CAS “RF for accelerators”

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Longitudinal beam dynamics II. Instabilities & intensity effects

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This lecture is continuation of Longitudinal beam dynamics I by *F. Tecker*
See also Beam tracking I and II by *H. Timko*
and Introduction to the corresponding Hands-on
by *S. Albright & A. Lasheen*

Outline

- Longitudinal beam dynamics II
  - Single RF System
  - Double RF system
- Equations of motion with intensity effects
- Single-bunch intensity effects
  - Potential well distortion
  - Loss of Landau damping
  - Instabilities
- Linearised Vlasov equation
- Derivation of Lebedev equation
- Multi-bunch instabilities (driven by HOM)
  - Thresholds
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  - Spectrum
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LONGITUDINAL BEAM DYNAMICS II.
SINGLE RF SYSTEM

Synchronous particle

In a synchrotron, the average particle orbit is constant during acceleration
→ magnetic field B and RF frequency ω_{RF} increase synchronously.

Design momentum p_s follows the magnetic field variation $p_s = qB\rho$,
Corresponding particle has revolution period $T_0 = \dfrac{2\pi R}{\beta_0 c}$ (R – average machine radius)

and angular revolution frequency $\omega_0 = 2\pi f_0 = 2\pi/T_0$

A particle synchronised with RF frequency $\omega_{RF} = h\,\omega_0$
is called synchronous particle, its energy gain per turn is

$$\boldsymbol{\delta} E = q\,V_0\,sin\phi_s,$$

where ϕ_s is synchronous phase, h - harmonic number,
and q - particle charge.

The acceleration rate of synchronous particle is

$$\frac{\boldsymbol{\delta} E}{T_0} = \frac{dE_s}{dt} = \frac{\omega_0}{2\pi}\,qV_0\,sin\phi_s$$

In this phase convention: no acceleration, if $\phi_s = 0$ or π.

Longitudinal motion
→ synchrotron motion

Non-synchronous particle. First equation of longitudinal motion

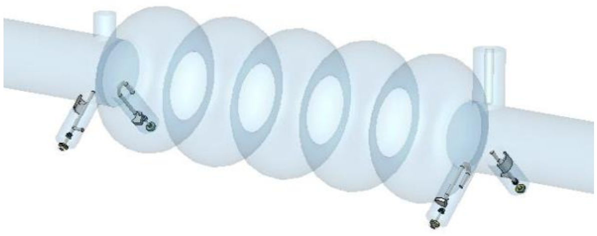
Non-synchronous particle with parameters *ω, ϕ, p, E* has small deviations *Δω, Δϕ, Δp, ΔE* from corresponding parameters ω_0, ϕ_s, p_s, E_s of the synchronous particle.

The energy gain for this particle is $q\,V_0 \sin\phi$, where $\phi = \phi_s + \Delta\phi$

and acceleration rate $$\frac{dE}{dt} = \frac{\omega_0}{2\pi}\, q\, V_0\, sin\phi$$

Subtracting similar equation for synchronous particle we obtain

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi}\, qV_0\, (sin\phi - sin\phi_s)$$



→ This is our 1st equation of longitudinal motion

Second equation of particle motion

The phase of any particle relative to the RF voltage is

$$\phi(t) = \int \omega_{RF} dt \; - h\theta(t),$$

where its azimuthal position $\theta = \int \omega dt$.

Then $\Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h}\frac{d\phi}{dt}$.

$$\frac{\Delta\omega}{\omega_0} = -\eta \frac{\Delta p}{p}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

Now, using the definition of slip factor η,

$$\frac{d\phi}{dt} = -h\omega_0 \frac{\Delta\omega}{\omega_0} = h\omega_0 \, \eta \, \frac{\Delta p}{p},$$

$$\frac{\Delta p}{p} = \frac{1}{\beta^2}\left(\frac{\Delta E}{E}\right)$$

and the 2nd equation of longitudinal motion is

$$\frac{d\phi}{dt} = \frac{h\omega_0^2\eta}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)$$

Phase equation.
Small amplitude oscillations

Combining two equations of motion in conjugate variables $\left(\frac{\Delta E}{h\omega_0}, \phi\right)$

$$\frac{d}{dt}\left(\frac{\Delta E}{h\omega_0}\right) = \frac{1}{2\pi h} qV_0(sin\phi - sin\phi_s)$$

&

$$\frac{d\phi}{dt} = \frac{\omega_{RF}^2 \eta}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right)$$

and assuming slow time-variation of E, ω_0, η, we obtain **phase equation:**

$$\frac{d^2\phi}{dt^2} = \frac{h\omega_0^2\eta}{2\pi\ \beta^2 E} qV_0(sin\phi - sin\phi_s)$$

For small amplitude particles with $\Delta\phi = \phi - \phi_s < 1$

(since $sin\phi = \sin(\phi_s + \Delta\phi) \simeq sin\phi_s + \Delta\phi\ cos\phi_s$) $\rightarrow$

$$\frac{d^2\Delta\phi}{dt^2} + \omega_{s0}^2\Delta\phi = 0$$

Where frequency of linear synchrotron oscillations is

$$\omega_{s0}^2 = -\frac{h\omega_0^2\eta\ cos\phi_s}{2\pi\ \beta^2 E} qV_0$$

Small amplitude oscillations. Phase stability

$$\frac{d^2\Delta\phi}{dt^2} + \omega_{s0}^2 \Delta\phi = 0$$

→ Equation of a harmonic oscillator

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\omega_{s0}^2 = -\frac{h\omega_0^2 \eta \cos\phi_s}{2\,\pi\,\beta^2 E} qV_0$$

Solutions: $\Delta\phi = \Delta\phi_m \cos\psi$

$\Delta\dot{\phi} = -\Delta\dot{\phi}_m \sin\psi$

where $\Delta\dot{\phi} = \frac{d\phi}{dt} \sim \Delta E$

$\Psi = \omega_{s0} t$ – synchronous angle,

$$\Delta\dot{\phi}_m = \Delta\phi_m \omega_{s0} = \frac{\omega_{RF}^2 \eta}{\beta^2 E}\left(\frac{\Delta E_m}{h\omega_0}\right)$$

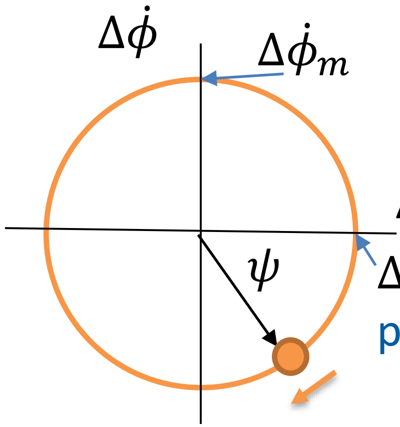
For phase stability:

$\omega_{s0}^2 > 0 \rightarrow \eta \cos\phi_s < 0$

No acceleration: $\sin\phi_s = 0$

→ $\phi_s = 0$ for $\gamma < \gamma_t$ $(\eta < 0)$

$\phi_s = \pi$ for $\gamma > \gamma_t$ $(\eta > 0)$



$\Delta\phi_m$ - amplitude of phase oscillations

Phase equation.
Large amplitude oscillations

The phase equation can be re-written as

$$\frac{d^2\phi}{dt^2} + \frac{\omega_{s0}^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0$$

Multiplying by $\dot{\phi} = \frac{d\phi}{dt}$ and integrating over t, we obtain an integral of motion

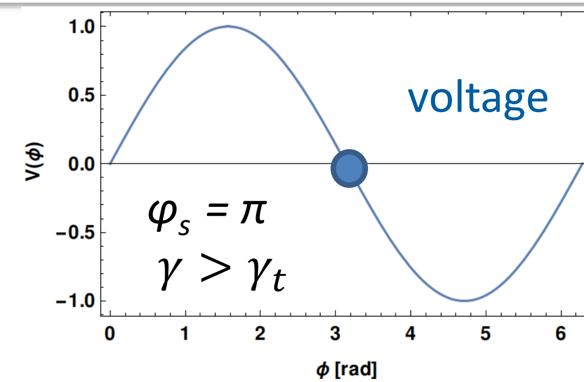
$$\frac{\dot{\phi}^2}{2\omega_{s0}^2} + U(\phi) = \mathcal{E}$$

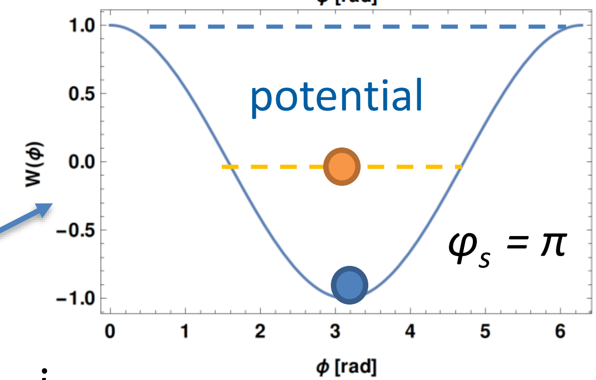
with energy of synchrotron oscillations $\mathcal{E}$ and RF potential

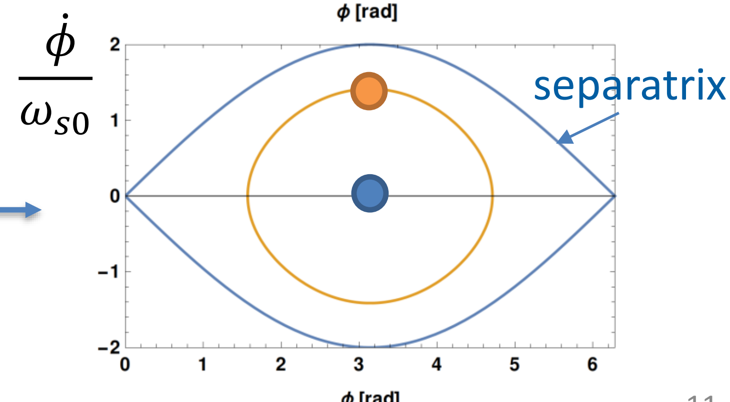
$$U(\phi) = -(\cos\phi + \phi\sin\phi_s)/\cos\phi_s$$

Phase trajectories are described by

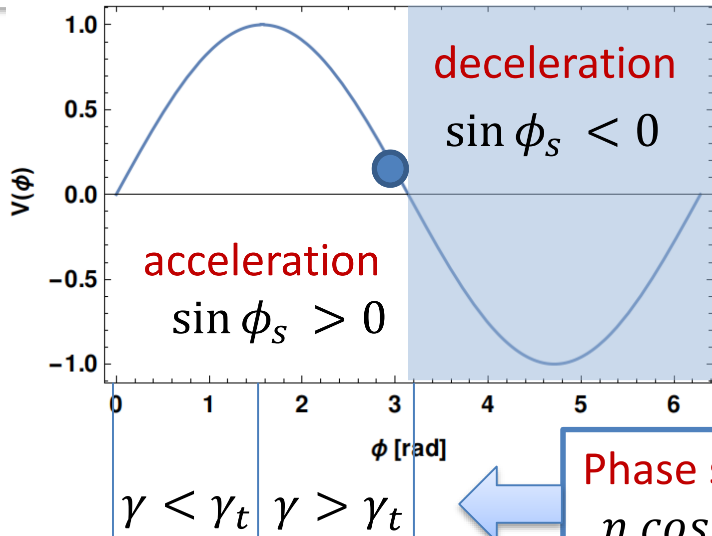
$$\dot{\phi} = \pm\omega_{s0}\sqrt{2[\mathcal{E} - U(\phi)]}$$



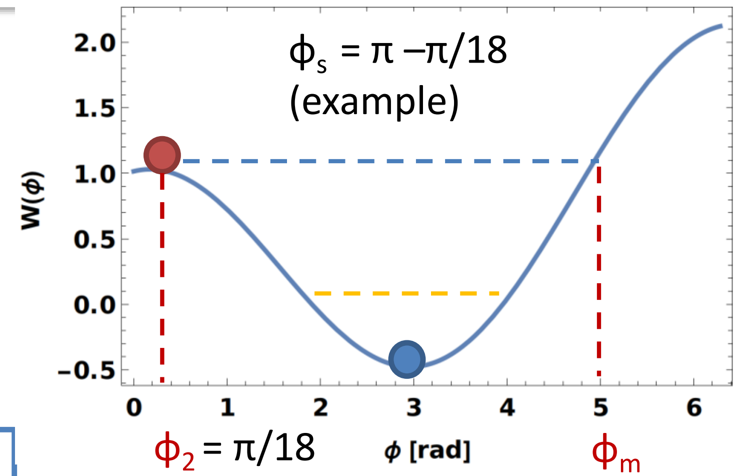




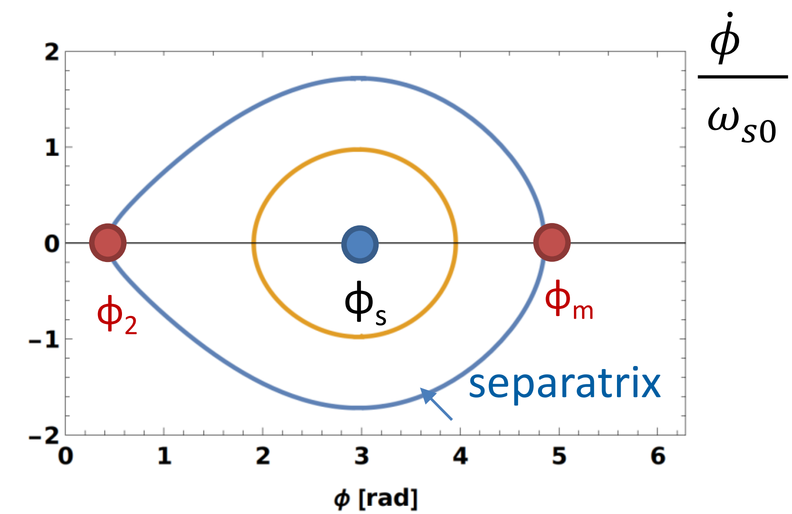
Acceleration. Separatrix



$\gamma > \gamma_t$



Phase stability
$\eta \cos\phi_s < 0$



Potential $U(\phi) = -(\cos\phi + \phi \sin\phi_s)/\cos\phi_s$.
ΔE = 0 at extremes of potential: $U'(\phi) = V(\phi) = 0$
→ $\sin\phi = \sin\phi_s$ has 2 solutions:

- $\phi_1 = \phi_s + 2\pi k$ (k = 0, 1,... h-1) - stable fixed point
- $\phi_2 = \pi - \phi_s + 2\pi k$ – unstable fixed point

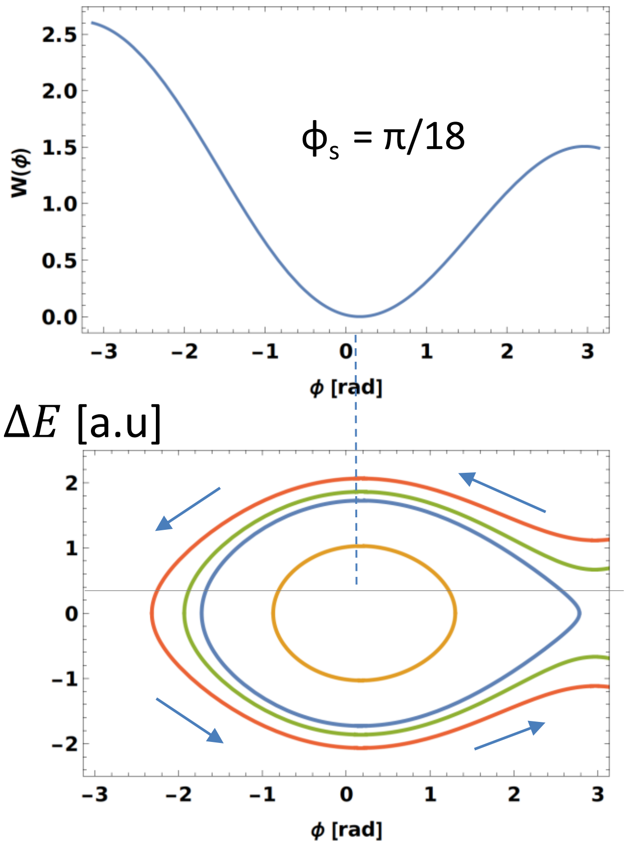
the 2[nd] turning point is ϕ_m defined by

$$U(\phi_m) = U(\phi_2) = U(\pi - \phi_s)$$

Acceleration. Transition crossing

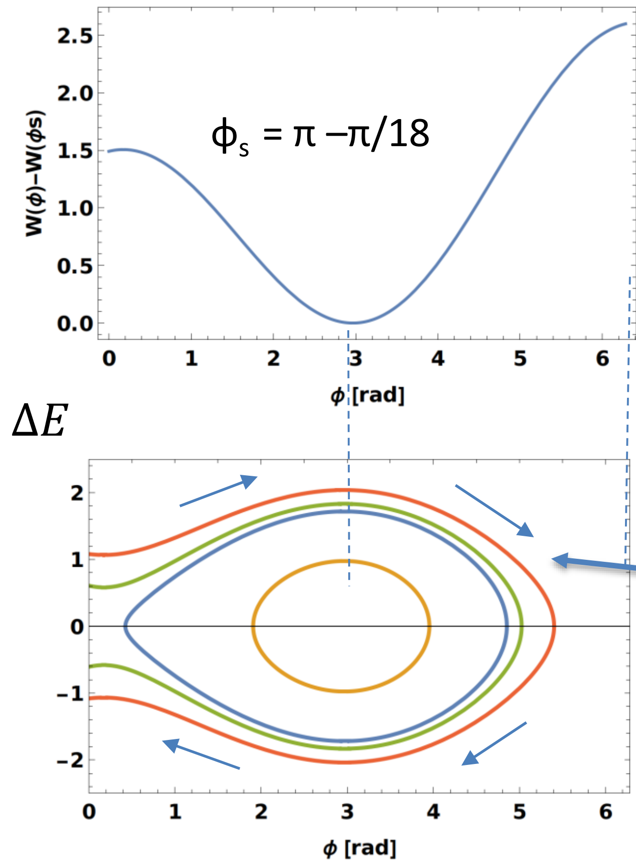
Below transition:

$\gamma < \gamma_t, \quad \eta < 0$



Above transition:

$\gamma > \gamma_t, \eta > 0$



Direction of particle motion. Since

$$\dot{\phi} = \frac{\omega_{RF}^2 \eta}{\beta^2 E} \left(\frac{\Delta E}{h\omega_0} \right)$$

→ ϕ increases ($\dot{\phi} > 0$) for

ΔE < 0 if η < 0

or

ΔE > 0 if η > 0

→ RF phase jump (π - 2ϕ_s) needed during transition crossing

Acceleration. RF bucket

The phase-space limited by the separatrix is the RF bucket:

$$\dot{\phi} = \frac{\omega_{RF}^2 \eta}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right) = \pm\omega_{s0}\sqrt{2[U(\pi-\phi_s) - U(\phi)]}$$

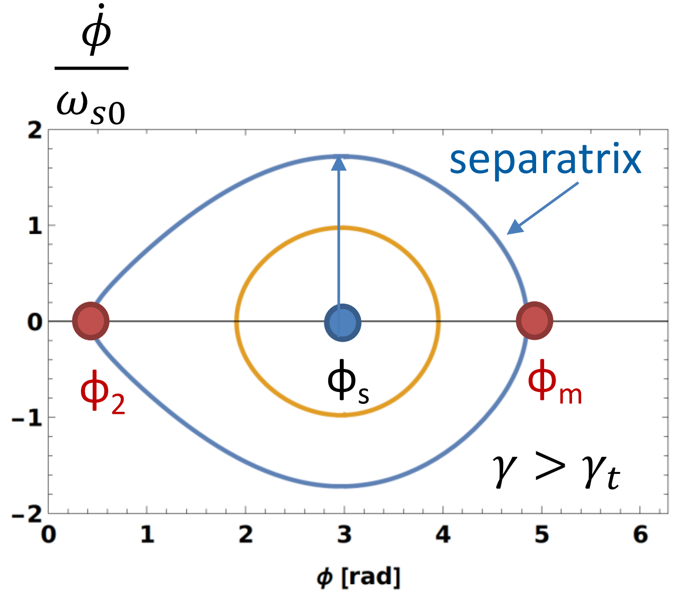
Bucket area [eVs] (also longitudinal acceptance): $A = \frac{2}{h\omega_0}\int_{\phi_2}^{\phi_m} \Delta E(\phi) d\phi$

Bucket length [rad] is $\phi_m - (\pi - \phi_s)$

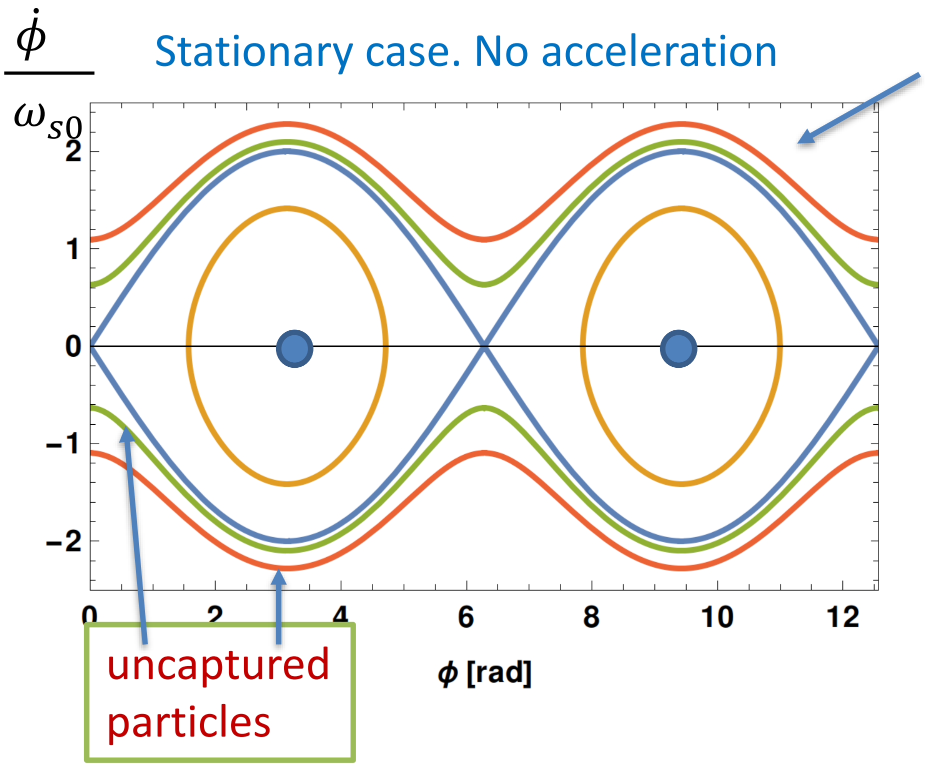
Bucket height (energy acceptance):

$$\frac{\Delta E_{max}}{E} = \frac{\nu_{s0}\,\beta^2}{h\eta}\sqrt{2[U(\pi-\phi_s) - U(\phi_s)]}$$

with synchrotron tune $\nu_{s0} = \frac{\omega_{s0}}{\omega_0}$



Longitudinal phase space. Bucket area



The separatrix separates the bound oscillations from unbound

No energy gain → $\sin\phi_s = 0$ and $U(\phi) = \cos\phi$ (for $\gamma > \gamma_t$). Then

$$\int_0^{2\pi} \sqrt{2[U(\pi-\phi_s) - U(\phi)]}\, d\phi$$

$$= 2\int_0^{2\pi} \sin\frac{\phi}{2}\, d\phi = 8$$

Due to acceleration the bucket area

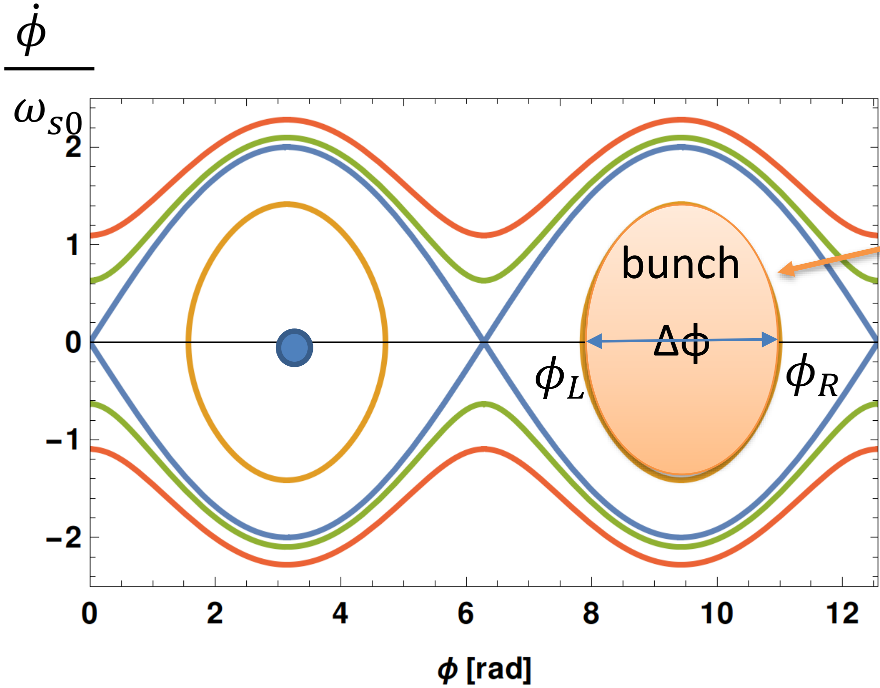
$$\mathrm{A} = \frac{16\beta}{\omega_{RF}}\sqrt{\frac{qVE}{2\pi h|\eta|}}\,\Gamma(\phi_s)$$

is reduced by a factor $\Gamma(\phi_s)$, which approximately is

$$\Gamma(\phi_s) \simeq \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

Note that A = 0 for $\phi_s = \pi/2$.

Longitudinal phase space.
Bunch emittance



Particles usually fill only some part of the bucket, a bunch

One can have up to $h = \omega_{RF}/\omega_0$ bunches in the ring

Longitudinal emittance [eV s] is the area inside the limiting particle trajectory

$$\epsilon = \frac{2}{h\omega_0}\int_{\phi_L}^{\phi_R} \Delta E_b(\phi)d\phi$$

with $U(\phi_L) = U(\phi_R)$.

For ions units are [eV s/charge]

Bunch length [rad] $\Delta\phi_b = \phi_L - \phi_R$

and in [s] $\tau = \Delta\phi_b/(h\omega_0)$

Longitudinal bunch emittance

Phase trajectory (reminder)

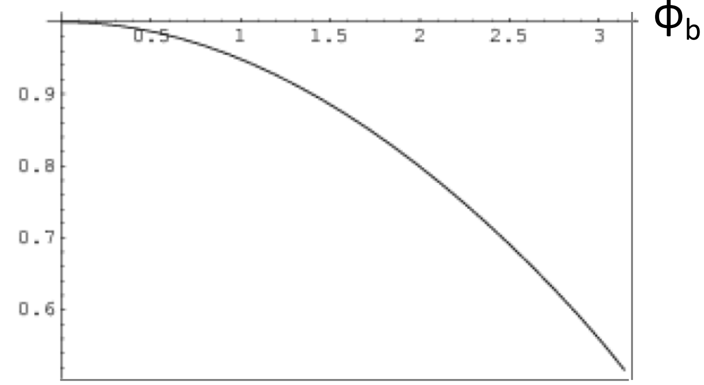
$$\frac{\omega_{RF}^2|\eta|}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right) = \omega_{s0}\sqrt{2[U(\phi_L) - U(\phi)]}$$

For non-accelerating bucket: $\phi_L - \phi_R = 2\phi_b$
and bunch emittance is

$$\epsilon_b = \frac{4\ E\ \omega_{s0}}{\omega_{RF}^2|\eta|}\int_0^{\phi_b}\sqrt{2(cos\phi - cos\phi_b)}d\phi$$

Short-bunch approximation $\frac{\pi\phi_b^2}{4}$

Ratio $\boldsymbol{\epsilon_b/\epsilon_b^{short}}$
(no acceleration)



→ Short-bunch approximation
$\epsilon_b = \pi\Delta E\ \phi_b/h = \pi\ \Delta E\ \tau/2$
should be used with caution
for $\phi_b > 1$

Energy loss

- Even without acceleration the average energy of particles may change due to synchrotron radiation, induced voltage, electron cloud…
- Energy lost by a particle per turn $\Delta E = U_0$ is compensated by the RF system

$$U_0 = q\, V \sin\phi_s$$

→ Bucket becomes accelerating with $B = const$.

→ Damping of synchrotron oscillations

Synchrotron radiation at LHC flat top:

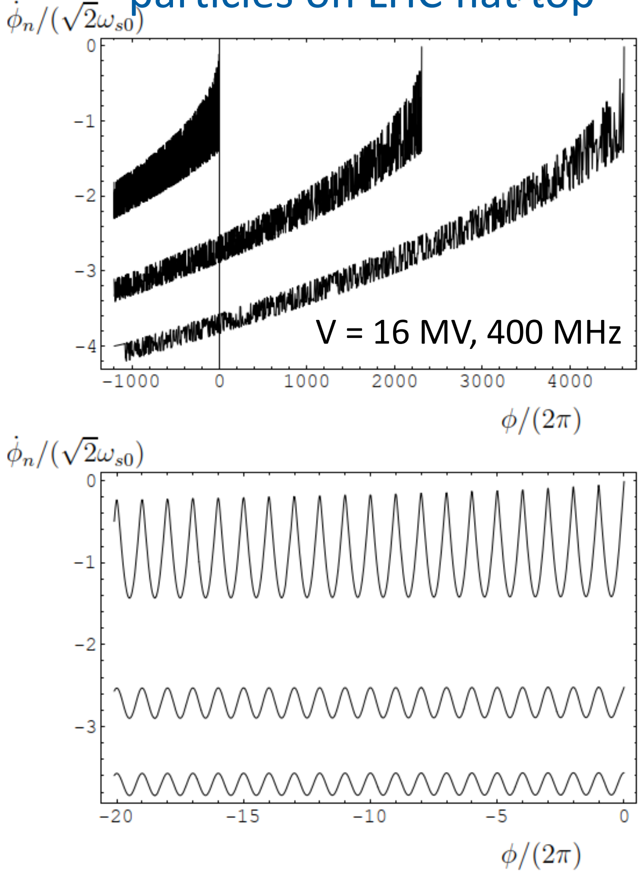
U_0 = 7 keV for protons and for $^{208}Pb^{82+}$ ions

$$\frac{U_{ion}}{U_p} \simeq \frac{Z^6}{A^4} \simeq 162$$ larger!

Damping time $\frac{\tau_{ion}}{\tau_p} \simeq \frac{A^4}{Z^5} \simeq 0.5$ is also smaller!

→ 6.3 h for longitudinal emittance

Phase trajectories of lost particles on LHC flat top



Synchrotron frequency

Due to non-linearity of RF voltage

$$\frac{d^2\phi}{dt^2} + \frac{\omega_{s0}^2}{cos\phi_s}(\sin\phi - \sin\phi_s) = 0$$

all particles oscillate with different frequencies $\omega_s = 2\pi f_s$ which depend on amplitude of oscillations.

Since

$$\frac{d\phi}{dt} = \dot{\phi} = \omega_{s0}\sqrt{2[U(\phi_L) - U(\phi)]}$$

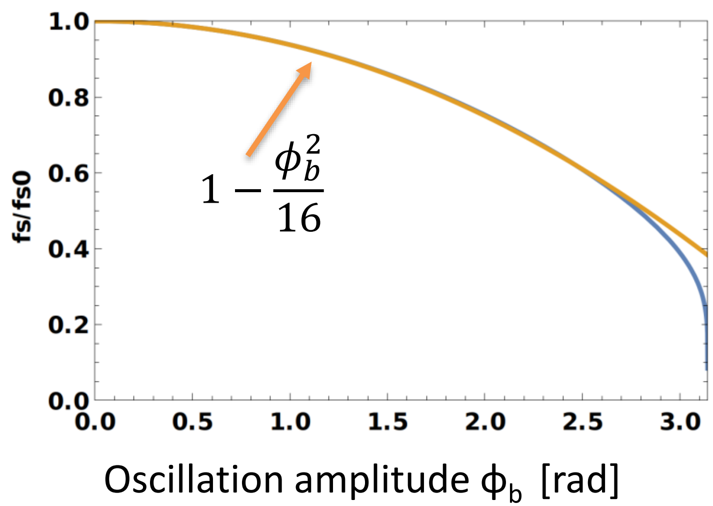
the period of synchrotron oscillations is

$$T_s = \oint \frac{d\phi}{\dot{\phi}} = \frac{2}{\omega_{s0}} \int_{\phi_L}^{\phi_R} \frac{d\phi}{\sqrt{2[U(\phi_L) - U(\phi)]}}$$

For non-accelerating bucket in single RF system

$$\frac{\omega_s}{\omega_{s0}} = \frac{\pi}{2\,K(\sin\frac{\phi_b}{2})} \simeq 1 - \frac{\phi_b^2}{16}$$

K(x) is the complete elliptical integral of the 1st kind



Hamiltonian of longitudinal motion

Using conjugate variables ϕ and *ΔE/(hω₀)*, the equations of longitudinal motion can be also presented as

$$\frac{\partial H}{\partial \phi} = -\left(\frac{\dot{\Delta E}}{h\omega_0}\right) \quad \text{and} \quad \frac{\partial H}{\partial\left(\frac{\Delta E}{h\omega_0}\right)} = \dot{\phi},$$

with the Hamiltonian

$$H = \frac{1}{2}\frac{h^2\omega_0^2\eta}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right)^2 - \frac{qV\cos\phi_s}{2\pi}\,[U(\phi) - U(\phi_s)],$$

It can be easily generalised for more RF systems (with different harmonic numbers *h*) and induced voltage (collective effects).

Adiabaticity: why is it important?

During beam acceleration, the Hamiltonian *H* of the system depends on time.
If changes are slow enough (adiabatic), *H* is considered to be quasi-static.

The parameter λ changes adiabatically if $T\frac{d\lambda}{dt} \ll \lambda$, where *T* is a period.

Applying for the synchrotron motion $\frac{dT_s}{dt} \ll 1$

Adiabatic invariant of motion (action)

$$J = \frac{1}{2\pi}\oint \frac{\Delta E(\phi)}{h\omega_0}\, d\phi = \frac{\epsilon}{2\pi}$$

→ Longitudinal bunch emittance is an invariant of motion during acceleration and RF manipulations, if parameter change is adiabatic

DOUBLE RF SYSTEM

Multi-harmonic RF system

Many rings have multiple RF systems (with different f_{rf}) for

- Beam acceleration. In low energy rings this is dictated by the fast change of particle velocity βc: $$f_{rf} = hf_0 = h\frac{\beta c}{2\pi R}$$
- Acceleration of different particles (leptons, protons, ions) → **6** different RF systems in the CERN SPS in the past: at 100 MHz, **200 MHz** (SW and **TW**), 352 MHz, 400 MHz and **800 MHz**.
- Beam transfer from one ring to another.
- RF manipulations (bunch splitting, merging, rotation, beam coalescing, batch compression, controlled emittance blow-up, …). → 5 RF systems used in the CERN PS (see lectures of *H. Damerau*).

Note: wide-band RF system (as Finemet in CERN PSB) allows simultaneous operation at different harmonics h

Higher harmonic RF system

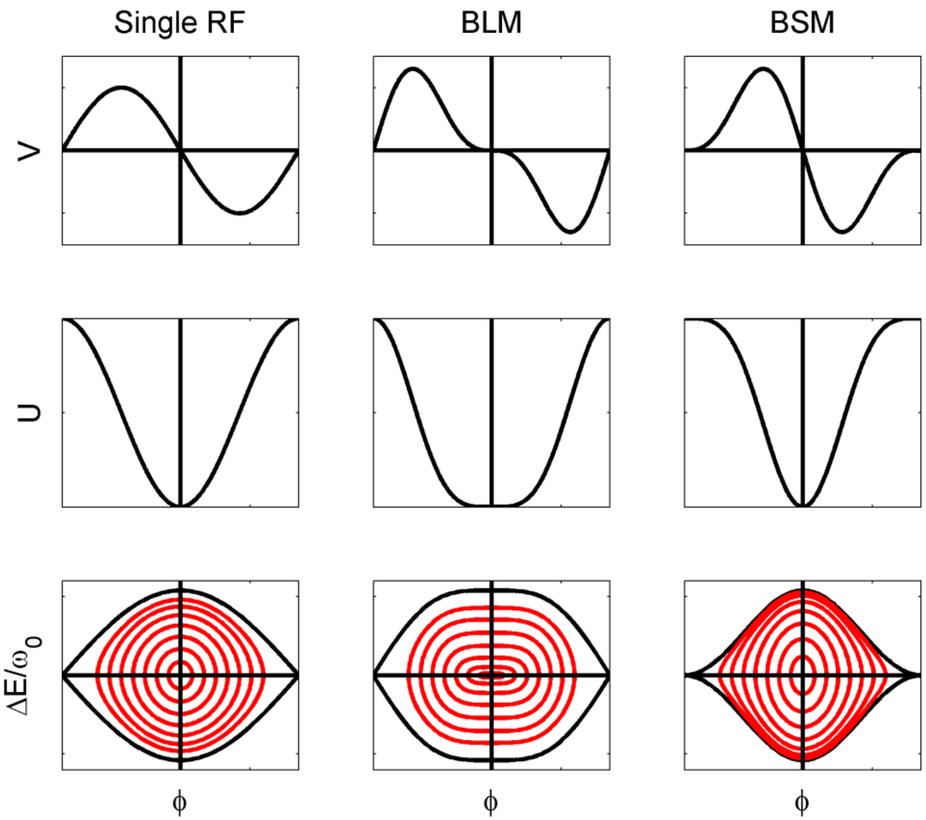
Main applications of a higher harmonic (HH) RF system used in addition to the main RF system

- Reduction of the peak line density - bunch flattening (to decrease space charge and other intensity effects)
- Increase of available bucket area
- Increase of synchrotron frequency spread → to cure beam instabilities (see next lecture)
- Modification of a zero-amplitude synchrotron frequency f_{s0}
- Control of bunch length
- Exotic: compensation of $(V_1 + V_2)$ during momentum slip-stacking, …

The HH RF system can be also passive, using the voltage induced by the beam, with an amplitude and a phase which are the functions of the beam parameters → applied mainly in the **lepton rings**.

Higher harmonic RF system: operation modes

Example for $n = 2$, $r = 1/2$, $\sin\phi_{s0} = 0$



above transition $\Phi_n = 0$ $\Phi_n = \pi$

below transition $\Phi_n = \pi$ $\Phi_n = 0$

The total voltage in double RF system

$$V = V_1\,[\sin\phi + r\sin(n\phi + \Phi_n)],$$

V_1 and $V_n = r\,V_1$ - voltage amplitudes of the main and HH RF system with harmonic $h_n = nh$.

Typically $V_n < V_1$ and even $V_n < V_1/n$.

The operation mode is defined by Φ_n :

(1) bunch-lengthening mode (BLM) or "out-of-phase"

(2) bunch-shortening (BSM) or "in-phase"

→ The choice of the mode is dictated by the application

→ for even n; opposite for odd n = 3, 5 ...

Double RF system: synchronous phase and synchrotron frequency

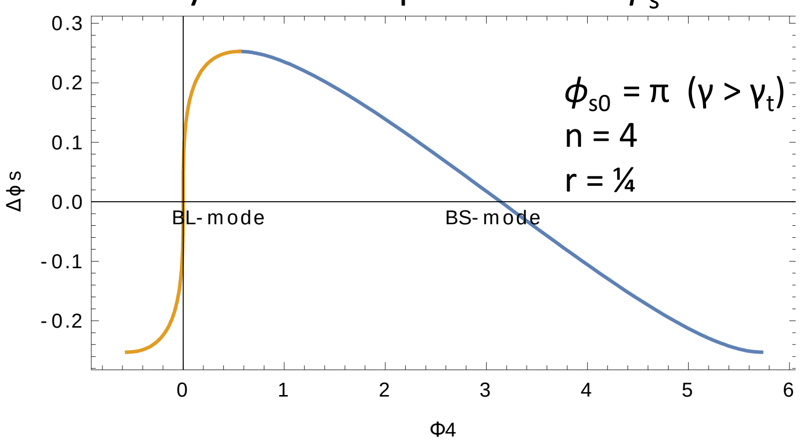
Synchronous phase ϕ_s

From definition of a synchronous particle

$$\sin \phi_{s0} = \sin \phi_s + r \sin (n\phi_s + \Phi_n)$$

$\rightarrow \phi_s = \phi_{s0}$ for $\Phi_n = -n\phi_s + \pi$ (or 0)

Synchronous phase shift $\Delta\phi_s$

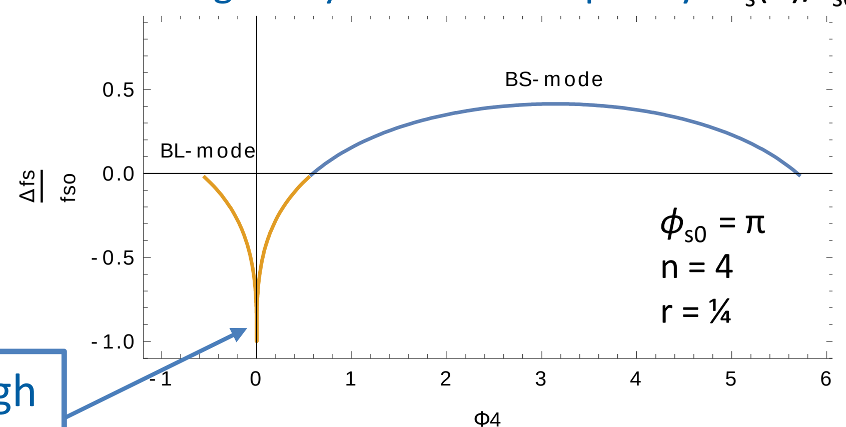


Synchrotron frequency $f_s(0)$

From V' = 0, the synchrotron frequency at zero amplitude of synchrotron oscillations

$$f_s(\phi_a = 0) = f_{s0} \sqrt{\frac{\cos \phi_s + rn \cos (n\phi_s + \Phi_n)}{\cos \phi_{s0}}}$$

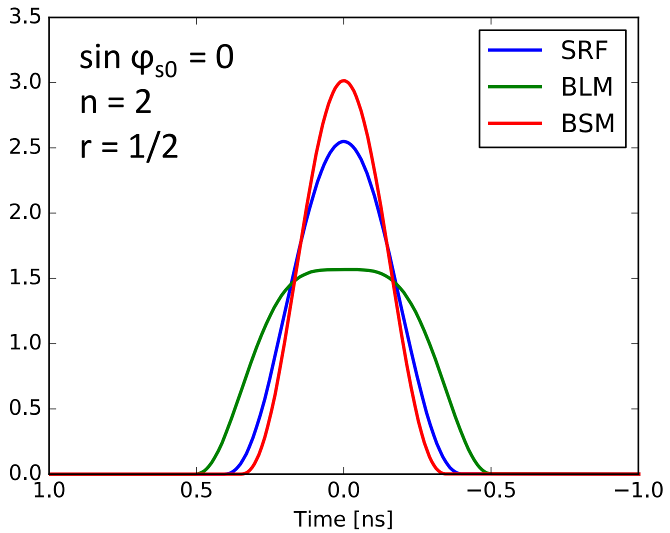
Relative change in synchrotron frequency $\Delta f_s(0)/f_{s0}$



Large frequency change, but also high sensitivity to phase shift in BL-mode

Double RF system: bunch shape

The peak line density is reduced in BL-mode and increased in BS-mode



Flat bunches (BL-mode)

The "flat" bunches are obtained when $V'(\varphi_s) = 0$ and $V''(\varphi_s) = 0$:

$$\cos\phi_s = -rn\cos\left(n\phi_s + \Phi_n\right),$$
$$\sin\phi_s = -rn^2\sin\left(n\phi_s + \Phi_n\right).$$

The HH voltage parameters **r** and Φ_n are defined for given n and φ_{s0}

$$r^2 = \frac{1}{n^2} - \frac{\sin^2\phi_{s0}}{n^2 - 1}$$

→ For $\sin\varphi_{s0} = 0$ (no acceleration): $r = 1/n$, $\Phi_n = \pi + (1-n)\varphi_s$ and $\varphi_s = \varphi_{s0} = 0$ or π,

where we also used relation: $\sin\phi_{s0} = \sin\phi_s + r\sin\left(n\phi_s + \Phi_n\right)$

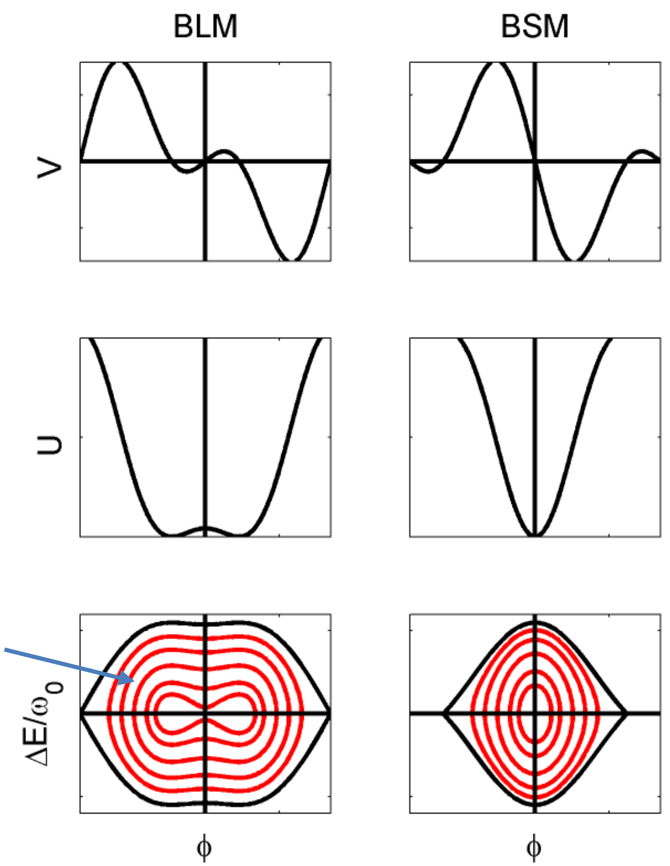
Double RF system: bucket area

Bucket area is increased in BL-mode and decreased in BS-mode. Higher is n, smaller is the effect.
The ratio of the stationary-bucket area in BL-mode (with n = 2) to one in a single RF system is a monotonic function for r < 1/n:

$$A(r)/A(r=0) = \left[\sqrt{1+2r} + \ln(\sqrt{1+2r} + \sqrt{2r})\right]/2.$$

For r > 1/n, the bucket area continues to shrink in BS-mode. In BL-mode 2 buckets start to form.

Minimum peak line density can be obtained for r >1/n (depending on particle distribution)

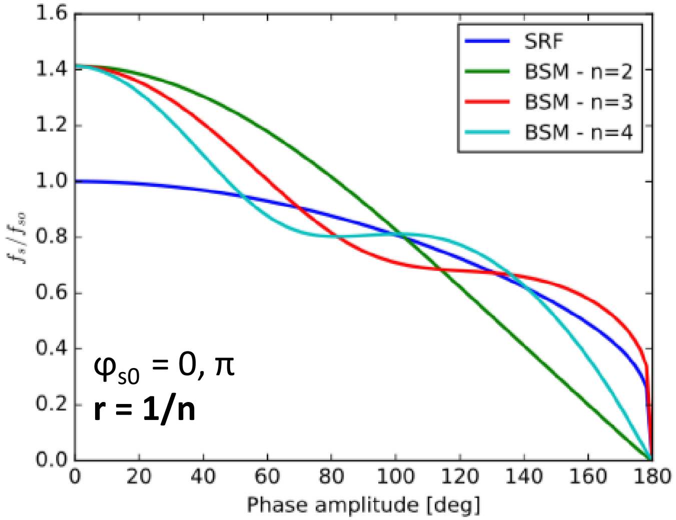


sin φ_{s0} = 0, **n = 2,** but **r = 0.75**

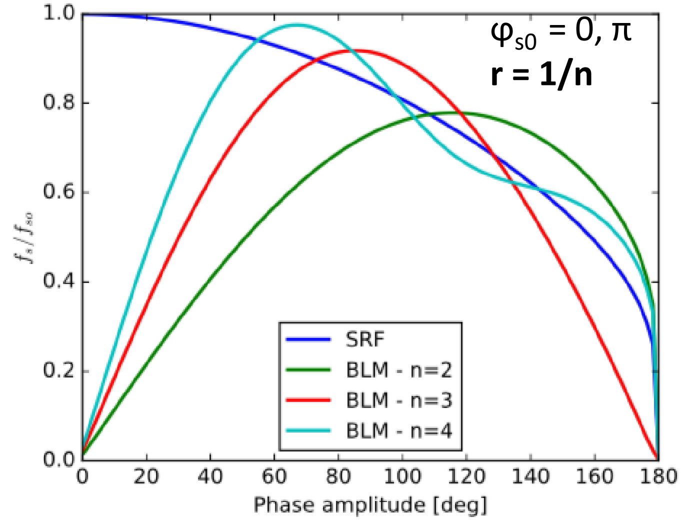
Double RF system: synchrotron frequency distribution

Beam stability is improved with a larger synchrotron frequency spread providing Landau damping. → In this application, the HH RF system is often called a "Landau cavity"

BS-mode



BL-mode

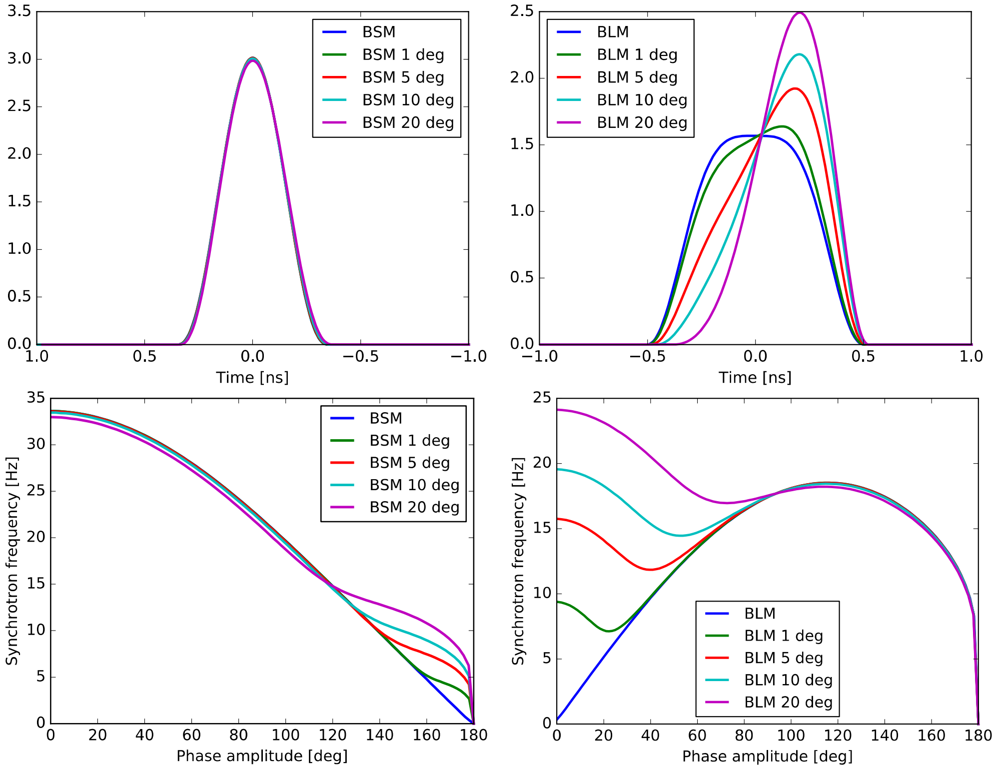


→ For the same HH voltage (ratio r), a larger spread Δf_s or change in $f_s(0)$ can be obtained for a higher n and in BL-mode.

Nevertheless BS-mode is used for beam stability in the CERN SPS and PS . Why?

Double RF system: phase shift Φ_n

BS-mode BL-mode



Examples for various errors in Φ_2 with $\sin \varphi_{s0} = 0$, $n = 2$, and $r = 1/2$ (800 MHz RF system in LHC).

Very accurate phase control is required in BLM during acceleration and in presence of intensity effects.

Transient beam loading (e.g. due to the bunch gaps) displaces bunches and modifies Φ_n seen by them.
→ Problems for beam stability and beam manipulations.

→ More RF power is required.
For example, for the 2nd harmonic RF system in LHC, more than 4 times power would be needed in BL-mode than in BS-mode.

Higher harmonic RF system: bunch-lengthening or shortening mode?

- **Bunch lengthening mode:**
  + reduced peak line density (flat bunch)
  + increased bucket area
  + for a given V_n (or r), larger increase in synchrotron frequency spread
  - high sensitivity to RF phase shift Φ_n (tilted bunches)
  - flat region in f_s - distribution (limitation on max bunch length) for all n
  - more RF power is required in presence of beam loading
- **Bunch shortening mode:**
  + good for beam stability (multi-bunch)
  + very robust (large allowed phase shift)
  + increased linear synchrotron frequency (TMC instability)
  - increased peak line density (can be mitigated by emittance blow-up)
  - reduced bucket size (as compared to single RF)
  - flat region in f_s distribution (limitation on max bunch length) for $n > 2$

EQUATIONS OF MOTION WITH INTENSITY EFFECTS

Longitudinal equations of motion with intensity effects

Equations of longitudinal motion in conjugate variables $\left(\frac{\Delta E}{h\omega_0}, \phi\right)$:

$$\frac{d\phi}{dt} = \frac{\omega_{RF}^2 \eta}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right)$$

this equation doesn't change

$$\frac{d}{dt}\left(\frac{\Delta E}{h\omega_0}\right) = \frac{qV_0}{2\pi h}\,(sin\phi - sin\phi_{s0})$$

$$\frac{d}{dt}\left(\frac{\Delta E}{h\omega_0}\right) = \frac{q}{2\pi h}\,[Vt(\phi) - V_0 sin\phi_{s0}]$$

Phase equation

$$\frac{d^2\phi}{dt^2} = \frac{h\omega_0^2 \eta q}{2\pi\,\beta^2 E} V_0 (sin\phi - sin\phi_{s0})$$

$$\frac{d^2\phi}{dt^2} = \frac{h\omega_0^2 \eta q}{2\pi\,\beta^2 E}[Vt(\phi) - V_0 sin\phi_{s0}],$$

where $V_t(\phi) = V_0 \sin\phi + V_{ind}(\phi)$ includes now induced voltage $V_{ind}(\phi)$,

but we keep here $V_0 \sin\phi_{s0}$ → why?

Synchronous particle

A synchronous particle is synchronised with RF frequency $\omega_{RF} = h\ \omega_0$

and its energy gain per turn is $\Delta E = qV_0 \sin\phi_{s0}$,

where ϕ_{s0} is synchronous phase in absence of intensity effects.

The total voltage V_t seen by a particle is the sum of the RF voltage V_{rf} and the voltage induced by beam V_{ind} :

$$V_{rf}(\phi) \rightarrow V_t = V_{rf}(\phi) + V_{ind}(\phi)$$

The acceleration rate of synchronous particle, defined by the magnetic ramp,

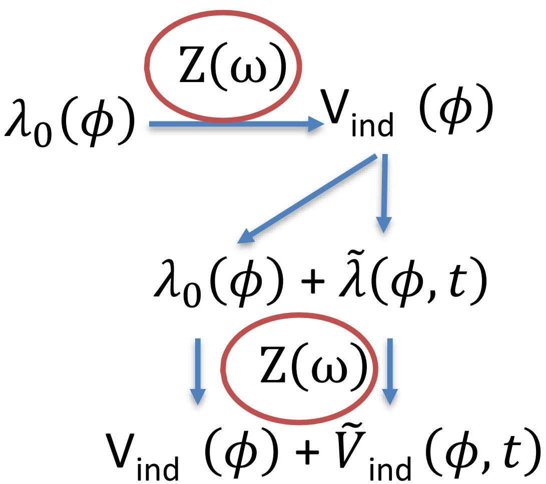
$\frac{\Delta E}{T_0} = \frac{dE_s}{dt} = \frac{\omega_0}{2\pi} qV_0 \sin\phi_{s0}$ and now $\frac{\Delta E}{T_0} = \frac{\omega_0}{2\pi} q[V_0 \sin\phi_s + Vi_{nd}(\phi_s)]$,

where ϕ_s is new synchronous phase, with intensity effects.

Therefore $$V_0 \sin\phi_{s0} = V_0 \sin\phi_s + Vi_{nd}(\phi_s)$$

Intensity effects

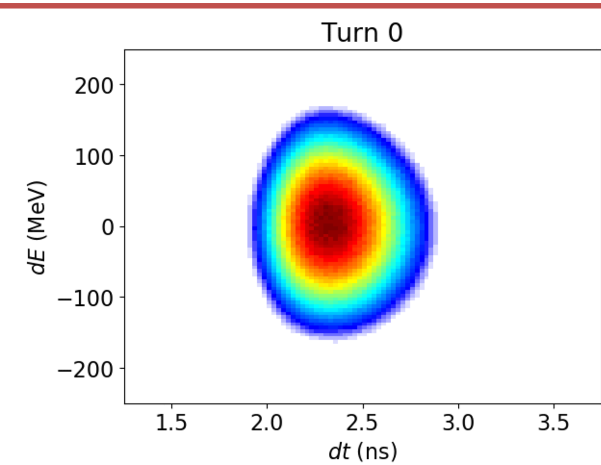
The total voltage V_t includes the voltages induced by a stationary λ_0 and the perturbed (by induced voltage!) $\tilde{\lambda}$ line density (beam current)



Initial particle distribution is modified
→ **new equilibrium** (stationary solution).
This effect is called potential well distortion
(usually considered as a single-bunch).

Instability: perturbations $\tilde{\lambda}$ are growing with time: $\tilde{\lambda}(\phi, t) = \tilde{\lambda}(\phi)\, e^{-i\Omega t}$

For multi-bunch beam, the effect of induced voltage due to cavity impedance is called beam loading (see lecture by H. Damerau)



Longitudinal impedance

Real and **imaginary** parts of impedance: $Z(\omega) = \mathrm{Re}Z(\omega) + i\ \mathrm{Im}Z(\omega)$

- Resistive impedance $\mathrm{Re}Z \rightarrow$ beam loading, instabilities, beam induced heating,…
- Reactive impedance $\mathrm{Im}Z \rightarrow$ potential well distortion, loss of Landau damping,…

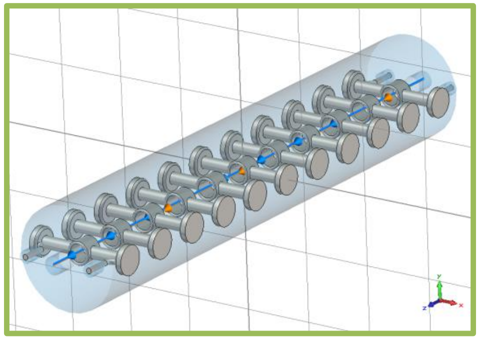
Resonant impedance: $$Z(\omega) = \frac{R_{sh}}{1 + iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$ with **bandwidth** $$\Delta\omega_r = \frac{\omega_r}{2Q}$$

- **Narrow-band impedance** (RF cavities and other cavity-like objects)
  $\rightarrow$ multi-bunch effects (beam loading, coupled-bunch instability)
- **Broad-band impedance** (space charge, cross-section changes): $\boldsymbol{\tau\Delta\omega_r > 1}$
  $\rightarrow$ single-bunch effects (potential well distortion, loss of Landau damping, single-bunch instability)

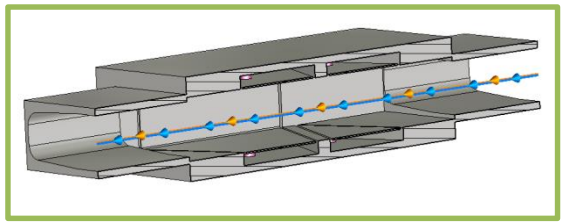
See lecture on Impedances and wakefields by *A. Mostacci*

Contributors to impedance model (CERN SPS)

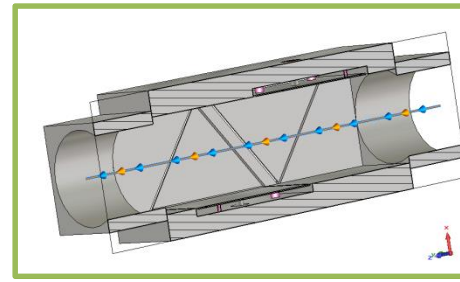
TW RF cavities:
200 MHz and 800 MHz



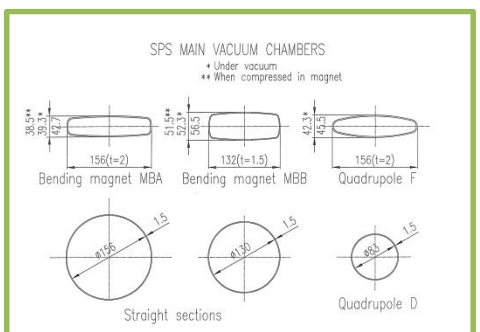
Beam position monitor H



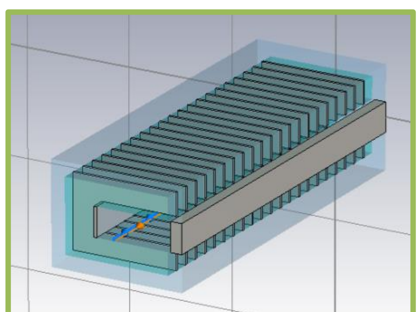
Beam position monitor V



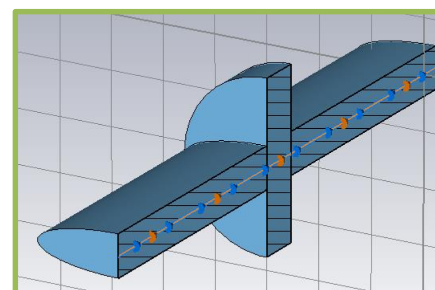
Vacuum chambers



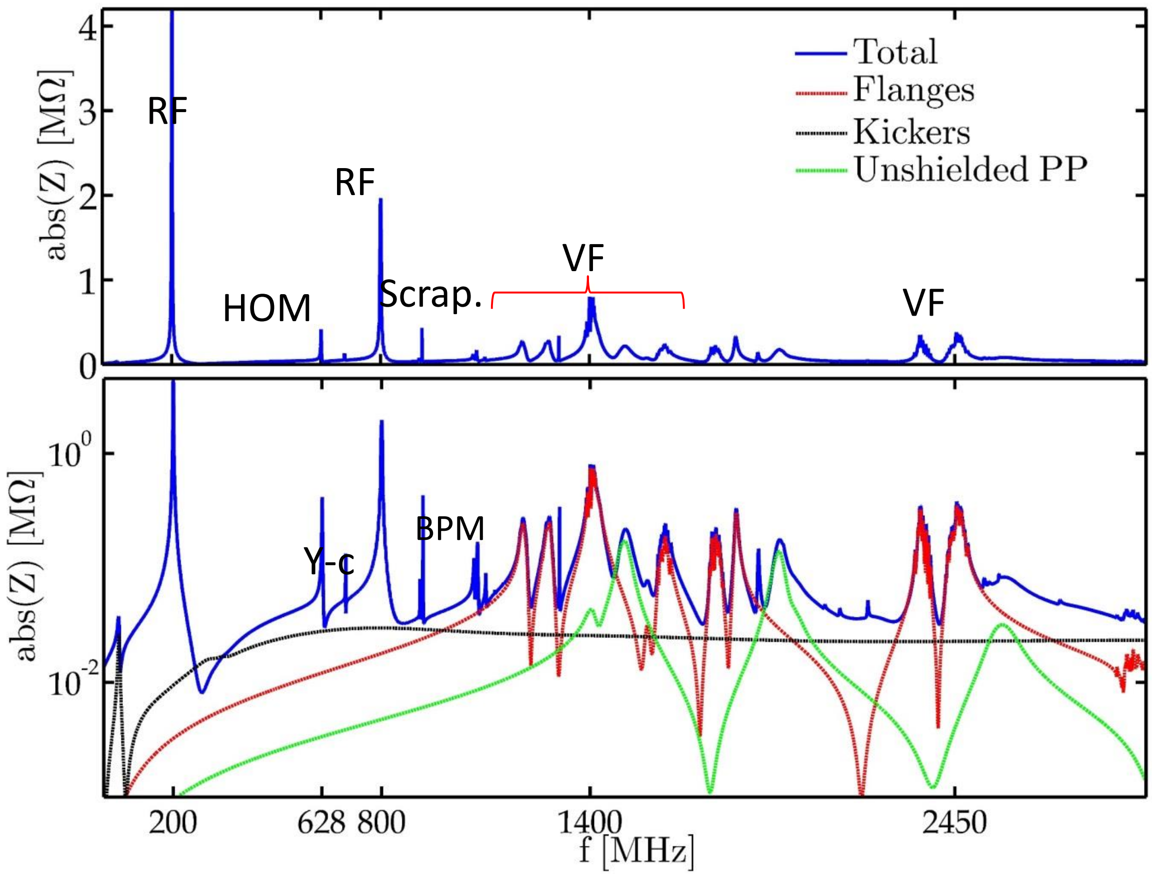
Kickers



Vacuum flanges (step transitions)



Example of realistic impedance model (CERN SPS)

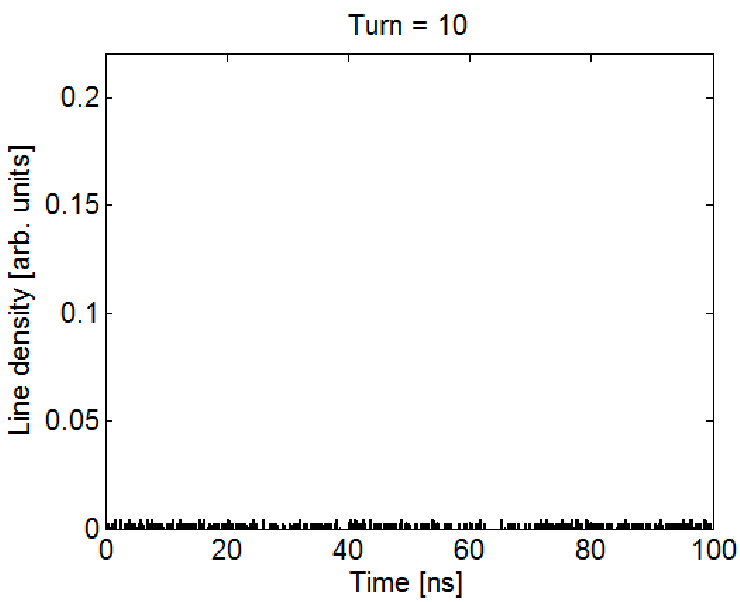


This model includes:

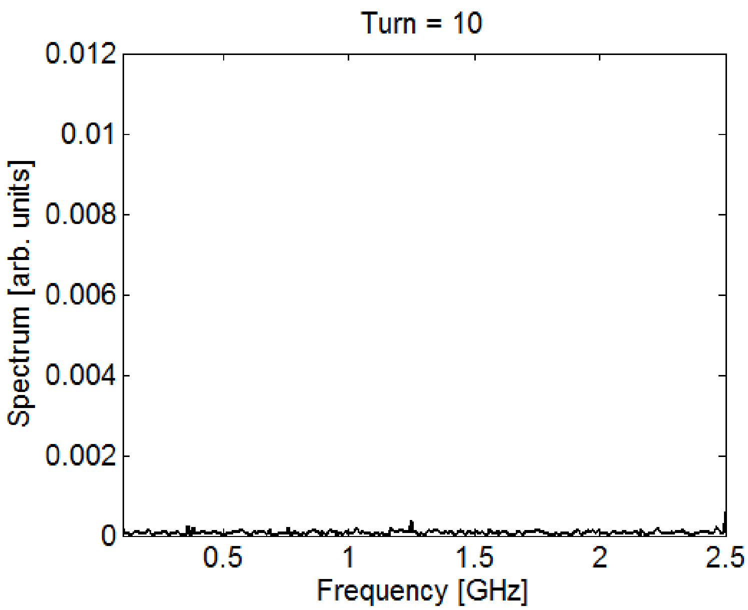
- 200 MHz cavities + HOMs
- 800 MHz cavities (2)
- Kicker magnets (8 MKEs, 4 MKPs, 5 MKDs, 2 MKQs)
- Vacuum flanges (~500)
- BPMs: BPH&BPV (~200)
- Unshielded pumping ports
- Beam scrappers
- Resistive wall
- 6 electrostatic septa ZS
- MSE/MST + PMs
- …

Spectrum of unstable bunches

Single bunches injected into the ring (CERN SPS 26 GeV/c) with RF off → slow debunching and fast instability

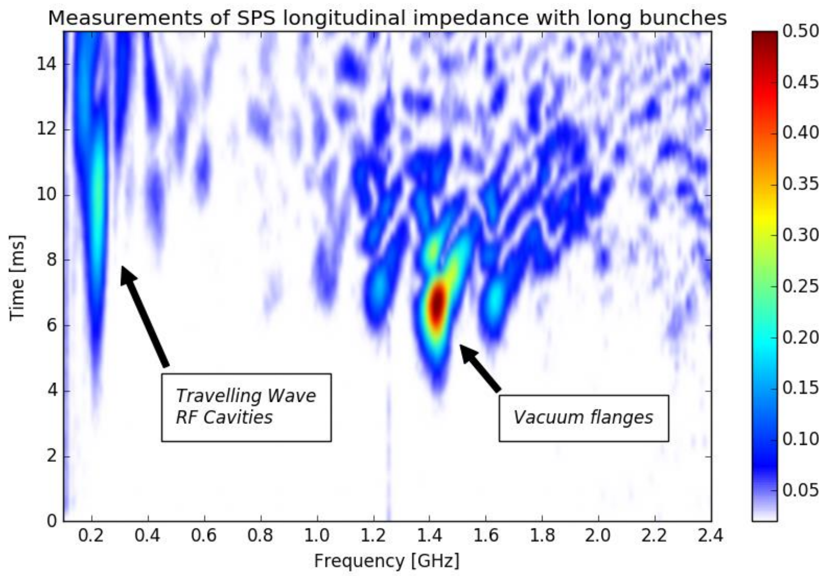


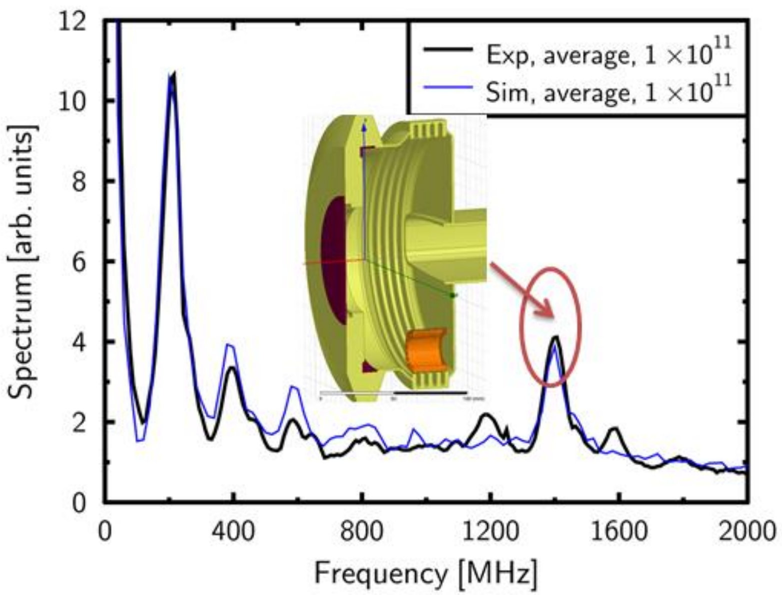
Bunch profile



Spectrum of unstable bunch

Spectrum of unstable bunches





Line density modulation growing at resonant frequencies of impedances with high R/Q → method of impedance identification and measurement (used for two impedance reduction campaigns in the CERN SPS)

SINGLE-BUNCH INTENSITY EFFECTS

Induced voltage

Using periodicity over azimuth θ and relation $\phi(t) = \int \omega_{RF} dt \; - h\theta(t)$,

we have $V_{ind}(\phi) = V_{ind}(\phi + 2\pi h)$.

Then beam induced voltage $V_{ind} = \sum_{k=-\infty}^{\infty} V_k e^{\frac{ik\phi}{h}}$, and it can also be presented

$$V_{ind} = -\sum_{k=-\infty}^{\infty} Z_k I_k e^{\frac{ik\phi}{h}} = -qh\omega_0 N_p \sum_{k=-\infty}^{\infty} Z_k \lambda_k e^{ik\theta}$$

where Z_k is the longitudinal impedance at frequency $k\omega_0$,
I_k and λ_k are the k-th Fourier harmonics of the beam current and line density with normalisation

$$\int_{-\pi h}^{\pi h} d\phi \lambda(\phi) = \int_{-\pi h}^{\pi h} d\phi \int_{-\infty}^{\infty} d\dot{\phi} \mathcal{F}(\phi, \dot{\phi}) = 1.$$

For beam current we used $I(\phi) = qh\omega_0 N_p \lambda(\phi)$

Phase equation with intensity effects

Using definition of linear synchrotron frequency, the phase equation is

$$\omega_{s0}^2 = -\frac{h\omega_0^2\eta\, qV_0 cos\phi_{s0}}{2\pi\, \beta^2 E}$$

$$\frac{d^2\phi}{dt^2} + \frac{\omega_{s0}^2}{V_0\, cos\,\phi_{s0}}(V_t - V_0\, sin\,\phi_{s0}) = 0$$

Defining $\phi = \phi_{s0} + \Delta\phi$, induced voltage can be presented as

for k Δϕ/h << 1

$$V_{ind}(\phi) = -2\pi h I_b \sum_{k=-\infty}^{\infty} Z_k\, \lambda_k\, e^{\frac{ik\Delta\phi}{h}} \simeq -2\pi h I_b \sum_{k=-\infty}^{\infty} Z_k\, \lambda_k\, [1 + \frac{ik\Delta\phi}{h} + \;...]$$

Stationary bunch: potential well distortion

For $\Delta\phi = \phi - \phi_{s0} << 1$
the phase equation $\rightarrow$

$$\frac{d^2\Delta\phi}{dt^2} + \omega_{s0}^2[\Delta\phi + \frac{V_{ind}(\phi)}{V_{rf}\cos\phi_{s0}}] = 0$$

Assuming symmetric bunch profile, so that $\lambda_k = \lambda_{-k}$,

$$V_{ind}(\phi) \simeq -2\pi h I_b \sum_{k=-\infty}^{\infty} \lambda_k [\mathrm{Re}Z_k - \frac{k\,\Delta\phi}{h}\mathrm{Im}Z_k + \ \dots]$$

phase shift

$$\Delta\phi_s \simeq \frac{2\pi h I_b}{V_{rf}\cos\phi_{s0}} \sum_{k=-\infty}^{\infty} \lambda_k \,\mathrm{Re}Z_k$$

frequency shift

$$\omega_s^2 \simeq \omega_{s0}^2[1 + \frac{2\pi I_b}{V_{rf}\ \cos\phi_{s0}} \sum_{k=-\infty}^{\infty} k\lambda_k \,\mathrm{Im}Z_k]$$

Bunch lengthening for constant emittance: $1 = (\tau/\tau_0)^4 + (\tau/\tau_0)\,[\omega_s^2(\tau_0) - \omega_{s0}^2]/\omega_{s0}^2$

Potential well distortion: synchronous phase shift

(1) This is the synchronous phase shift of the potential well, also valid for particles with small oscillation amplitude, but difficult to measure

$$\Delta\phi_s = \frac{2\pi h I_b}{V_{rf}\cos\phi_{s0}} \sum_{k=-\infty}^{\infty} \lambda_k \,\mathrm{Re}Z_k$$

(2) The shift of the bunch centre with respect to RF voltage can be found from the energy loss (per turn and particle) ***U*** – loss factor

$$V_0 \sin\phi_{s0} \simeq V_0 \sin\phi_{s0} + \Delta\phi_b \cos\phi_{s0} + U/q \rightarrow \Delta\phi_b \simeq -\frac{\boldsymbol{U}}{q\,V_{rf}\cos\phi_{s0}}$$

$$\boldsymbol{U} = -q h^2 I_b \sum_{k=-\infty}^{\infty} |\lambda_k|^2 \,\mathrm{Re}Z_k$$

→ Measurements of $\Delta\phi_b$ for various $\boldsymbol{\tau}$ give estimation of Re*Z(ω)* of the ring

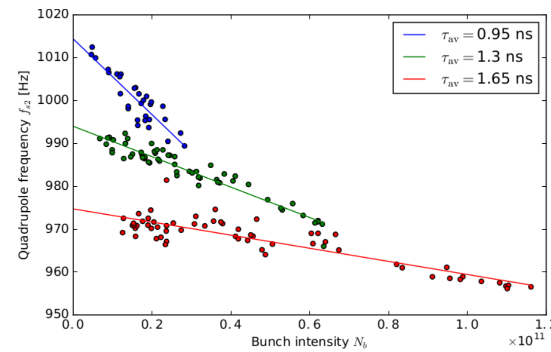
Potential well distortion: synchrotron frequency shift

We have obtained frequency

$$\omega_s^2 \simeq \omega_{s0}^2 [\, 1 + \frac{2\pi I_b}{V_{rf}\ \cos\phi_{s0}} \sum_{k=-\infty}^{\infty} \lambda_k\ k^2 \frac{\mathrm{Im} Z_k}{k}]$$

For shift $\Delta\omega_s = \omega_s - \omega_{s0} \ll \omega_{s0}$ and ImZ/k = *const* over stable bunch spectrum:

$$\Delta\omega_s \simeq \omega_{s0} \frac{\pi I_b}{V_{rf}\cos\phi_{s0}} \frac{\mathrm{Im} Z}{k} \sum_{k=-\infty}^{\infty} \lambda_k\, k^2$$



Since $\lambda(\phi) = \sum_k \lambda_k e^{\frac{ik\phi}{h}} \rightarrow \sum_{k=-\infty}^{\infty} \lambda_k\ k^2 = -h^2 \frac{d^2\lambda}{d\phi^2}\Big|_{\phi=0}$

For a parabolic bunch

$$\lambda(\phi) = \lambda_0 \left(1 - \frac{\phi^2}{\phi_b^2}\right)$$

with $\phi_b = h\omega_0\tau/2$

$$\omega_s^2 = \omega_{s0}^2 [1 + \frac{3\ I_b}{\pi^2\, V_{rf}\, h \cos\phi_{s0}\, (f_0\tau)^3} \frac{\mathrm{Im} Z}{k}]$$

Strong dependence on bunch length

Defocusing effect above transition ($\cos\varphi_s < 0$) for ImZ/k > 0 (inductive impedance) → more RF voltage needed

VLASOV EQUATION

Vlasov equation

The Vlasov equation can be derived from the Liouville' theorem $\frac{d\mathcal{F}}{dt} = 0$

In variables $(\mathcal{E}, \psi)$ the Vlasov equation is
$$\frac{\partial \mathcal{F}}{dt} + \frac{d\mathcal{E}}{dt}\frac{\partial \mathcal{F}}{\partial \mathcal{E}} + \frac{d\psi}{dt}\frac{\partial \mathcal{F}}{\partial \psi} = 0$$

→ for a stationary case $\mathcal{F} = \mathcal{F}(\mathcal{E})$

For beam stability study, we should analyse the time behaviour of small perturbations $\tilde{\mathcal{F}}(\mathcal{E}, \psi, t)$, $\tilde{\lambda}(\phi, t)$ and $\tilde{V}_{\text{ind}}(\phi, t)$ of $\mathcal{F}(\mathcal{E})$, $\lambda(\phi)$ and $V_{ind}(\phi)$,

assuming the dependence on time as
$$\tilde{\mathcal{F}}(\mathcal{E}, \psi, t) = \tilde{\mathcal{F}}(\mathcal{E}, \psi, \Omega)e^{-i\Omega t}$$

If perturbations grow with time (Im$\Omega > 0$) → beam is unstable

The linearised Vlasov equation
$$\frac{\partial \tilde{\mathcal{F}}}{\partial t} + \frac{d\mathcal{E}}{dt}\frac{d\mathcal{F}}{d\mathcal{E}} + \frac{d\psi}{dt}\frac{\partial \tilde{\mathcal{F}}}{\partial \psi} = 0$$

Equations of unperturbed motion

In variables $(\mathcal{E}, \psi)$

$$\mathcal{E} = \frac{\dot{\phi}^2}{2\omega_{s0}^2} + U_t(\phi),$$

energy of synchrotron motion

$$\psi = \mathrm{sgn}(\eta \Delta E) \frac{\omega_s(\mathcal{E})}{\sqrt{2}\omega_{s0}} \int_{\phi_{\max}}^{\phi} \frac{d\phi'}{\sqrt{\mathcal{E} - U_t(\phi')}}$$

phase of synchrotron motion

with potential

$$U_t(\phi) = \frac{1}{V_0 \cos \phi_{s0}} \int_{\Delta\phi_s}^{\phi} d\phi' [V_t(\phi') - V_0 \sin \phi_{s0}]$$

the equations of unperturbed particle motion are simply

$$\dot{\psi} = \omega_s(\mathcal{E}), \qquad \dot{\mathcal{E}} = 0.$$

Equations of perturbed motion

In presence of perturbation (induced voltage) the phase equation is

$$\frac{d\dot{\phi}}{dt} + \frac{\omega_{s0}^2}{V_0 \cos\phi_{s0}} \left[V_{\mathrm{tot}}(\phi) - V_0 \sin\phi_{s0}\right] = -\frac{\omega_{s0}^2}{V_0 \cos\phi_{s0}} \tilde{V}_{\mathrm{ind}}(\phi, t).$$

In variables $(\mathcal{E}, \psi)$, after multiplication by $\dot{\phi}$

$$\frac{d\mathcal{E}}{dt} = -\frac{d\phi}{dt} \frac{\tilde{V}_{\mathrm{ind}}(\phi, t)}{V_0 \cos\phi_{s0}} = -\omega_s(\mathcal{E}) \frac{\partial \tilde{U}_{\mathrm{ind}}(\phi, t)}{\partial \psi}$$

Here the same definition of potential (as for unperturbed motion) was used:

$$\tilde{U}_{\mathrm{ind}}(\phi) = \frac{1}{V_0 \cos\phi_{s0}} \int_{\Delta\phi_s}^{\phi} d\phi' \tilde{V}_{\mathrm{ind}}(\phi').$$

Linearised Vlasov equation

The Vlasov equation can be now rewritten

$$\frac{\partial \tilde{\mathcal{F}}}{\partial t} + \frac{d\mathcal{E}}{dt}\frac{d\mathcal{F}}{d\mathcal{E}} + \frac{d\psi}{dt}\frac{\partial \tilde{\mathcal{F}}}{\partial \psi} = 0$$

Taking into account that

$$-\omega_s(\mathcal{E})\frac{\partial \tilde{U}_{\mathrm{ind}}(\phi, t)}{\partial \psi}$$

$$\dot{\psi} = \omega_s(\mathcal{E})$$

The linearised Vlasov equation

$$\left[\frac{\partial}{\partial t} + \omega_s \frac{\partial}{\partial \psi}\right]\tilde{\mathcal{F}} = \omega_s(\mathcal{E})\frac{\partial \tilde{U}_{\mathrm{ind}}}{\partial \psi}\frac{d\mathcal{F}}{d\mathcal{E}}$$

LEBEDEV EQUATION

Lebedev equation (1/4)

Solutions of Vlasov equation should be periodic in ψ and can be expanded in azimuthal Fourier harmonics *(m)*

$$\tilde{U}_{\text{ind}}(\mathcal{E},\psi,\Omega) = \sum_{m=-\infty}^{\infty} \tilde{U}_{\text{ind},m}(\mathcal{E},\Omega)e^{im\psi}$$

$$\left[\frac{\partial}{\partial t} + \omega_s \frac{\partial}{\partial \psi}\right]\tilde{\mathcal{F}} = \omega_s(\mathcal{E})\frac{\partial \tilde{U}_{\text{ind}}}{\partial \psi}\frac{d\mathcal{F}}{d\mathcal{E}}$$

$$\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega) = \sum_{m=-\infty}^{\infty} \tilde{\mathcal{F}}_m(\mathcal{E},\Omega)e^{im\psi}$$

(m = 1, 2, 3, … – dipole, quadrupole, sextupole, …)

Then

$$\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega) = -\omega_s(\mathcal{E})\frac{d\mathcal{F}}{d\mathcal{E}}\sum_{m=-\infty}^{\infty}\frac{m\tilde{U}_{\text{ind},m}(\mathcal{E},\Omega)}{\Omega - m\omega_s(\mathcal{E})}e^{im\psi}$$

$$\tilde{U}_{\text{ind}}(\phi) = \frac{1}{V_0\cos\phi_{s0}}\int_{\Delta\phi_s}^{\phi} d\phi' \tilde{V}_{\text{ind}}(\phi')$$

Since voltage harmonics $\tilde{V}_k(\Omega) = -qN_p h\omega_0 Z_k(\Omega)\tilde{\lambda}_k(\Omega)$

the Fourier harmonics of perturbed potential can be written as

$$\tilde{U}_{\text{ind},m}(\mathcal{E},\Omega) = \frac{i\,qN_p\omega_0 h}{V_0\cos\phi_{s0}}\sum_{k=-\infty}^{\infty}\frac{Z_k(\Omega)}{k}\tilde{\lambda}_k(\Omega)I_{mk}(\mathcal{E})$$

where $Z_k(\Omega) = Z_k(k\omega_0 + \Omega)$

Lebedev equation (2/4)

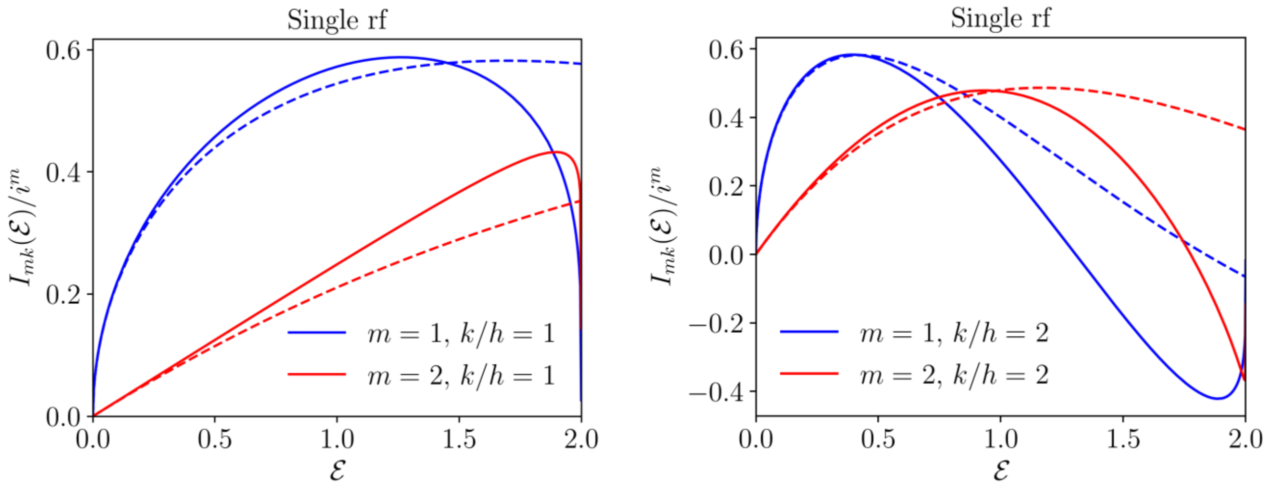
Functions $I_{mk}(\mathcal{E})$ are Fourier harmonics of the expansion of the azimuth plane wave over harmonics of synchrotron motion $m\psi$

$$I_{mk}(\mathcal{E}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\psi\, e^{i\frac{k}{h}\phi(\mathcal{E},\psi) - im\psi}$$
$$= \frac{1}{\pi}\int_{0}^{\pi} d\psi\, e^{i\frac{k}{h}\phi(\mathcal{E},\psi)} \cos\, m\psi$$

$$\phi(\mathcal{E},-\psi) = \phi(\mathcal{E},\psi)$$

For short bunches in single RF $\phi(\mathcal{E},\psi) \approx \sqrt{2\mathcal{E}}\cos\psi$ and (J_m is the 1st kind Bessel function of the order m)

$$I_{mk}(\mathcal{E}) \approx i^m J_m\left(\frac{k}{h}\sqrt{2\mathcal{E}}\right).$$



Exact (solid lines) and approximate (dashed) functions

Lebedev equation (3/4)

The line-density harmonic λ_k is related to distribution function $\mathcal{F}(\mathcal{E})$ as

$$\tilde{\lambda}_k(\Omega) = \frac{1}{2\pi h}\int_{-\pi h}^{\pi h} d\phi \tilde{\lambda}(\phi) e^{-i\frac{k}{h}\phi} = \frac{\omega_{s0}^2}{2\pi h}\int_{-\pi}^{\pi} d\psi \int_0^{\mathcal{E}_{\max}} d\mathcal{E} \frac{\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega)}{\omega_s(\mathcal{E})} e^{-i\frac{k}{h}\phi(\mathcal{E},\psi)}$$

where the transformation of variables $d\phi d\dot{\phi} = \omega_{s0}^2\, d\psi\, d\mathcal{E}/\omega_s(\mathcal{E})$ was used.

The harmonics of line density perturbation $\tilde{\lambda}_k(\Omega)$ are related to the perturbation of distribution function $\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega)$ in similar way

$$\tilde{\lambda}_k(\Omega) = \frac{\omega_{s0}^2}{2\pi h}\int_{-\pi}^{\pi} d\psi \int_0^{\mathcal{E}_{\max}} d\mathcal{E} \frac{\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega)}{\omega_s(\mathcal{E})} e^{-i\frac{k}{h}\phi(\mathcal{E},\psi)}$$

Last step: insert the obtained solution of Vlasov equation for $\tilde{\mathcal{F}}(\mathcal{E},\psi,\Omega)$

Lebedev equation (4/4)

General system of equations for line density harmonics **(A. N. Lebedev, 1968)**

$$\tilde{\lambda}_p(\Omega) = -i\xi \sum_{k=-\infty}^{\infty} G_{pk}(\Omega) \frac{Z_k(\Omega)}{k} \tilde{\lambda}_k(\Omega),$$

where

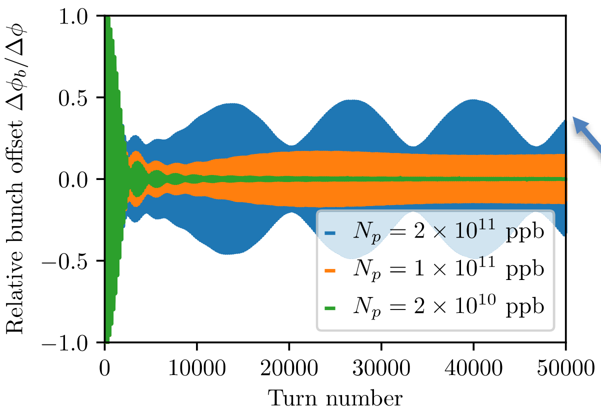
$$\xi = \frac{\omega_{s0}^2 q N_p\, h\, \omega_0}{V_0 \cos\phi_{s0}} = \frac{2\pi\, \omega_{s0}^2 I_0 h}{V_0 \cos\phi_{s0}}$$

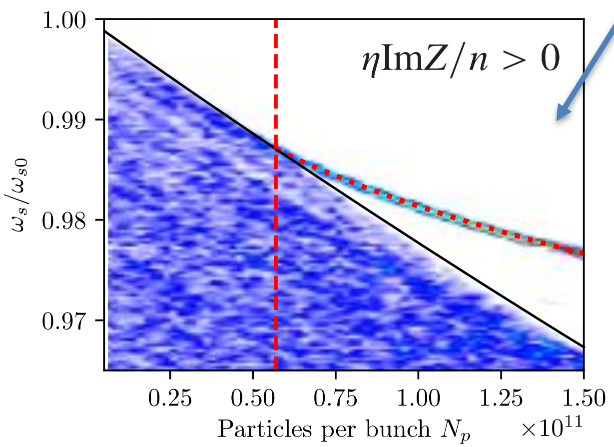
ξ - is intensity parameter (I_0 the average beam current)

and beam transfer functions $G_{pk}(\Omega)$ are defined as

$$G_{pk}(\Omega) = \sum_{m=-\infty}^{\infty} \int_0^{\mathcal{E}_{\max}} d\mathcal{E} \frac{d\mathcal{F}(\mathcal{E})}{d\mathcal{E}} \frac{I_{mk}(\mathcal{E}) I_{mp}^*(\mathcal{E})}{\Omega/m - \omega_s(\mathcal{E})} = 2\sum_{m=1}^{\infty} \int_0^{\mathcal{E}_{\max}} d\mathcal{E} \frac{d\mathcal{F}(\mathcal{E})}{d\mathcal{E}} \frac{I_{mk}(\mathcal{E}) I_{mp}^*(\mathcal{E}) \omega_s(\mathcal{E})}{\Omega^2/m^2 - \omega_s^2(\mathcal{E})}$$

Loss of Landau damping in longitudinal plane





Landau damping in longitudinal plane is provided by synchrotron frequency spread.
→ If it is lost, bunch oscillations are not damped anymore: observed in Tevatron, LHC, SPS, …
→ This happens when coherent mode is out from the synchrotron frequency spread.

The solution for **Ω** exists if $\det \left| \delta_{pk} + \xi G_{pk}(\Omega) \frac{Z_k(\Omega)}{k} \right| = 0$

The analytic threshold $\xi_{\mathrm{th}} \simeq - \left[\sum_{k=-\infty}^{\infty} G_{kk}(\Omega) \frac{Z_k(\Omega)}{k} \right]^{-1}$

Low frequency approximation $f < 1/(\pi\tau)$ for reactive impedance with ImZ/k = const and parabolic bunches

$$\left[\frac{I_0 h^2}{V_0 \cos\phi_{s0}} \frac{\mathrm{Im}Z}{k} \right]_{th} \simeq \frac{\phi_{max}^4}{48} \frac{h}{k_{max}}$$

k_{max} – is cut-off frequency of impedance

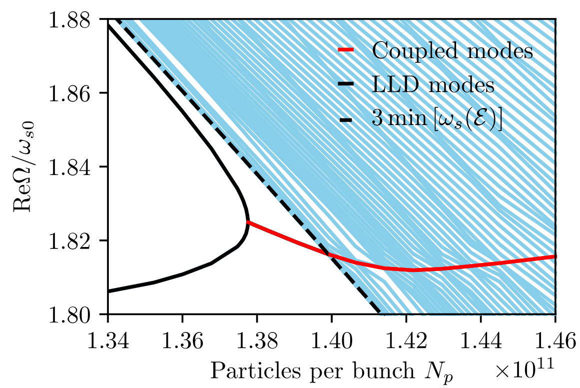
Single-bunch instabilities

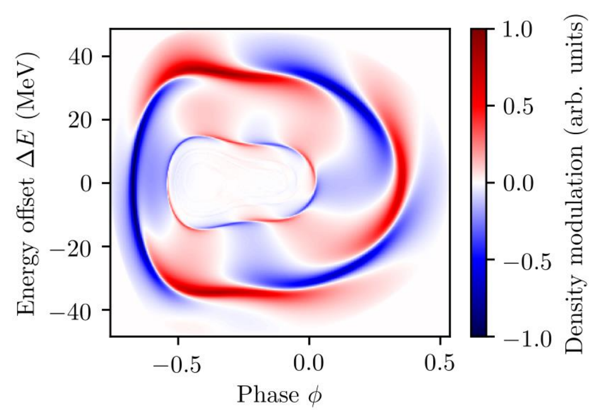
Vlasov equation is solved using $\tilde{\mathcal{F}}(\mathcal{E},\psi,t) = e^{-i\Omega t} \sum_{m=1}^{\infty} C_m(\mathcal{E},\Omega) \left[\cos m\psi + \frac{i\Omega}{m\omega_s(\mathcal{E})} \sin m\psi\right]$

The solutions in 2-D phase space are coherent modes characterized by two mode numbers: radial and azimuthal.

Main types of instability (all present in the CERN SPS!):

- azimuthal mode coupling (in reality more rare): 1st mechanism proposed!
- radial mode coupling (Oide?)
- microwave (simultaneous excitation of many radial or azimuthal modes)



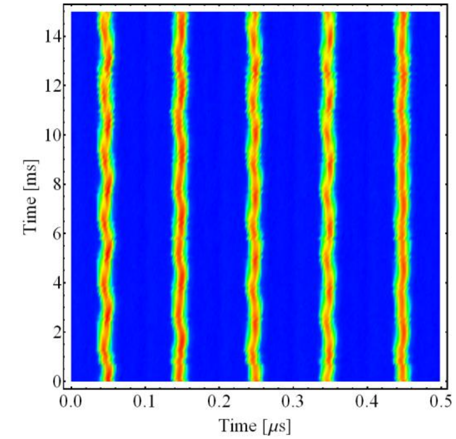


→ Potential well distortion is important in all cases

MULTI-BUNCH INSTABILITIES

Multi-bunch instabilities

The multi-bunch instability is driven by a resonator impedance with a narrow bandwidth $\Delta\omega_r$, so that one bunch sees the wake from the previous bunch(es) → coupled-bunch instability



→ Only one (unstable) term with harmonic $k = l\,M + n$ close to $k_r = \omega_r/\omega_0$, can be kept in Lebedev equation (M is a number of equidistant bunches with the phase shift $2\pi n/M$ between bunches, $n = 0,1\ldots M\text{-}1$ and $-\infty < l < \infty$), assuming $\Delta\omega_r << M\omega_0$.

$$\tilde{\lambda}_k(\Omega) = -i\xi G_{kk}(\Omega)\frac{Z_k(\Omega)}{k}\tilde{\lambda}_k(\Omega), \quad \Longrightarrow \quad \frac{k}{Z_k} = -i\xi\, G_{kk}(\Omega)$$

G_{pk} can be re-written using the principal value $\mathcal{P}$ for $\Omega = m\omega_s(\mathcal{E}_m)$

$$G_{pk}(\Omega) = 2\sum_{m=1}^{\infty}\left[\mathcal{P}\int_0^{\infty}\mathcal{F}'(\mathcal{E})\frac{I_{mp}(\mathcal{E})I^*_{mk}(\mathcal{E})\omega_s(\mathcal{E})d\mathcal{E}}{\Omega^2/m^2-\omega_s^2(\mathcal{E})} + i\frac{\pi}{2}\mathcal{F}'(\mathcal{E}_m)\frac{I_{mp}(\mathcal{E}_m)I^*_{mk}(\mathcal{E}_m)}{|\omega'_s(\mathcal{E}_m)|}\right]$$

Multi-bunch instability: stability diagrams

$$\frac{k}{Z_k} = -i\xi\, G_{kk}(\Omega)$$

Stability diagram for $\mathrm{Im}\Omega \to +0$

$$\mathrm{Re}G_{kk}(\Omega) = 2\sum_{m=1}^{\infty} \mathcal{P}\int_0^{\infty} \mathcal{F}'(\mathcal{E}) \frac{|I_{mk}(\mathcal{E})|^2 \omega_s(\mathcal{E}) d\mathcal{E}}{\Omega^2/m^2 - \omega_s^2(\mathcal{E})}$$

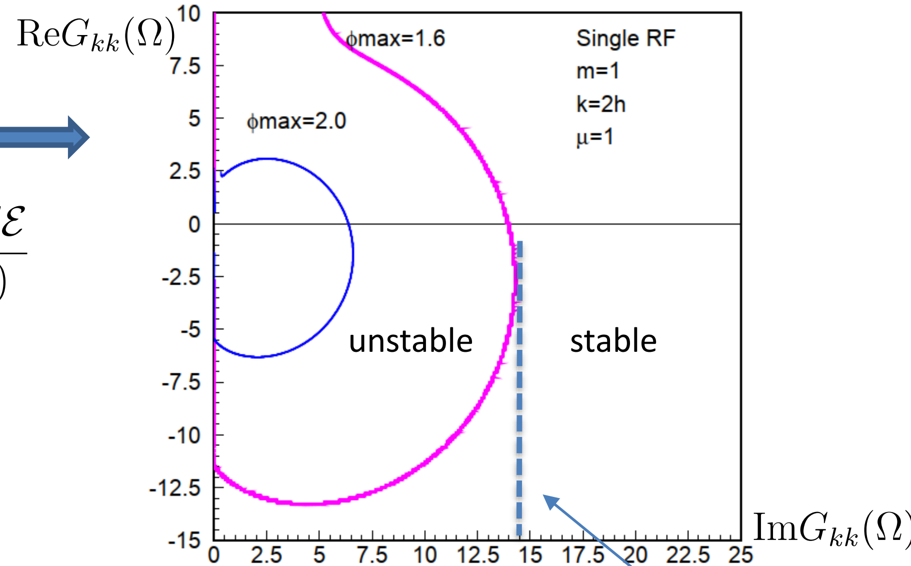
$$\mathrm{Im}G_{kk}(\Omega) = \pi \sum_{m=1}^{\infty} \frac{\mathcal{F}'(\mathcal{E}_m) I_{mk}^2(\mathcal{E}_m)}{\omega_s'(\mathcal{E}_m)}$$

From stability diagram the threshold is

$$\frac{k}{Z_k} > \xi\ \mathrm{Im}G_{kk}^{max}$$

Stability diagram for

$$\mathcal{F}(\mathcal{E}) = \mathcal{F}_0(1 - \mathcal{E}/\mathcal{E}_{max})^{\mu}$$



→ Beam is stable if vertical line $1/R_{sh}$ is inside stability region

$$Z_k^{-1} = 1/R_{sh} + i\, Q(\omega/\omega_r - \omega_r/\omega)/R_{sh}$$

Multi-bunch instability threshold

Requirements for HOM damping for given I_0

$$\frac{R_{sh}}{n_r} < \frac{V_0 \cos\phi_{s0}}{2\pi^2 \omega_{s0}^2 I_0 \, h} \min\left[\sum_{m=1}^{\infty} \frac{\omega_s'(\mathcal{E}_m)}{\mathcal{F}'(\mathcal{E}_m) I_{mk}^2(\mathcal{E}_m)}\right]$$

In single RF system without acceleration

$$R_{sh} < \frac{V\,(\pi f_{rf}\tau)^3}{32\, I_0} W_\mu(f_r\tau)$$

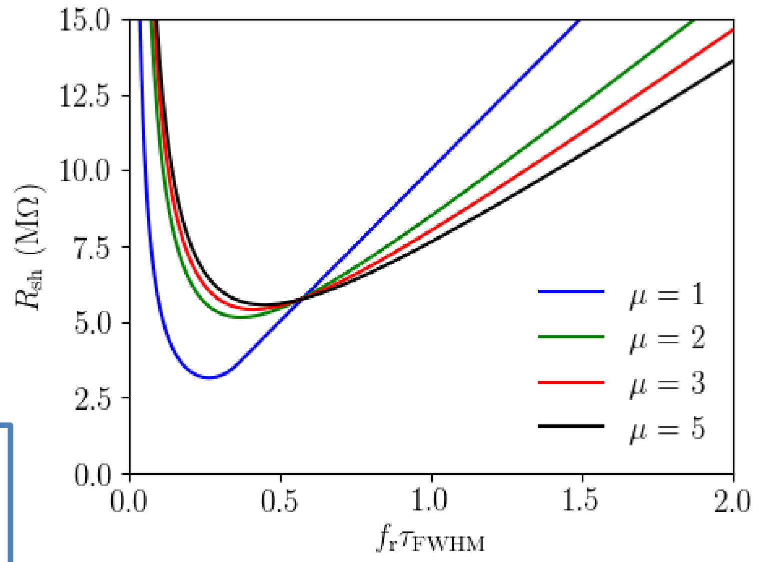
where for distribution $\mathcal{F}(\mathcal{E}) = \mathcal{F}_0(1-\mathcal{E}/\mathcal{E}_{max})^\mu$

$$W_\mu(x) = \frac{x}{\mu(\mu+1)} \min_{y\in[0,1]} \left[\sum_{m=1}^{\infty}(1-y^2)^{\mu-1} J_m^2(\pi x y)\right]^{-1}$$

Threshold R_{sh} for coupled-bunch instability

Example for FCC-hh at 50 TeV with

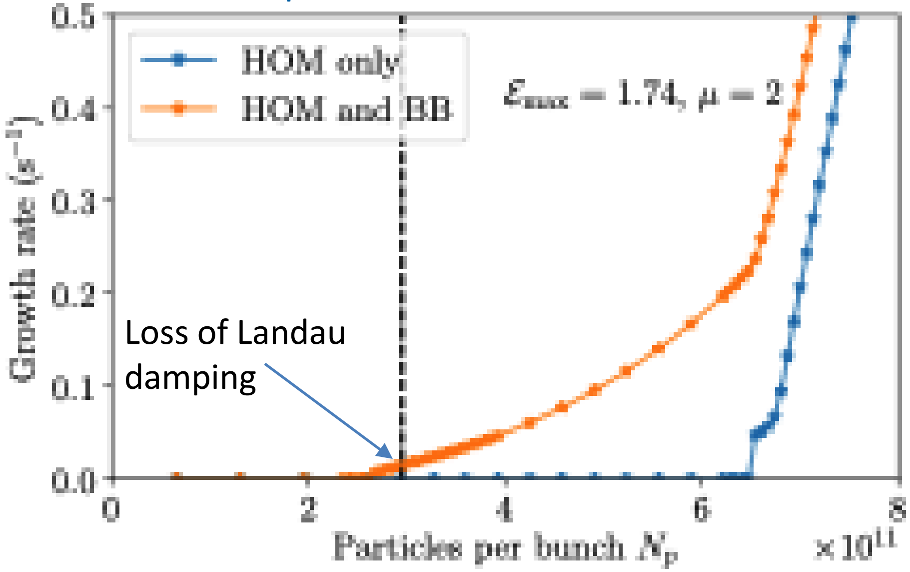
$N_b=10^{11}$ p/b, V = 38 MV, γ_t =99.3



→ The FWHM bunch length is important

Multi-bunch instability threshold: impact of broad-band impedance

Example for LHC at 450 GeV/c



Loss of Landau damping

$$\frac{1}{\xi^{BB}(\Omega)} \simeq \frac{\text{Im}Z}{k} \sum_{k<|k_c|} \text{Im}G_{kk}(\Omega)$$

$$\xi_{\text{th}}^{BB} = \xi^{BB}(\Omega_1)$$

Coupled-bunch instability

$$\frac{1}{\xi^{NB}(\Omega)} \simeq -\frac{\text{Re}Z_{k_r}}{k_r} G_{k_r k_r}(\Omega)$$

$$\xi_{\text{th}}^{NB} = \xi^{NB}(\Omega_2)$$

Using again equation

$$\det \left| \delta_{pk} + \xi G_{pk}(\Omega) \frac{Z_k(\Omega)}{k} \right| = 0$$

→ Coupled-bunch instability threshold in presence of both narrow- and broad- band impedances

$$\frac{1}{\xi_{\text{th}}(\Omega_3)} = \frac{1}{\xi^{NB}(\Omega_3)} + \frac{1}{\xi^{BB}(\Omega_3)}$$

Multi-bunch instability: growth rates

The instability growth rate ImΩ for separate multipole m (no coupling) can be easily found neglecting synchrotron frequency spread $\Delta\omega_{s0}$ for ImΩ >> $\Delta\omega_{s0}$

$$\frac{k}{Z_k} = -i\xi\, G_{kk}(\Omega)$$

$$G_{kk}(\Omega) = \sum_{m=-\infty}^{\infty} \int_0^{\mathcal{E}_{\max}} d\mathcal{E}\, \frac{\mathcal{F}'(\mathcal{E}) I_{mk}^2(\mathcal{E})}{\Omega/m - \omega_s(\not{\mathcal{E}})}$$

where for binomial function

$$\mathcal{F}(\mathcal{E}) = \frac{1}{2\pi\omega_{s0} A_N} \left(1 - \frac{\mathcal{E}}{\mathcal{E}_{\max}}\right)^{\mu}$$

with normalisation A_N

$$(\Omega/m - \omega_{s0}) = -i\xi \frac{Z_k}{k} \int_0^{\mathcal{E}_{\max}} d\mathcal{E}\, \mathcal{F}'(\mathcal{E}) I_{mk}^2(\mathcal{E})$$

$$A_N = \omega_{s0} \int_0^{\mathcal{E}_{max}} d\mathcal{E}\, \frac{(1 - \mathcal{E}/\mathcal{E}_{max})^{\mu}}{\omega_s(\not{\mathcal{E}})} \simeq \frac{\mathcal{E}_{max}}{\mu + 1}$$

$$\xi = \frac{2\pi h \omega_{s0}^2 I_0}{V_0 \cos\phi_{s0}}$$

$$I_{mk}(\mathcal{E}) \simeq i^m J_m(k/h\sqrt{2\mathcal{E}})$$

Multi-bunch instability growth rates

After substitution of ξ, $\mathcal{F}'(\mathcal{E})$ and I_{mk} we obtain

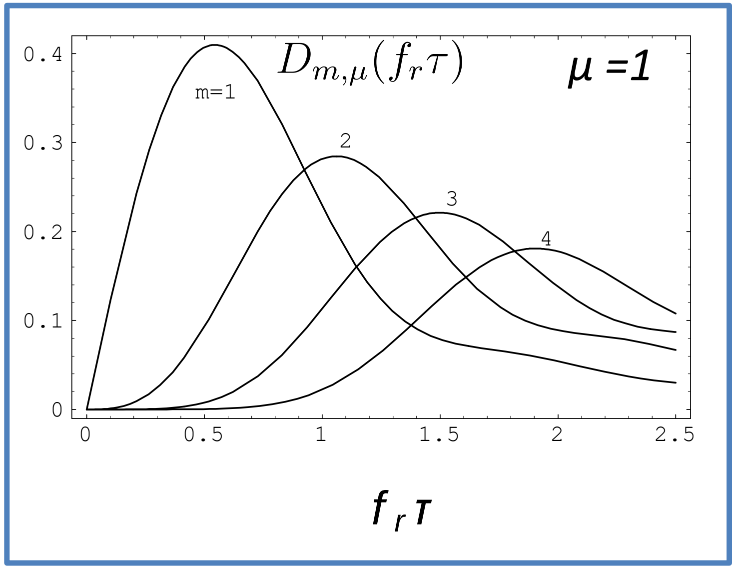
$$\Omega - m\omega_{s0} = im\frac{2\pi h\,\omega_{s0}^2 I_0}{V_0\cos\phi_{s0}}\frac{\mathrm{Re}Z_k}{k_r}\frac{\mu(\mu+1)}{2\pi\omega_{s0}\mathcal{E}_{max}^2}\int_0^{\mathcal{E}_{max}} d\mathcal{E}\left(1-\frac{\mathcal{E}}{\mathcal{E}_{max}}\right)^{\mu} J_m^2\left(\frac{k_r}{h}\sqrt{2\mathcal{E}}\right)$$

Taking into account that $\mathcal{E}_{max} = \phi_{max}^2/2$ and $\phi_{max} = h\omega_0\tau/2$ we finally have

$$\frac{\mathrm{Im}\Omega}{\omega_{s0}} = \frac{4}{\pi^2}\frac{I_0\,\mathrm{Re}Z_k}{hV_0\,|\cos\phi_{s0}|}\frac{D_{m,\mu}(f_r\tau)}{f_0\tau}$$

with formfactor

$$D_{m,\mu}(f_r\tau) = \frac{m\mu(\mu+1)}{f_r\tau}\int_0^1 x(1-x^2)^{\mu-1}J_m^2(\pi f_r\tau x)\,dx$$



Robinson instability

For fundamental impedance we need to keep two terms in Lebedev equation: with $k_1 = n + l_1 M$ and $k_2 = n + l_2 M$, where $l_2 = -l_1$ or $l_2 = -l_1 - 1$. This is true also if $k_r \sim LM/2$

Then we obtain the system of two equations

$$\gamma\lambda_1 = -i\xi(g_{11}\lambda_1 + g_{12}\lambda_2)$$
$$\gamma\lambda_2 = -i\xi(g_{21}\lambda_1 + g_{22}\lambda_2)$$

where $g_{12} \equiv g^m_{k_1 k_2} = \int_0^{\mathcal{E}_{\max}} d\mathcal{E}\, \mathcal{F}'(\mathcal{E}) I_{mk_1}(\mathcal{E}) I^*_{mk_2}(\mathcal{E})$ and $\gamma = \dfrac{\Omega - m\omega_s}{m\omega_s}$

Taking into account that $g_{11} \simeq g_{22}$ and $g_{12} \simeq g_{21} = g_{11}(-1)^m$

We have for the growing mode

$$\gamma \simeq -i\xi g_{11}\left(\frac{Z_{k_1}}{k_1} + \frac{Z_{k_2}}{k_2}\right)$$

There is no instability for $n = 0$ if $k_1 = -k_2$, except Robinson type, where we should take into account that $Z_k(\Omega) = Z_k(k\omega_0 + \Omega)$ and $\mathrm{Re}\Omega = m\omega_s$

Multi-bunch instability: spectrum

For a narrowband resonant impedance at unknown $\omega_r = \omega_0\, p_r$, the instability spectrum has components at $\omega = (n + l\, M)\omega_0 + m\omega_s$, $-\infty < l < \infty$,

(n = 0, 1… M-1 is the coupled-bunch mode number, M is number of equidistant bunches in the ring and m=1, 2,… is the multipole number).

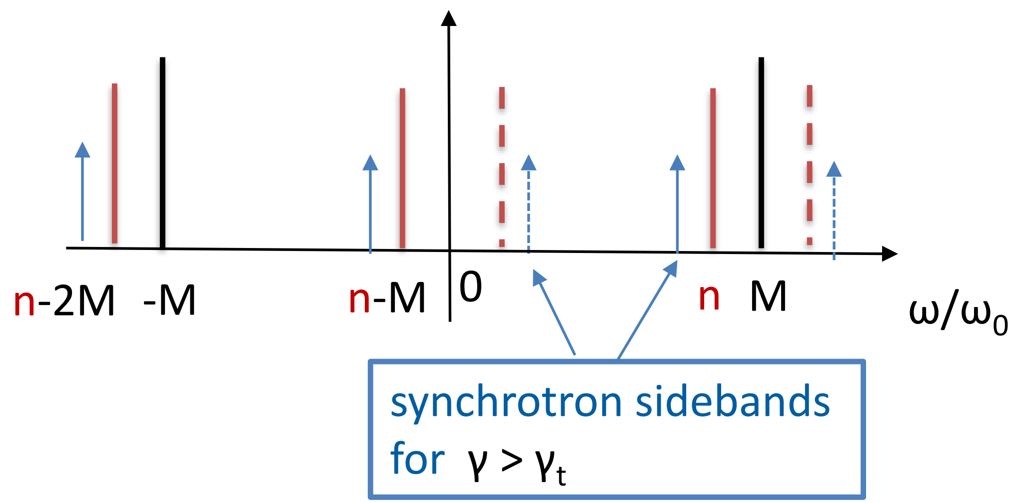
On the spectrum analyzer negative ω appear at $[(l+1)\, M - n)]\omega_0 - m\omega_s$

→ Measured mode n is not sufficient to determine ω_r since $n + l\, M \approx \pm\, p_r$

Lines at $n + l\, M$ and $(l+1)\, M - n$.

→ Measure n for different M

→ Measure f_{max} of the envelope



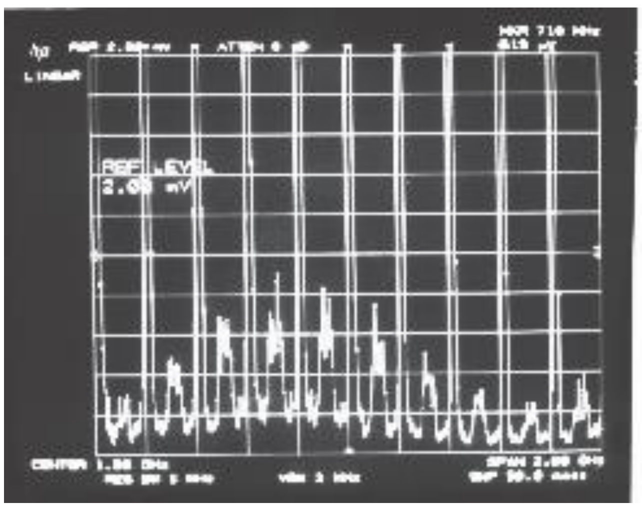
Multi-bunch instability: spectrum

Spectrum envelope with maximum at f_{max}:

if $f_{max}\,\tau < 1 \;\rightarrow\; f_r < 1/\tau$

if $f_{max}\,\tau > 1 \;\rightarrow\; f_r \sim f_{max}$

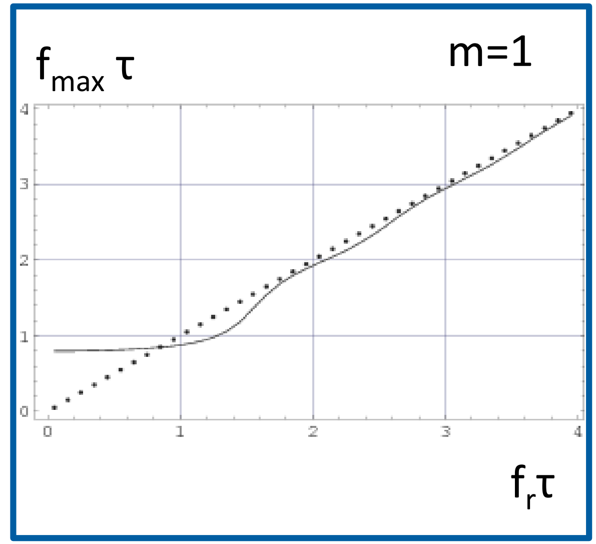
SPS: known HOMs at 623 & 912 MHz



0 2 GHz

200 MHz beam lines

5 ns spaced bunches



→ f_r can be identified from spectrum

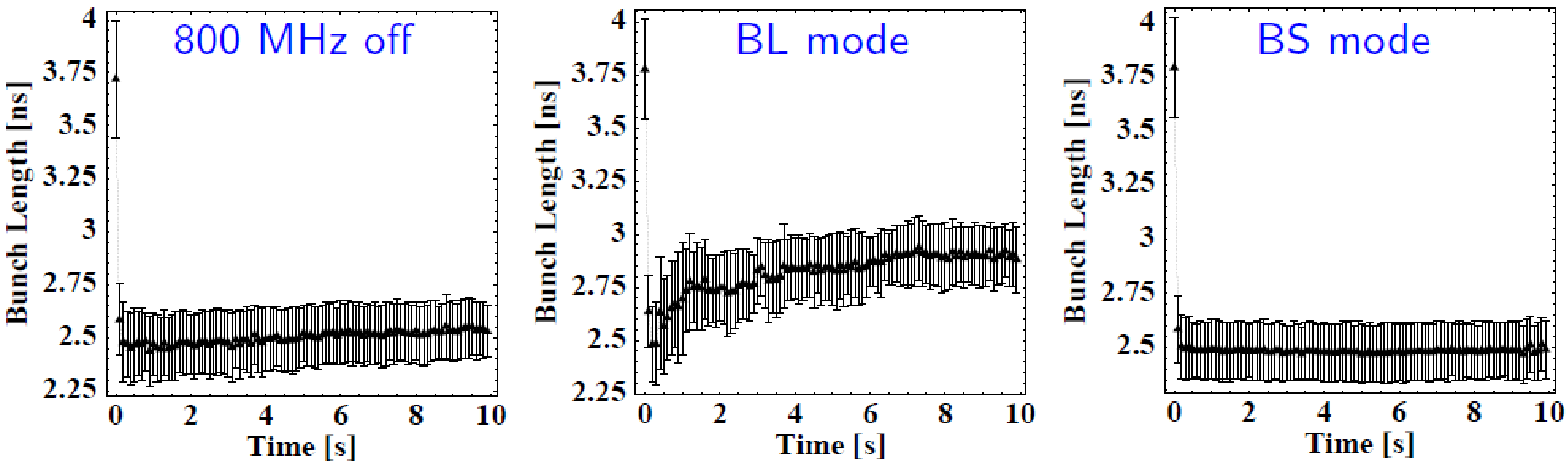
→ R_{sh} from growth rate measurements

Multi-bunch instability: cures

- Active
  - Feedback systems
  - Higher harmonic RF system
  - Controlled emittance blow-up
- Passive
  - HOM damping (couplers),
  - HOM-free cavity design,
  - Impedance reduction (modification of machine elements),
  - Change of optics (of gamma transition)
  - Synchrotron radiation damping in lepton rings

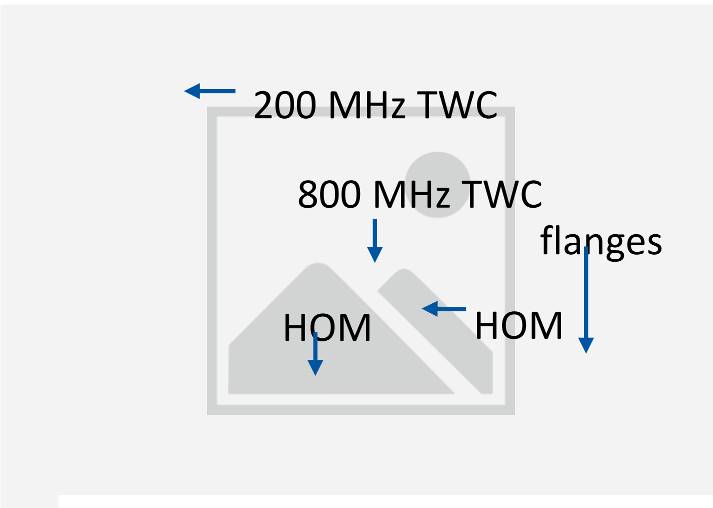
Longitudinal beam instability cures: Higher harmonic RF system

Measured averaged bunch length in (200 + 800) MHz RF system for $V_4/V_1 = 0.23$
Nominal LHC beam on the 26 GeV/c flat bottom of the CERN SPS



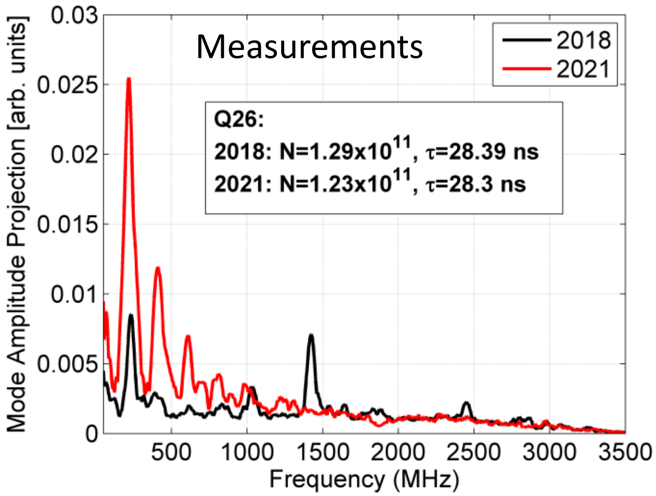
In double RF system the CB instability threshold is 5 times higher.
Only BS-mode works (phase control, flat portion in f_s - distribution)

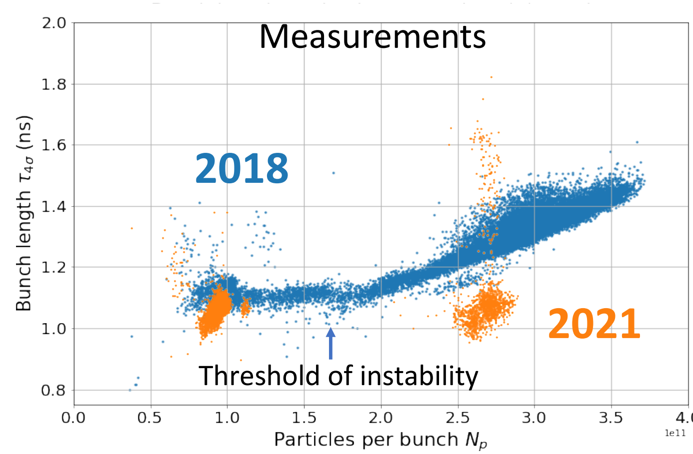
Beam instability cures: Impedance reduction in the CERN SPS



- 200 MHz RF: 4 long → 6 short structures
- 623 MHz HOM damping
- Vacuum flanges shielding (~100)

→ Smaller bunch lengthening,
reduced potential well distortion
→ Stable HL-LHC beam





Thank you for your attention!

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Further reading

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