

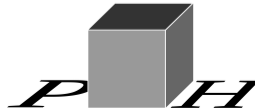
**DETAILS ON THE CALCULATION OF NRQCD MATCHING COEFFICIENTS
FOR $\eta_c \rightarrow \gamma\gamma$ USING FEYNCALC+FEYNONIUM**

Vladyslav Shtabovenko

University of Siegen
Center for Particle Physics Siegen (CPPS)

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- **FEYNCALC** is a **MATHEMATICA** package for symbolic Feynman diagram calculations
- Open-source and publicly available: <https://feyncalc.github.io>
- Toolbox-oriented approach to symbolic Feynman diagram calculations
- Not foolproof: correctness of the results \propto user's understanding of QFT
- Most useful at tree- and 1-loop level (WIP: extension to more loops using a **FORM** [Vermaseren, 2000] library)
- Since 2020 native support for noncovariant calculations (Cartesian vectors, Pauli matrices etc.) \Rightarrow NRQCD, pNRQCD, NR Dark matter

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- **FEYNCALC** alone is not sufficient to do loop calculations
- **FEYNHELPERS** is a collection of interfaces to other tools/packages
- Publicly available <https://github.com/FeynCalc/feynhelpers> and documented
 - **FEYNHELPERS** manual as PDF:
<https://github.com/FeynCalc/feynhelpers-manual/releases/tag/dev-manual>
 - The PDF is automatically generated from the Markdown docu
 - Online version: <https://feyncalc.github.io/referenceFeynHelpersDev>
- 🔗 Original release featured only interfaces to **PACKAGE-X** [Patel, 2017] and **FIRE** (**MATHEMATICA** version)
- 🔗 Current development version also includes connectors to **QGRAF** [Nogueira, 1993], **LOOPTOOLS** [Hahn & Perez-Victoria, 1999], **PYSECDEC** [Borowka et al., 2018], **C++ FIRE** and **FERMAT** [Lewis]

- **C++ FIRE** interface works with loop integrals and topologies
- `FIREPrepareStartFile`: prepares a **MATHEMATICA** script that generates the relevant **FIRE** .start file
- `FIRECreateConfigFile`: creates the **FIRE** .config file
- `FIRECreateIntegralFile`: creates list of integrals that need to be reduced
- The resulting scripts can be evaluated on any machine that has **FIRE** installed (ideally a server/cluster)
- Alternatively (not recommended), one can generate the .start file and start the reduction directly from a notebook
 - `FIRECreateStartFile`
 - `FIRERunReduction`
- `FIREImportResults`: converts reduction results from .table files to **FEYNCALC**'s **GLIs**

- **PYSECDEC** interface allows for a seamless numerical evaluation of integrals obtained in **FEYNCALC**
- Need to specify the integral, requested order in ϵ and the numerical values
- `PSDCreatePythonScripts` generates two **PYTHON** scripts for **PYSECDEC**
 - `generate_int.py`: to generate the integration libraries
 - `integrate_int.py`: to evaluate the integral numerically

- Following [Jia, Yang, Sang and Xu, 2011] (arXiv:1104.1418, very pedagogical)
- NRQCD factorization for the decay $\eta_c \rightarrow \gamma\gamma$ at relative order- v^2

$$\Gamma(\eta_c \rightarrow \gamma\gamma) = \frac{F(^1S_0)}{m_c^2} \langle \eta_c | \psi^\dagger \chi | 0 \rangle \langle 0 | \chi^\dagger \psi | \eta_c \rangle + \frac{G(^1S_0)}{m_c^4} \left(\frac{1}{2} \langle \eta_c | \psi^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \chi | 0 \rangle \langle 0 | \chi^\dagger \psi | \eta_c \rangle + \text{h.c.} \right)$$

- $\psi (\chi)$: Pauli field annihilating (creating) a quark (antiquark); $\psi^\dagger \overleftrightarrow{\mathbf{D}} \chi = \psi^\dagger (D^i \chi) - (D^i \psi)^\dagger \chi$
- Wilson coefficients** $F(^1S_0)$ and $G(^1S_0)$ from the *perturbative* matching between QCD and NRQCD
- Exclusive electromagnetic process \Rightarrow factorization holds at the *amplitude level*
- NRQCD amplitude for $\eta_c \rightarrow \gamma\gamma$

$$\mathcal{A}_{\text{NRQCD}} = \hat{\mathbf{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \left[c_0 \langle 0 | \chi^\dagger \psi | \eta_c \rangle + \frac{c_2}{m_c^2} \langle 0 | \chi^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \psi | \eta_c \rangle + \mathcal{O}(v^2) \right]$$

- $\hat{\mathbf{k}}_1$ is the direction of the 3-momentum of the 1st photon, ε_i^* are the photon polarization vectors.
- c_0 and c_2 are **short distance coefficients**.

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- Relation between **Wilson** and **short distance coefficients**?

- Square $\mathcal{A}_{\text{NRQCD}}$ and integrate over the $1 \rightarrow 2$ phase space. Then compare the result to $\Gamma(\eta_c \rightarrow \gamma\gamma)$:

$$F(^1S_0) = \frac{m_c^2}{8\pi} |c_0|^2, \quad G(^1S_0) = \frac{m_c^2}{4\pi} \text{Re}(c_0 c_2^*)$$

- So our goal is to calculate c_0 and c_2 !

- The matching is done between perturbative QCD and perturbative NRQCD amplitudes in the limit $v \ll c$

- Perturbative NRQCD amplitude: replace η_c (nonpert. bound state) by $c\bar{c}(^1S_0)$ (S -wave charm-anticharm pair in the spin and color singlet configuration)

- Tree-level perturbative NRQCD amplitude

$$\begin{aligned} \mathcal{A}_{\text{NRQCD}}^{\text{pert.,(0)}} &= \hat{\mathbf{k}}_1 \cdot \boldsymbol{\varepsilon}_1^* \times \boldsymbol{\varepsilon}_2^* \left[c_0 \langle 0 | \chi^\dagger \psi | c\bar{c}(^1S_0) \rangle + \frac{c_2}{m_c^2} \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right)^2 \psi | c\bar{c}(^1S_0) \rangle \right] \\ &= \sqrt{2N_c} \hat{\mathbf{k}}_1 \cdot \boldsymbol{\varepsilon}_1^* \times \boldsymbol{\varepsilon}_2^* (\boldsymbol{\eta}^\dagger \boldsymbol{\xi}) \left[c_0 + c_2 \frac{\mathbf{q}^2}{m_c^2} \right] \end{aligned}$$

with $\boldsymbol{\eta}^1 = \boldsymbol{\xi}^1 = (1, 0)^T$, $\boldsymbol{\eta}^2 = \boldsymbol{\xi}^2 = (0, 1)^T$

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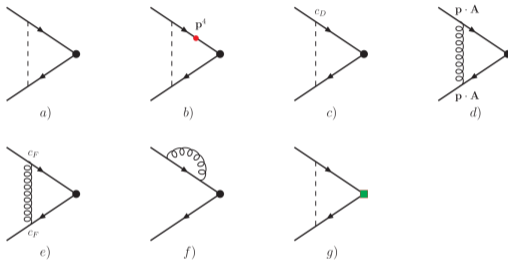
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- In principle, when doing matching at NLO, we need NLO corrections both in QCD and NRQCD!
- There are indeed 1-loop corrections to the tree-level NRQCD amplitude [Jia, Yang, Sang and Xu, 2011]



- However, 1-loop contributions to the UV-renormalized NRQCD amplitude do not contribute to the matching coefficients!

- Perturbative NRQCD 1-loop amplitude

$$\mathcal{A}_{\text{NRQCD}}^{\text{pert.},(1)} = \sqrt{2N_c} \hat{\mathbf{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* (\eta^\dagger \xi) \left[c_0 + c_2 \frac{\mathbf{q}^2}{m_c^2} + \frac{4C_F e_Q^2 \alpha \alpha_s}{m_c} \left\{ \frac{2v^2}{3} \left(\frac{1}{\varepsilon_{\text{IR}}} - \gamma_E + \ln(4\pi) \right) + \frac{1}{4v} \left(1 + \frac{5}{6} v^2 \right) \left[\pi^2 + i\pi \left(-\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E + \ln \frac{\mathbf{q}^2}{\pi \mu^2} \right) \right] \right\} \right]$$

- Still contains $1/v$ -Coulomb singularities and IR divergences: both must cancel in the matching!
- The explicit cancellation is a nontrivial cross check for the correctness of the calculation
- However, one can also turn the argument around and use it to simplify the calculation. Effectively

$$\mathcal{A}_{\text{NRQCD}}^{\text{pert.},(1)} \rightarrow \sqrt{2N_c} \hat{\mathbf{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* (\eta^\dagger \xi) \left[c_0 + c_2 \frac{\mathbf{q}^2}{m_c^2} \right]$$

- On the QCD side we can completely avoid the Coulomb singularities by expanding in \mathbf{q} before calculating the loop integrals [Butenschön 2009]
- Quick and dirty: drop the remaining IR-divergences in the UV-renormalized QCD amplitude and match that to $\mathcal{A}_{\text{NRQCD}}^{\text{pert.},(1)}$ to get our short distance coefficients c_0 and c_2

- Kinematics for the QCD process $c(p_1)\bar{c}(p_2) \rightarrow \gamma(k_1)\gamma(k_2)$

$$p_1 = \frac{1}{2}P + q, \quad p_2 = \frac{1}{2}P - q$$

with

$$P = (2E, 0)^T, \quad q = (0, \mathbf{q})^T, \quad k_1 = E(1, \hat{\mathbf{k}}_1)^T, \quad k_2 = E(1, -\hat{\mathbf{k}}_1)^T$$

and $E = \sqrt{m_c^2 + \mathbf{q}^2}$

- We need to expand the 1-loop amplitude up to $\mathcal{O}(\mathbf{q}^2)$

- We can do reproduce this calculation in 2 ways:
 - Up to $\mathcal{O}(q^2)$ using Passarino-Veltman
 - Up to $\mathcal{O}(q^0)$ using multi-loop techniques

- We also need to project out the spin singlet color singlet S -wave component [Bodwin and Petrelli, 2002]

$$\mathcal{A}_{\text{QCD}}^{\text{pert.},(1)} = \bar{v}(p_2) X \bar{u}(p_1) \rightarrow \text{Tr}(\Pi_1^{(1)} X)$$

with

$$\Pi_1^{(1)} = \frac{1}{8\sqrt{2}E^2(E + m_c)} (\not{p}_1 + m_c)(\not{P} + 2E)\gamma^5(\not{p}_2 - m_c) \otimes \frac{1_c}{\sqrt{N_c}}$$

and use

$$q^\mu q^\nu \rightarrow \frac{\mathbf{q}^2}{3} \left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2} \right)$$

- Traces with γ^5 in dim reg: use the t'Hooft-Veltman-Breitenlohner-Maison scheme

• Final result

$$F(^1S_0) = 2\pi e_Q^4 \alpha^2 \left[1 + \frac{C_F \alpha_s}{\pi} \left(\frac{\pi^2}{4} - 5 \right) \right]$$
$$G(^1S_0) = -\frac{8\pi e_Q^4 \alpha^2}{3} \left[1 + \frac{C_F \alpha_s}{\pi} \left(\frac{5\pi^2}{16} - \frac{49}{12} - \ln \frac{\mu^2}{4m_c^2} \right) \right]$$

• Let us see how we can reproduce this using **FEYNONIUM!**