mathrm{t}) \end{Bmatrix} = 0 \text{ on } \mathbf{\Gamma}_\mathrm{wg}$$

$$\phi(\mathbf{r}_\mathrm{t}) = \text{const. on } \mathbf{\Gamma}_\mathrm{wg} \text{ for TEM modes}$$

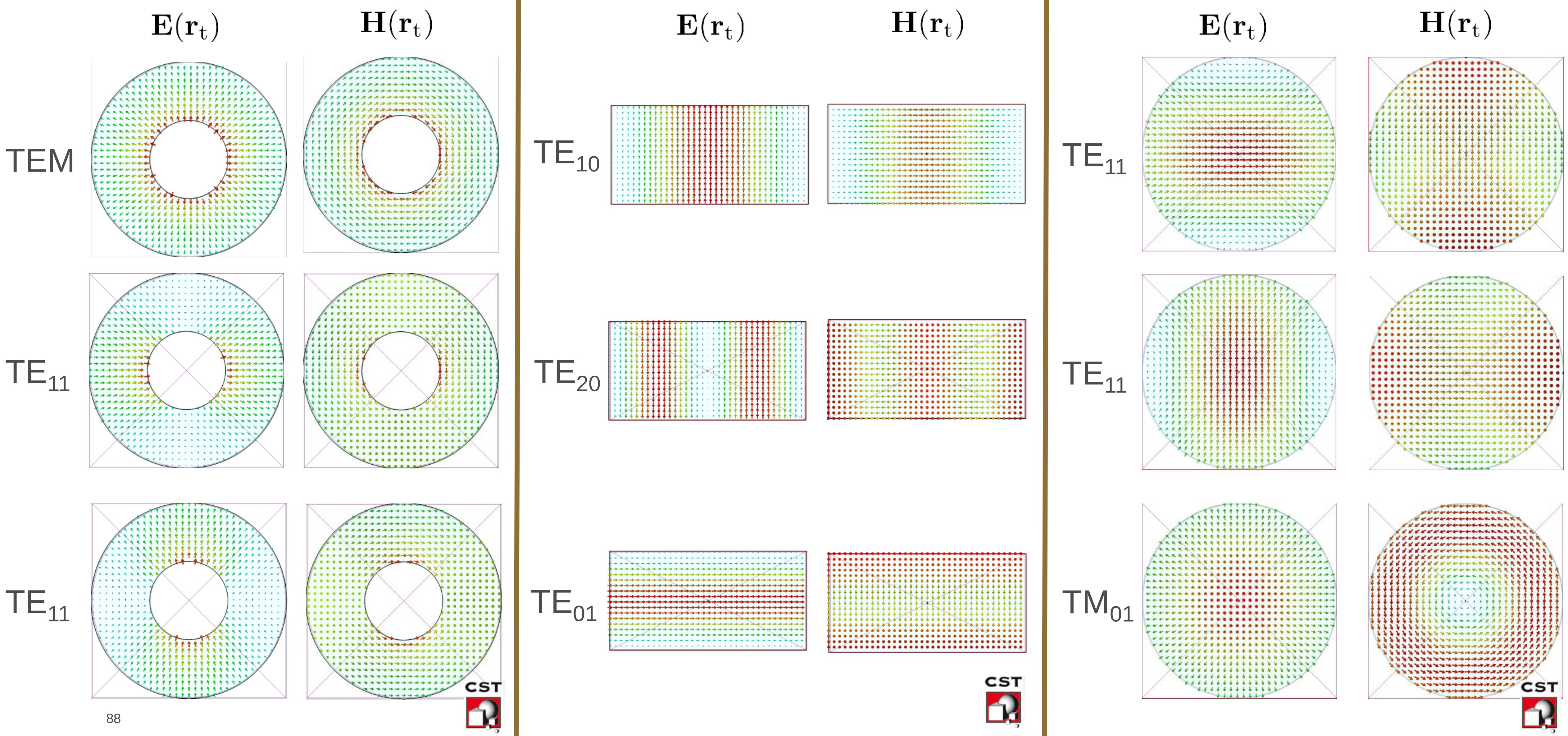
$$\phi(\mathbf{r}_\mathrm{t}) = 0 \text{ on } \mathbf{\Gamma}_\mathrm{wg} \text{ for TM modes}$$

$$\frac{\partial}{\partial \mathbf{n_c}} \psi(\mathbf{r}_\mathrm{t}) = 0 \text{ on } \mathbf{\Gamma}_\mathrm{wg} \text{ for TE modes}$$

Cutoff angular frequency: $k_\mathrm{t}/\sqrt{\varepsilon\mu} = \omega_\mathrm{co}$

Propagation constant: $k_z = \sqrt{\varepsilon\mu}\sqrt{\omega^2 - \omega_\mathrm{co}^2}$

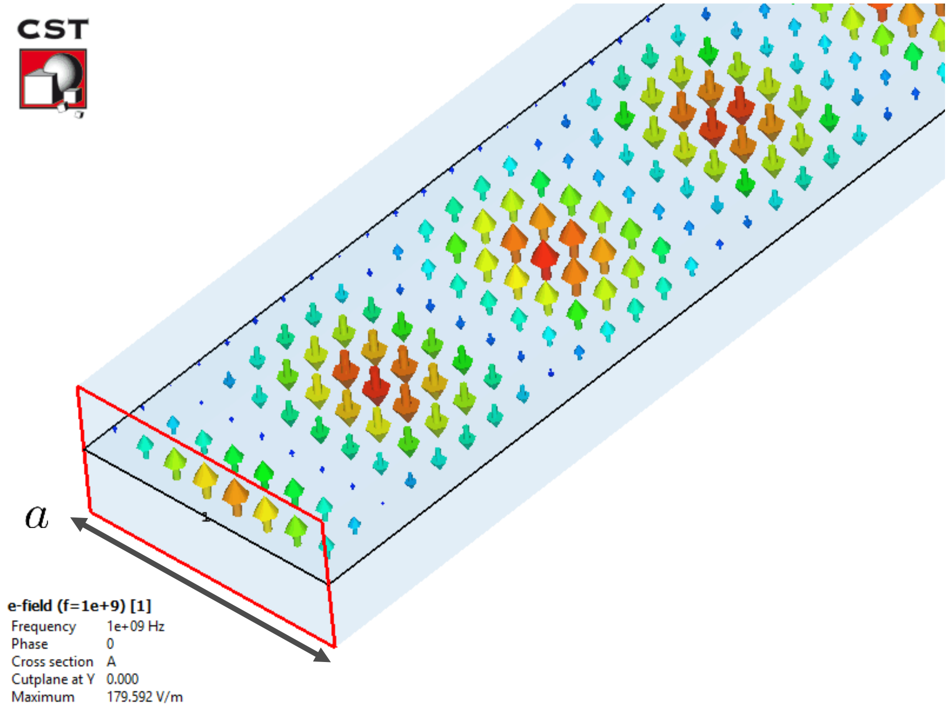
First Three Waveguide Modes of Considered Waveguides

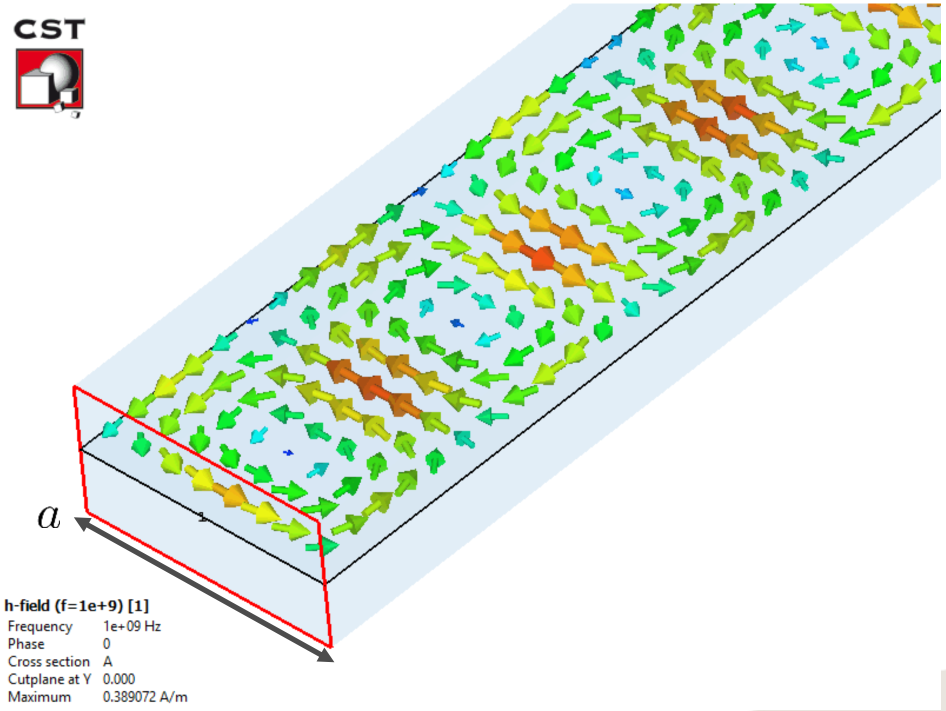


Rectangular Waveguide – TE_{10} Mode (Above Cutoff)

$$\psi(x,y) = E_0 \frac{a}{\pi} \cos\left(\frac{\pi}{a}x\right), \quad k_\mathrm{t} = \frac{\pi}{a}, \quad f_\mathrm{co} = \frac{c}{2a}$$

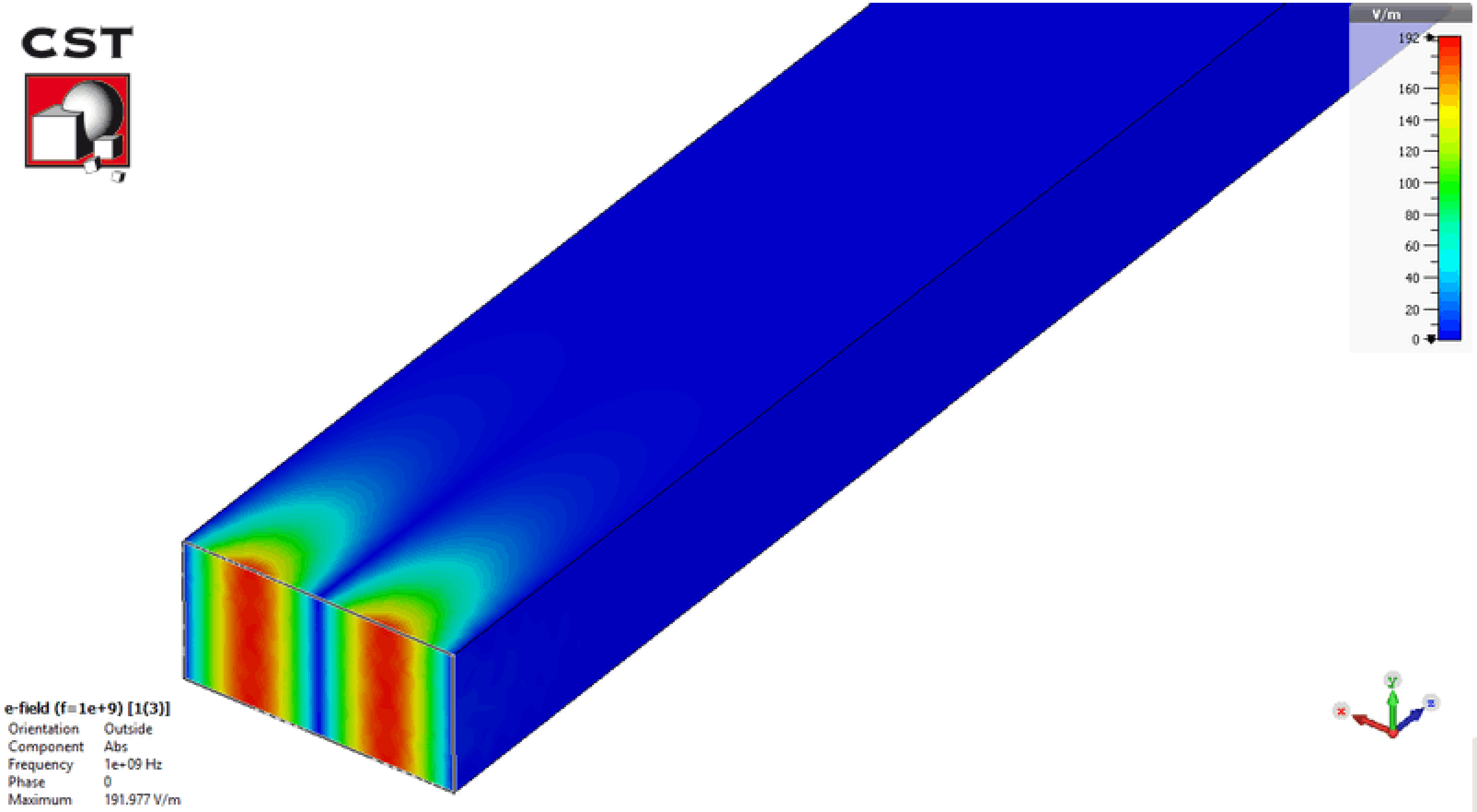
$$\underline{\mathbf{E}}(\mathbf{r}) = \begin{pmatrix} 0 \\ E_0 \sin\left(\frac{\pi}{a}x\right) \\ 0 \end{pmatrix} \mathrm{e}^{j(\omega t - k_z z)} \qquad \underline{\mathbf{H}}(\mathbf{r}) = \begin{pmatrix} -E_0 \frac{k_z}{\omega\mu} \sin\left(\frac{\pi}{a}x\right) \\ 0 \\ -\frac{1}{j\omega\mu} \frac{\pi}{a} E_0 \cos\left(\frac{\pi}{a}x\right) \end{pmatrix} \mathrm{e}^{j(\omega t - k_z z)}$$





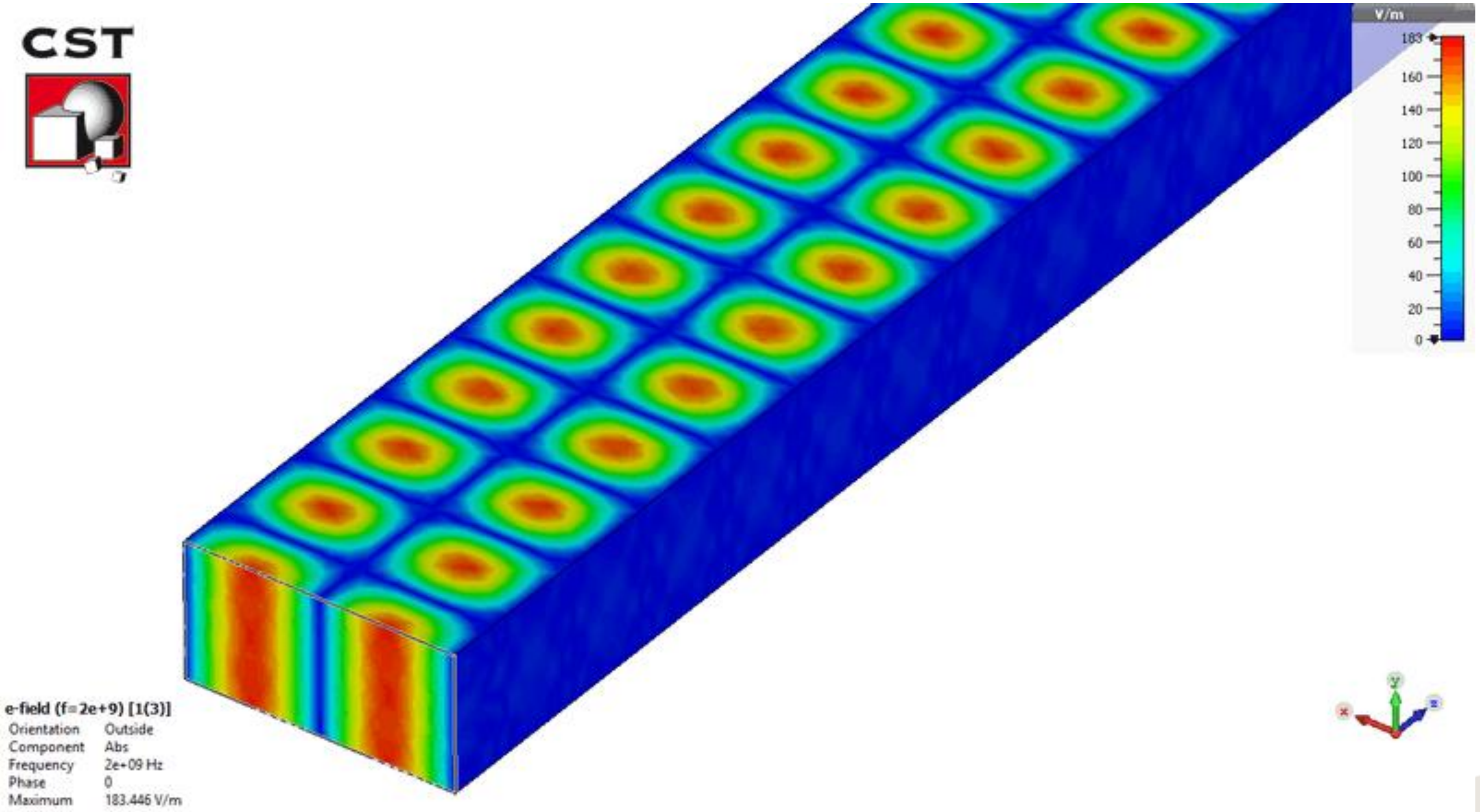
Rectangular Waveguide – TE_{20} Mode (Below Cutoff)

… the wave decays exponentially, *k* is purely imaginary …



Rectangular Waveguide – TE_{20} (Above Cutoff)

… excitation above cutoff frequency, mode can propagate, *k* is again real …



Eigenmodes – Standing Wave Solutions of the Homogeneous Wave Equation

Eigenmodes are solutions of the wave equation for the non-excited, loss-free and charge-free case:

$$\Delta \mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mathbf{0}$$ assuming time-harmonic fields

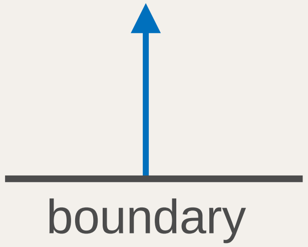
$$\Delta \mathbf{E}(\mathbf{r})\cos(\omega t - \varphi) + \varepsilon\mu\omega^2 \mathbf{E}(\mathbf{r})\cos(\omega t - \varphi) = \mathbf{0}$$ division by cosine term

Helmholtz equation in 3D

$$\Delta \mathbf{E}(\mathbf{r}) + \underbrace{\varepsilon\mu\omega^2}_{k^2} \mathbf{E}(\mathbf{r}) = \mathbf{0} \text{ in } \mathbf{\Omega}$$

with PEC boundary

$$\mathbf{n} \times \mathbf{E}(\mathbf{r},t) = \mathbf{0} \text{ on } \partial\mathbf{\Omega}$$

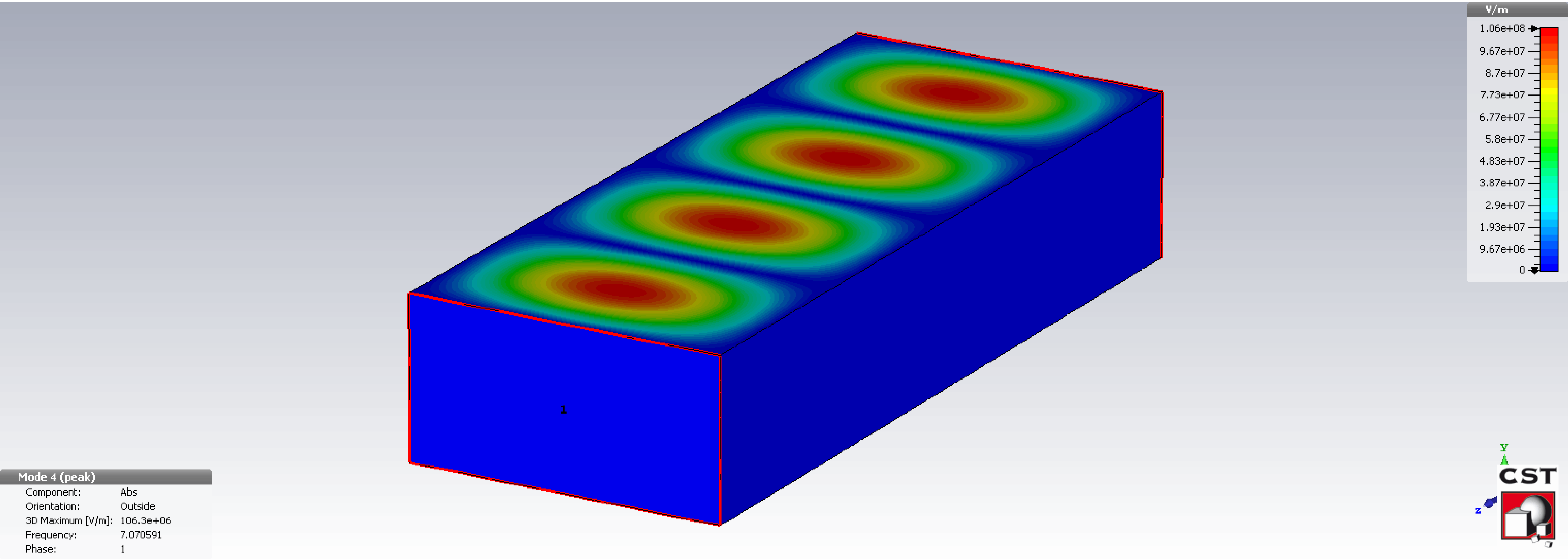


and/or PMC boundary

$$\mathbf{n} \cdot \mathbf{E}(\mathbf{r},t) = 0 \text{ on } \partial\mathbf{\Omega}$$

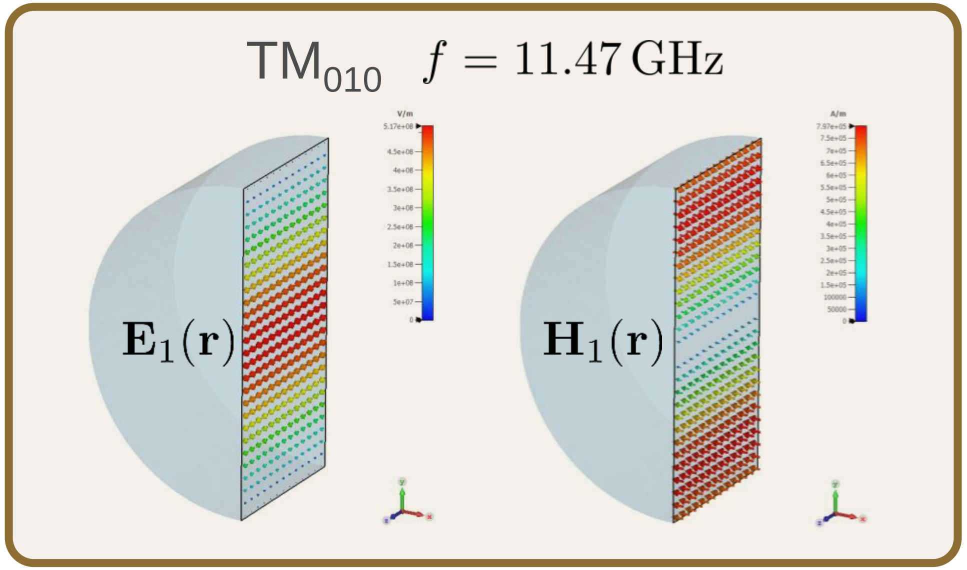
Infinite number of solutions characterized by field pattern $\mathbf{E}_n(\mathbf{r})$ and resonant frequency $\omega_n = k_n/\sqrt{\varepsilon\mu}$!

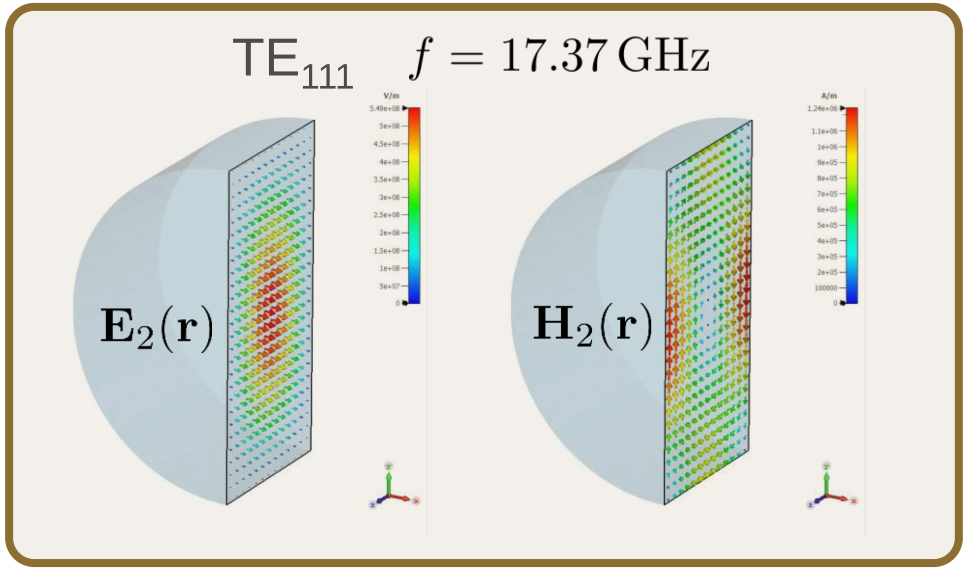
Electromagnetic Waves – Standing Wave in a Rectangular Waveguide (TE_{104})

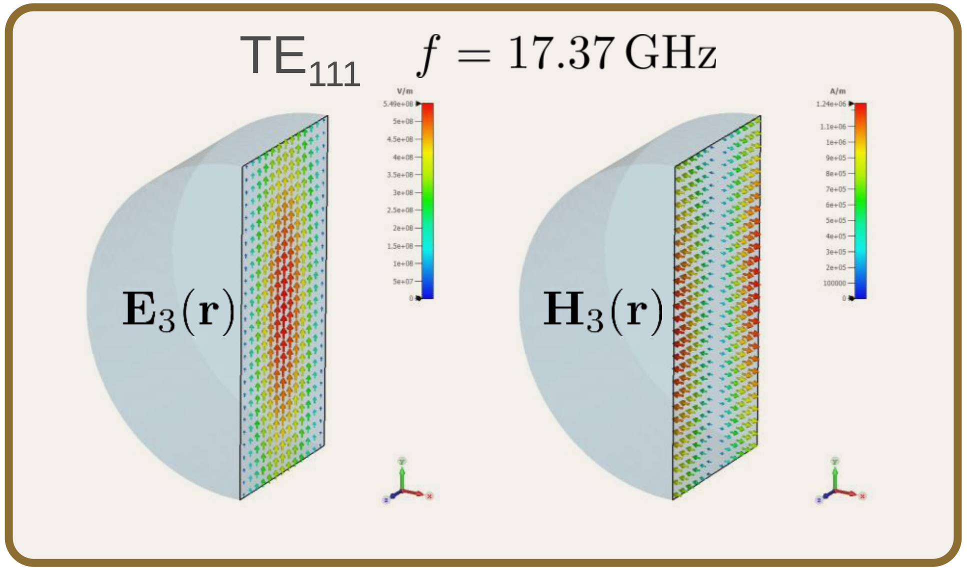


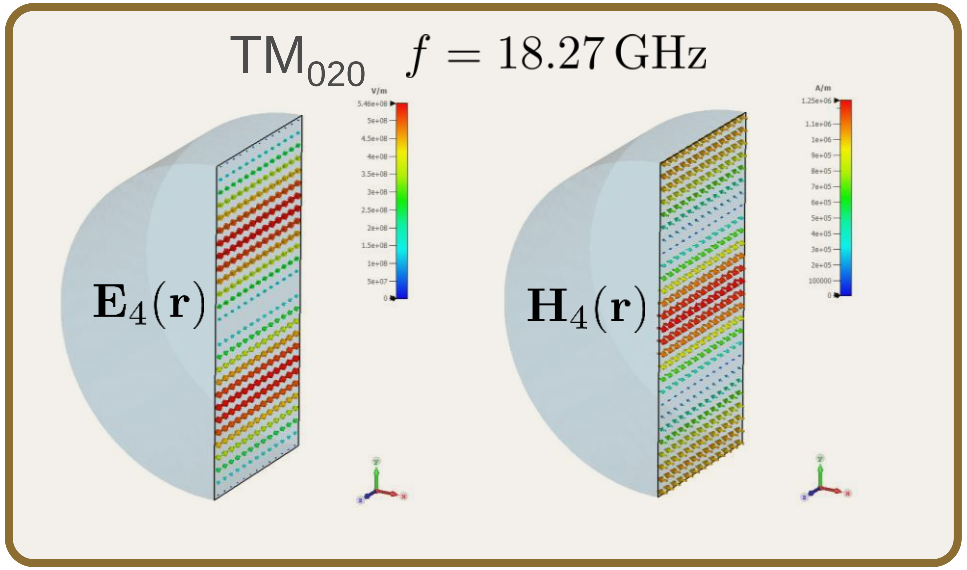
Simulated with CST Studio Suite®

Some Eigenmodes in a “Pillbox” Resonator (R = 10 mm, L = 10 mm)









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Backup Slides

TM_{mn} Modes

$$H_r = -C\frac{m}{r}J_m\left(j_{mn}\frac{r}{a}\right)\sin(m\varphi)\,e^{-jk_z z}$$

$$H_\varphi = -C\frac{j_{mn}}{a}J'_m\left(j_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$E_r = -CZ^E\frac{j_{mn}}{a}J'_m\left(j_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$E_\varphi = CZ^E\frac{m}{r}J_m\left(j_{mn}\frac{r}{a}\right)\sin(m\varphi)\,e^{-jk_z z}$$

$$E_z = C\left(\frac{j_{mn}}{a}\right)^2\frac{1}{j\omega\varepsilon}J_m\left(j_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$k_z^2 = k^2 - \left(\frac{j_{mn}}{a}\right)^2$$

$$Z^E = \frac{k_z}{\omega\varepsilon}$$

nth root of mth Bessel function

$$\omega_{c\,mn} = \frac{j_{mn}}{a}c$$

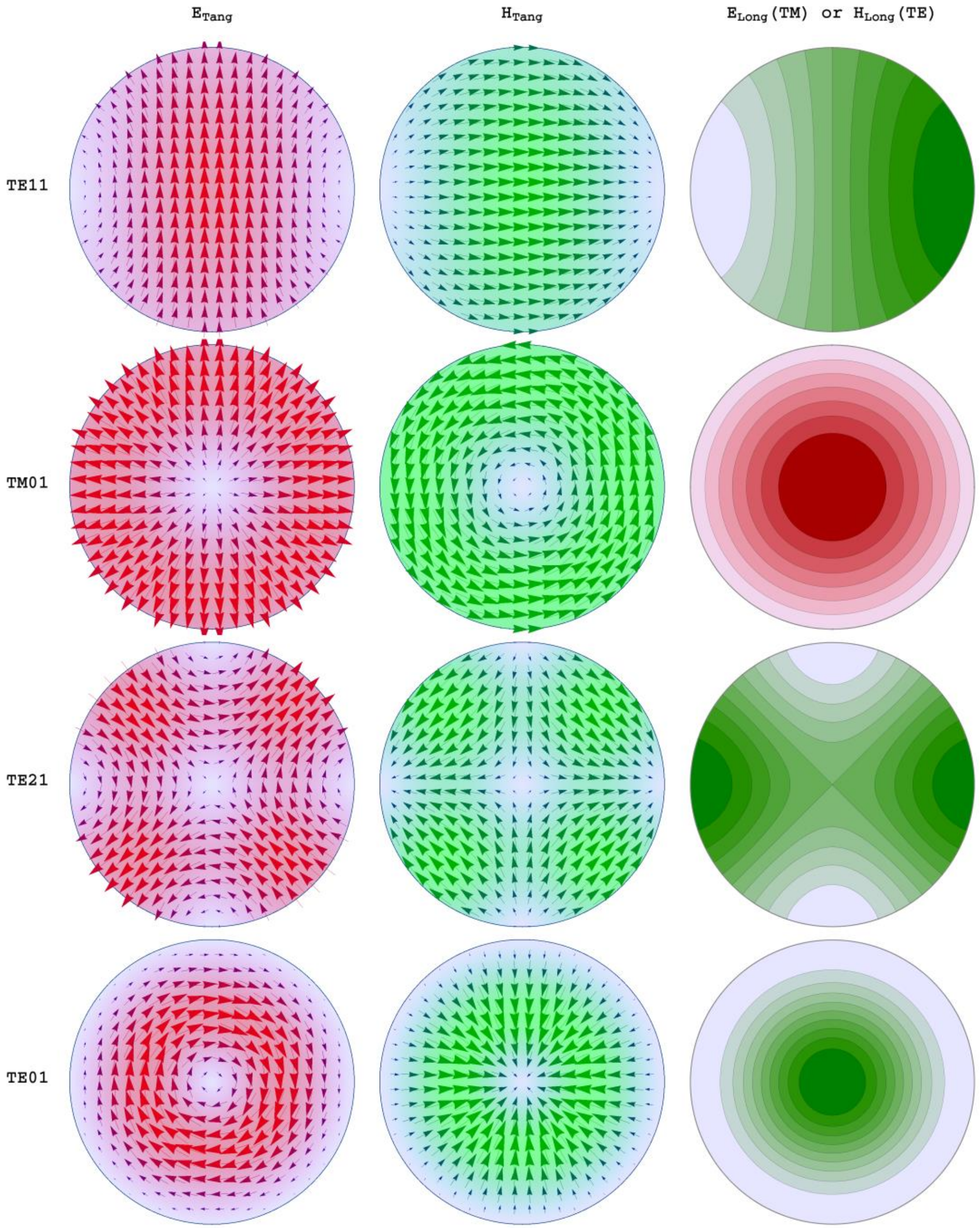

Figure courtesy of K. Brackebusch

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TE_{mn} Modes

$$E_r = -C\frac{m}{r}J_m\left(j'_{mn}\frac{r}{a}\right)\sin(m\varphi)\,e^{-jk_z z}$$

$$E_\varphi = -C\frac{j'_{mn}}{a}J'_m\left(j'_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$H_r = C\frac{1}{Z^H}\frac{j'_{mn}}{a}J'_m\left(j'_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$H_\varphi = -C\frac{1}{Z^H}\frac{m}{r}J_m\left(j'_{mn}\frac{r}{a}\right)\sin(m\varphi)\,e^{-jk_z z}$$

$$H_z = -C\left(\frac{j'_{mn}}{a}\right)^2\frac{1}{j\omega\mu}J_m\left(j'_{mn}\frac{r}{a}\right)\cos(m\varphi)\,e^{-jk_z z}$$

$$k_z^2 = k^2 - \left(\frac{j'_{mn}}{a}\right)^2$$

$$Z^H = \frac{\omega\mu}{k_z}$$

nth root of derivative of mth Bessel function

$$\omega_{c\,mn} = \frac{j'_{mn}}{a}c$$

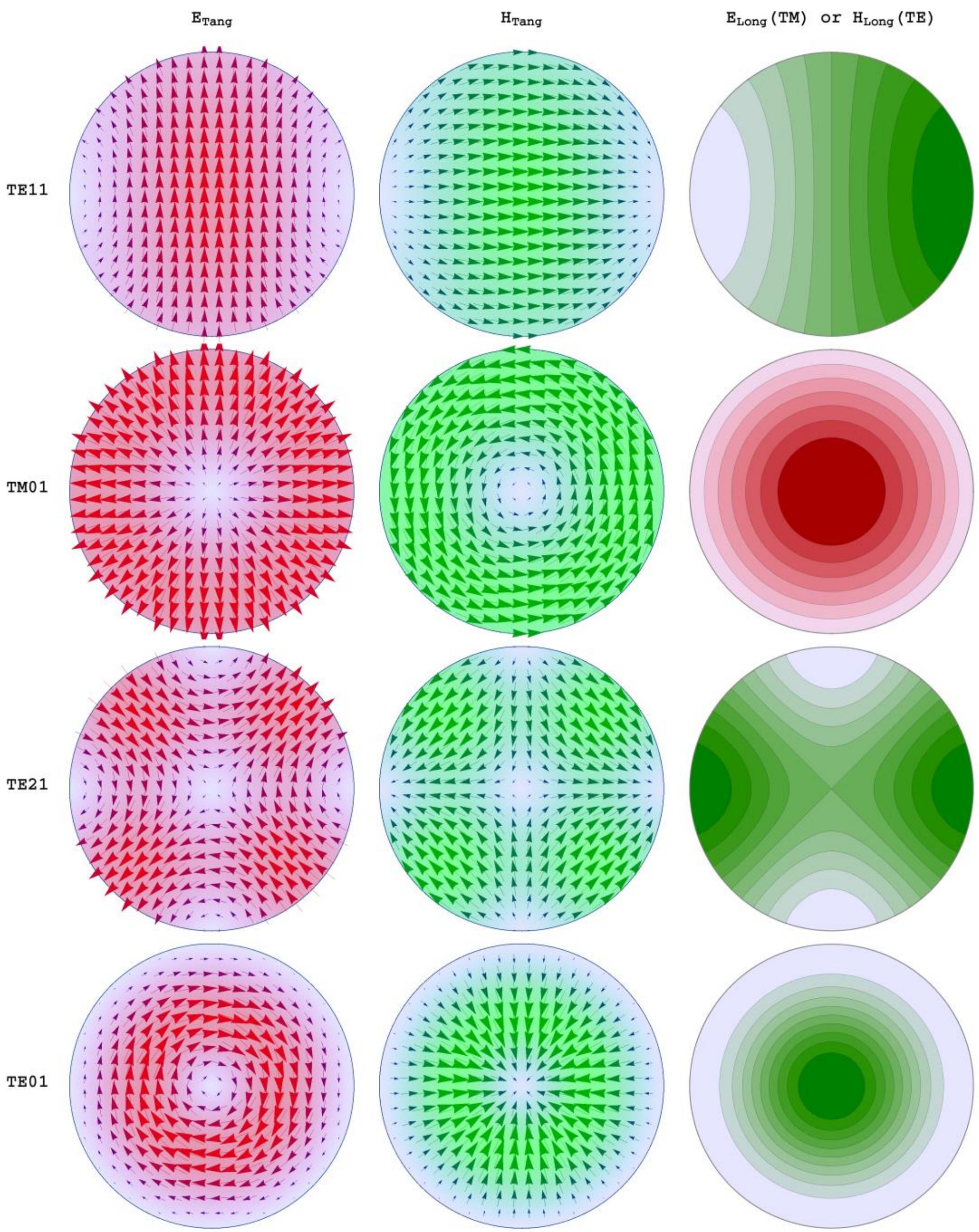
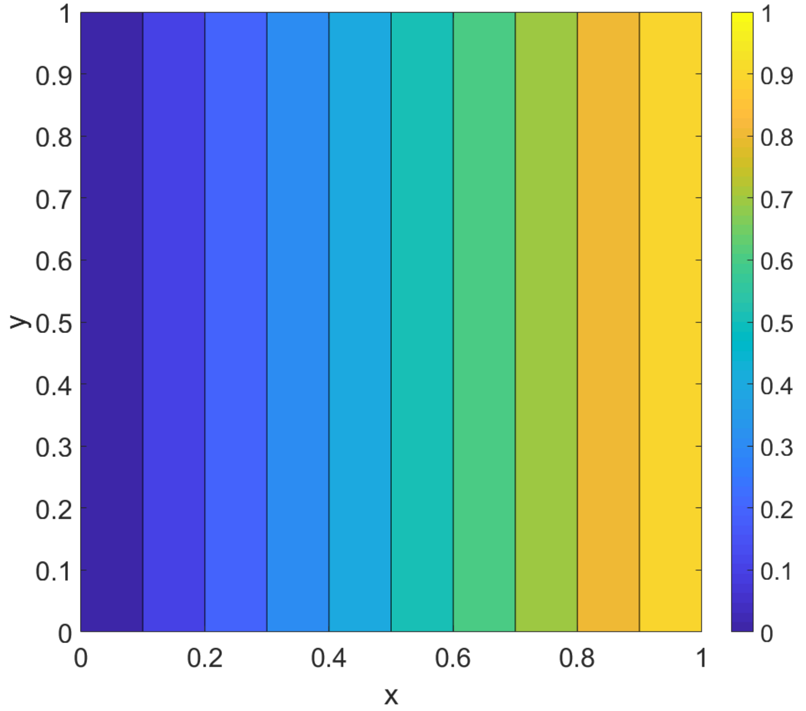


Figure courtesy of K. Brackebusch

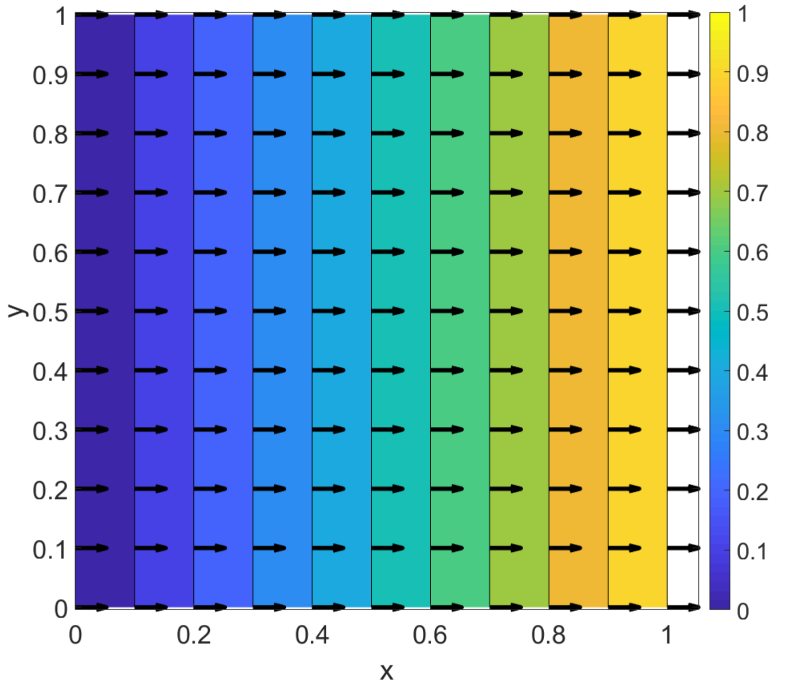
Gradient Operator

$$\nabla \phi(\mathbf{r}, t) = \operatorname{grad} \phi(\mathbf{r}, t) = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x}\phi(x,y,z,t) \\ \frac{\partial}{\partial y}\phi(x,y,z,t) \\ \frac{\partial}{\partial z}\phi(x,y,z,t) \end{pmatrix}}_{\text{for Cartesian coordinate system}}$$

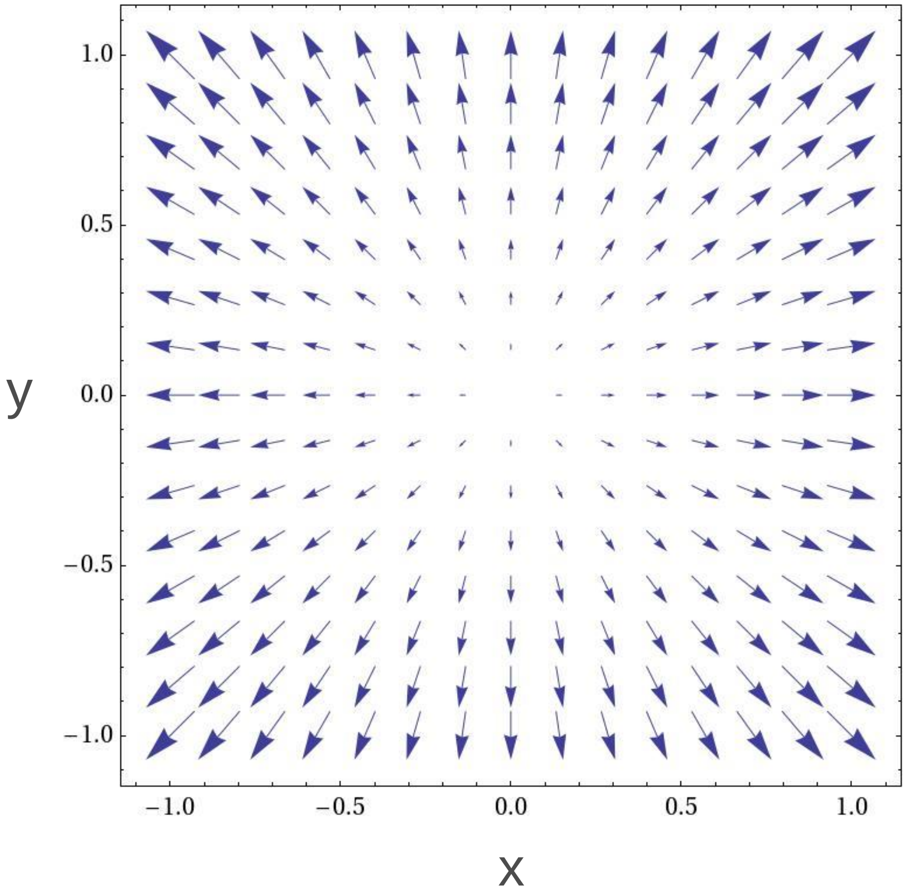
$\phi(x, y, z) = x$



$\nabla \phi(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$



Example Divergence



$$\mathbf{F}(x,y,z) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}0 = 2 > 0$$

Phase and Group Velocity

Wave pulses contains more than one frequency. In general, between frequency and wavenumber (wavelength) we have the Dispersion relation

$$\omega = \omega(k)$$

for free-space waves $\omega = ck$

we define

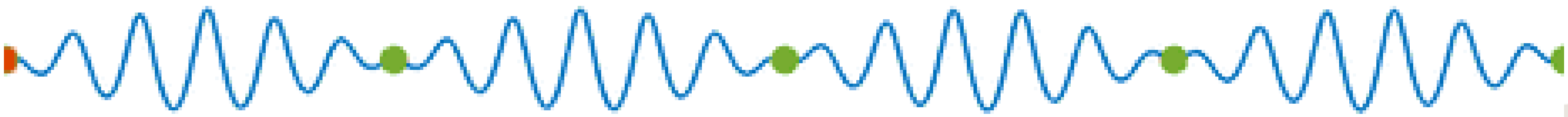
Phase velocity $v_{\text{ph}} = \dfrac{\omega(k)}{k}$

$$v_{\text{ph}} v_{\text{gr}} = c^2$$

Group velocity $v_{\text{gr}} = \dfrac{\mathrm{d}\omega(k)}{\mathrm{d}k}$

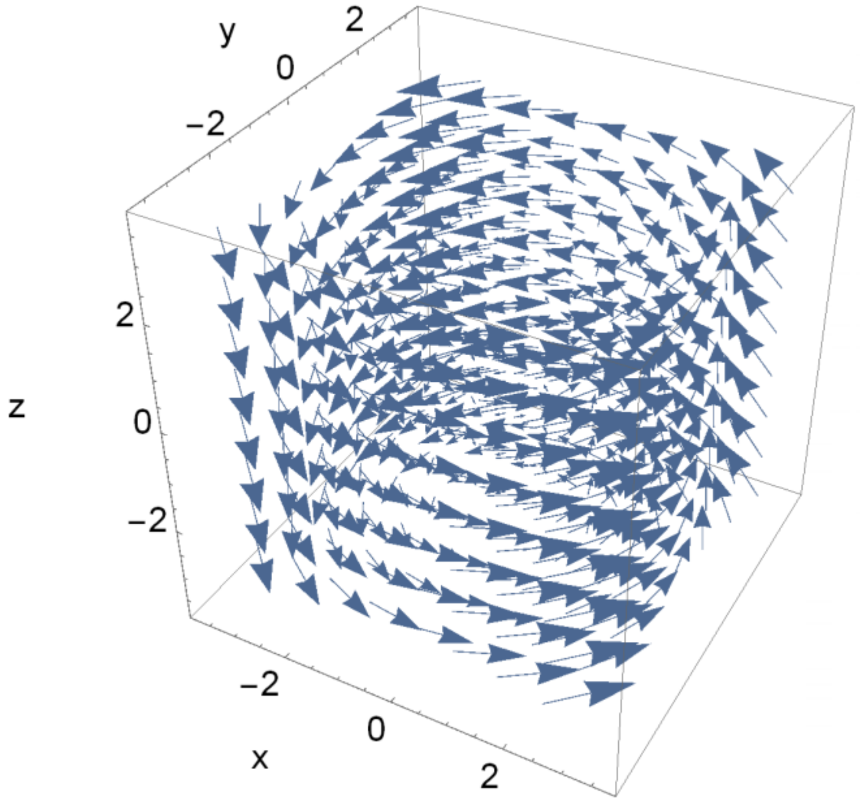
the velocity of a wave crest

the velocity of the wave group

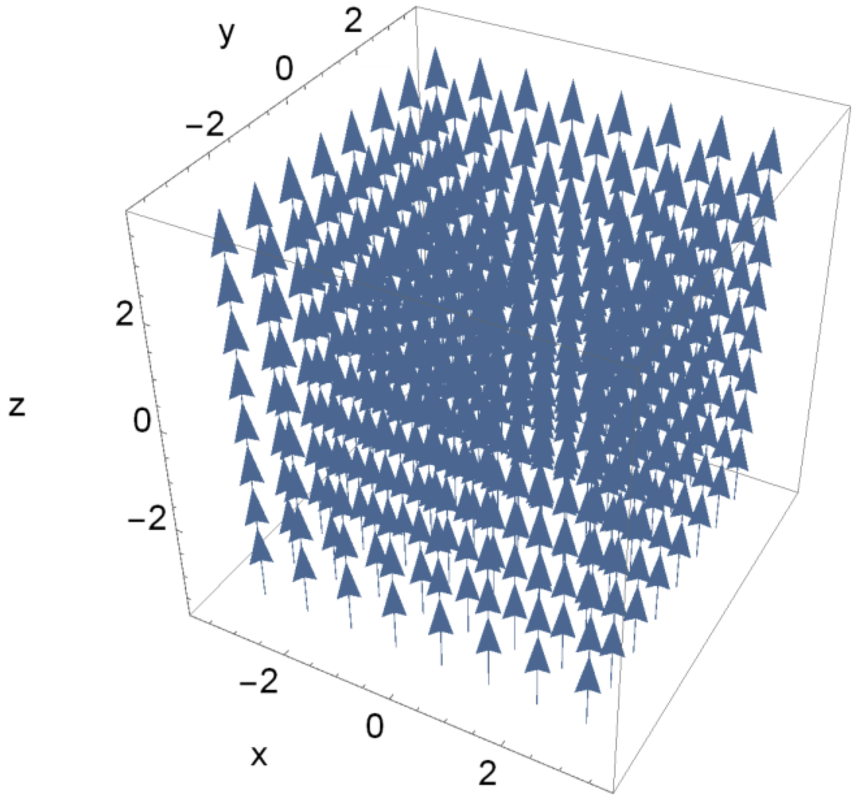


https://en.wikipedia.org/wiki/Group_velocity#/media/File:Wave_group.gif

Example Curl of 3D Field



$$\mathbf{F}(x,y,0) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$



$$\nabla \times \mathbf{F}(x,y,0) = \begin{pmatrix} -\frac{\partial}{\partial z} x \\ -\frac{\partial}{\partial z} y \\ \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

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Some Remarks in Material Modelling

Often it is not sufficient to consider the material parameters as constants, because matter can be

- inhomogeneous $\varepsilon_r = \varepsilon_r(\mathbf{r})$ $\mu_r = \mu_r(\mathbf{r})$

- dispersive, so that the material parameters are complex-valued and frequency-dependent:

$$\varepsilon_r = \underline{\varepsilon}_r(j\omega) \qquad \mu_r = \underline{\mu}_r(j\omega)$$

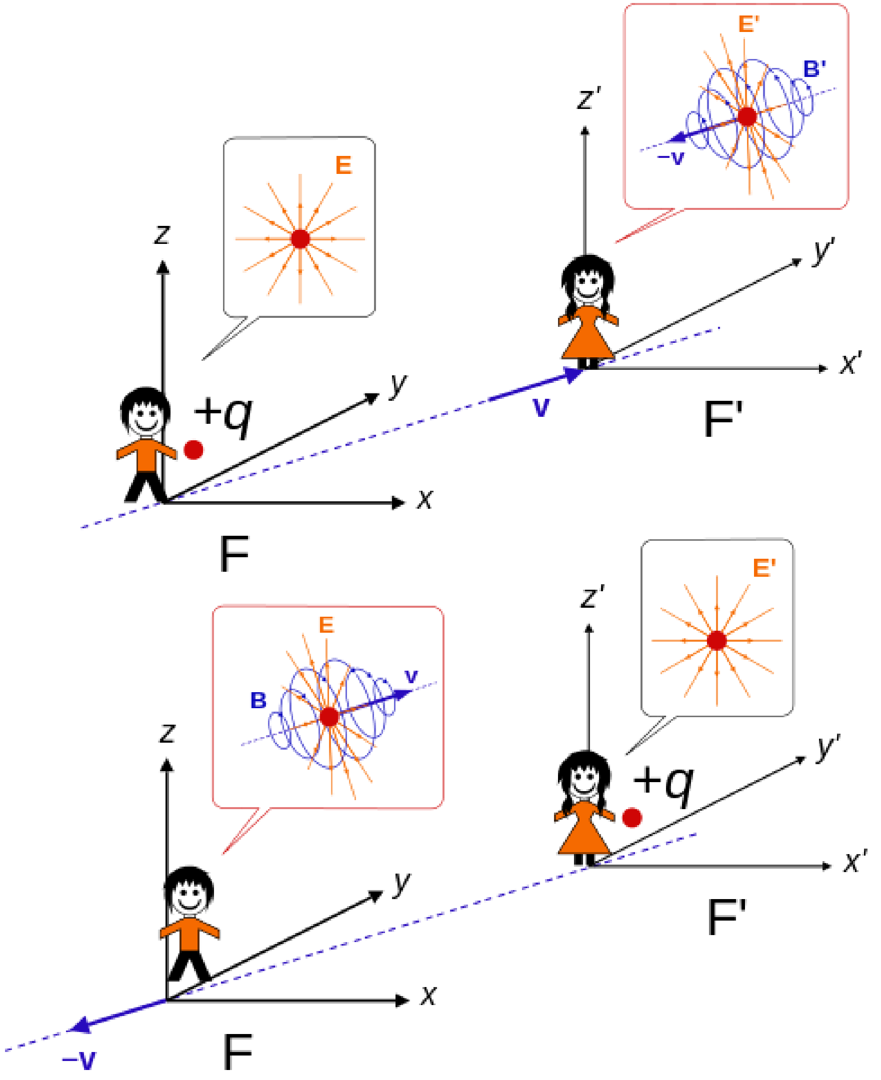
- anisotropic (directional dependent), so that the material parameters become tensors

$$\varepsilon_r = \begin{pmatrix} \varepsilon_{xx,r} & \varepsilon_{xy,r} & \varepsilon_{xz,r} \\ \varepsilon_{yx,r} & \varepsilon_{yy,r} & \varepsilon_{yz,r} \\ \varepsilon_{zx,r} & \varepsilon_{zy,r} & \varepsilon_{zz,r} \end{pmatrix} \qquad \mu_r = \begin{pmatrix} \mu_{xx,r} & \mu_{xy,r} & \mu_{xz,r} \\ \mu_{yx,r} & \mu_{yy,r} & \mu_{yz,r} \\ \mu_{zx,r} & \mu_{zy,r} & \mu_{zz,r} \end{pmatrix}$$

- non-linear (and possibly having a hysteresis in addition), so that the material parameters are functions on the field strength itself

$$\varepsilon_r = \varepsilon_r(\mathbf{E}) \qquad \mu_r = \mu_r(\mathbf{H})$$

Lorentz Transformations and Fields



$$\mathbf{E}_{\parallel'} = \mathbf{E}_{\parallel}$$

$$\mathbf{B}_{\parallel'} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}_{\perp'} = \gamma\left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}_{\perp}\right) = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})_{\perp},$$

$$\mathbf{B}_{\perp'} = \gamma\left(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}\right) = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E})_{\perp},$$

https://en.wikipedia.org/wiki/Lorentz_transformation