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TMDs from double J/ψ production

Quarkonia as Tools 2023 workshop (QaT2023+)

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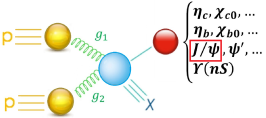
1. Introduction
2. Gluon TMDs
3. TMD and LHCb in the collider mode
4. Azimuthal modulations
5. Numerical results
6. Conclusions

General introduction

Inclusive production of J/ψ pairs in pp collisions (gluon fusion)

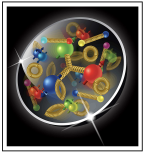
$\Downarrow$

Azimuthal modulations of the cross section for inclusive production of quarkonium pairs in hadronic collisions



↪ **understanding the internal structure of nucleons**

↪ gluon dynamics poorly known



Results → future measurements at LHC fixed-target experiments

↪ unexplored Transverse Momentum Dependent PDFs (TMDs)

Our knowledge of the internal structure of the proton

PDFs → great precision
Collinear QCD phenomenology

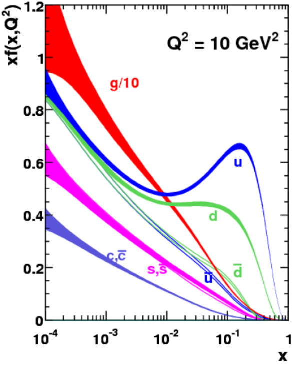
↪ only 1D information
↪ x dependence

⇓

3D structure of the nucleon
Beyond collinear factorisation

⇓

Transverse dynamics!



- Nucleon structure in terms of **TMDs** → quark TMDs
  → gluon TMDs

TMDs → quark and gluon ones

TMDs → 3D structure of the nucleon

Correlations between k_T and the polarisation of the nucleon/parton

2 components ▸ collinear (x)

▸ transversal ($\vec{k}_\perp$) → generate q_T (final-state)

Quark TMDs extracted from data

↪ SIDIS, DY processes

↪ Precision era!

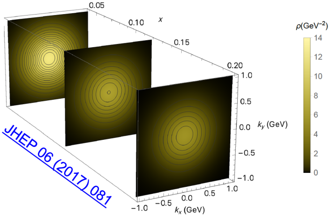
Gluon TMDs → lack of data

↪ Extremely poorly know!

↪ How to measure them?

Inclusive **quarkonium** production

A. Bacchetta et al. (JHEP 08 (2008) 023)



		Quark U	Quark L	Quark T	
Nucleon	U	f_1		$h_1^\perp$	
Nucleon	L		g_{1L}	$h_{1L}^\perp$	
Nucleon	T	$f_{1T}^\perp$	g_{1T}	h_1	$h_{1T}^\perp$

Quarkonium production processes

Experimental point of view:

- quarkonium production observed in different experiments
- J/ψ: easy to produce and detect
  $\hookrightarrow$ plenty of experimental data

Theoretical point of view:

- Not clear how to treat quarkonium production
- 3 common models → Colour Singlet Model (CSM)
  → Colour Octet Mechanism (COM)
  → Colour Evaporation Model (CEM)
- not complete agreement with experimental data
- however for J/ψ-pair production: **CSM** good description

TMD factorisation

Study of gluon TMDs → TMD factorisation ($q_T \ll Q$)

General factorised cross section
↪ partonic scattering amplitude (*perturbative*)
↪ k_T-dependent correlators (*non-perturbative*)

$$d\sigma = \int dx_1 dx_2 d^2\vec{k}_{T1} d^2\vec{k}_{T2} \delta^{(2)}(\vec{k}_{T1} + \vec{k}_{T2} - \vec{q}_T)$$

$$\times\, \Phi_g^{\mu\nu}(x_1, \vec{k}_{T1}) \Phi_g^{\rho\sigma}(x_2, \vec{k}_{T2}) \left[\hat{\mathcal{M}}_{\mu\rho}\, \hat{\mathcal{M}}^{*}_{\nu\sigma} \right]\Big|_{\substack{k_1 = x_1 P_1 \\ k_2 = x_2 P_2}} + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)$$

- In order to stay in TMD regime: $q_T \leq Q/2$

Gluon TMDs

2 independent collinear partonic distributions:

- $f_1^g(x)$ *"unpolarised"*
- $g_1^g(x)$ *"circular"*

Unpolarised protons ⟶ 2 TMDs:

- f_1^g: unpolarised gluon TMD
- $h_1^{\perp g}$: linearly polarised gluon TMD

Nucleon \ Gluon	U	C	L
U	f_1		$h_1^{\perp}$
L		g_{1L}	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	$h_1, h_{1T}^{\perp}$

Gluon TMDs and correlators

TMD correlator parametrisation for an unpolarised proton

- ▷ unpolarised: $f_1^g \longrightarrow$
- ▷ linearly polarised: $h_1^{\perp g} \longrightarrow$

Nucleon \ Gluon	U	C	L
U	f_1		$h_1^\perp$
L		g_{1L}	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	$h_1, h_{1T}^\perp$

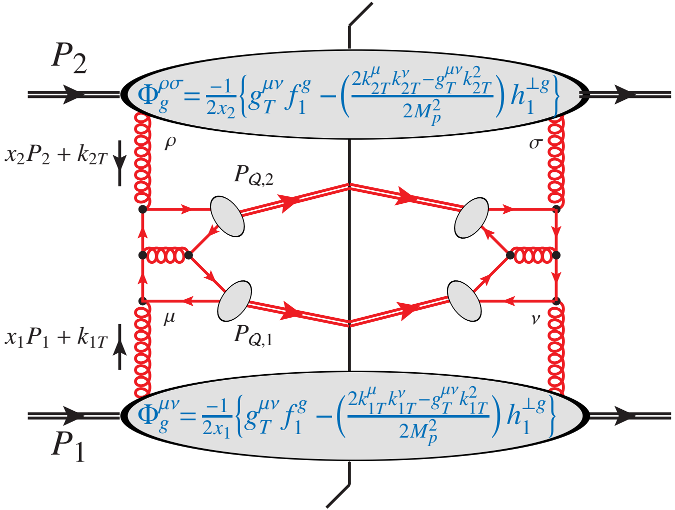
$$\Phi_g^{\mu\nu}(x,\vec{k}_T) = -\frac{1}{2x}\Bigg[g_T^{\mu\nu} f_1^g(x,\vec{k}_T^2)$$

$$- \left(\frac{k_T^\mu k_T^\nu}{M_H^2} + g_T^{\mu\nu}\frac{\vec{k}_T^2}{2M_H^2}\right) h_1^{\perp g}(x,\vec{k}_T^2)\Bigg]$$

↪ Second term goes to 0 if $k_T = 0$

P.J. Mulders and J. Rodrigues (Phys.Rev.D 63 (2001) 094021)

LO Feynman diagram for $p(P_1) + p(P_2) \longrightarrow \mathcal{Q}(P_{Q,1}) + \mathcal{Q}(P_{Q,1}) + X$



Hadronic cross section

The general formula for the cross section of gluon fusion is:

$$
\begin{aligned}
d\sigma^{gg} \propto\ & F_1 \times \mathcal{C}[f_1^g f_1^g] \\
& + F_2 \times \mathcal{C}[w_2 h_1^{\perp g} h_1^{\perp g}] \\
& + (F_3 \times \mathcal{C}[w_3 f_1^g h_1^{\perp g}] + F_3' \times \mathcal{C}[w_3' h_1^{\perp g} f_1^g]) \cos(2\Phi_{CS}) \\
& + (F_4 \times \mathcal{C}[w_4 h_1^{\perp g} h_1^{\perp g}]) \cos(4\Phi_{CS})
\end{aligned}
$$

Where the convolutions are:

$$
\begin{aligned}
\mathcal{C}[w\, f\, g](x_1, x_2, \vec{q}_T) = & \int d^2\vec{k}_{T1} \int d^2\vec{k}_{T2}\, \delta^{(2)}(\vec{k}_{T1} + \vec{k}_{T2} - \vec{q}_T) \\
& \times w_{n,m}(\vec{k}_{T1}, \vec{k}_{T2})\, f(x_1, \vec{k}_{T1})\, g(x_2, \vec{k}_{T2})
\end{aligned}
$$

Why di-J/ψ production?

- Single J/ψ production: a lot of data at low p_T ✓
  ↪ but gluon in the final state → presence of soft gluons (non-perturbative) between Initial State Interactions (ISIs) and Final State Interactions (FSIs) can be problematic
  ↪ **no** TMD factorisation ✗
- Single η_c production: no gluon in the final state ✓
  ↪ but no data at low p_T ✗
- Double J/ψ production:
  - data at low $p_T^{\psi\psi}$ ✓
  - no gluon in the final state ✓

  ↪ gluon fusion: ISI can be encapsulated in the TMDs
  ↪ consider CSM: no FSIs

→ **Safe TMD factorisation**

PhD Thesis F. Scarpa (10.33612/diss.128346301)

$J/\psi + J/\psi$ at low $P_T^{\psi\psi}$: where LHCb can contribute

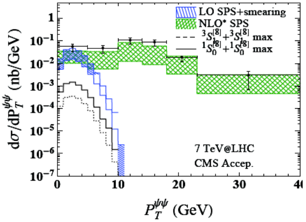
- J/ψ: relatively easy to detect. Already studied by LHCb, CMS, ATLAS & D0; NA3

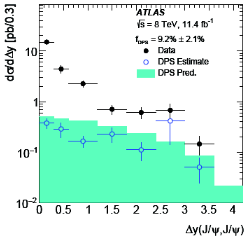
  LHCb PLB 707 (2012) 52; JHEP 1706 (2017) 047; CMS JHEP 1409 (2014) 094; ATLAS EPJC 77 (2017) 76; D0 PRD 90 (2014) 111101; NA3 PLB 158 (1985) 85
- Negligible $q\bar{q}$ contributions even at AFTER@LHC ($\sqrt{s}$ = 115 GeV) energies

  J.P.L., H.S. Shao NPB 900 (2015) 273
- At lower energies (AMBER, SPD), $q\bar{q}$ contributions need to computed
- Negligible CO contributions, in particular at low $P_T^{\psi\psi}$ [black/dashed curves vs. blue; log. plot]

  JPL, H.S. Shao PLB 751 (2015) 479; P. Ko, C. Yu, and J. Lee, JHEP 01 (2011) 070; Y.-J. Li, G.-Z. Xu, K.-Y. Liu, and Y.-J. Zhang, JHEP 07 (2013) 051
- No final state gluon needed for the Born contribution: pure colourless final state

  JPL, H.S. Shao PRL 111, 122001 (2013)
- In the CMS & ATLAS acceptances (P_T cut), small DPS effects, but required by the data at large Δy



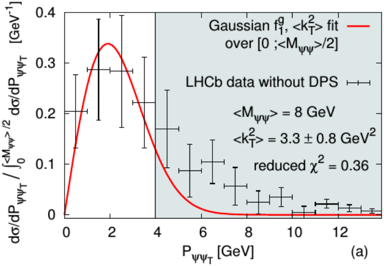


- DPS in LHCb data [kinematical distributions a priori under-control : independent scatterings]

TMD modelling: f_1^g and the relevance of the LHCb data

JPL, C. Pisano, F. Scarpa, M. Schlegel, PLB 784(2018)217

- f_1^g modelled as a Gaussian in $\vec{k}_T$: $f_1^g(x, \vec{k}_T^2) = \frac{g(x)}{\pi\langle k_T^2\rangle} \exp\left(\frac{-\vec{k}_T^2}{\langle k_T^2\rangle}\right)$ where $g(x)$ is the usual collinear PDF
- First experimental determination [with a pure colorless final state] of $\langle k_T^2\rangle$ by fitting $\mathcal{C}[f_1^g f_1^g]$ over the normalised LHCb $d\sigma/dP_{\psi\psi_T}$ spectrum at 13 TeV from which we have subtracted the DPS yield determined by LHCb

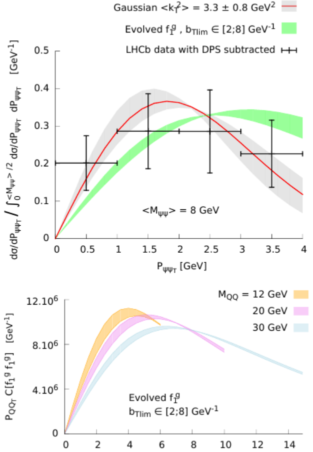


- Integration over $\phi \Rightarrow \cos(n\phi)$-terms cancel out
- $F_2 \ll F_1 \Rightarrow$ only $\mathcal{C}[f_1^g f_1^g]$ contributes to the cross-section
- No evolution so far: $\langle k_T^2\rangle \sim 3$ GeV^2 accounts both for non-perturbative and perturbative broadenings at a scale close to $M_{\psi\psi} \sim 8$ GeV
- Disentangling such (non-)perturbative effects requires data at different scales

Switching on TMD evolution

- With a fit we obtained $\langle k_T^2 \rangle \sim 3\ \text{GeV}^2$
- Let us compare such a value with what a proper NLL evolution up to the scale $M_{\psi\psi} \sim 8$ GeV would give
- Evolution effects are measurable
- So far, no x dependence information

F. Scarpa, D. Boer, M.G. Echevarria, JPL, C. Pisano, M. Schlegel, EPJC (2020) 80:87



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& + (F_3 \times \mathcal{C}[w_3 f_1^g h_1^{\perp g}] + F_3' \times \mathcal{C}[w_3' h_1^{\perp g} f_1^g]) \cos(2\Phi_{CS}) \\
& + (F_4 \times \mathcal{C}[w_4 h_1^{\perp g} h_1^{\perp g}]) \cos(4\Phi_{CS})
\end{aligned}
$$

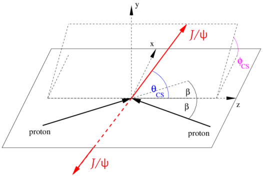
- first two members: azimuthally independent
- third member: $\cos(2\Phi_{CS})$-modulation
- fourth member: $\cos(4\Phi_{CS})$-modulation

Computation of azimuthal modulations (average)

The corresponding expressions for $\cos(2\Phi_{CS})$ and $\cos(4\Phi_{CS})$:

$$\langle\cos(2\phi_{CS})\rangle = \frac{1}{2}\frac{F_3\mathcal{C}[w_3 f_1^g h_1^{\perp g}] + F_3'\mathcal{C}[w_3' h_1^{\perp g} f_1^g]}{F_1\mathcal{C}[f_1^g f_1^g] + F_2\mathcal{C}[w_2 h_1^{\perp g} h_1^{\perp g}]}$$

$$\langle\cos(4\phi_{CS})\rangle = \frac{1}{2}\frac{F_4\mathcal{C}[w_4 h_1^{\perp g} h_1^{\perp g}]}{F_1\mathcal{C}[f_1^g f_1^g] + F_2\mathcal{C}[w_2 h_1^{\perp g} h_1^{\perp g}]}$$



- The hard-scattering coefficients (F_1, F_2, F_3, F_3', F_4) give the explicit dependence on $M_{\psi\psi}$ and θ_{CS}
- Modulations due to $h_1^{\perp g}$
- Set scale $Q^2 = M^2_{\psi\psi}$ and consider $M_{\psi\psi} = 8, 16$ GeV
- TMD evolution applied within the convolutions

Asymmetric kinematics

Goal: study x dependence $\longrightarrow$ general $x_1 \neq x_2$

Implementation: **ex-novo code** in Python
TMD factorisation, TMD convolutions, TMD evolution
(use of LHAPDF package for PDF parametrisation)

$\Downarrow$

Code validation: reproduced published results (collider mode)

$\Downarrow$

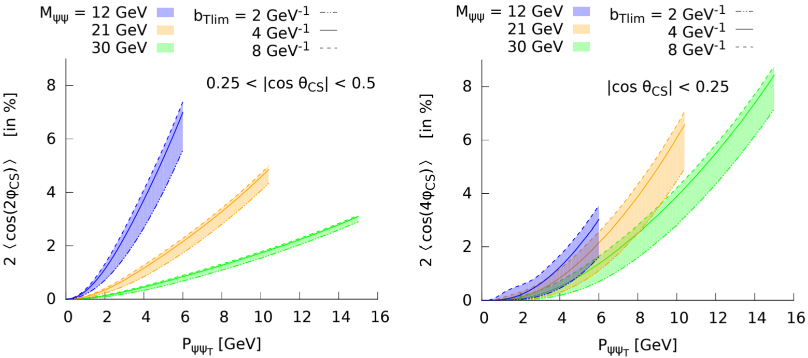
NEW: **first studies** with $x_1 \neq x_2$
(two sets of x_1, x_2 but same rapidity $y = \frac{1}{2} \ln \frac{x_1}{x_2}$)

$J/\psi + J/\psi$	$\langle x_2 \rangle \sim \frac{M_{\psi\psi}}{\sqrt{s}} e^{-Y^{\text{c.m.s.}}_{\psi\psi}}$	$\sigma_{gg} \times \mathcal{B}^2_{\mu\mu}$ [fb]	$\sigma_{q\bar{q}} \times \mathcal{B}^2_{\mu\mu}$ [fb]	Counts/year
$4.5 < Y^{\text{lab.}}_{\psi\psi} < 5.0$	0.13	$\mathcal{O}(5)$	$\mathcal{O}(1)$	$\mathcal{O}(50)$
$4.0 < Y^{\text{lab.}}_{\psi\psi} < 4.5$	0.29	$\mathcal{O}(50)$	$\mathcal{O}(10)$	$\mathcal{O}(500)$
$3.5 < Y^{\text{lab.}}_{\psi\psi} < 4.0$	0.45	$\mathcal{O}(50)$	$\mathcal{O}(10)$	$\mathcal{O}(500)$
$3.0 < Y^{\text{lab.}}_{\psi\psi} < 3.5$	0.60	$\mathcal{O}(10)$	$\mathcal{O}(10)$	$\mathcal{O}(100)$
$2.5 < Y^{\text{lab.}}_{\psi\psi} < 3.0$	0.77	$\mathcal{O}(5)$	$\mathcal{O}(2)$	$\mathcal{O}(70)$

Results in collider mode: $\cos(2\Phi_{CS})$, $\cos(4\Phi_{CS})$

Plots considering:

- two ranges of $\cos(\theta_{CS})$: [0; 0.25] and [0.25; 0.50]
- three values for the invariant mass: 12, 21, 30 GeV; **x1=x2**

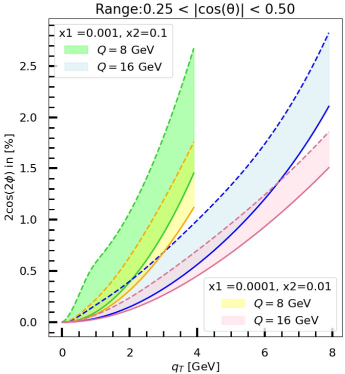


Eur.Phys.J.C 80 no.2,(2020) 87, arXiv:1909.05769 [hep-ph]

Preliminary: predictions for $\cos(2\Phi_{CS})$

Plots considering:

- range of $\cos(\theta_{CS})$: [0.25; 0.50], $Q = M_{\psi\psi}$
- two different sets $(x_1; x_2)$: $(10^{-3}; 10^{-1})$ and $(10^{-4}; 10^{-2})$



▷ asymmetry up to 3% (measurable)

▷ $q_T < Q/2$ considered

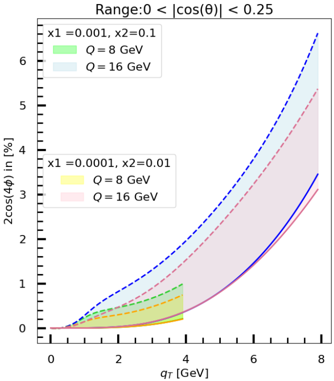
▷ big overlap in the low q_T region, not for large q_T

▷ ∼ same magnitude for low and high Q (lower for lower $x_{1,2}$)

Preliminary: predictions for $\cos(4\Phi_{CS})$

Plots considering:

- range of $\cos(\theta_{CS})$: $[0;\ 0.25]$, $Q = M_{\psi\psi}$
- two different sets $(x_1; x_2)$: $(10^{-3}; 10^{-1})$ and $(10^{-4}; 10^{-2})$



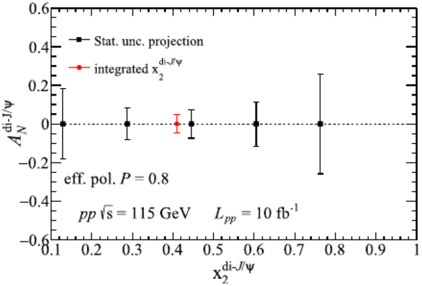
- max asymmetry $5-6\%$ (measurable)
- $q_T < Q/2$ considered
- overlap $\forall q_T$ (no x dependence)
- much higher amplitude for high Q (at high q_T)

Summary

- Quarkonium production is a great tool for many purposes
  $\hookrightarrow$ exploration of nucleon structure through gluon TMDs
- Double J/ψ production gives the possibility to investigate gluon TMD induced effects
  $\hookrightarrow$ k_T dependent effects
  $\hookrightarrow$ azimuthal modulations
  $\hookrightarrow$ spin effects
- **NEW** Fixed-target mode: lower azimuthal modulations for $\frac{x1}{x2} \neq 1$ ($x1 \simeq x2$ seems to be favoured)

Further studies

- **FUTURE** Studies can be made in the (near) future considering polarised protons → access to more gluon TMDs
- Di-J/ψ production: most promising → gluon Sivers function



Backup slides

Results in collider mode: $\cos(2\Phi_{CS})$

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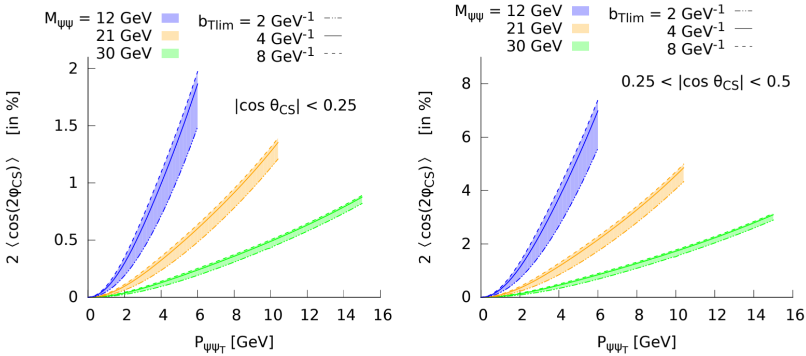

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Results in collider mode: $\cos(4\Phi_{CS})$

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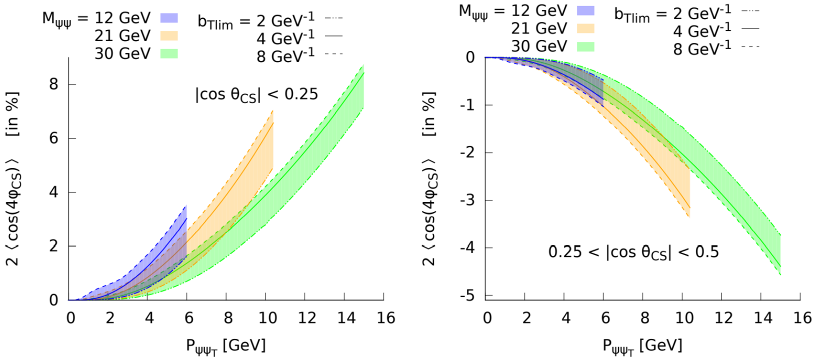

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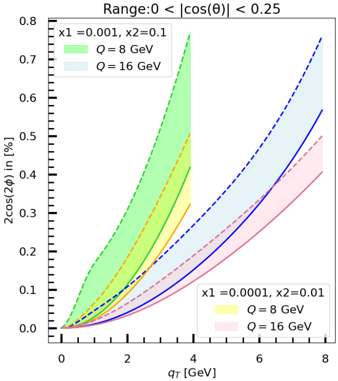

- ▹ contribution below 1%
- ▹ $q_T < Q/2$ considered
- ▹ big overlap in the low q_T region, not for large q_T
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Preliminary: predictions for $\cos(4\Phi_{CS})$

Plots considering:

- range of $\cos(\theta_{CS})$: [0.25; 0.50], $Q = M_{\psi\psi}$
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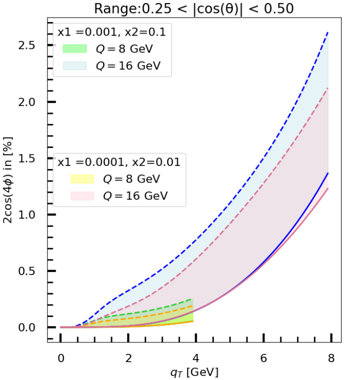

- ▹ contribution up to 3% (measurable)
- ▹ $q_T < Q/2$ considered
- ▹ overlap $\forall q_T$
- ▹ higher amplitude for high Q (low Q negligible)

Introduction Evolution (1)

- Beyond tree level, the TMDs and hard factors F become scale dependent J. Collins (ISBN: 9781107645257)
- Implementing evolution is more easily done in impact parameter space (b_T), where convolutions become simple products:

$$d\sigma_{UU}^{gg} \propto \int d^2\mathrm{b}_T \, e^{-i\mathrm{b}_T\cdot \mathrm{q}_T} \, \hat{W}(\mathrm{b}_T, Q) + \mathcal{O}(\mathrm{q}_T^2/Q^2)$$

$$\hat{W}(\mathrm{b}_T, Q) = \hat{f}(x_1, \mathrm{b}_T; \zeta_f, \mu)\, \hat{g}(x_2, \mathrm{b}_T; \zeta_g, \mu)\, \mathcal{H}(Q; \mu)\,.$$

- The convolutions are rewritten by Fourier transforming:

$$\mathcal{C}[w\, f\, g](x_1, x_2, \vec{q}_T) = \int d^2\vec{k}_{T1} \int d^2\vec{k}_{T2}\, \delta^{(2)}(\vec{k}_{T1} + \vec{k}_{T2} - \vec{q}_T) \times w_{n,m}(\vec{k}_{T1}, \vec{k}_{T2})\, f(x_1, \vec{k}_{T1})\, g(x_2, \vec{k}_{T2})$$

$$\Rightarrow \int_0^\infty \frac{db_T}{2\pi}\, b_T^n\, J_m(b_T\, q_T)\, \hat{f}(x_1, \mathrm{b}_T)\, \hat{g}(x_2, \mathrm{b}_T)$$

Introduction Evolution (2)

$$\mathcal{C}[w\, f\, g](x_1, x_2, \vec{q}_T; Q) = \int_0^\infty \frac{db_T}{2\pi}\, b_T^n\, J_m(b_T\, q_T)$$
$$\times\; e^{-S_A(b_T; Q^2, Q)}\, \hat{f}(x_1, \mathrm{b}_T; \mu_b^2, \mu_b)\, \hat{g}(x_2, \mathrm{b}_T; \mu_b^2, \mu_b)$$

- S_A contains ln Qb_T
- Expressions (based on pQCD) are valid when: $b_0/Q \leq b_T \leq b_{T,\mathrm{max}}$
- At lower limit $\mu_b = \; b_0/b_T$ becomes larger than Q, i.e. evolution should stop ($S_A = 0$)
- At upper limit perturbation theory starts to fail, which is not exactly known. Common to take $b_{T,\mathrm{max}} = 0.5\,\mathrm{GeV}^{-1}$ or $b_{T,\mathrm{max}} = 1.5\,\mathrm{GeV}^{-1}$.
- This effectively boils down to a different resummation: $\mu_b(b_T)/Q \rightarrow \mu_b(b_T^*)/Q$

Introduction Evolution (3)

- We need to add a component that takes over as $b_T > b_{T,\max}$:

$$\hat{W}(b_T, Q) \equiv \hat{W}(b_T^*, Q) e^{-S_{NP}(b_T, Q)}$$

- There are different parameterizations for S_{NP} in the literature, but typically it is chosen to be a Gaussian:

$$S_{NP}(b_T; Q) = A \ln \frac{Q}{Q_{NP}} b_T^2 \quad \text{with} \quad Q_{NP} = 1\,\text{GeV}$$

- We obtain the following expression for the convolutions:

$$\mathcal{C}[w\, f\, g](x_1, x_2, \vec{q}_T; Q) = \int_0^\infty \frac{db_T}{2\pi}\, b_T^n\, J_m(b_T\, q_T)\, e^{-S_A(b_T^*; Q^2, Q)}\, e^{-S_{NP}(b_T; Q)} \times \hat{f}(x_1, \mathrm{b}_T^*; \mu_b^2, \mu_b)\, \hat{g}(x_2, \mathrm{b}_T^*; \mu_b^2, \mu_b)$$

The Sudakov Factor and Scales

- The solution of the evolution equations results in:

$$\hat{f}_1^g(x_1, b_T; \zeta, \mu) = e^{-\frac{1}{2}S_A(b_T;\zeta,\mu)}\hat{f}_1^g(x_1, b_T; \mu_b^2, \mu_b)$$
$$\hat{h}_1^{\perp g}(x_1, b_T; \zeta, \mu) = e^{-\frac{1}{2}S_A(b_T;\zeta,\mu)}\hat{h}_1^{\perp g}(x_1, b_T; \mu_b^2, \mu_b)$$

- $\mu \sim Q$ avoids large logarithms in $\mathcal{H}$
- TMDs should be evaluated at their natural scale: $\sqrt{\zeta_0} \sim \mu_0 \ll \sqrt{\zeta} \sim \mu$
- $\Rightarrow$ take $\sqrt{\zeta_0} \sim \mu_0 \sim \mu_b \equiv b_0/b_T$ (with $b_0 = 2e^{-\gamma_E}$), in order to minimize both logarithms of μb_T and ζb_T^2 in S_A, and then evolved up to $\sqrt{\zeta} \sim \mu \sim Q$

Perturbative tails

- The large transverse momentum perturbative tail of the TMDs can be written as:

$$\hat{f}_1^g(x, b_T; \mu_b^2, \mu_b) = f_{g/P}(x; \mu_b) + \mathcal{O}(\alpha_s) + \mathcal{O}(b_T \Lambda_{\mathrm{QCD}})$$

$$\hat{h}_1^{\perp g}(x, b_T; \mu_b^2, \mu_b) = -\frac{\alpha_s(\mu_b)}{\pi} \int_x^1 \frac{dx'}{x'} \left(\frac{x'}{x} - 1\right) \Big\{ C_A\, f_{g/P}(x'; \mu_b) + C_F \sum_{i=q,\bar{q}} f_{i/P}(x'; \mu_b) \Big\} + \mathcal{O}(\alpha_s^2) + \mathcal{O}(b_T \Lambda_{\mathrm{QCD}})$$

P. Sun et al. (Phys.Rev.D 84 (2011) 094005)

b_T-Domains

- To ensure $b_0/Q \leq b_T$ we take:

$$b_c(b_T) = \sqrt{b_T^2 + \left(\frac{b_0}{Q}\right)^2}$$

- For $b_T \leq b_{T,\text{max}}$:

$$b_T^*(b_c(b_T)) = \frac{b_c(b_T)}{\sqrt{1 + \left(\frac{b_c(b_T)}{b_{T,\text{max}}}\right)^2}}$$

J. Collins et al. (Phys.Rev.D 94 (2016) 3, 034014)

The Non-perturbative Sudakov Factor

$$S_{NP}(b_T; Q) = A \ln \frac{Q}{Q_{NP}} b_T^2 \quad \text{with} \quad Q_{NP} = 1\,\text{GeV}$$

$b_{T,\text{lim}}$ (GeV^{-1})	r (fm $\sim 1/(0.2$ GeV$)$)	A (GeV^2)
2	0.2	0.64
4	0.4	0.16
8	0.8	0.04

Table 1: Values of the parameter A for $b_{T,\text{lim}}$ and r determined at $Q = 12$ GeV. A is defined at which $\exp(-S_{NP})$ becomes negligible ($\sim 10^{-3}$). To estimate the uncertainty associated with the S_{NP} we vary $b_{T,\text{lim}}$ spanning roughly from $b_{T,\text{max}} = 1.5$ GeV^{-1} to the charge radius of the proton. r is the range over which the interactions occur from the centre of the proton.

Hard scattering coefficients

$$F_1 = \frac{\mathcal{N}}{\mathcal{D} M_\Psi^2} \sum_{n=0}^{6} f_{1,n} (\cos\theta_{CS})^{2n} \qquad F_2 = \frac{2^4 3 M_\Psi^2 \mathcal{N}}{\mathcal{D} M_{\Psi\Psi}^4} \sum_{n=0}^{4} f_{2,n} (\cos\theta_{CS})^{2n}$$

$$F_3' = F_3 = \frac{-2^3(1-\alpha^2)\mathcal{N}}{\mathcal{D} M_{\Psi\Psi}^2} \sum_{n=0}^{5} f_{3,n} (\cos\theta_{CS})^{2n}$$

$$F_4 = \frac{(1-\alpha^2)^2 \mathcal{N}}{\mathcal{D} M_{\Psi\Psi}^2} \sum_{n=0}^{6} f_{4,n} (\cos\theta_{CS})^{2n} \tag{1}$$

with: $\alpha = \frac{2M_\Psi}{M_{\Psi\Psi}}$, $\mathcal{N} = 2^{11} 3^{-4} (N_c^2 - 1)^{-2} \pi^2 \alpha_s^4 |R_\Psi(0)|^4$, $\mathcal{D} = M_{\Psi\Psi}^4 (1 - (1-\alpha^2)\cos\theta_{CS}^2)^4$ and $R_\Psi(0)$ is the J/ψ radial wave function at the origin and $N_c = 3$.