Integrand-based methods at one loop

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Tree-level matrix elements without expressions

- At the end of the 90s we realized we need to be able to calculate cross sections for the multi-jet events that would be abundant at LHC.

- The textbook approach to get an *expression* for $|\mathcal{M}|^2$ as function of just external momenta $p_2, p_2, \dots, p_n$ seems hopeless.

N	ϕ^3	$\phi^3 + \phi^4$
4	3	4
5	15	25
6	105	220
7	945	2,485
8	10,395	34,300
12	654,729,075	5,348,843,500
13	13,749,310,575	140,880,765,025
14	316,234,143,225	4,063,875,715,900

number of graphs

Exploding QCD, Draggiotis 2002

Do we actually need complete expressions for matrix elements?

- There is no other option than the Monte Carlo method to calculate cross sections:
  - we are facing arbitrary phase space cuts
  - we want to produce (plots of) differential cross sections of arbitrary observables
- we only need the numerical value (single real number) of $|\mathcal{M}|^2$ provided numerical values of $p_1, p_2, \dots, p_n$ for each event.

Tree-level matrix elements without expressions

The fact that we only need numerical values of $|\mathcal{M}|^2$ creates some opportunities

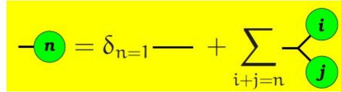
- use spin/helicity amplitudes

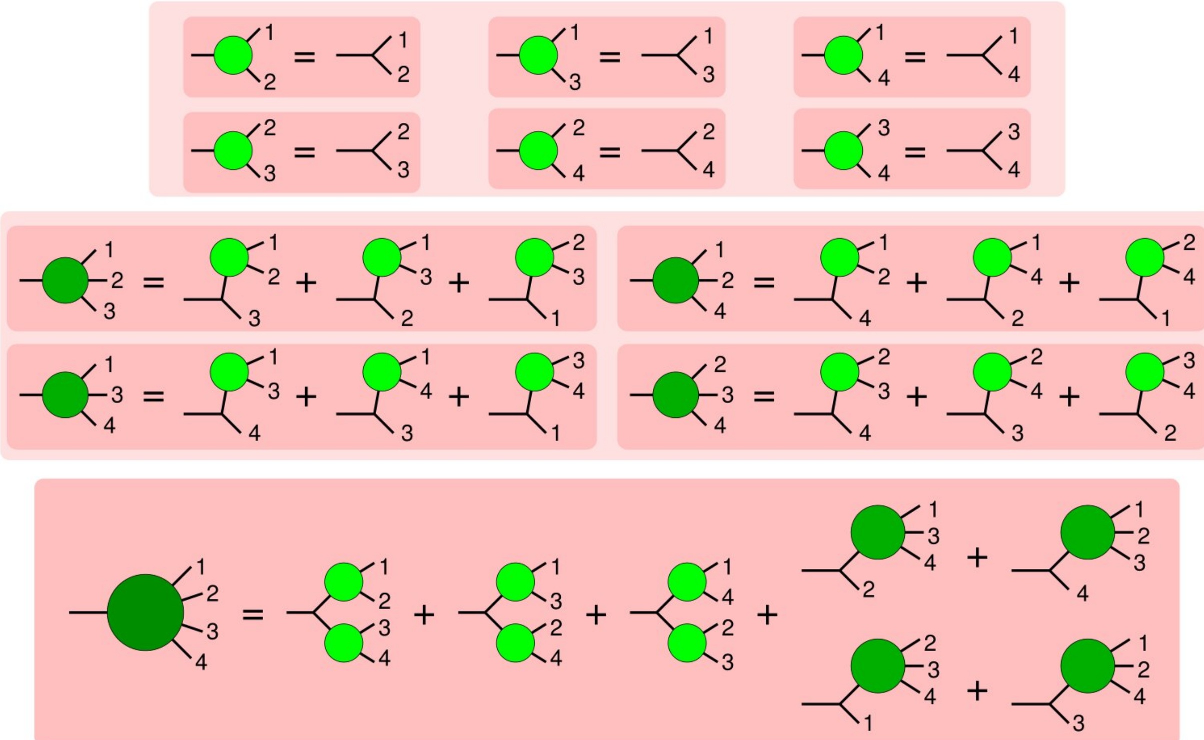
$$|\mathcal{M}|^2\left(\{p_i\}_{i=1}^n\right) = \sum_{\{\lambda\}} \left|\mathcal{M}\left(\{p_i,\lambda_i\}_{i=1}^n\right)\right|^2$$

  and evaluate $\mathcal{M}\left(\{p_i,\lambda_i\}_{i=1}^n\right)$ numerically (roughly 2^n complex numbers). So only need to be able to calculate (spin/helicity) *amplitudes* instead of matrix elements. The sum can be made part of the Monte Carlo procedure, have polarized events.

- given external momenta p_i (real numbers), calculate
  - external spinors/polarization vectors u_i or ϵ_i (complex numbers)
  - currents $\bar{u}_i\gamma^\mu u_j$ (complex numbers)
  - propagators *eg.* $\gamma_\mu(p_i^\mu + p_j^\mu)/(p_i+p_j)^2$ (complex numbers)
  - whole graph (complex number)
- Of course, this need to be automated into a computer program.
- Avoid organization in terms of graphs.

Recursive computation





Caravaglios, Moretti 1995

How to proceed to one loop?

We know a one-loop amplitude involving a set of ℓoop-momentum-dependent denominators $\mathcal{D}_i(\ell) = (\ell + K_i)^2 - m_i^2 + i\eta$, with some enumeration $i = 0, 1, \ldots$, can be decomposed as

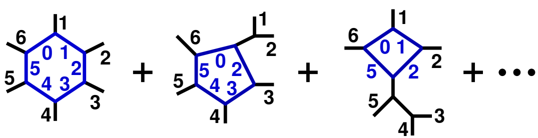
$$\mathcal{A}^{\text{loop}} = \sum_{i,j,k,l} C_{ijkl}\,\text{Box}_{ijkl} + \sum_{i,j,k} C_{ijk}\,\text{Tri}_{ijk} + \sum_{i,j} C_{ij}\,\text{Bub}_{ij} + \sum_{i} C_i\,\text{Tad}_i + \mathcal{R} + \mathcal{O}(\varepsilon)\ ,$$

where the *master integrals* are defined as

$$\text{Box}_{ijkl} = \int \frac{d^{4-2\varepsilon}\ell}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)\mathcal{D}_l(\ell)}\ , \qquad \text{Tri}_{ijk} = \int \frac{d^{4-2\varepsilon}\ell}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)}\ ,$$

$$\text{Bub}_{ij} = \int \frac{d^{4-2\varepsilon}\ell}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)}\ , \qquad \text{Tad}_i = \int \frac{d^{4-2\varepsilon}\ell}{\mathcal{D}_i(\ell)}\ .$$

In practice, we should imagine that $\mathcal{A}^{\text{loop}}$ is an *ordered* subamplitude, and we can write it with a common denominator as



$$= \quad \mathcal{A}^{\text{loop}} = \int d^{4-2\varepsilon}\ell\, \frac{\mathcal{N}(\ell)}{\prod_i \mathcal{D}_i(\ell)}$$

How to find the coefficients $C_{ijkl}, C_{ijk}, C_{ij}, C_i$ and $\mathcal{R}$ as complex numbers in a strategy compatible with the strategy at tree-level?

Integrand-based approach

Does the one-loop decomposition into scalar integrals hold at the integrand level in 4-dim?

$$\frac{\mathcal{N}(\ell)}{\prod_i \mathcal{D}_i(\ell)} \stackrel{?}{=} \sum_{i,j,k,l} \frac{C_{ijkl}}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)\mathcal{D}_l(\ell)} + \sum_{i,j,k} \frac{C_{ijk}}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)} + \sum_{i,j} \frac{C_{ij}}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)} + \sum_i \frac{C_i}{\mathcal{D}_i(\ell)}$$

Integrand-based approach

Ossola, Papadopoulos, Pittau 2007
Ellis, Giele, Kunszt 2008

At the integrand-level in 4-dim, terms appear that integrate to zero

$$\frac{\mathcal{N}(\ell)}{\prod_i \mathcal{D}_i(\ell)} = \sum_{i,j,k,l} \frac{C_{ijkl} + \tilde{\mathcal{C}}_{ijkl}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)\mathcal{D}_l(\ell)} + \sum_{i,j,k} \frac{C_{ijk} + \tilde{\mathcal{C}}_{ijk}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)} + \sum_{i,j} \frac{C_{ij} + \tilde{\mathcal{C}}_{ij}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)} + \sum_{i} \frac{C_{i} + \tilde{\mathcal{C}}_{i}(\ell)}{\mathcal{D}_i(\ell)}$$

The coefficients C are the same as at integral-level, and the polynomials $\tilde{\mathcal{C}}$ have finite rank

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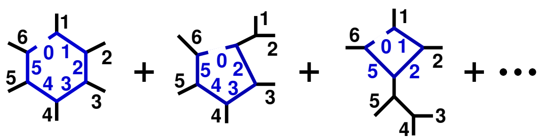
$\Longrightarrow$ problem solved! (in principle)

- choose as many values of ℓ as there are coefficients (including the ones inside the polynomials $\tilde{\mathcal{C}}$)

- left-hand side can be evaluated for each value of ℓ using tree-level techniques

$$\frac{\mathcal{N}(\ell)}{\prod_i \mathcal{D}_i(\ell)} =$$



- invert linear system numerically

Integrand-based approach

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The coefficients C are the same as at integral-level, and the polynomials $\tilde{\mathcal{C}}$ have finite rank

Choose values of ℓ such that denominators $\mathcal{D}_i(\ell)$ vanish:

- in 4-dim, at most 4 can vanish simultaneously, with 2 solutions $\ell_{1,2}$. So the right-hand side only involves the term with those 4 denominators (say i, j, k, l) (multiply left and right with those) $\Longrightarrow$ equation with only 2 unknown coefficents $C_{ijkl}, \tilde{C}_{ijkl}$

- find all box-coefficients for all combinations of i, j, k, l, and proceed with triangle coefficients $C_{ijk}, \tilde{C}^{(\rho)}_{ijk}$, $\rho = 1, \ldots, 6$ using

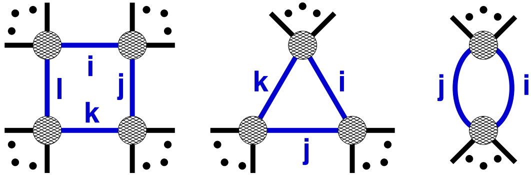
$$\frac{\mathcal{N}(\ell)}{\prod_i \mathcal{D}_i(\ell)} - \sum_{i,j,k,l} \frac{C_{ijkl} + \tilde{\mathcal{C}}_{ijkl}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)\mathcal{D}_l(\ell)} = \sum_{i,j,k} \frac{C_{ijk} + \tilde{\mathcal{C}}_{ijk}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\mathcal{D}_k(\ell)} + \sum_{i,j} \frac{C_{ij} + \tilde{\mathcal{C}}_{ij}(\ell)}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)} + \sum_{i} \frac{C_{i} + \tilde{\mathcal{C}}_{i}(\ell)}{\mathcal{D}_i(\ell)}$$

- etc.

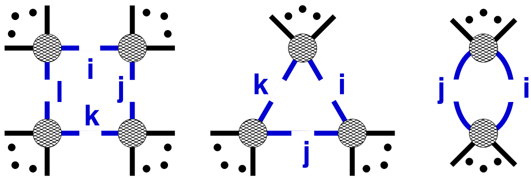
Integrand cuts

What happens to the left-hand side, the one-loop integrand, when denominators vanish?

The set of all one-loop graphs that must involve denominators i, j, k, l, (or denominators i, j, k, or denominators i, j), look like



Choosing integration momenta ℓ at the amplitude level such that those denominators vanish puts the lines on-shell. The blobs become tree-level amplitudes with 2 extra on-shell lines.



The left-hand side of the equation can be constructed in each step by sewing together *on-shell* tree-level amplitudes.

A loop momentum choice ℓ that puts a number of denominators to zero is called a *cut*, and is a solution to the *cut equations*.

The terms of the one-loop amplitude proportional to the coefficients C is called the *cut-constructable* part.

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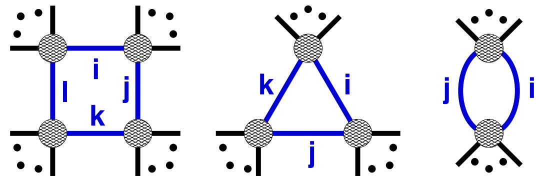

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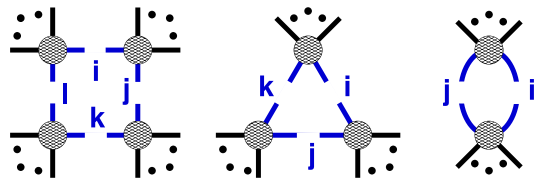

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Rational terms

$$\mathcal{A}^{\rm loop} = \sum_{i,j,k,l} C_{ijkl}\,{\rm Box}_{ijkl} + \sum_{i,j,k} C_{ijk}\,{\rm Tri}_{ijk} + \sum_{i,j} C_{ij}\,{\rm Bub}_{ij} + \sum_{i} C_i\,{\rm Tad}_i + \boxed{\mathcal{R}} + \mathcal{O}(\varepsilon)\ ,$$

They do not contain (di)logarithms of invariants, and appear due to the regularization of divergencies, in dimensional regularization from integrals of the type

$$\int d^{4-2\varepsilon}\ell\,\frac{(\ell_{-2\varepsilon}^2)^n}{\mathcal{D}_i(\ell)\mathcal{D}_j(\ell)\cdots}$$

The terms can be separated in two groups: Ossola, Papadopoulos, Pittau 2008

$\mathcal{R}_1$ appears due to the mismatch between 4-dim and $(4-2\varepsilon)$-dim denominators. It can be calculated by interpreting the extra-dimensional components as extra mass

$$\mathcal{D}(\ell_{4-2\varepsilon}) = (\ell_4 + K_j)^2 - m_j^2 - \ell_{-2\varepsilon}^2$$

$\mathcal{R}_2$ appears due to treating numerators strictly 4-dimensional. These terms follow the structure of counter-terms in renormalization, and can be calculated as tree-level amplitudes with a single special vertex (Garzelli, Malamos, Pittau 2010,2011).

$\mathcal{R}$ can also be obtained by including 5-point integrals in the basis, and performing the cut-construction in *explicit* higher dimensions. Giele, Kunszt, Melnikov 2008

Tensor coefficient recursion

AvH 2009

OpenLoops Cascioli, Maierhofer Pozzorini, 2012

Recola Actis, Denner, Hofer Scharf, Uccirati 2013

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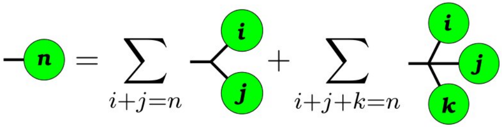
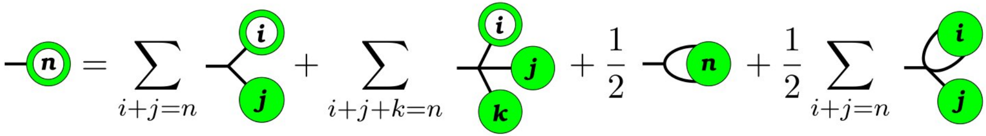
Tree-level recursion



A blob with label j has j on-shell external lines

One-loop recursion



Tensor coefficient recursion

AvH 2009 | OpenLoops (Cascioli, Maierhofer, Pozzorini, 2012) | Recola (Actis, Denner, Hofer, Scharf, Uccirati 2013)

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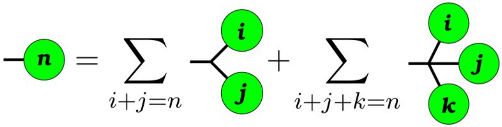

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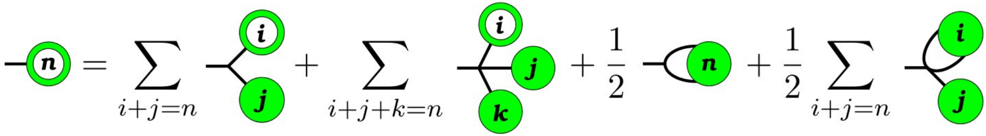
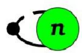


How to deal with the explicit loops in the recursion? Assume all tensor integrals with arbitrary denominators and with arbitrary rank are available (eg. via del Aguila, Pittau 2004)

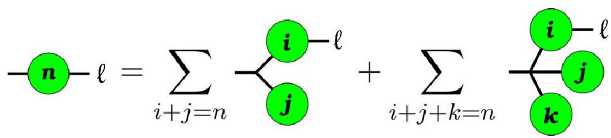
$$\mathcal{T}\big(\{i_j\},\{\nu_j\}\big) = \int d^{4-2\varepsilon}\ell\, \mathcal{J}\big(\ell;\{i_j\},\{\nu_j\}\big) \quad , \quad \mathcal{J}\big(\ell;\{i_j\}_{j=1}^{p},\{\nu_j\}_{j=1}^{r}\big) = \frac{\ell_4^{\nu_1}\ell_4^{\nu_2}\cdots\ell_4^{\nu_r}}{\mathcal{D}_{i_1}(\ell)\mathcal{D}_{i_2}(\ell)\cdots\mathcal{D}_{i_p}(\ell)}$$

The cut of the explicit loop has a decomposition in terms of tensor integrands.



$$= \sum_{\text{sets}\{\mathcal{D}\},\text{sets}\{\nu\}} \mathcal{G}\big(\{\mathcal{D}\},\{\nu\}\big)\,\mathcal{J}\big(\{\mathcal{D}\},\{\nu\}\big)$$

It also follows a tree-level recursion with an extra special let with momentum ℓ.



Using the vertex relation $V^{\mu\nu\rho}(p,q+\ell) = V^{\mu\nu\rho}(p,q) + X^{\mu\nu\rho}_{\sigma}\ell^{\sigma}$ we find recursive relations for the coefficients $\mathcal{G}\big(\{\mathcal{D}\},\{\nu\}\big)$.

Public Software

Several programs apply integrand-based methods to calculate full 1-loop amplitudes or virtual contributions (user chooses process, and provides numerical values of extenal momenta):

- Helac-NLO Comput.Phys.Commun. 184 (2013) 986-997 • e-Print: 1110.1499
- MadLoop JHEP 05 (2011) 044 • e-Print: 1103.0621
- Njet Comput.Phys.Commun. 184 (2013) 1981-1998 • e-Print: 1209.0100
- GoSam Eur.Phys.J.C 74 (2014) 8, 3001 • e-Print: 1404.7096
- Recola Comput.Phys.Commun. 214 (2017) 140-173 • e-Print: 1605.01090
- OpenLoops Eur.Phys.J.C 79 (2019) 10, 866 • e-Print: 1907.13071

Programs that perform the 1-loop decomposition into scalar integrals for "arbitrary" user-provided integrands (user provides numerical values of denominator momenta, masses, and a function that evaluates the numerator as function of the loop-momentum):

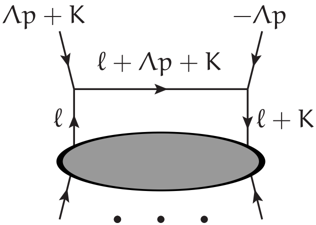
- CutTools JHEP 03 (2008) 042 • e-Print: 0711.3596 (inside Helac-NLO and MadLoop)
- Samurai JHEP 08 (2010) 080 • e-Print: 1006.0710 (the Sam in GoSam)
- Ninja JHEP 03 (2014) 115 • e-Print: 1312.6678

Linear Denominators

Integrand methods need to be constructed in conjunction with the regularization for the linear denominators.

AvH 1710.07609:

$$\frac{1}{p\cdot(\ell+K)} = \lim_{\Lambda\to\infty} \frac{2\Lambda}{(\ell+\Lambda p+K)^2}$$



Works fine (divergence of at most $\ln^2\Lambda$) for box-integrals, most triangle integrals, but

$$\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon} r_\Gamma}\int d^{4-2\varepsilon}\ell\, \frac{2\Lambda}{\ell^2(\ell+\Lambda p+K)^2} = 2\Lambda\left[\frac{1}{\varepsilon} + 2 - \ln\left(\frac{-2\Lambda p\cdot K}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right]$$

This behavior may cancel between all graphs, but then numerators need to be expanded to sub-leading orders in Λ.

eg. quark-line : $\dfrac{\gamma_\mu(\ell^\mu + \Lambda p^\mu + K^\mu)}{(\ell+\Lambda p+K)^2} \longrightarrow \dfrac{\Lambda\gamma_\mu p^\mu}{(\ell+\Lambda p+K)^2}$ can give wrong result