CAS RF School
Matching & Power coupling

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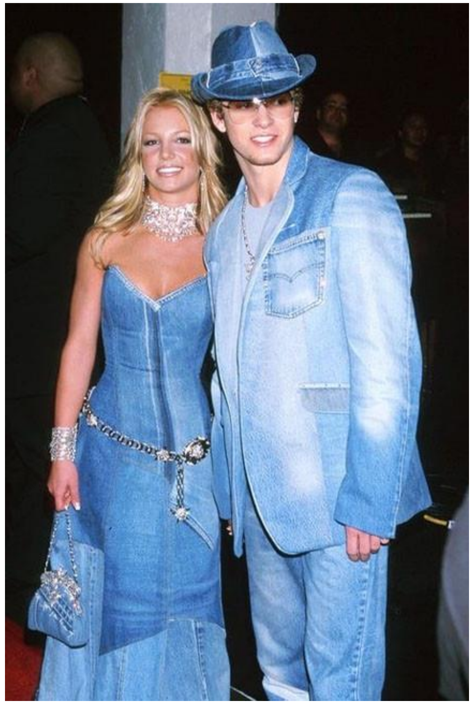
Matching and Power coupling

- If we type matching power couple into google this is what you find

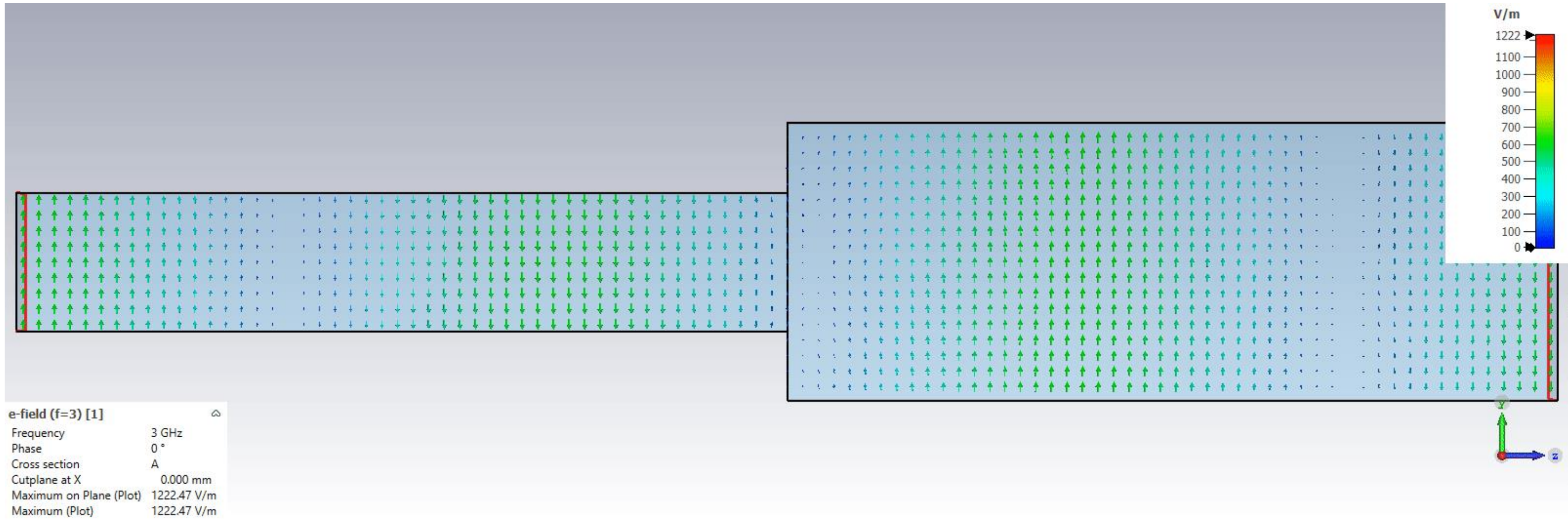


- This lecture is not about that, its about matching the input and output impedances in order to minimize reflections and maximise the power entering the cavity

Mode matching

- At an interface the **fields must be continuous** at the interface (ie the same on both sides)
- On any metal walls the metallic boundary conditions must be preserved ($E_{||}=H_T=0$)
- For this to be true the **sum of the modes on each side of the interface should create identical E and H fields**, where $\mathbf{a_n}$ is the amplitude of forward mode n and $\mathbf{b_n}$ is the amplitude of reflected mode n

$$\sum_{all\ modes}^{region1} a_n E_n + \sum_{all\ modes}^{region1} b_n E_n = \sum_{all\ modes}^{region2} a_m E_m$$

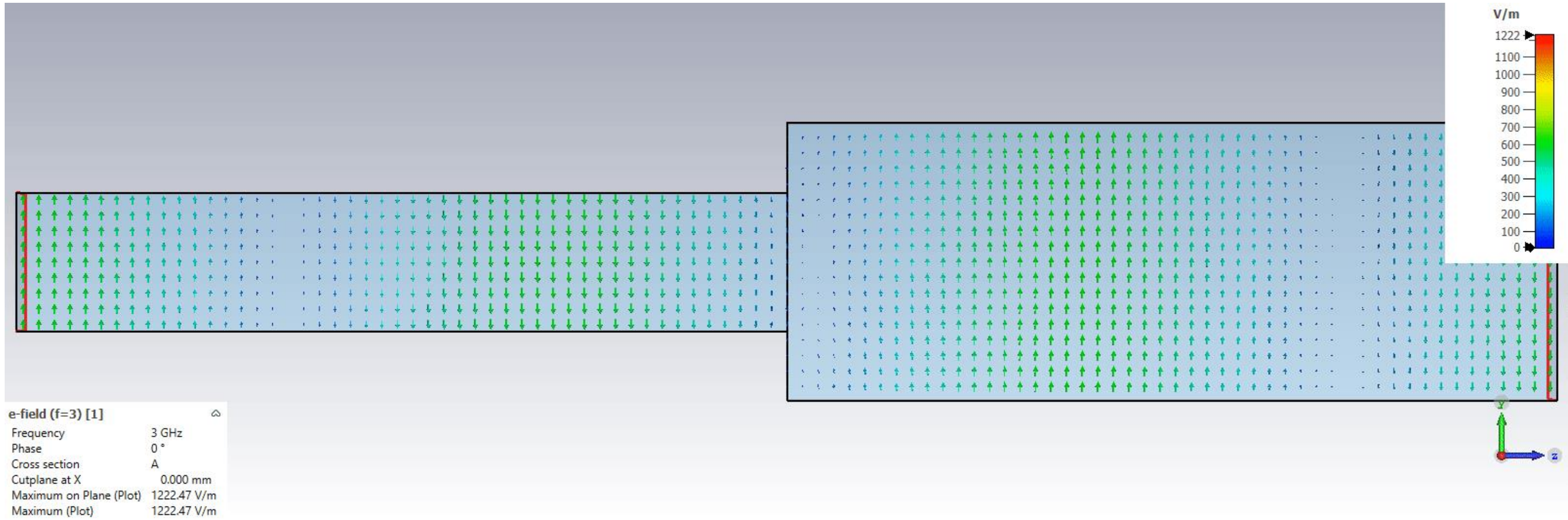


Mode matching

- We have the **same for magnetic fields**

$$\sum_{all\,modes}^{region1} a_n H_n + \sum_{all\,modes}^{region1} b_n H_n = \sum_{all\,modes}^{region2} a_m H_m$$

- In the example below we have a TE10 mode waveguide with a step in height. There is only one propagating mode
- The sum of the forward and reflected TE10 mode in the left side of the interface cannot equal the TE10 mode on right hand side as the field patterns are different, so the **field must scatter into a higher order non-propagating mode** to match the fields



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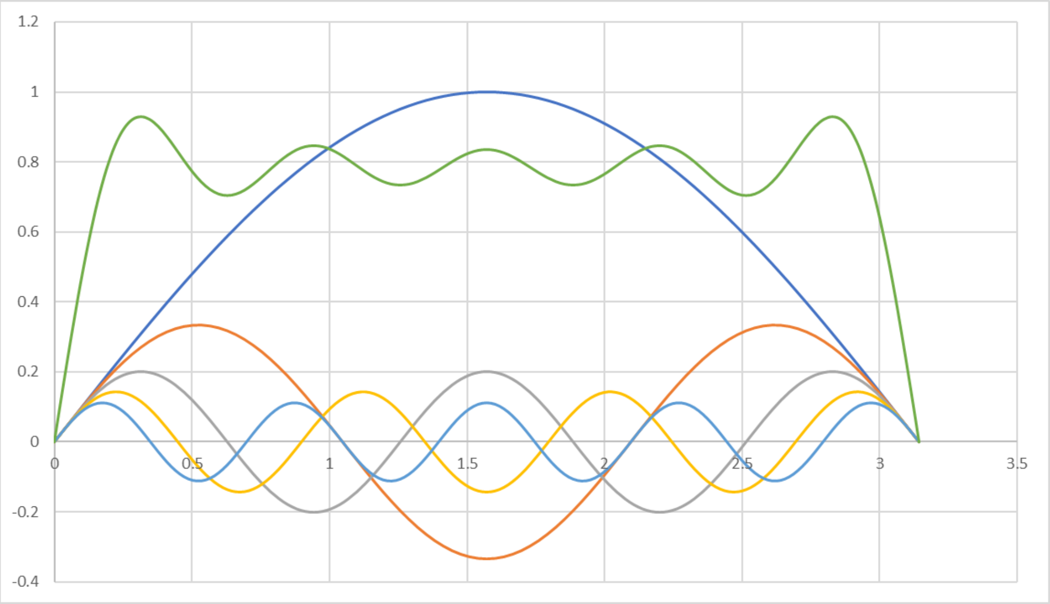
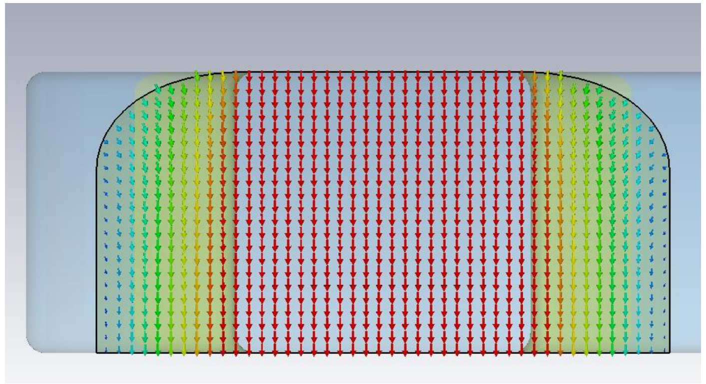
Mode matching and Fourier series

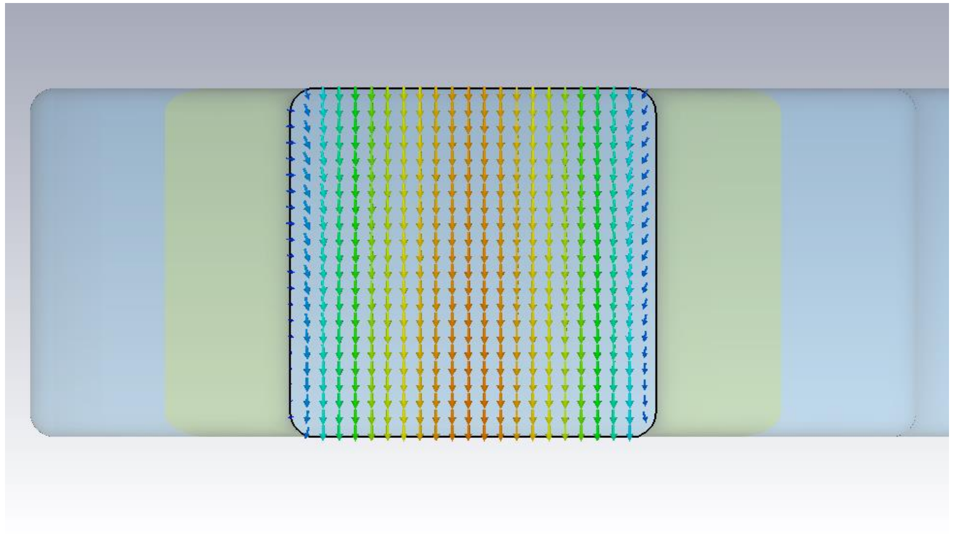
- In rectangular waveguide the modes vary in each cartesian dimension as a **sine or cosine wave**, meeting the boundary conditions at the walls forcing one component to zero
- Fourier theory says any shape can be created by a **superposition of sinusoidal waves of different harmonics**
- Therefore any field pattern can be created by a superposition of modes
- As we can have forwards and reverse waves this would not be single valued, but the **requirement to match both E and H reduces it to a single solution** comprising of a finite number of mode amplitudes



Field in the cavity (square)



Field in the coupler (sinusoidal)



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Mode matching

- In the case of couplers the field of the cavity must be matched in the coupler by summing over all coupler modes

$$\mathbf{E}_{\mathbf{cav}} = \sum_{n=1} a_n \mathbf{E}_{\mathbf{n,coup}} \qquad \mathbf{B}_{\mathbf{cav}} = \sum_{n=1} b_n \mathbf{B}_{\mathbf{n,coup}}$$

- If our coupler mode can **perfectly match the cavity field at the interface in both E and H we get perfect coupling**
- But it almost **impossible to match the fields exactly** so other methods must be used
- But a cavities field amplitude changes with time as it fills so matching is a **time dependent problem**, typically we aim for perfect coupling in steady state (ie after its filled)

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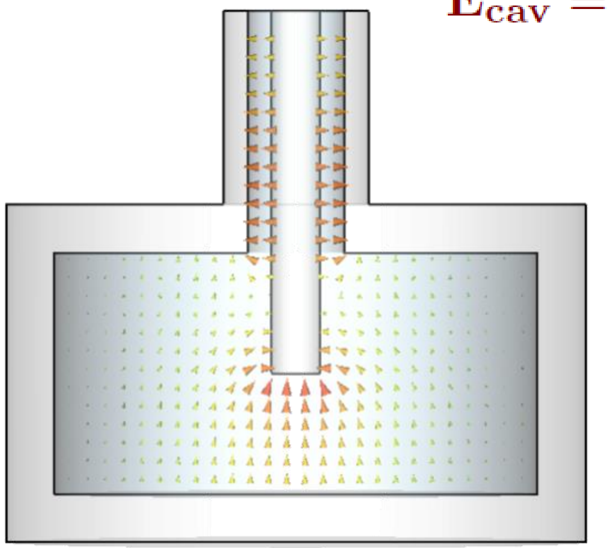
Electric coupling

$$\mathbf{E}_{\mathbf{cav}} = \sum_{n=1} a_n \mathbf{E}_{\mathbf{n,coup}}$$

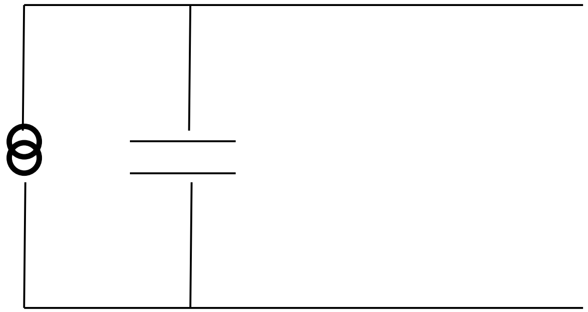
For coaxial lines it is more convenient to use equivalent circuits, and represent the coupling as a current/voltage source and a capacitor/inductor

For capacitive coupling we leave the inner conductor unterminated and use the **capacitance between the conductor and the cavity walls** to couple



The equivalent circuit for electric coupling is a **current source** in parallel with a capacitance.

The top rail represents the inner conductor and the ground plane is the cavity body and outer conductor.



We get the charge by integrating Gauss' law over the tip of the inner conductor.

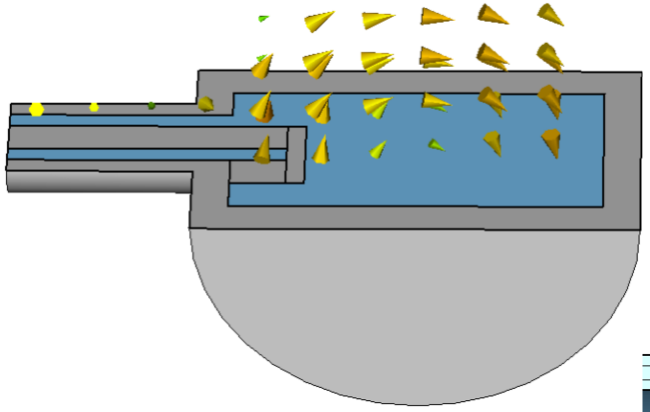
$$Q = \varepsilon_0 \int E.dS$$

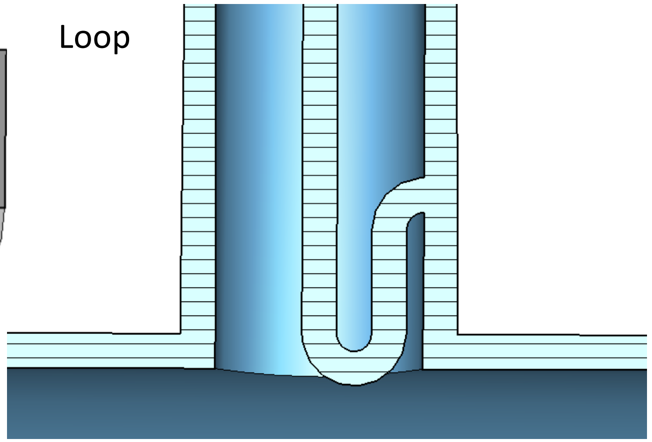
$$I = -\frac{dQ}{dt}$$

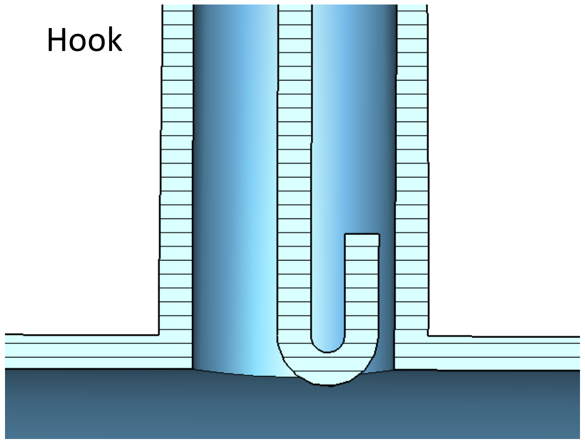
Magnetic coupling

From Faradays law

$$V = -\int \frac{dB}{dt}.dS$$





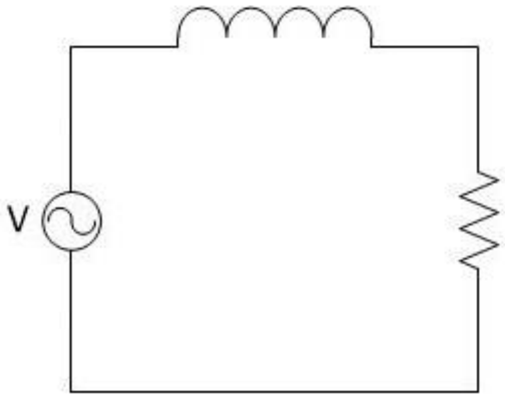


The equivalent circuit for magnetic coupling where the inner conductor loops round and is connected to the outer conductor is a **voltage source** in **series** with an **inductance**

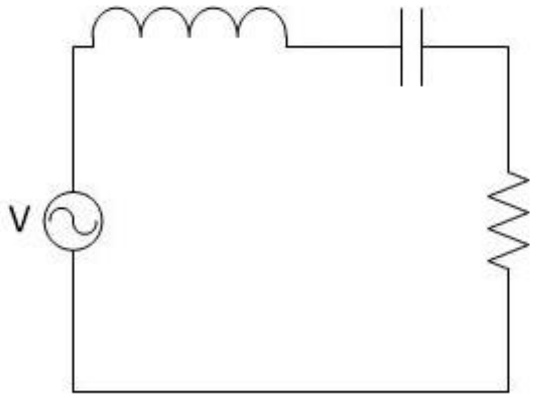
.

If the loop isn't physically connected then there is also a **series capacitance** making a band pass filter.

$$\mathrm{B_{cav}} = \sum_{n=1} b_n \mathrm{B_{n,coup}}$$



Circuit for magnetic field coupling with a loop



Circuit for magnetic field coupling with a hook

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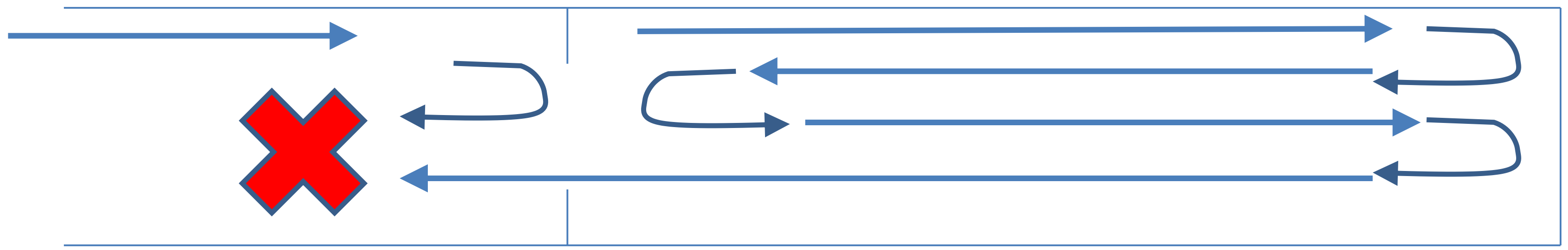
Cavity matching

- In the case of a cavity we have two reflective elements
  - the coupling iris the joins the waveguide to the cavity
  - the back of the cavity



- The wave will **bounce between those two elements** and hence the reflections will **change with time** as the fields build up.
- Ideally in **steady state** the reflections from each element are equal and opposite **cancelling each other** out and hence **matching**

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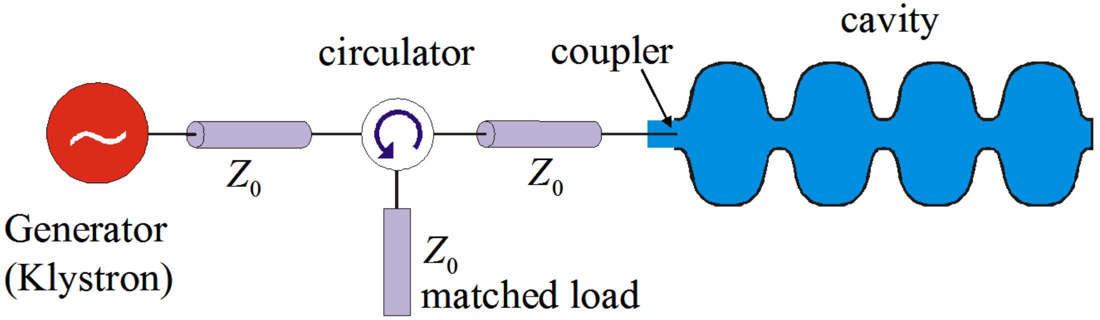
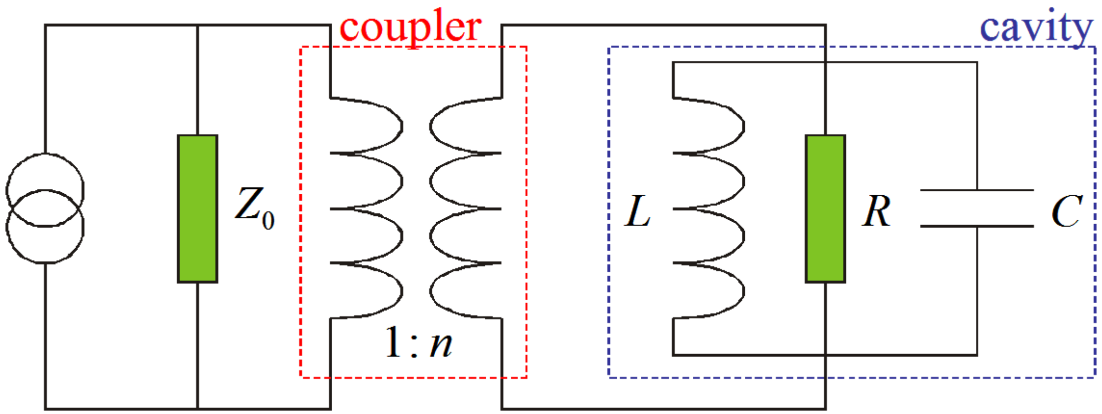
Couplers





To model how a cavity fills in time we can use an equivalent circuit

The coupling can be represented as a **transformer** (ie a mutual inductance).

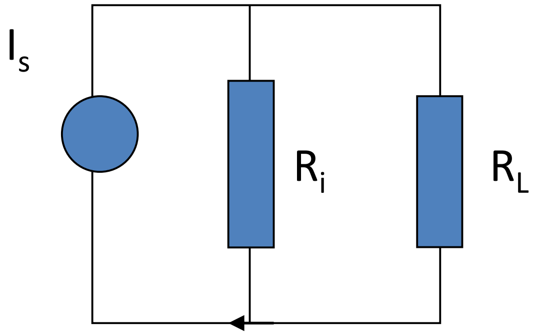
The RF source is represented by a **ideal current source** in parallel to a parallel resistance to represent the line and the coupler is represented as an n:1 turn transformer.

$$Q_e = \frac{\omega U}{P_e} = \omega n^2 Z_0 C$$

$$\beta = \frac{P_e}{P_c} = \frac{Q_0}{Q_e} = \frac{R}{n^2 Z_0}$$

Typically Z_0 is 50 Ohms but the cavity shunt impedance, R, is ~MOhms and hence the transformer changes the cavity impedance seen by the source to match.

Power Transfer Theory

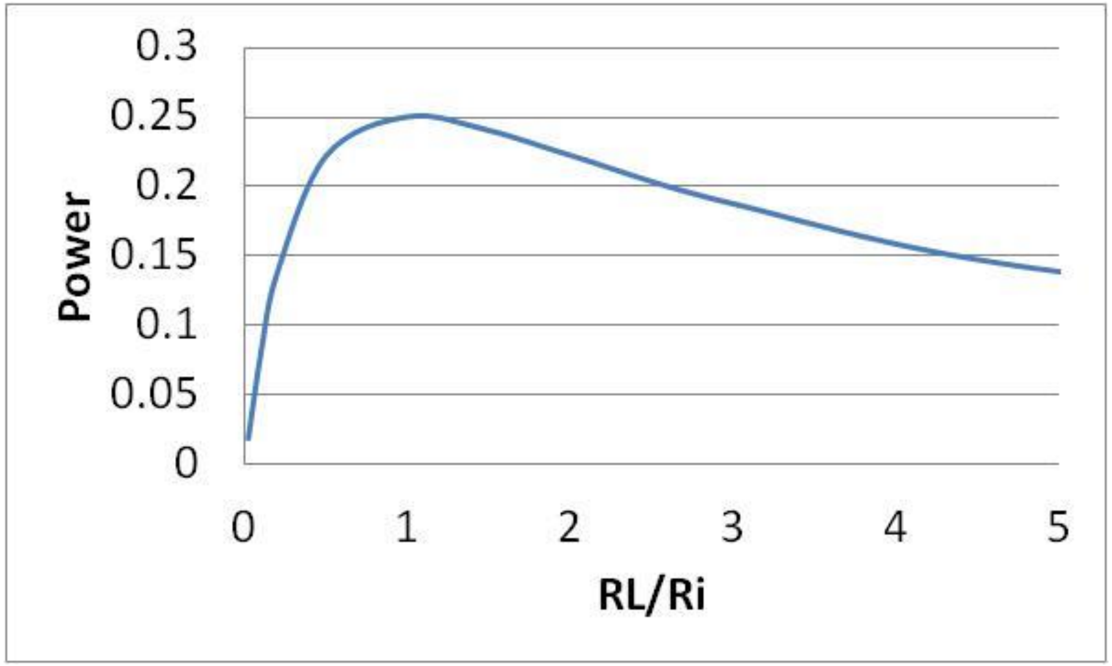


$V_s = I_s R_i R_L/(R_i+R_L)$

$P_L=V_s^2 / R_L$ (the power delivered to the load)

$P_L = I_s^2 R_i^2 R_L/ (R_i+R_L)^2$

$P_L= I_s^2 R_i^2 / (R_i^2/R_L +2R_s+R_L)$



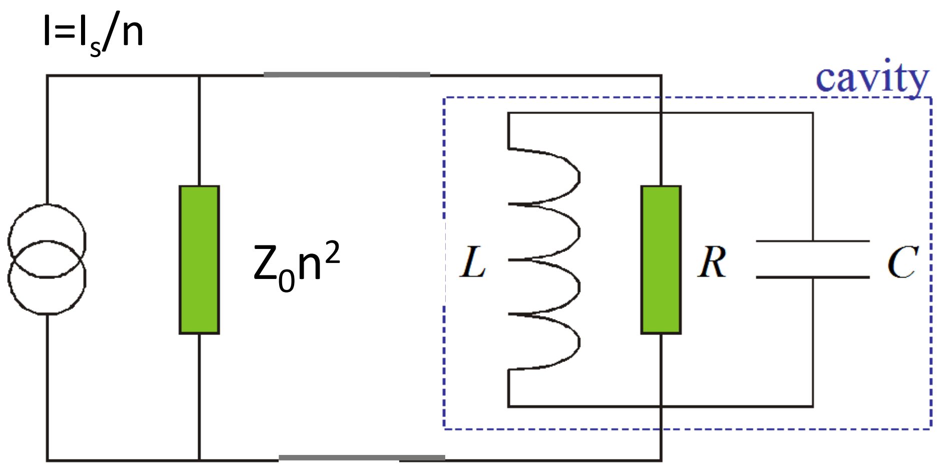
The maximum peak power transferred occurs when

$$\frac{dP_L}{dR_L} = \frac{I_s^2 R_i^2 (R_i - R_L)}{(R_i + R_L)^3} = 0$$

- Which occurs when $R_L=R_i$
- In this case the load dissipates a peak power $P_L=I_s^2R_i/4$

Coupler circuit representation

- The transformer makes the load impedance seen **smaller by a factor of n^2**
- By varying the coupling we vary n allowing us to set the load impedance.



- The reflection (ie match) depends on the difference in the coupler and cavity impedance

$$P_R = P_g \left(\frac{(Z_0 - Z_c)^2}{(Z_0 + Z_c)^2} \right)$$

- When on resonance the cavity impedance equals the shunt impedance (Z_c=R)
- We can then simply model the circuit as two parallel resistors.

Generator Power

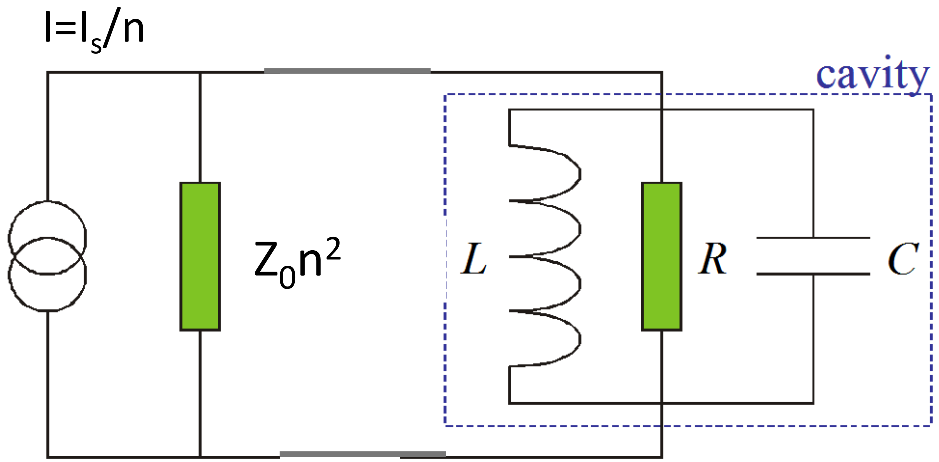
- An **RF amplifier** has a **fixed forward power rather than instantaneous** ie reflections do not change the forward power
- It makes sense that the **generator power** from the RF source is the **maximum power that can be delivered to the cavity** rather than power delivered from the current source.
- This occurs when

$$Z_0 n^2 = R$$

- In this case the **rms** power delivered, P_+, is

$$P_+ = \frac{I_s^{\,2} R}{8n^2} = \frac{I_s^{\,2} Z_0}{8}$$

- As the coupler (n) changes or the cavity impedance changes **the power from the current source will vary but we ignore this**. We only care about the power dissipated in the cavity and the generator power which is fixed as above.



$$V_c = IR_{tot} = \frac{IRZ_0 n^2}{R + n^2 Z_0} = \frac{IR}{1+\beta}$$

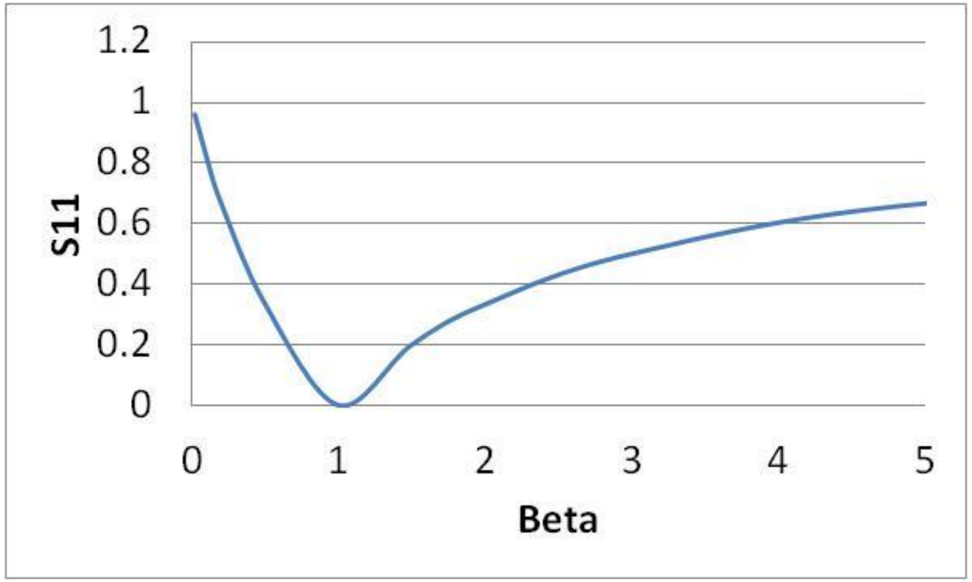
$$V_c = \frac{2\sqrt{2RP_+\beta}}{1+\beta}$$

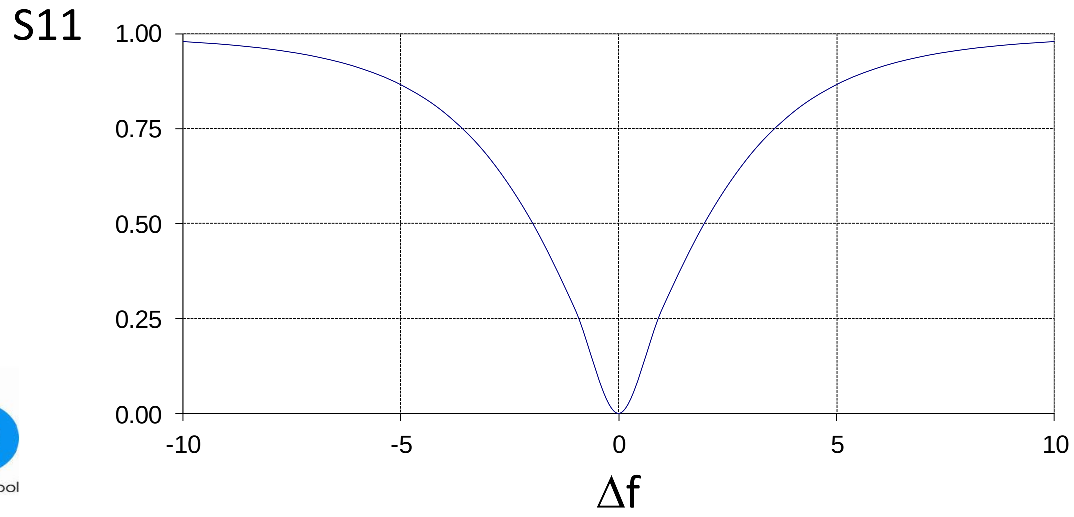
Reflected Voltage

We can also define the reflection in terms of β and we find the **match occurs at β=1**.
The reflection also depends on the **frequency difference** between the source and the cavity resonance

$$P_R = P_g\left(\frac{(Z_0-Z_c)^2}{(Z_0+Z_c)^2}\right) = P_g\left(\frac{1-\beta-i\delta}{1+\beta+i\delta}\right)^2$$

where $$\delta = Q\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)$$

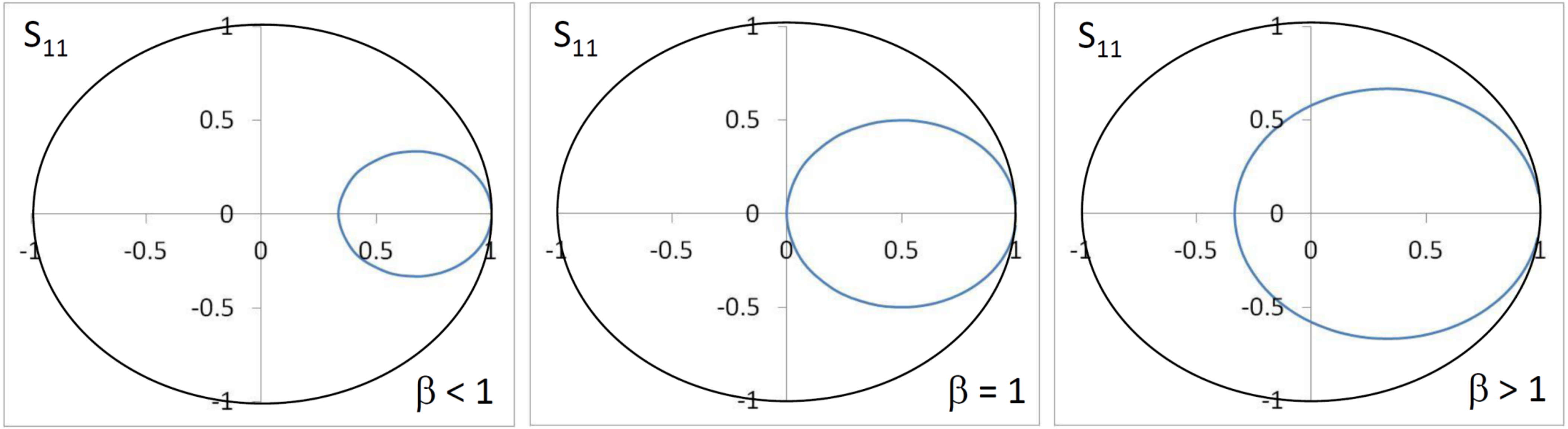




$$V_- = \sqrt{P_R} = V_+\frac{1-\beta-i\delta}{1+\beta+i\delta}$$

$$S_{11} = \frac{V_-}{V_+} = \frac{1-\beta-i\delta}{1+\beta+i\delta}$$

Under and Over coupled



We can tell if β is greater or less than 1 from the **polar plot of S_{11}**.
If the circle **doesn't encompass** the origin then beta is **less than 1 (undercoupled)**
If the circle **does encompass** the origin then beta is **greater than 1 (overcoupled)**

$$V_- = \sqrt{P_R} = V_+ \frac{1-\beta-i\delta}{1+\beta+i\delta}$$

$$S_{11} = \frac{V_-}{V_+} = \frac{1-\beta-i\delta}{1+\beta+i\delta}$$

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Cavity filling

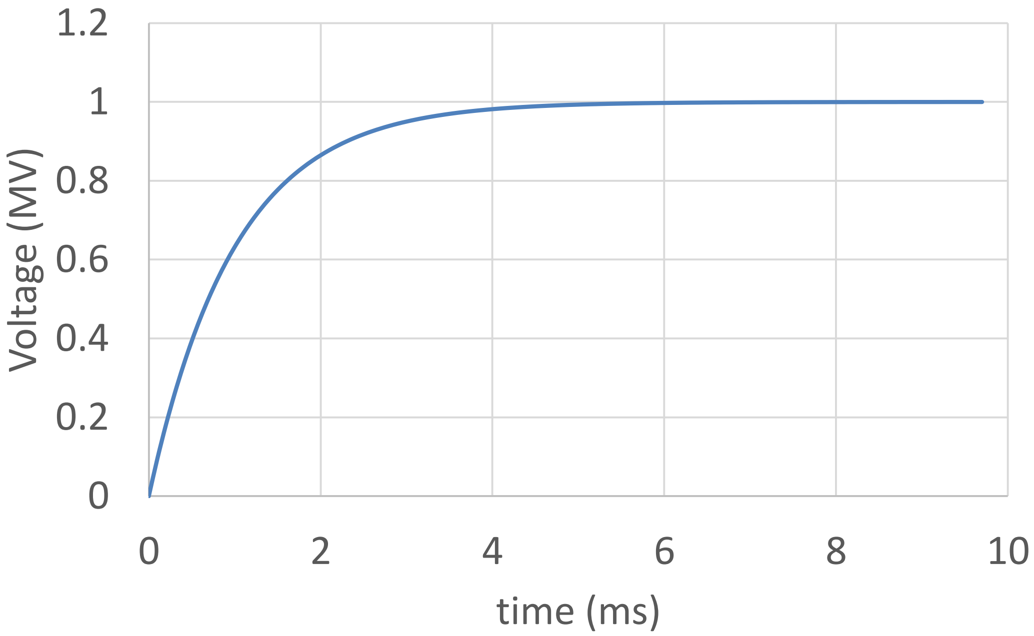
The most important behaviour we must understand is when the cavity is in steady state (ie when the cavity stored energy is constant and $U=U_0$). We can use the definitions of β and Q to derive,

$$U_0 = \frac{4\beta P_f}{(1+\beta)^2}\frac{Q_0}{\omega}$$

- The cavity fills with time as $1-e^{-t/\tau}$ where τ is the filling constant/time.

$$V_c = \frac{2\sqrt{2RP_{+}\beta}}{1+\beta}\left(1-e^{-t/\tau}\right)$$

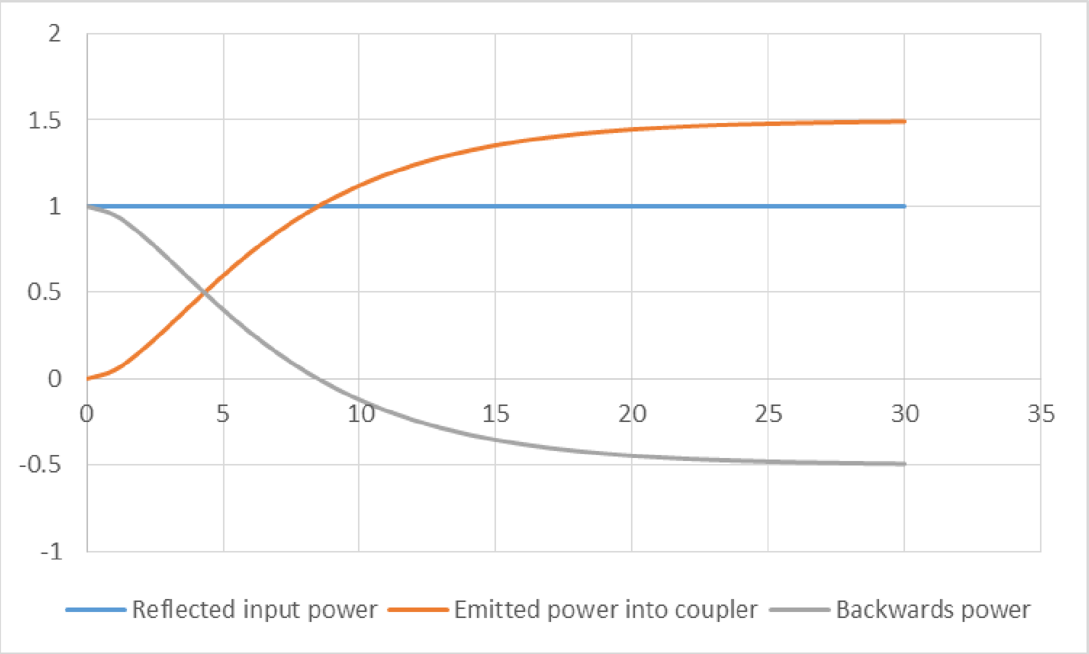
- The filling constant is
- $\tau=2Q_L/\omega$
- The **higher the loaded Q** the higher the stored energy but the **longer it takes to fill**.



Filling of cavity with τ=1ms

Transient reflections

When filling, the stored energy in a resonant cavity varies with time and hence so does the reflections.



- As we vary the external Q of a cavity the filling behaves differently.
- Initially **all power is reflected** from the cavity at the interface due to the difference in impedance
- As the cavities fill **the reflections reduce** as we get **cancellation** from waves coming from inside the cavity.

The cavity is only matched if the external Q of the cavity is equal to the ohmic Q (you may include beam losses in this).

Transient reflections

$$V_c = \frac{2\sqrt{2RP_+\beta}}{1+\beta}$$

As we know the voltage increases as $1-e^{-t/\tau}$ with time

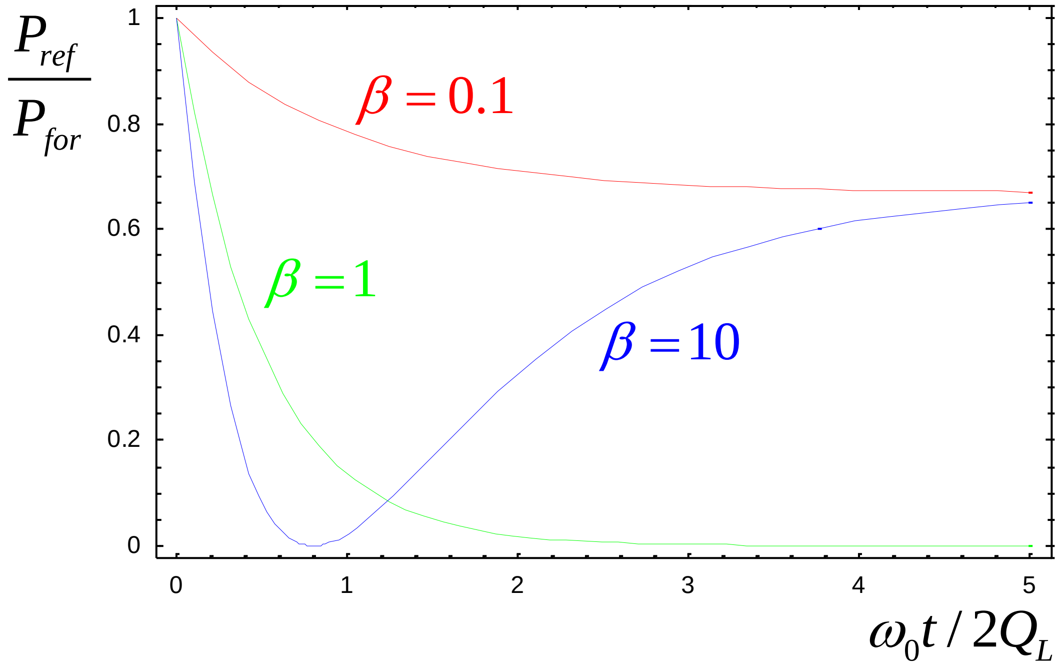
$$V_c = \frac{2\sqrt{2RP_+\beta}}{1+\beta}\left(1-e^{-t/\tau}\right)$$

Using the definition of reflected voltage and knowing that V_+ is constant

$$V_- = V - V_+$$

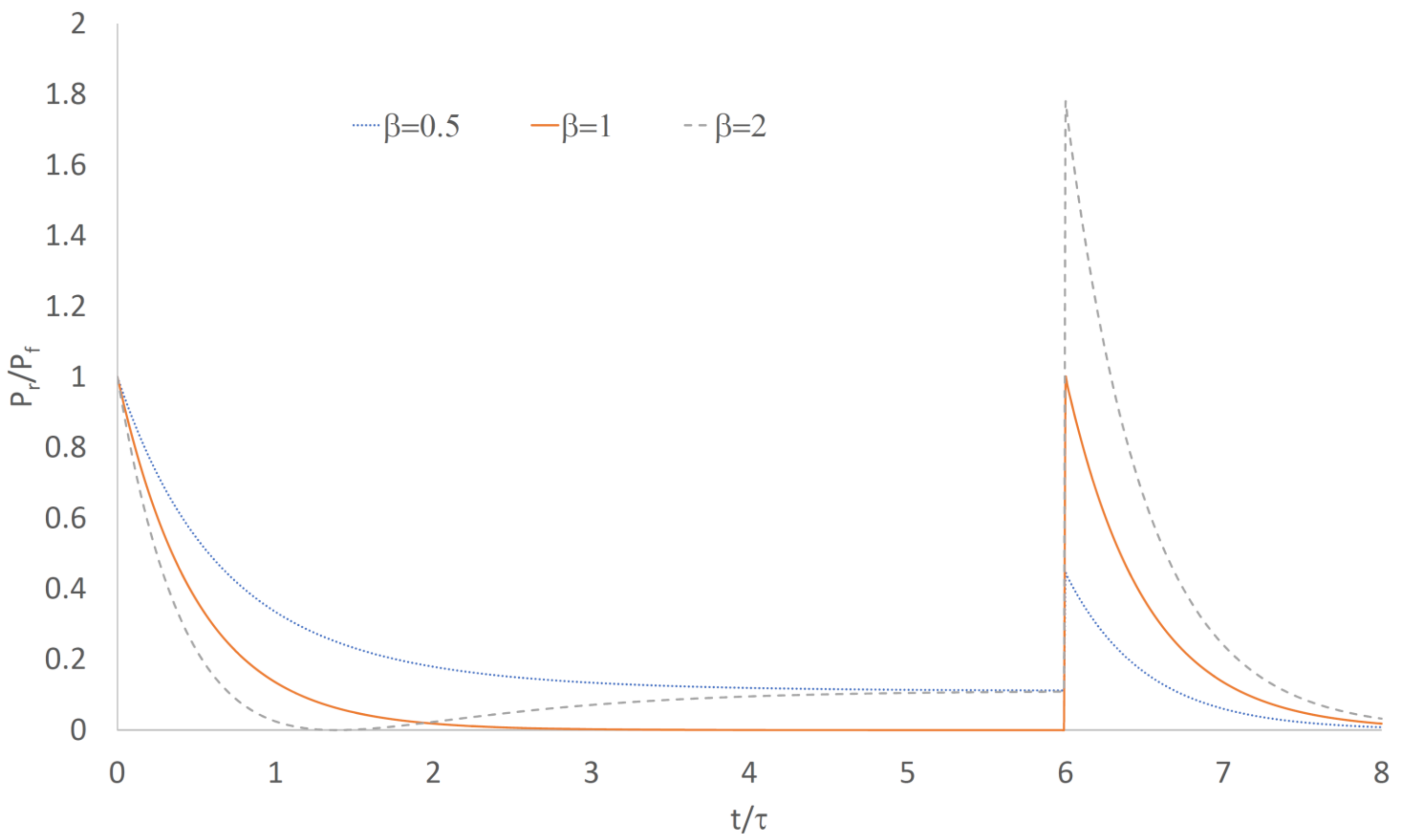
$$V_- = V_+\left[\frac{2\sqrt{\beta}}{1+\beta}\left(1-e^{-t/\tau}\right)-1\right]$$

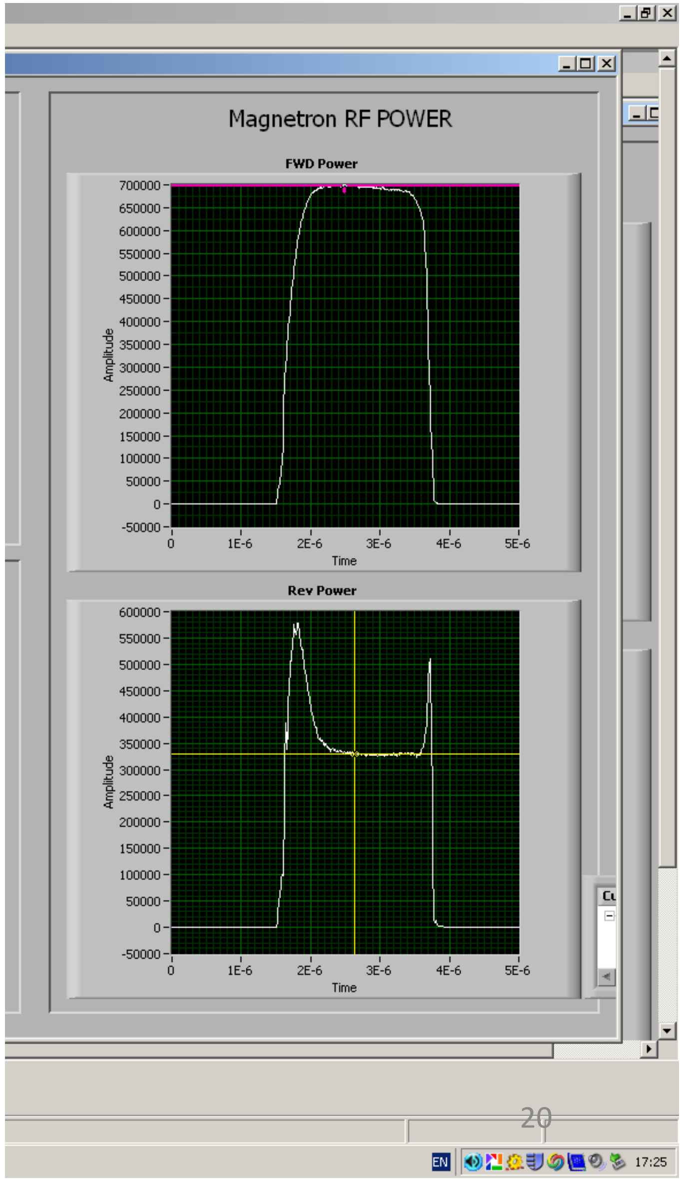
- The reflections can also be calculated from circuit theory



Reflections from a square wave

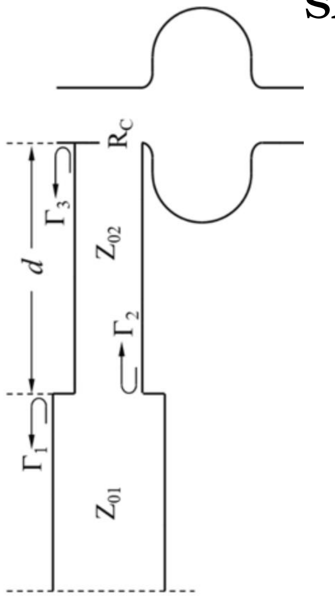
- Excited by a square pulse steady state reflections depends on β as we have seen, but when the RF is turned off the **emitted power is also dependant on β**

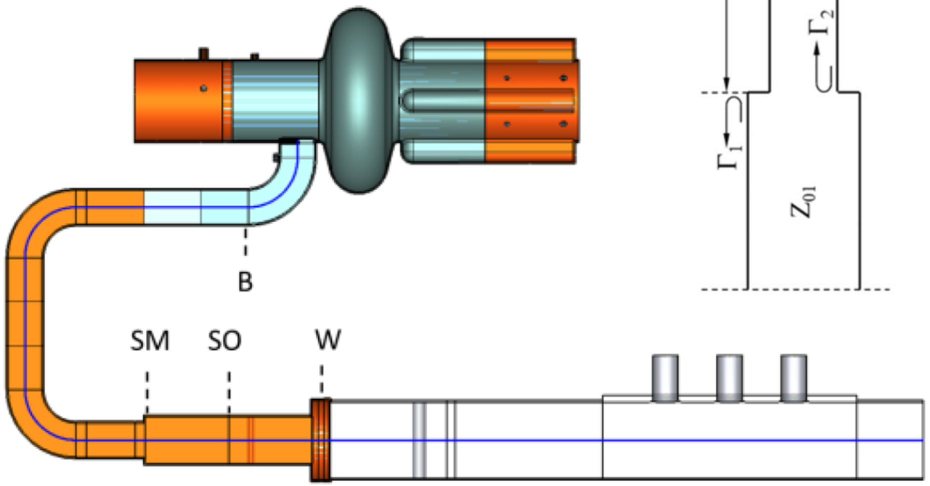


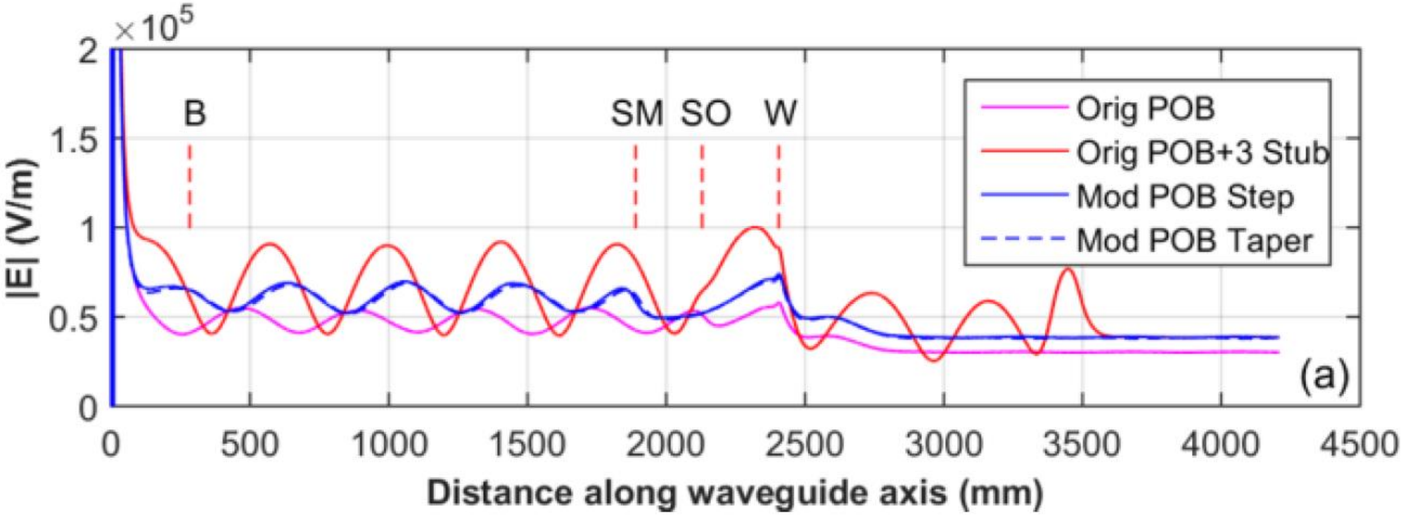


Stub Matching

- Sometimes it is necessary to change the Q of the cavity, this can be done using a 3-stub tuner to match.
- If the natural Qe of the coupling is not matched there is a reflection, however this can be **canceled out** with **another reflection 180 degrees out of phase**.
- Three stubs spaced apart by 0.375 wavelengths can provide any reflection phase/amplitude by varying the stub insertion.



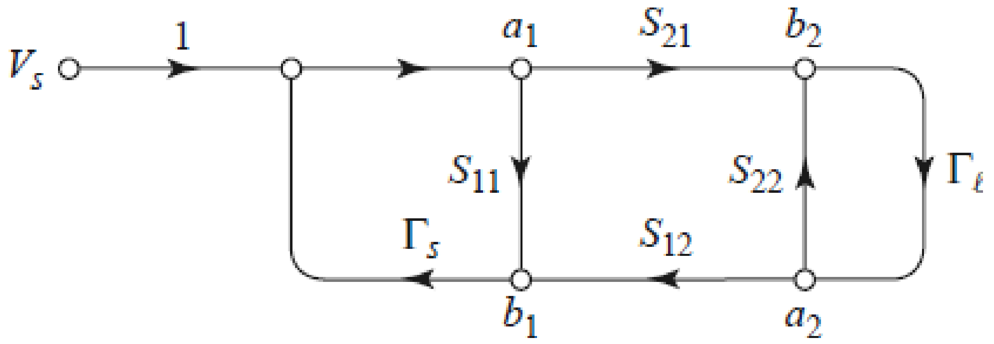




- The downside is there is a **standing wave** bouncing back and forth between the two reflective surfaces
- This creates a standing wave and hence **higher peak fields**

Matching with stubs

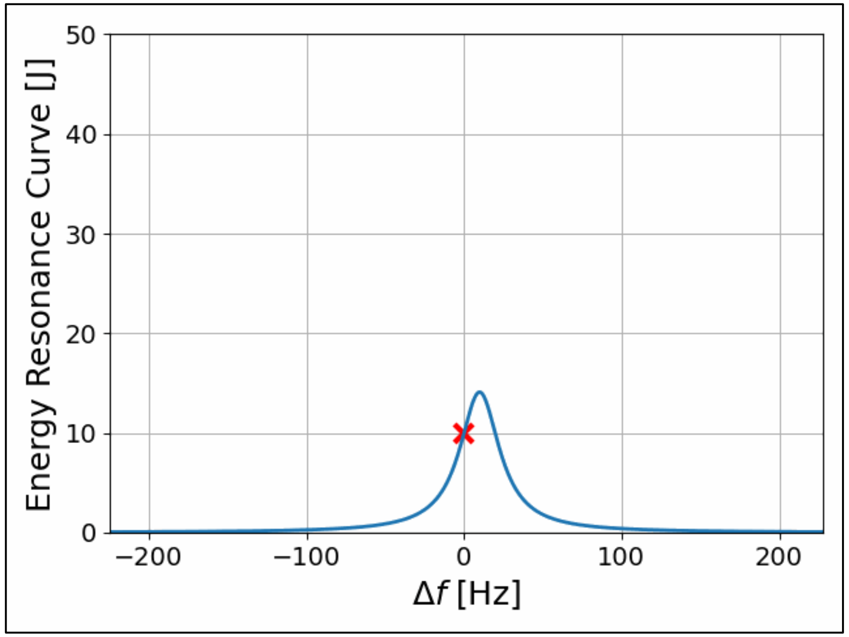
- Imagine we have two mismatched interfaces to a coaxial line.
- The coaxial line is represented by the S matrix and the two reflection coefficients are the stub and the cavity
- Lets assume the line is constant impedance so S11=S22=0
- Lets also assume the line is lossless so S21=S12=exp(-jkL) where k is the propagation constant and L is the distance between them.
- The cavity reflection is (1-β)/1+β) so the stub reflection needs to be the negative of



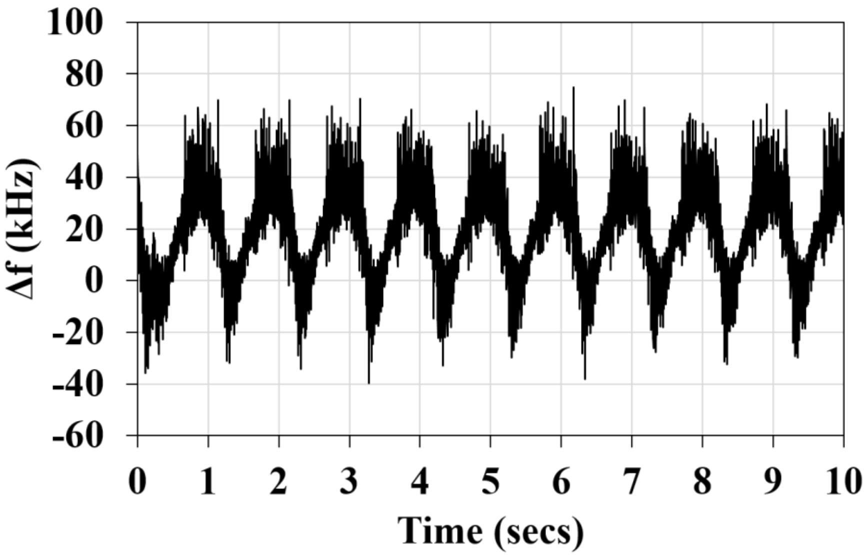
$$\Gamma_{\text{in}} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\Gamma_\ell}{1 - S_{22}\Gamma_\ell}.$$

$$\Gamma_{in} = \frac{1-\beta}{1+\beta} exp(-j2kL)$$

Microphonics



- Microphonics cause the cavity resonant frequency to vary in time by anywhere from 10 Hz to 10 kHz.
- As we have seen before the **reflections are dependent on the difference** between the drive and natural frequencies.

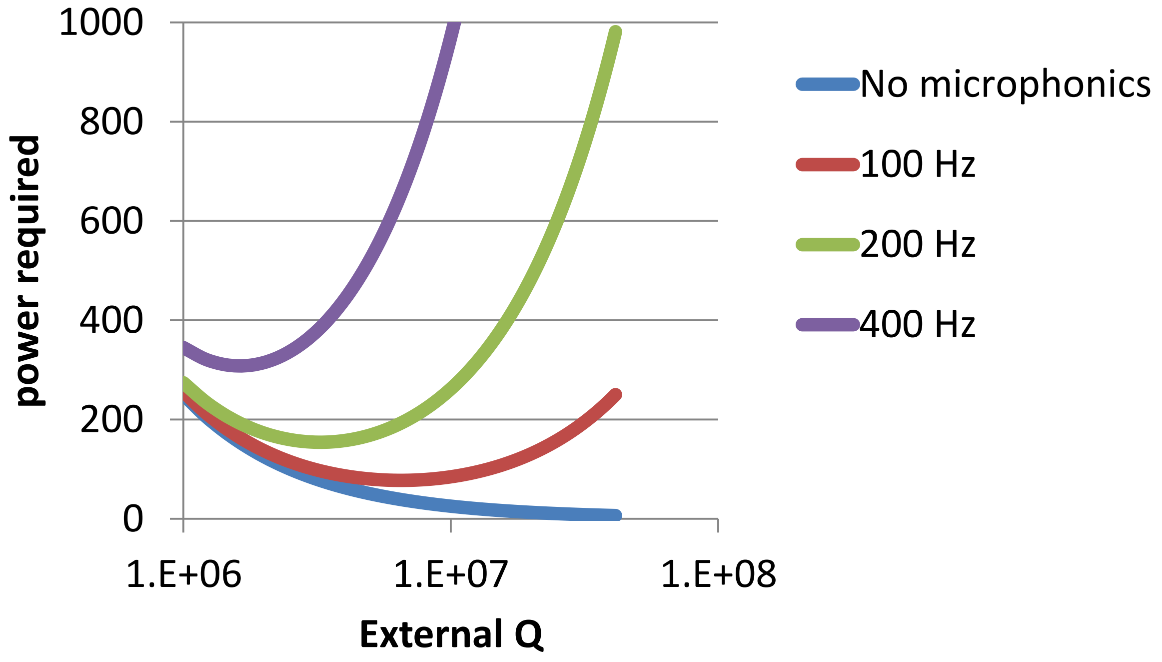


$$S_{11} = \frac{V_-}{V_+} = \frac{1-\beta-i\delta}{1+\beta+i\delta}$$

- As the detuning angle is proportional to the Q factor it leads to an **poor coupling** for SRF cavities, which have **high Q's**.

Microphonics

$$P_+ = \frac{V_c^{\,2}(1+\beta)^2}{8R\beta}\left(1+4Q_L^{\,2}\frac{\Delta\omega^2}{\omega^2}\right)$$

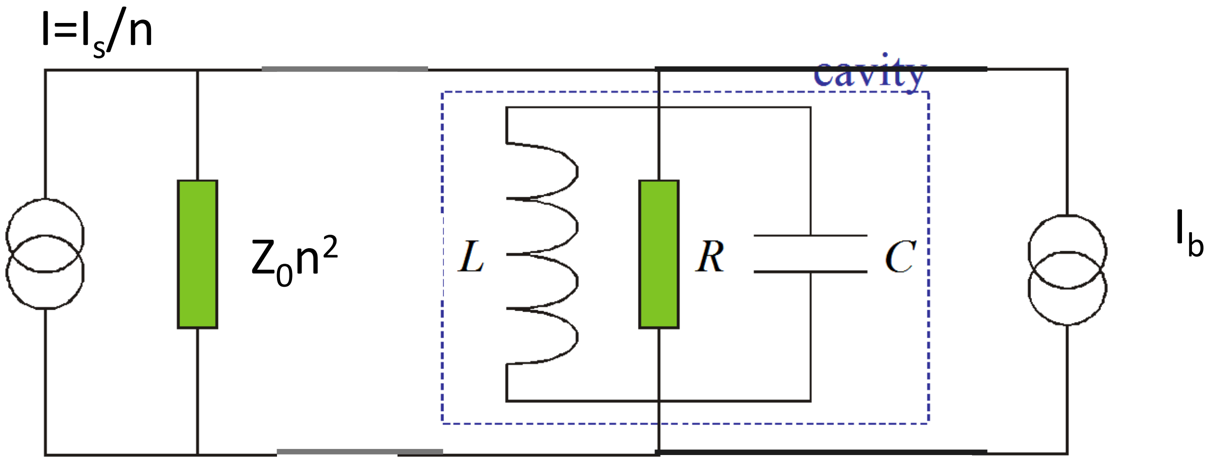


Power demand for 1 watt ohmic heating in ALICE

- As can be seen if the microphonics cause a detuning larger than the cavity bandwidth the **power demand to keep the cavity at the required voltage is large**.
- To avoid this a **lower external Q** is chosen than the ohmic Q, ie the coupler is not matched.
- This leads to higher reflections without microphonics but much lower reflections in practice

Beam Loading

- In addition to ohmic losses we must also consider the power extracted from the cavity by the beam.
- The beam draws a power $P_b=V_c\ I_{beam}$ from the cavity.
- I_{beam}=q f, where q is the bunch charge and f is the repetition rate
- This acts as a **current source in series** with the cavity



Coupling with Beam Loading

- The rf source will not see any difference between the power dissipated in the cavity walls and the power extracted by the beam hence we can calculate a new Q factor, Q_{cb}.

$$Q_{cb} = \frac{\omega U}{P_c + P_b}$$

- this Q_{cb} will replace Q_0 when calculating cavity filling. This means the match will change as well as needing more power.

$$\beta_{eb} = \frac{Q_{cb}}{Q_e} \qquad U_0 = \frac{4\beta_{eb} P_f}{(1+\beta_{eb})^2} \frac{Q_{cb}}{\omega}$$

- Normally we aim for β=1 with beam and have reflections when filling.

$$\beta = \frac{Q_0}{Q_e} = 1 + \frac{Pb}{Pc}$$

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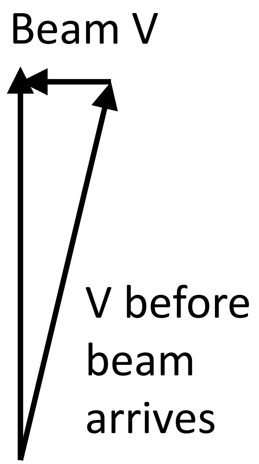
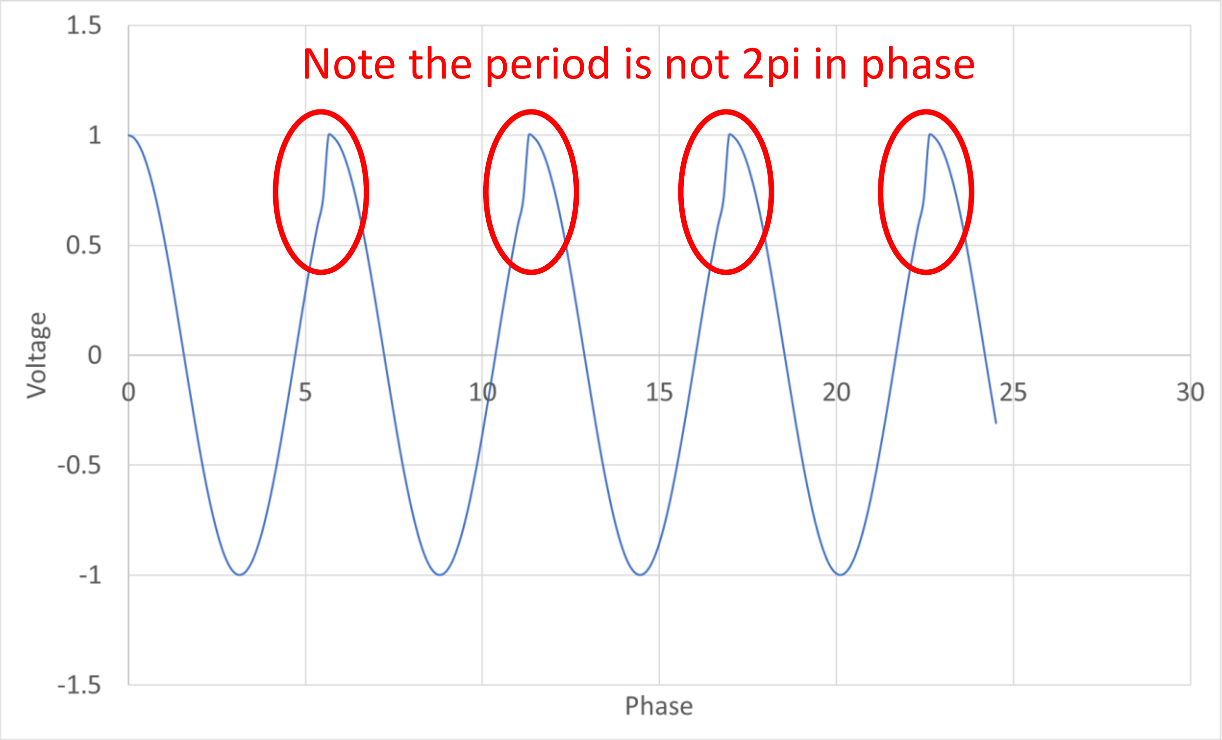
Detuning

- In many RF systems the beam is not accelerated on crest, but at a phase slightly above or below the maximum voltage.
- In this case the **beam induces a current** in the cavity which is **out of phase** with the main RF and hence **changes the cavity phase**, which must be controlled.





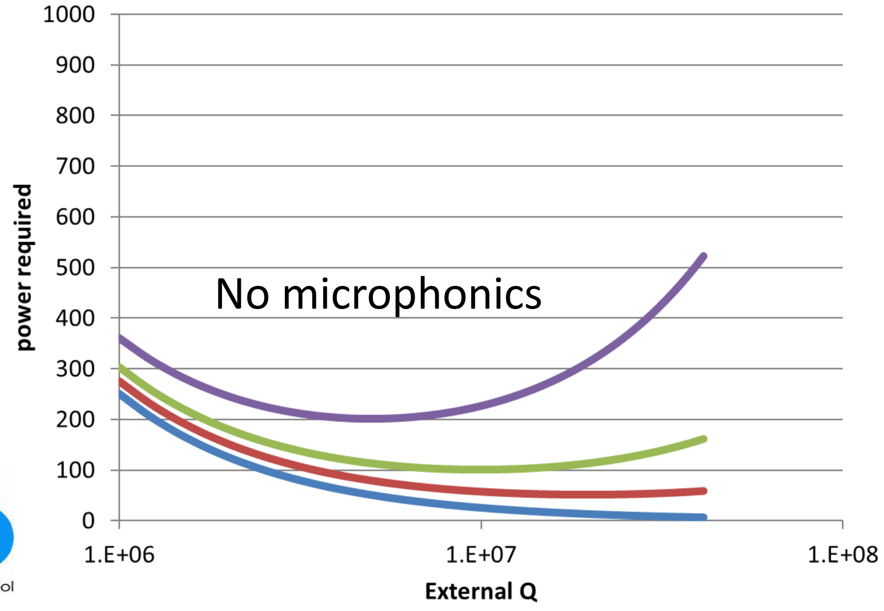
- This causes a phase (and frequency) shift as the imaginary part remains the same.
- Phase shift = atan $[(I_b\ R/Q)/V_g]$

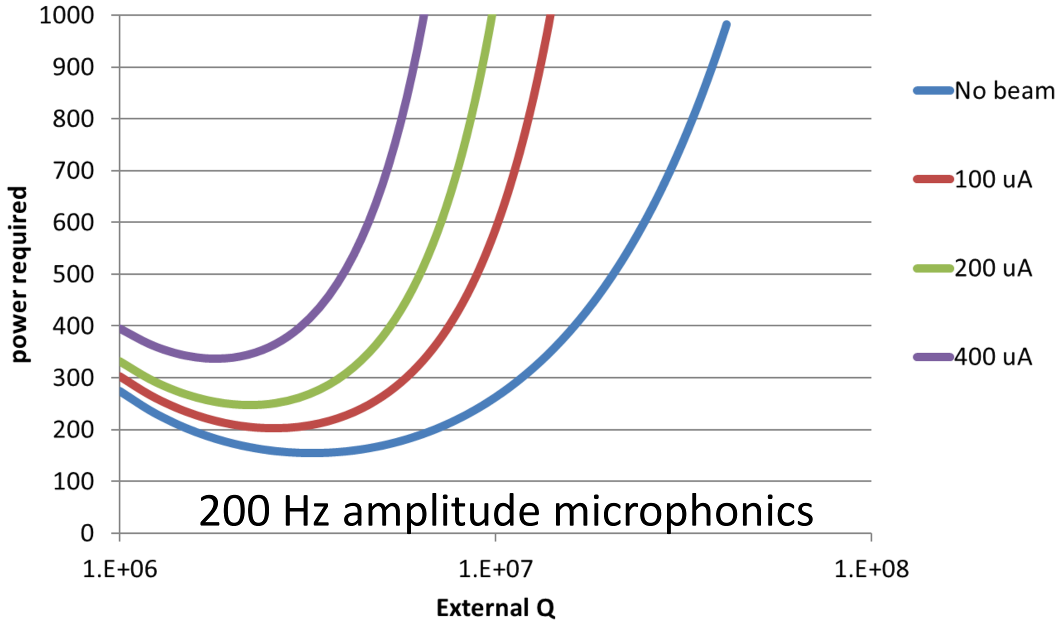
Detuning

- Correcting the phase take **more (reactive) power** than simply replacing power lost to the beam
- The phase/frequency shift is effectively adding a capacitance to the cavity circuit.
- This can be fixed by **detuning the cavity** so that we reduce the cavities capacitance accordingly.
- The cavity phase is advanced between bunches because it is at a higher frequency (real part becomes finite and negative).
- The two **phase shifts cancel if**
- $$\Delta\omega \approx \omega_0 \left(\frac{I_b R}{2 V_g Q} \right)$$
- However without beam the cavity has the wrong frequency.

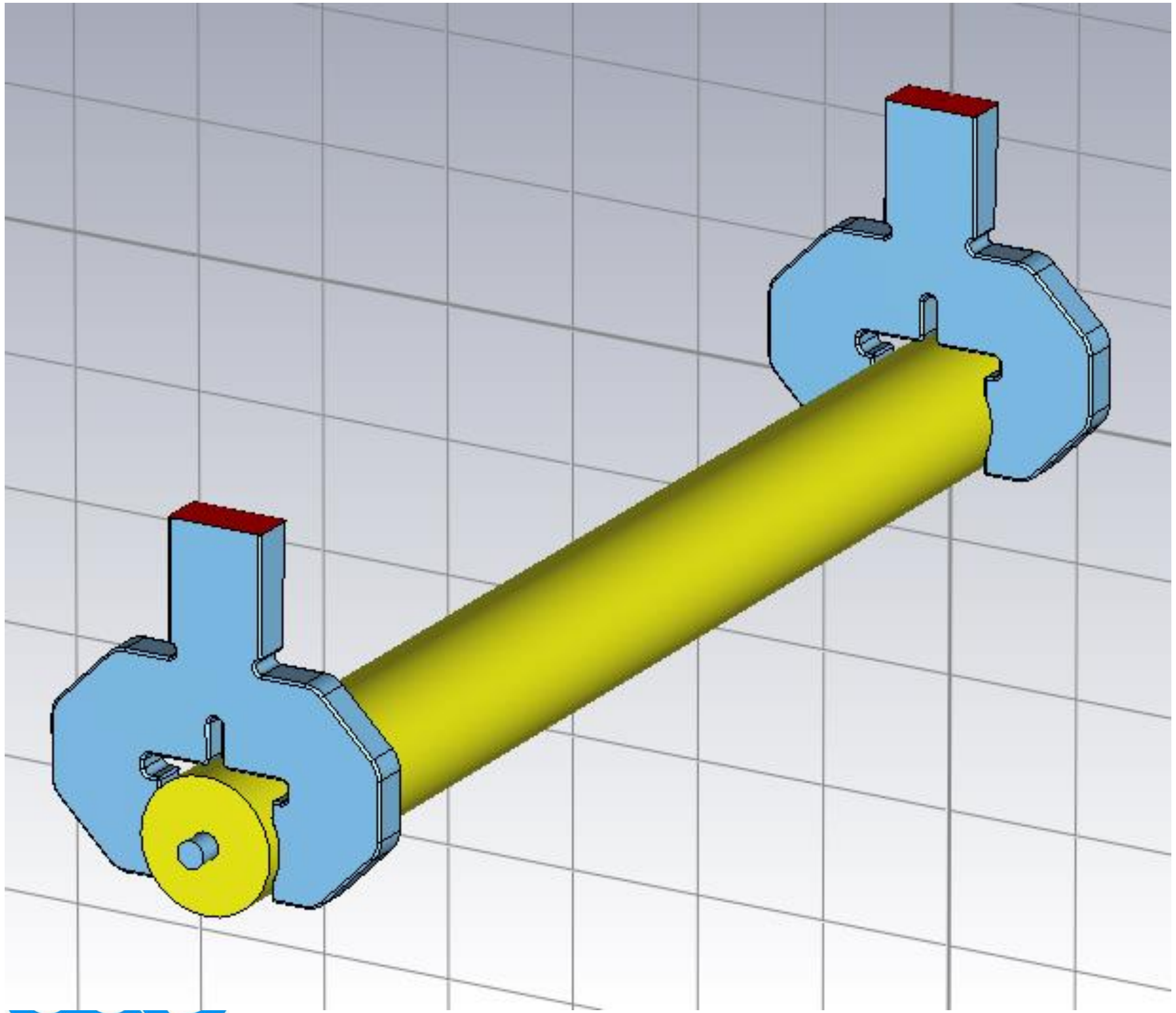
Required Power with beam

- In this case the power is $$P_g = \frac{\left[(1+\beta)P_c + P_b\right]^2}{4\beta P_c}$$
- Where $$P_b \approx I_b\left(V_g \sin\phi_s - \frac{1}{2} q\omega \frac{R}{Q}\right)$$
- Normally the 2nd term on the RHS is neglected.
- In an ERL the loading due to the injected and spent beam exactly cancel. This means that the beamloading can be neglected





TW Matched Couplers



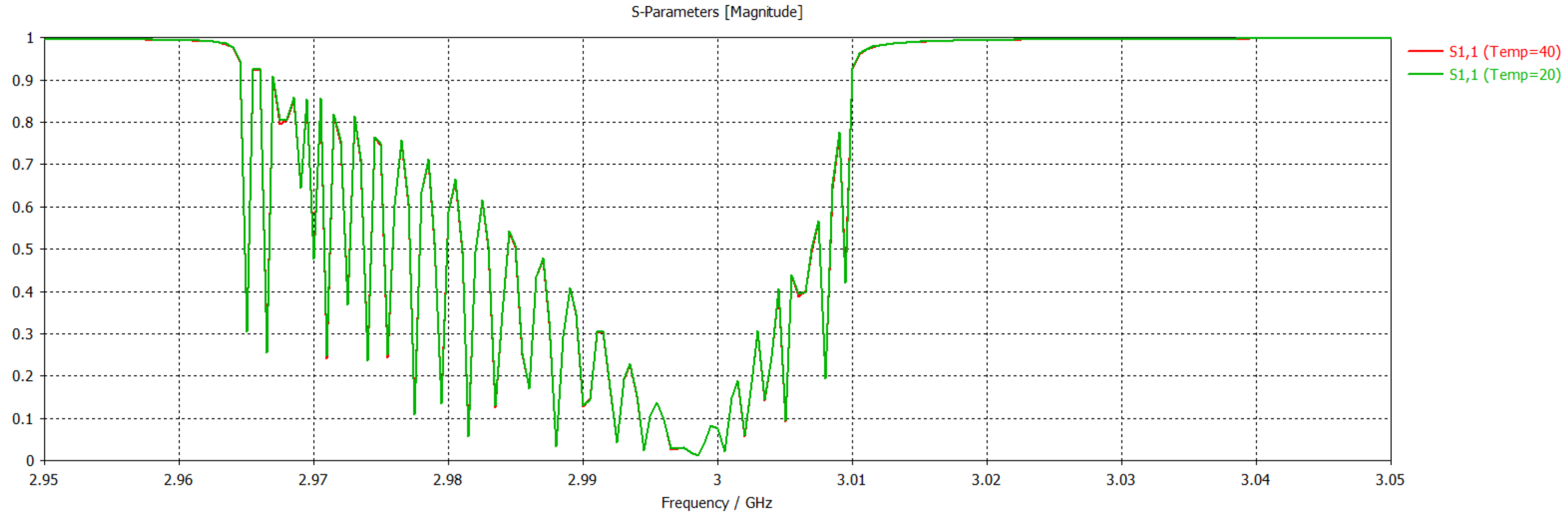
- A travelling wave structure has two (or four) couplers, and input and an output.

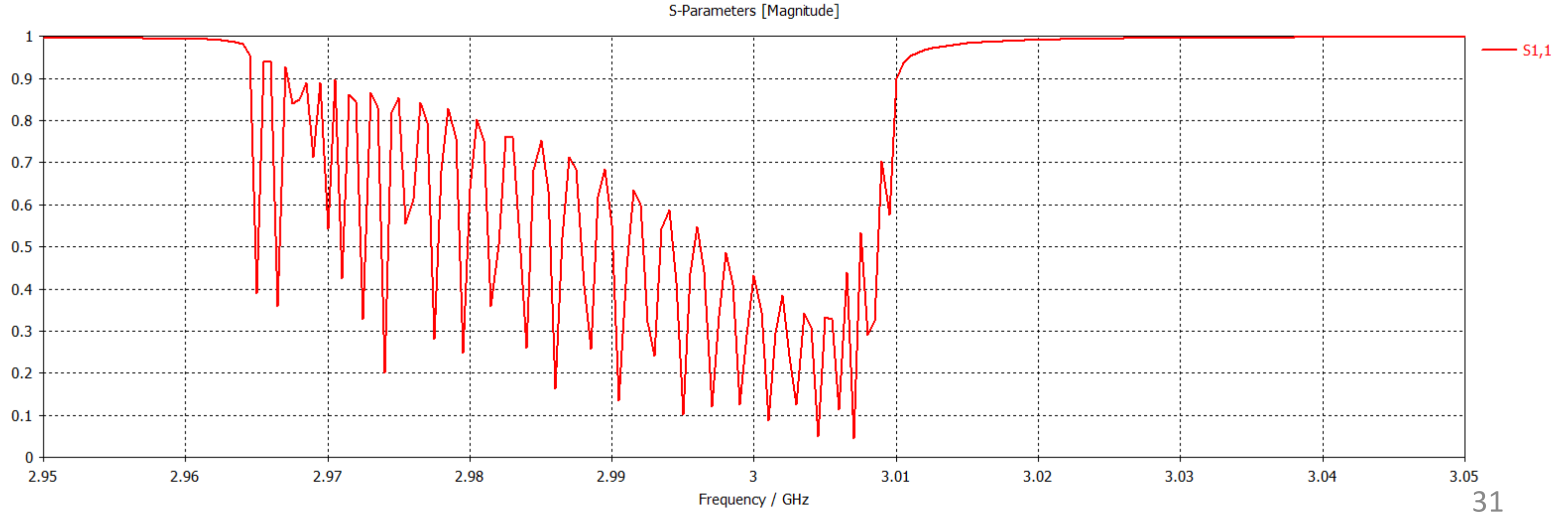
- The power travels from the input to the output

Matching travelling wave structures

- The two S11 plots are for the same cavity and both have S11<0.1 at 2.9985 GHz but they have different Q's for the couplers.

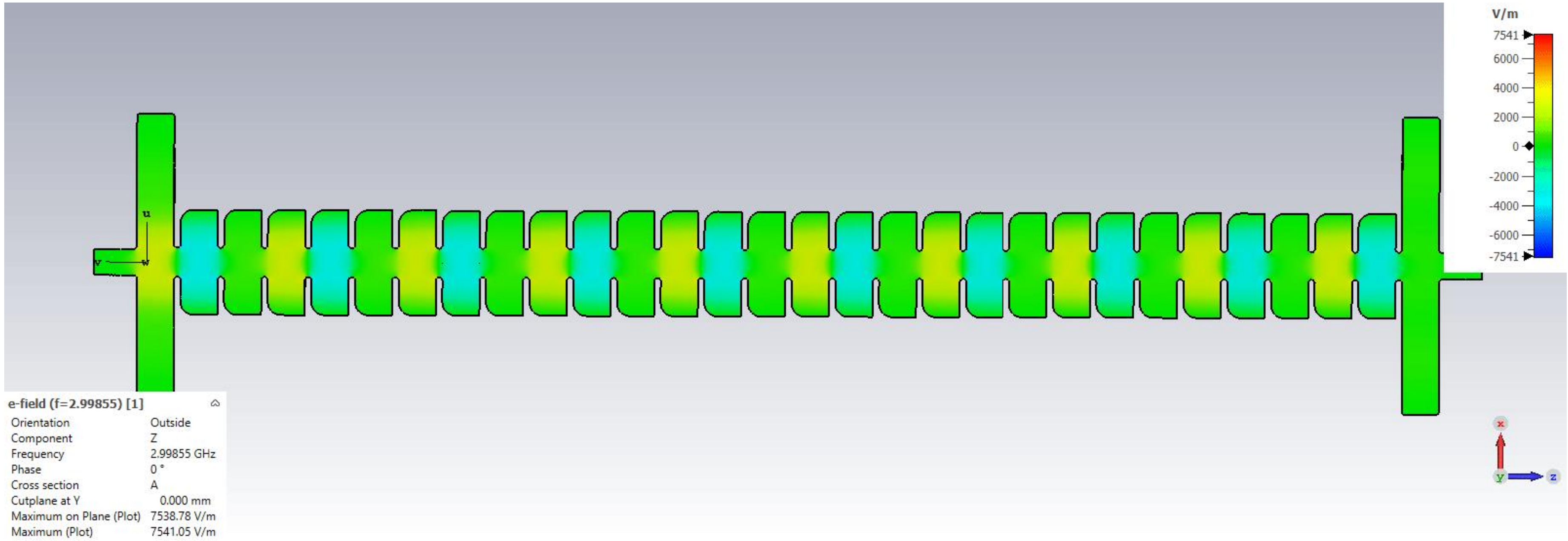
- **If S_{11}=0 is the cavity matched?**





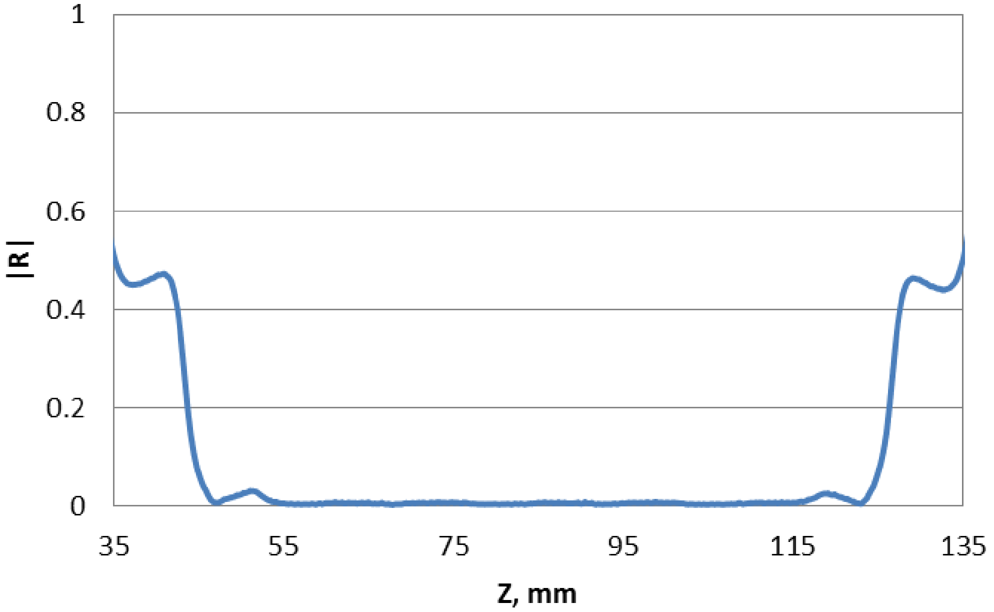
Is it matched?

- It depends what you mean by matched?
- The **reflections are zero in steady state** for both cavities but if we wish a **travelling wave** then there should be **no reflections at any time** not just in steady state.
- If there is a mismatch between each coupler and the cavity then the reflections from each coupler could cancel, similar to stub matching, giving S11=0
- But the downside is that there is a standing wave between the couplers with waves going in both directions inside the coupler
- This increases the losses without increasing acceleration as well as creating “hot” cells where the field exceeds breakdown limits



Matching TWS

- **<u>This is not the same as S_{11}=0 (global matching) .</u>** Here we have no **internal reflections**
- For a true travelling wave there is also **no reflection at the start of the pulse**.
- In order to verify a structure is matched we must measure the fields inside the cavity.



By measuring how the field varies between a cell and its nearest two neighbours we can use **Floquet theorem** to calculate the phase and internal reflections. For a TWS |R|=0.

$$\Delta(z) = \frac{E(z+P) - E(z-P)}{E(z)}$$

$$\Sigma(z) = \frac{E(z+P) + E(z-P)}{E(z)}$$

$$R = \frac{2\sin\psi - i\Delta(z)}{2\sin\psi + i\Delta(z)}.$$

$$\psi = \cos^{-1}\frac{\Sigma(z)}{2}$$

E = field
P = cell length
R = internal reflection coefficient
ψ = phase advance

Matched Couplers

- For a matched coupler we must consider the **matching of each cell** rather than the entire structure, ie considering the power flow between cells. As before a coupler is matched if $Q_e=Q_{internal}$
- Here

$$\frac{1}{Q_e}=\frac{1}{Q_0}+\frac{1}{Q_f}=\frac{P_c}{\omega U}+\frac{v_g(1-e^{-2\alpha L})}{\omega L}$$

Where Q_f is calculated from **the power flow from cell to cell**. If $Q_0 >> Q_f$

$$Q_e=\frac{\omega L}{v_g}=\frac{c\varphi}{v_g}$$

Example: CLARA S-band TWS

Phase advance = $2\pi/3$

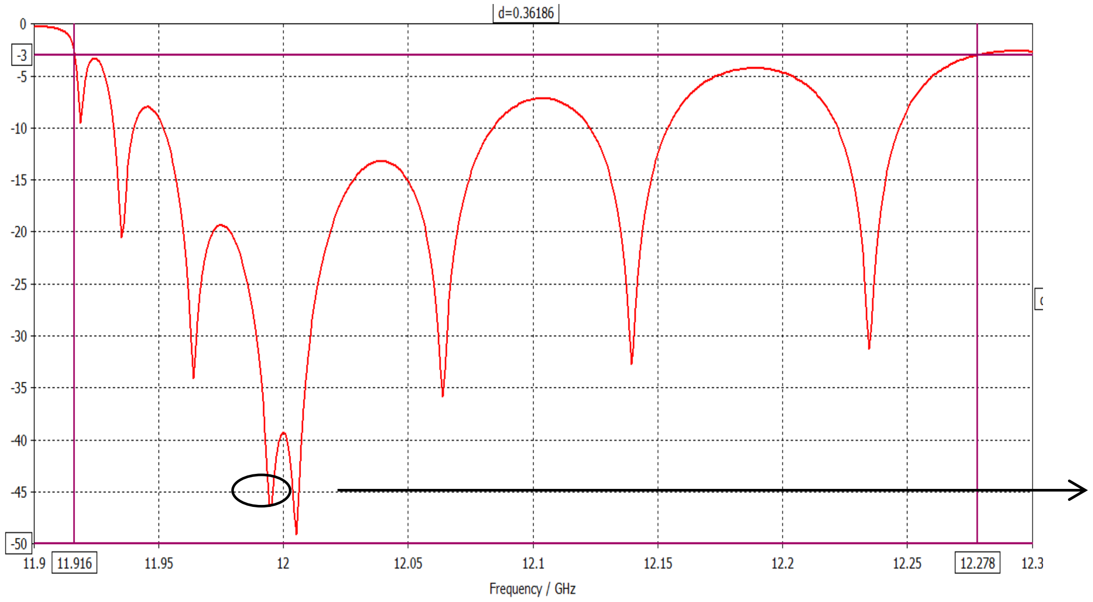
Group velocity = 3 % c at the input

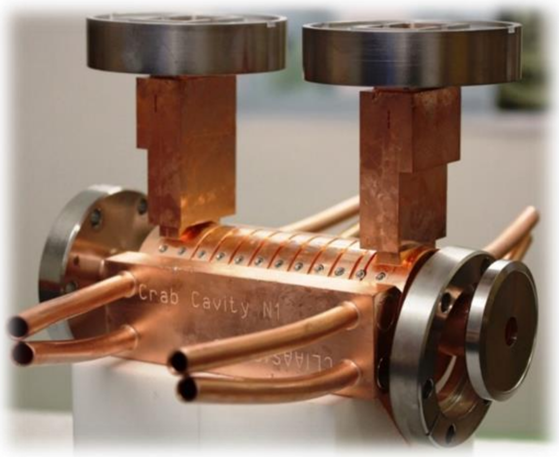
Q_e = 70

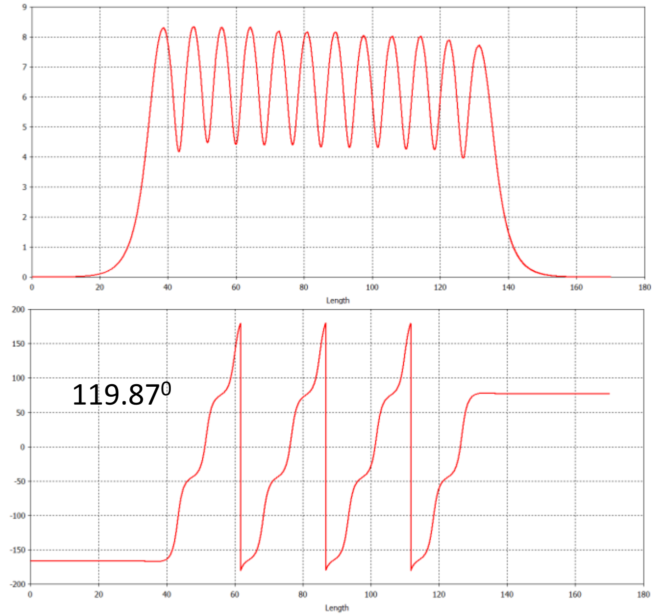
This is a very low Q compared to that of a standing wave cavity which gives the fast filling time and large bandwidth.

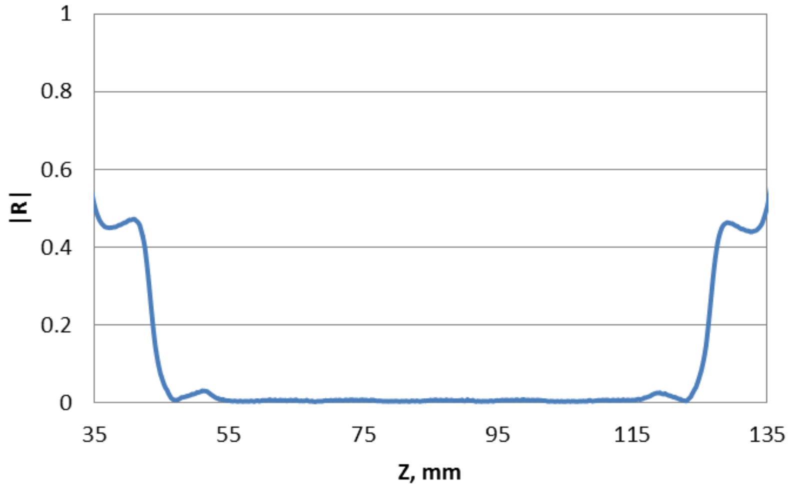
This is not power loss but power flow so the structure can still be efficient if made sufficiently long.

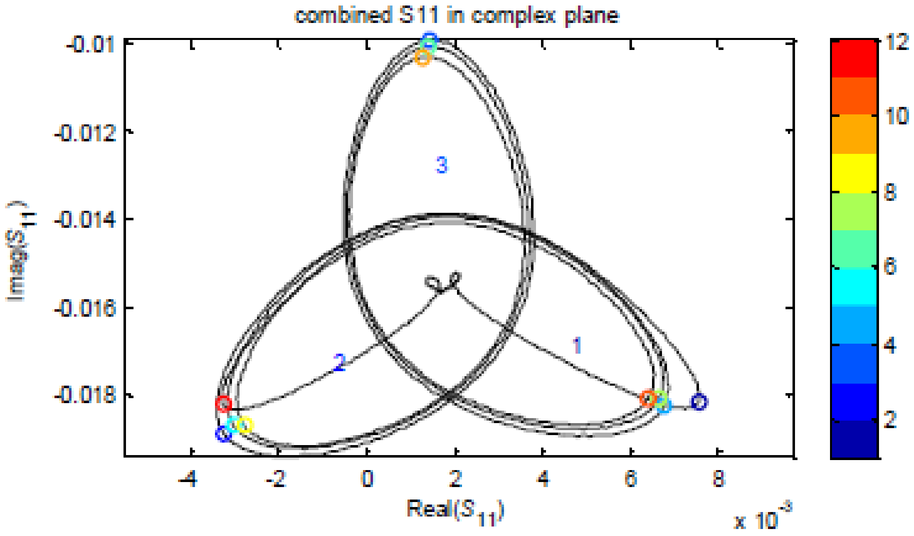
Example CLIC crab cavity





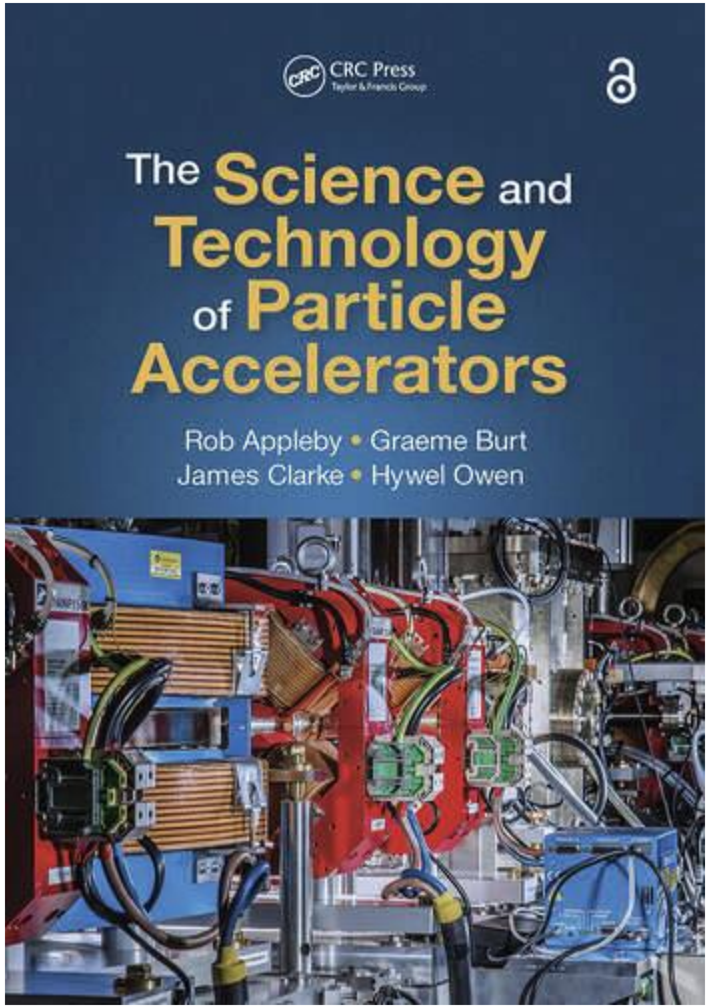






- If we look at the phase of reflections from a beadpull measurement we can find the amplitude and phase of each cell.

References



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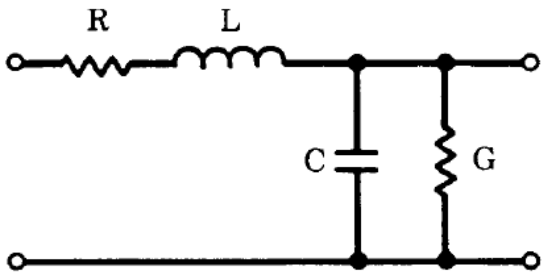
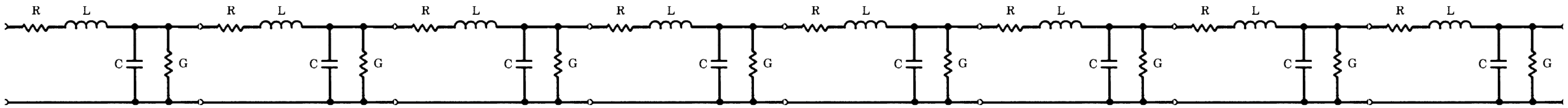
Additional material

Transmission line circuit

- All pairs of wires (go and return) have capacitance between them, and each wire has an inductance.
- This capacitance and inductance is not at a discrete point it is distributed along the wire. We represent this by a **chain** of low pass filters.





- The approximation is only valid if the cell lengths (Δz) are much smaller than the wavelength being modelled, this ensures the corner frequency is much higher than the frequency being modelled so that it doesn't act like a filter at all.
- C is capacitance per unit length so capacitance is $C\Delta z$
- L is inductance per unit length so inductance is $L\Delta z$

Transmission Line Circuit

- If we neglect losses to simplify…
- Kirchoff's voltage law gives us:

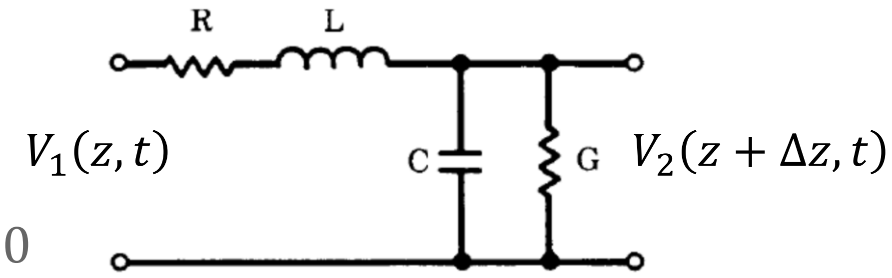
$$V_1(z,t) - L\Delta z \frac{\delta i(z,t)}{\delta t} - V_2(z+\Delta z,t) = 0$$

- Kirchoff's current law gives us:

$$i_1(z,t) - C\Delta z \frac{\delta v(z+\Delta z,t)}{\delta t} - i_2(z+\Delta z,t) = 0$$

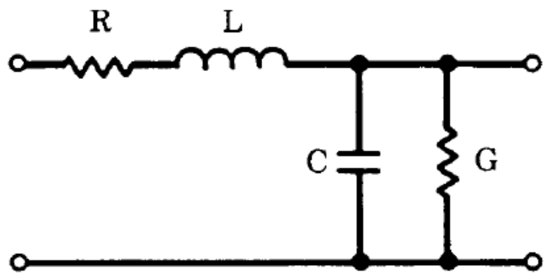
- Divide by Δz and take the limit $\Delta z \rightarrow 0$:

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$



Transmission line circuit

- The equation for an inductor is $$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$
- Differentiating with respect to z is $$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial^2 I}{\partial z \partial t}$$
- The equation for a capacitor is $$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$
- Inserting (3) into (2) gives $$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$$



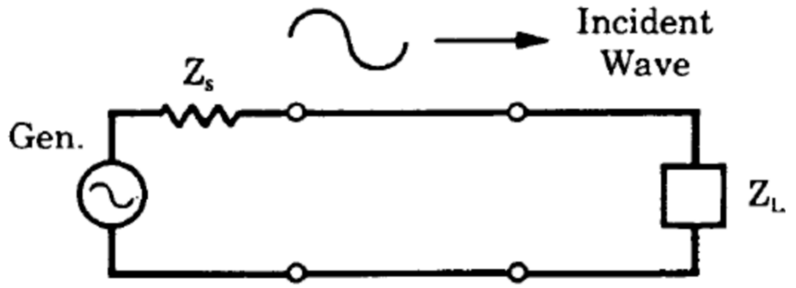
This is known as the wave equation and its solution is:

$$V_0 = V^+ \exp\left(i\omega\left[\frac{z}{\sqrt{LC}} - t\right]\right) + V^- \exp\left(i\omega\left[\frac{z}{\sqrt{LC}} + t\right]\right)$$

forward backward

Matched Transmission lines

- If we have an RF generator terminated by a matched load (at the same impedance as the source and line) at the end of a long transmission line we find at any instant in time the voltage varies sinusoidally along the line
- **the voltage isn't constant along the wire**



- However as we advance in time the phase of the wave varies in time. We call this a travelling wave. The voltage is given by.

$$V = V^+ \sin(kz - \omega t)$$

- Where k is $2\pi/\lambda$ = ω/sqrt(LC)
- The peak moves according to

$$n\pi = kz - \omega t$$

- Which implies the velocity is

$$v = \frac{1}{\sqrt{LC}} = \frac{\omega}{k}$$

- At low frequencies k is close to zero.

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Impedance

- Take the current equation $\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$
- Integrate both sides in space to get $I = -C\int dz \frac{\partial V}{\partial t}$
- dz/dt is velocity $v = \frac{1}{\sqrt{LC}}$ hence $I = \sqrt{\frac{C}{L}}\,V$
- Giving the characteristic impedance of the line. . It tells us that when the co-axial transmission line just carries a forward wave then the voltage is everywhere in phase with the current and they have the ratio

$$Z_o = \frac{V}{I} = \sqrt{\frac{L}{C}}$$

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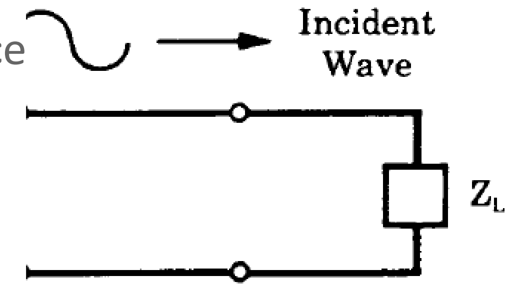
Interfaces

- If there is an interface between two lines (or a line and a source or a load). The **voltage and current must be continuous** (i.e. they must be the same at both sides of the interface)
- However this is not possible with just the forward travelling wave unless the impedance (Z=V/I) is the same on both sides.
- We must return to the initial solution with forward and backwards waves.

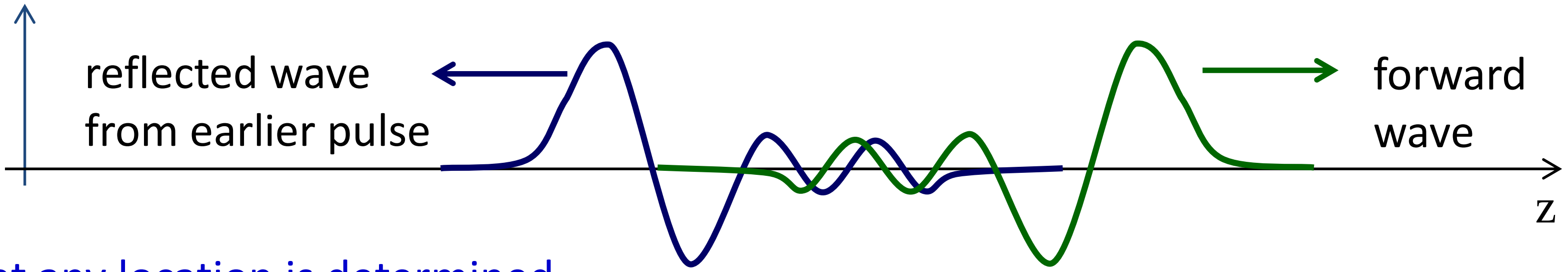
$$V_0 = V^+ \exp(ikz - i\omega t) + V^- \exp(ikz + i\omega t)$$

- By combining forwards and backwards waves we can find a solution with both V and I constant because the currents combine as:

$$I_0 = I^+ \exp(ikz - i\omega t) - I^- \exp(ikz + i\omega t)$$



Forward and Backward Waves



Voltage at any location is determined by summing the forward wave voltage with the backward wave voltage.

$$V_0 = V^+ \exp(ikz - i\omega t) + V^- \exp(ikz + i\omega t)$$
$$= F\left(z - \frac{t}{\sqrt{LC}}\right) + R\left(z + \frac{t}{\sqrt{LC}}\right)$$

Using the line impedance Z_o then current at any location is determined by

$$I(z,t) = \frac{1}{Z_o}\left\{F\left(z - \frac{t}{\sqrt{LC}}\right) - R\left(z + \frac{t}{\sqrt{LC}}\right)\right\}$$

Solving for the forward wave voltage F gives

$$2F\left(z - \frac{t}{\sqrt{LC}}\right) = V + I\,Z_o$$

Solving for the backward wave voltage R gives

$$2R\left(z + \frac{t}{\sqrt{LC}}\right) = V - I\,Z_o$$

Hence

$$\text{forward wave voltage} = \frac{V + I\,Z_o}{2} \qquad \text{backward wave voltage} = \frac{V - I\,Z_o}{2}$$

Reflection coefficient

- If we have a forwards wave travelling to an interface we will have a reflected wave and a transmitted (forwards wave).
- V & I must be the same on both sides but V+ and V- can be different. The voltage and current equations are hence:

$$V_2^+ = V_1^+ + V_1^-$$

$$I_2^+ = I_1^+ - I_1^-$$

$$Z_2 = \frac{V_2^+}{I_2^+} = \frac{V_1^+ + V_1^-}{I_1^+ - I_1^-}$$

- If we introduce a reflection coefficient Γ:

$$V_1^- = \Gamma V_1^+$$

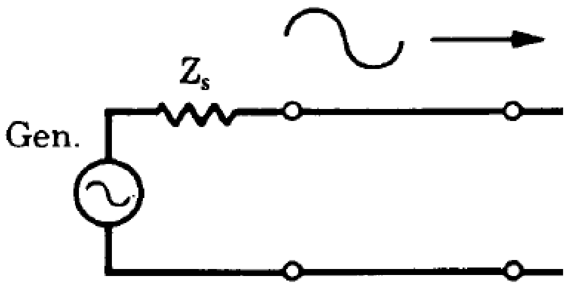
$$I_1^- = \Gamma I_1^+$$

$$Z_2 = \frac{V_1^+(1+\Gamma)}{I_1^+(1-\Gamma)}$$

$$= Z_1 \frac{(1+\Gamma)}{(1-\Gamma)}$$

$$\Gamma = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)}$$

RF sources



- RF sources are not current or voltage sources, they are forward travelling wave sources.
- They have an internal impedance Z_S.
- If a wave is reflected then the voltage and current at the source will change (as $V=V^{+}+V^{-}$) hence why it isn't a current or voltage source.
- However the forwards travelling wave never changes at $Z_s=V^{+}/I^{+}$ always.
- This means the **forward** voltage, current and power of an RF source is fixed.
- Reflected power is absorbed by the internal impedance in the simple case but in reality its much more complex and can make the source unstable.

Power waves

- When we have several components of different impedance it is not clear if it is better to use voltages or currents to tell what percentage of the signal is transmitted.
- However energy is conserved in all physical processes. Power is the change of energy with time hence it makes sense to use power.
- The transmitted power, $\mathbf{P_t}$, from an incoming wave of power $\mathbf{P_{in}}$ where the reflected power is, $\mathbf{P_{ref}}$, is hence:

$$P_t = P_{in} - P_{ref}$$

- However we want to know amplitudes, a, so we instead take the square root of the power:

$$a = \sqrt{P} = \sqrt{\frac{{V^+}^2}{Z}} = \frac{V^+}{\sqrt{Z}}$$

Terminated lines

- At high frequency the phase varies along the length of the line, L, dependant on the wavelength
- The phase change is kL=2πL/lamda
- For low frequencies k is very small and hence so is the phase change unless L is very large.
- At high frequencies as the wavelength is small k is large and we get phase changes even at short lengths of line.
- The phase change will change the components effective impedance.

$$Z_{in} = Z_0 \frac{Z_L + iZ_0 \tan(kl)}{Z_0 + iZ_L \tan(kl)}$$

$$S_{11} = \frac{Z_{in} - Z_G}{Z_{in} + Z_G}$$

We can change the phase of the reflection but not its amplitude. This means we can change the reactance to any value with a length of line but the resistance remains constant.