0.1.4 On the choice of the measure

- Always multiply by $e^{\varepsilon\gamma_E L}$, where L is the number of loops, to remove the nasty γ_E from the result.
- Putting $1/(i\pi^{d/2})$ into the measure removes this factor on the right hand side of the Feynman parametrization formula.
- So it is convenient to use the special measure

$$\left(\frac{e^{\varepsilon\gamma_E}}{i\pi^{d/2}}\right)^L d^dk_1 \dots d^dk_L$$

- The relation to the standard measure is obviously

$$\frac{1}{(2\pi)^{dL}} d^dk_1 \dots d^dk_L = \frac{(-1)^L}{2^{dL}} \left(\frac{e^{-\varepsilon\gamma_E}}{i\pi^{d/2}}\right)^L \left[\left(\frac{e^{\varepsilon\gamma_E}}{i\pi^{d/2}}\right)^L\right] d^dk_1 \dots d^dk_L$$

- So to convert from the special measure to the standard measure we need to multiply with

$$\frac{(-1)^L}{2^{dL}} \left(\frac{e^{-\varepsilon\gamma_E}}{i\pi^{d/2}}\right)^L$$

- An to convert from the standard measure to the special measure we multiply by

$$2^{dL}(-1)^L(i\pi^{d/2})^L e^{\varepsilon\gamma_E L}$$

0.1.5 Notes on conventions used in books and papers

Smirnov's book uses integrals with the "Euclidean" dependence,

$$\begin{aligned} -k^2 &\equiv -k^2 - i\eta, \\ 2k\cdot p &\equiv 2k\cdot p - i\eta \end{aligned}$$

but the integrals themselves are Minkowskian, not Euclidean. Writing them this way helps to eliminate $(-1)^\lambda$ prefactors. This is more natural when powers of propagators are complex numbers. So

$$\begin{aligned} \int \frac{d^dk}{(-k^2+m^2-i\eta)^\lambda} = (-1)^\lambda \int \frac{d^dk}{(k^2-m^2+i\eta)^\lambda} &\overset{\text{Smirnov}}{=} i\pi^{d/2}\frac{\Gamma(\lambda+\varepsilon-2)}{\Gamma(\lambda)}\frac{1}{(m^2-i\eta)^{\lambda+\varepsilon-2}} \\ \Rightarrow \int \frac{d^dk}{(k^2-m^2+i\eta)^\lambda} &= i\pi^{d/2}(-1)^\lambda\frac{\Gamma(\lambda+\varepsilon-2)}{\Gamma(\lambda)}\frac{1}{(m^2-i\eta)^{\lambda+\varepsilon-2}} \end{aligned}$$

On the other hand, if we define all propagators in such a way, that they appear with this dependence, i. e.

$$\begin{aligned} \frac{1}{p^2+i\eta} &\to -\frac{1}{-p^2-i\eta} \\ \frac{1}{p^2-m^2+i\eta} &\to -\frac{1}{m^2-p^2-i\eta} \end{aligned}$$

then we can directly employ Smirnov's formula without worrying about an additional $(-1)^\lambda$. Moreover, this way we can easily Wick-rotate to Euclidean space without introducing additional minus signs outside of the propagator.

There is, however, also another advantage. Comparing

$$\text{E:} \int \frac{d^dk}{(k^2+m^2-i\eta)^n} = \pi^{d/2}\frac{\Gamma(\lambda+\varepsilon-2)}{\Gamma(\lambda)}\frac{1}{(m^2-i\eta)^{\lambda+\varepsilon-2}}$$
$$\text{M:} \int \frac{d^dk}{(-k^2+m^2-i\eta)^\lambda} = i\pi^{d/2}\frac{\Gamma(\lambda+\varepsilon-2)}{\Gamma(\lambda)}\frac{1}{(m^2-i\eta)^{\lambda+\varepsilon-2}}$$

we see that Euclidean and Minkowski integrals become identical, up to an overall i, which can be absorbed into the prefactor. When recovering the prefactor, we multiply by $1/(2\pi)^d$ for Euclidean and $i/(2\pi)^d$ for Minkowski integrals.

Similarly, for massless integrals

$$\int \frac{d^dk}{(-k^2-i\eta)^{\lambda_1}[-(q-k)^2-i\eta]^{\lambda_2}} = (-1)^{\lambda_1+\lambda_2}\int \frac{d^dk}{(k^2+i\eta)^{\lambda_1}[(q-k)^2+i\eta]^{\lambda_2}}$$
$$\overset{\text{Smirnov}}{=} i\pi^{d/2}\frac{\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)}\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{(-q^2-i\eta)^{\lambda_1+\lambda_2+\varepsilon-2}}$$

$$\Rightarrow \int \frac{d^dk}{(k^2+i\eta)^{\lambda_1}[(q-k)^2+i\eta]^{\lambda_2}} = i\pi^{d/2}(-1)^{\lambda_1+\lambda_2}\frac{\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)}\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{(-q^2-i\eta)^{\lambda_1+\lambda_2+\varepsilon-2}}$$

so that

$$\text{E:} \int \frac{d^dk}{(k^2-i\eta)^{\lambda_1}[(q-k)^2-i\eta]^{\lambda_2}} = \pi^{d/2}\frac{\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)}\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{(q^2-i\eta)^{\lambda_1+\lambda_2+\varepsilon-2}}$$
$$\text{M:} \int \frac{d^dk}{(-k^2-i\eta)^{\lambda_1}[-(q-k)^2-i\eta]^{\lambda_2}} = i\pi^{d/2}\frac{\Gamma(2-\varepsilon-\lambda_1)\Gamma(2-\varepsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\varepsilon)}\frac{\Gamma(\lambda_1+\lambda_2+\varepsilon-2)}{(-q^2-i\eta)^{\lambda_1+\lambda_2+\varepsilon-2}}$$

Here we obviously have to keep in mind the difference between q^2 (E) and $-q^2$ (M)!

Another advantage of using the Euclidean sign convention is that when integrating out subloops of multiloop integrals we can directly apply the master formulas without the need to pull out factors of $(-1 \pm \eta)$.

0.1.6 Notes on the sign of the Feynman prescription

In practical calculations there is indeed a subtle difference between propagators written as $1/(p^2-m^2)$ and $1/(-p^2+m^2)$ even though both are Minkowskian

- FIESTA assumes that the propagators are entered as $[-(p+q)^2+m^2-i\eta]^{-1}$ (cf. e. g. Eq. 1 in arXiv:1511.03614).
- This means that if we have propagators of the form $[(p+q)^2-m^2+i\eta]^{-1}$, we must pull out the minus sign by hand, otherwise the imaginary part will come out wrong.
- If we naively enter $[(p+q)^2-m^2]^{-1}$ in FIESTA, what we actually get is $[(p+q)^2-m^2-i\eta]^{-1}$, which is definitely not what we want!
- pySecDec assumes that the propagators are entered as $[(p+q)^2-m^2+i\eta]^{-1}$ (cf. e. g. Eq. 1 in arXiv:1502.06595).
- This means that if we have propagators of the form $[-(p+q)^2+m^2-i\eta]^{-1}$, we must pull out the minus sign by hand, otherwise the imaginary part will come out wrong.

- If we naively enter $[-(p+q)^2+m^2]^{-1}$ in FIESTA, what we actually get is $[-(p+q)^2+m^2+i\eta]^{-1}$, which is definitely not what we want!
- To summarize, we have

Symbolic expression	Meaning in FIESTA	Meaning in pySecDec
$(p+q)^2$	$(p+q)^2-i\eta$	$(p+q)^2+i\eta$
$-(p+q)^2$	$-(p+q)^2-i\eta$	$-(p+q)^2+i\eta$
$(p+q)^2-m^2$	$(p+q)^2-m^2-i\eta$	$(p+q)^2-m^2+i\eta$
$-(p+q)^2+m^2$	$-(p+q)^2+m^2-i\eta$	$-(p+q)^2+m^2+i\eta$