Non-perturbative Amplituhedron geometry

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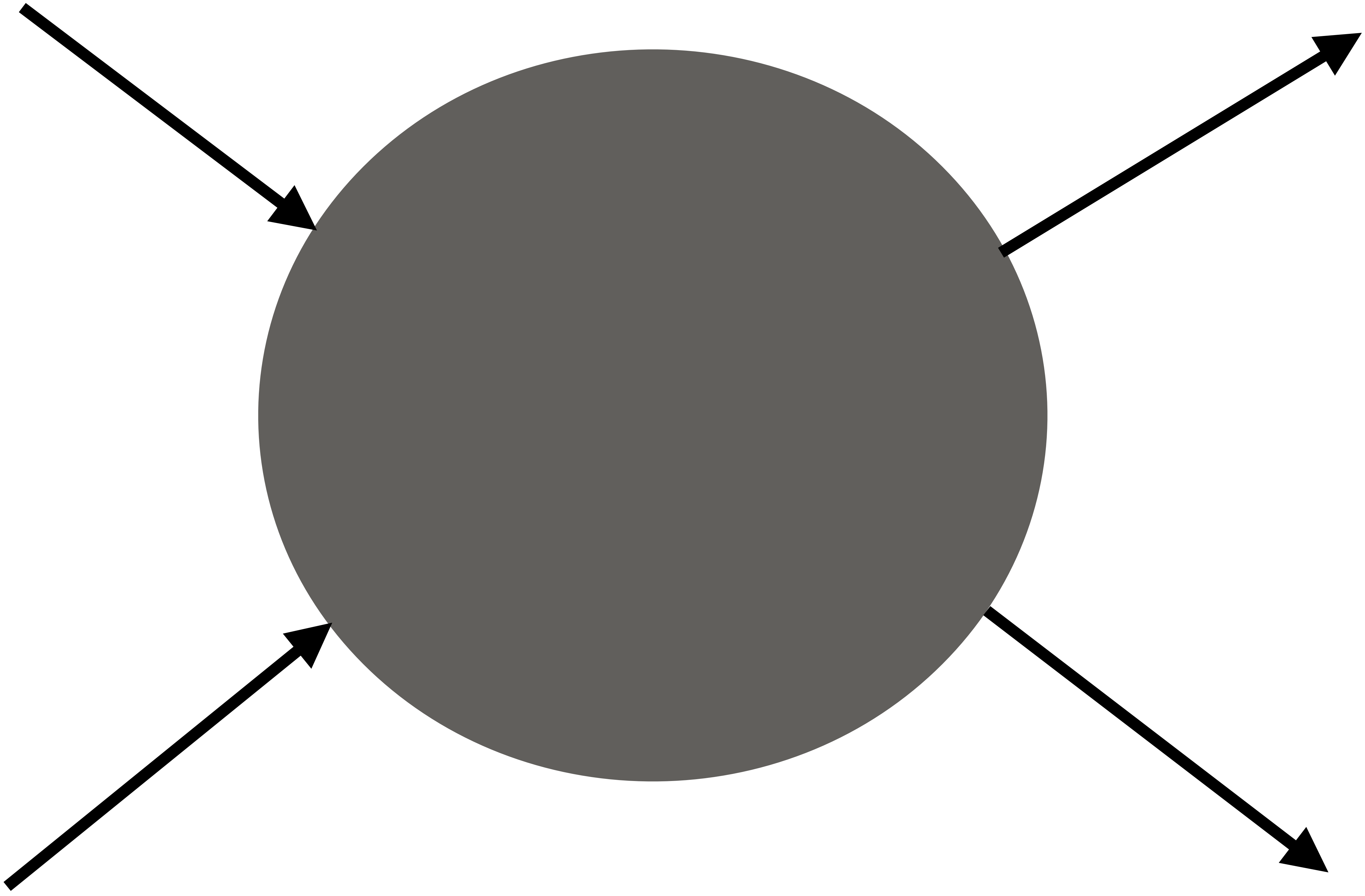
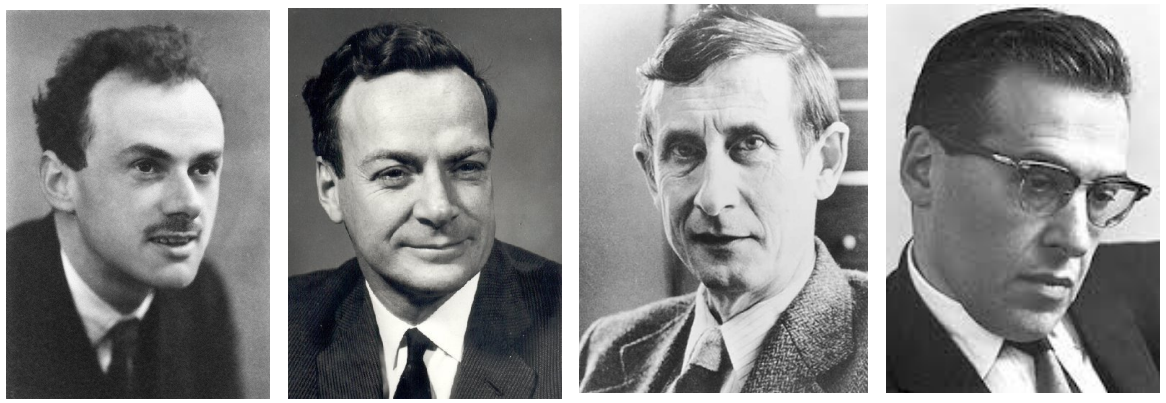
work with **Nima Arkani-Hamed** and **Johannes Henn**

CERN, March 8, 2023

How to define/calculate
the perturbative S-matrix in QFT?



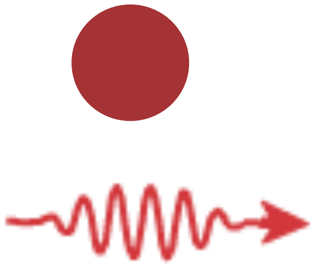
Quantum Field Theory



Dirac, Feynman, Dyson, Schwinger (1926-1950s)

- Elementary particles described by fields, their interactions (physical forces) by Lagrangian.

- Example: Quantum Electrodynamics - QFT for EM



e Electron ψ

γ Photon A

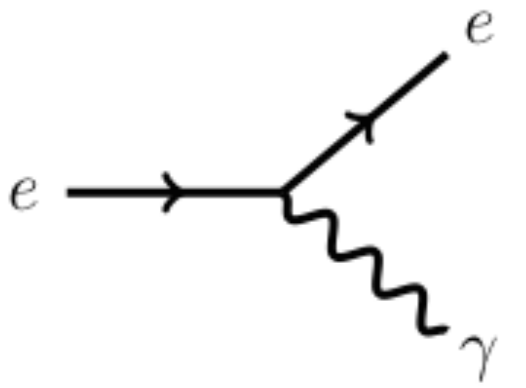
force carrier

$$\mathcal{L} = q \cdot \psi \, \psi \, A$$

Lagrangian

strength
of interaction

We associate a picture



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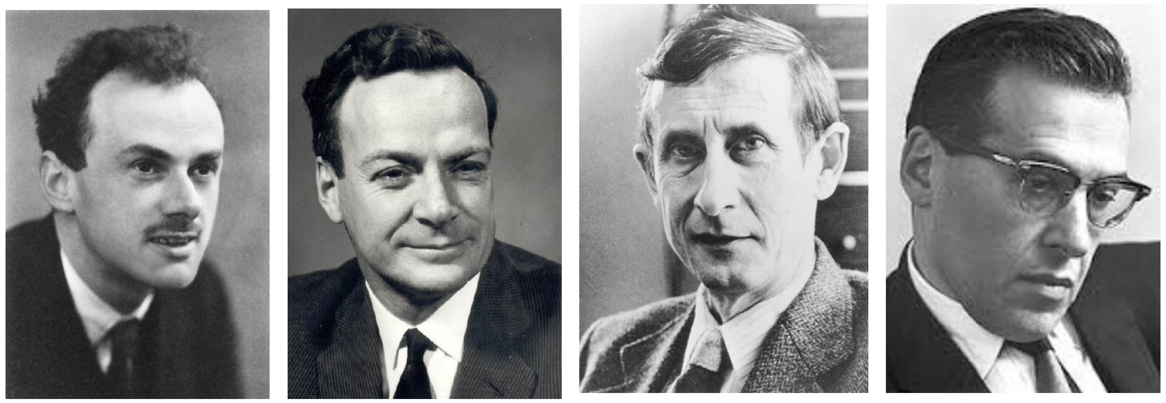
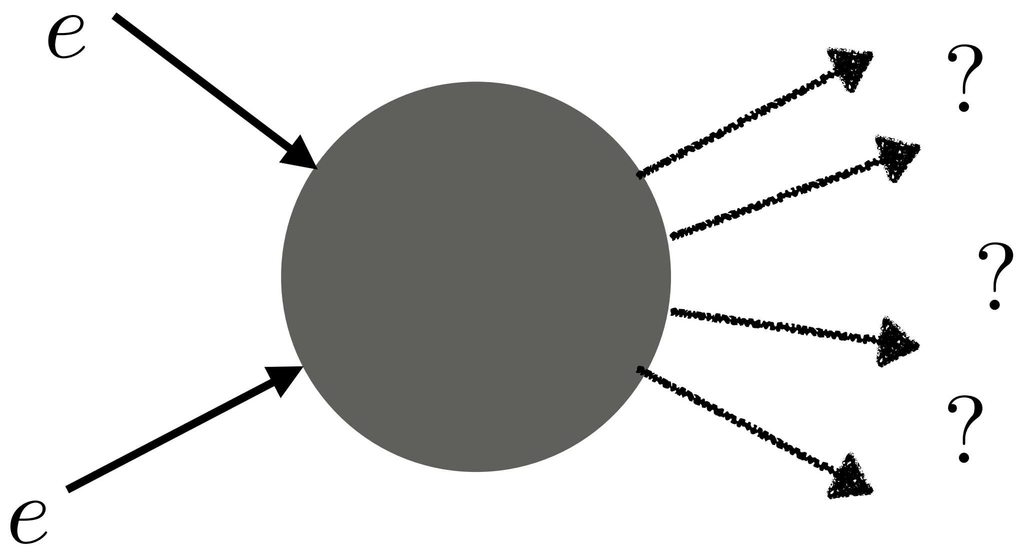


Dirac, Feynman, Dyson, Schwinger (1926-1950s)

- Use QFT to predict outcomes of particle experiments



initial
state

final
state

scattering process

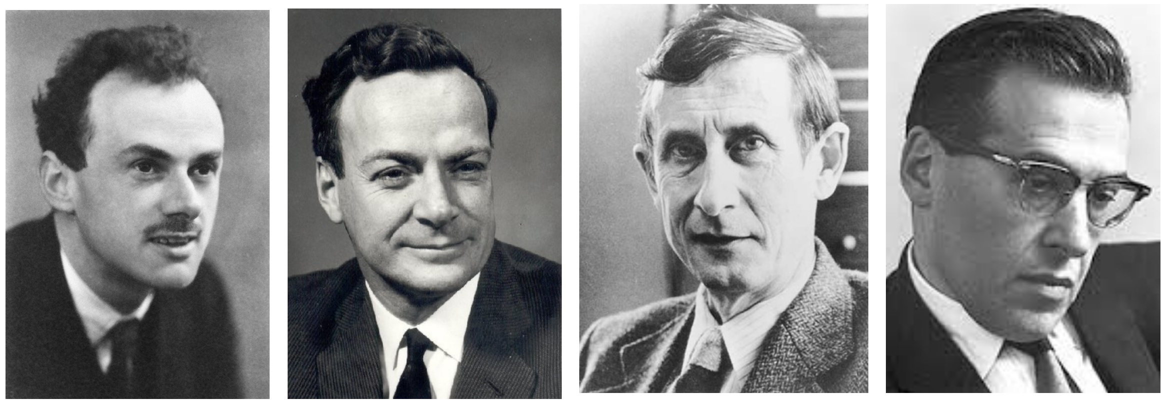
Quantum mechanics:
all final states possible

↓

Probabilities given by

Scattering amplitudes
$A(in, out)$

Quantum Field Theory



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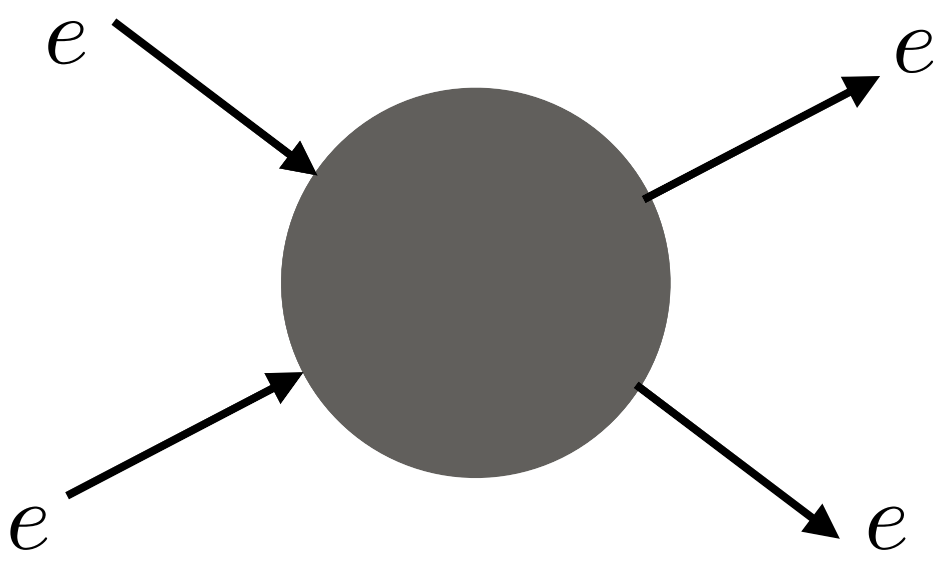
- Use QFT to predict outcomes of particle experiments

initial state

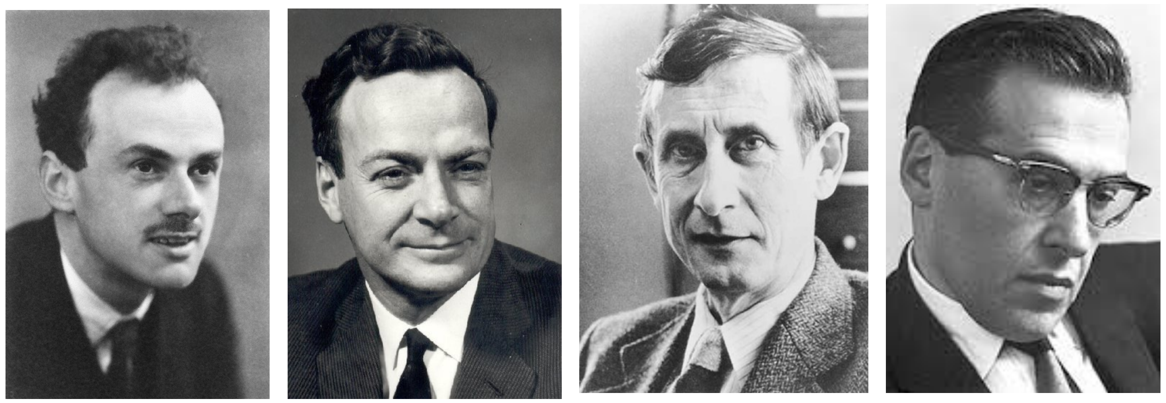


scattering process

$$A(ee \to ee)$$

Function of energies and angles of particles

Quantum Field Theory



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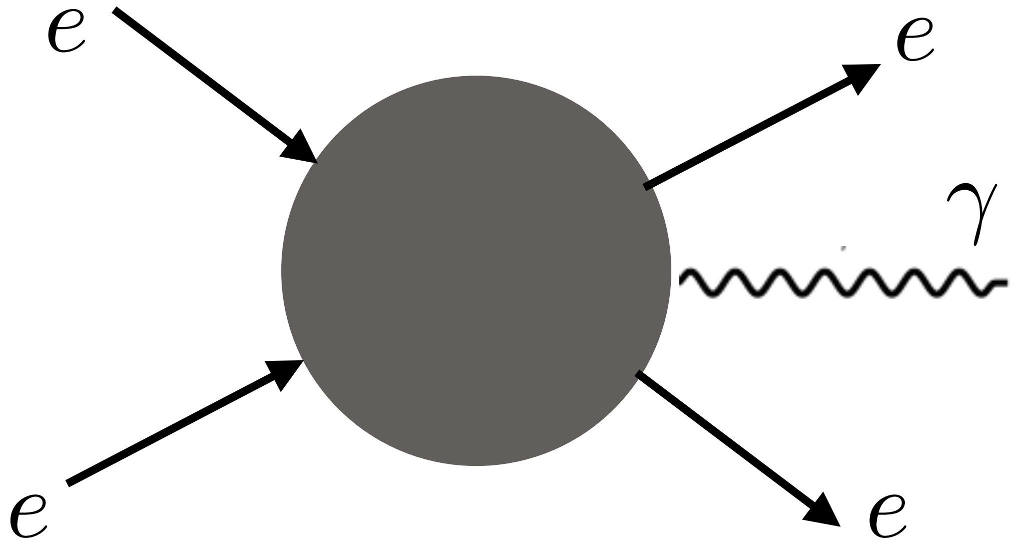


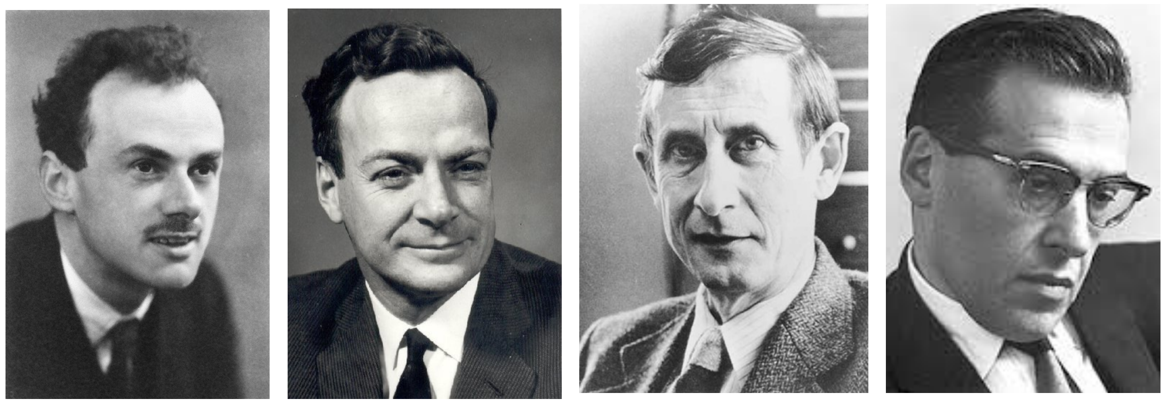
initial state

scattering process

$$A(ee \to ee\gamma)$$

Function of energies and angles of particles

Quantum Field Theory



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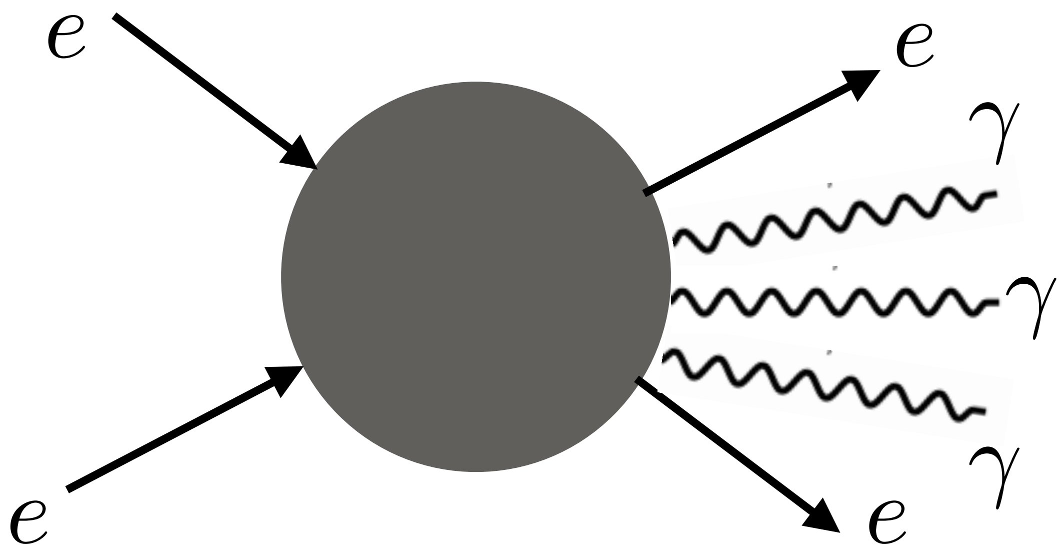


initial state

scattering process

$$A(ee \to ee\gamma\gamma\gamma)$$

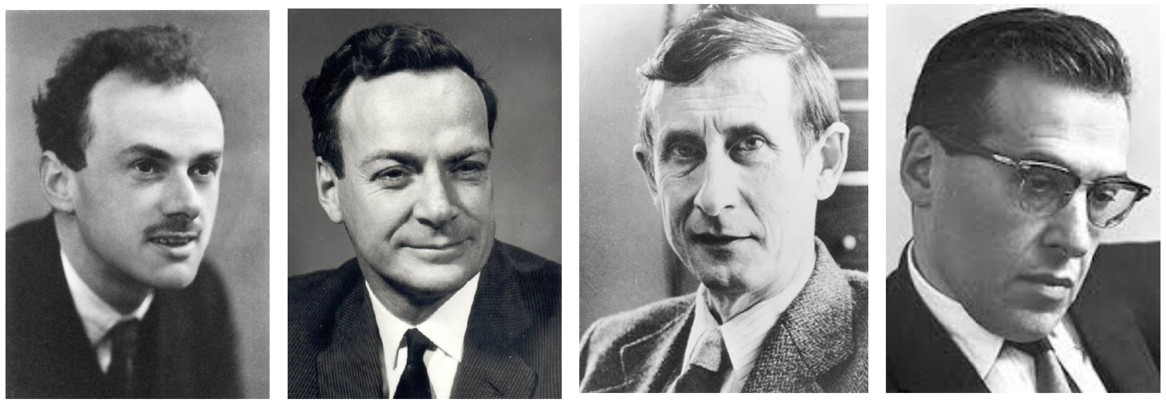
Function of energies and angles of particles

probability: square of amplitude

$$p_i \sim |A|^2$$

Unitarity: some of probabilities $\sum_i p_i = 1$

Quantum Field Theory



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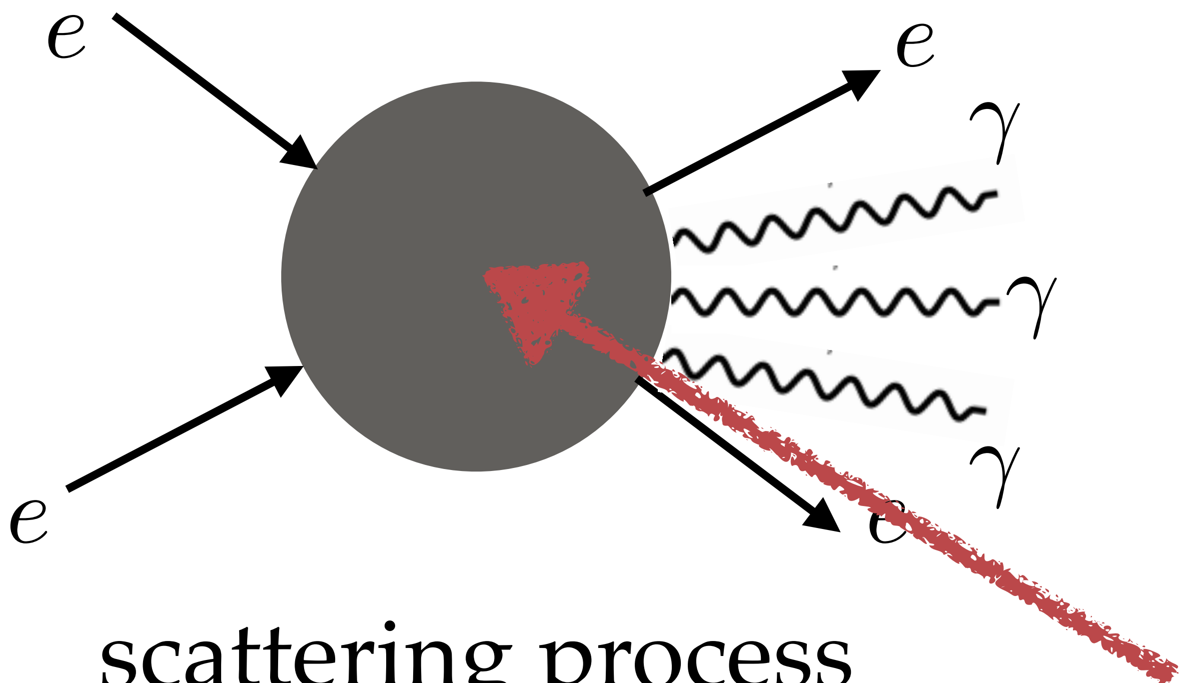


initial state

scattering process

$$A(ee \to ee\gamma\gamma\gamma)$$

Function of energies and angles of particles

Big question:

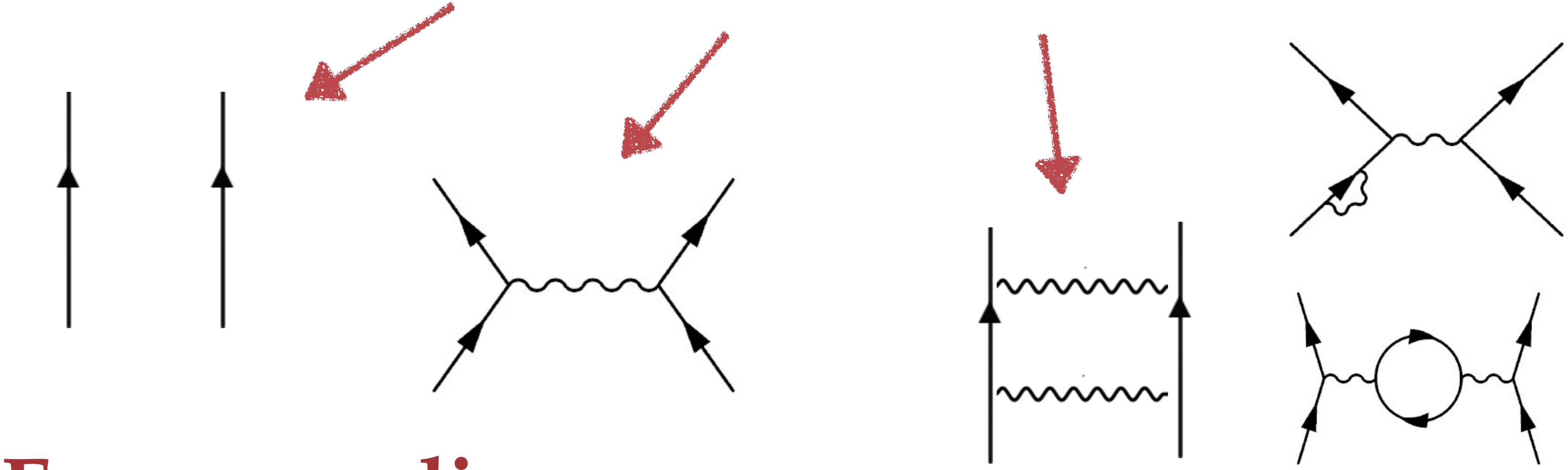
What happens inside?

Perturbative QFT

Feynman, Dyson, Schwinger (1940s-1950s)

- Expansion of the amplitude for small q: weak coupling

$$A(ee \to ee) = 1 + q^2 \cdot A_0 + q^4 \cdot A_1 + q^6 \cdot A_2 + \ldots$$



and many others

Feynman diagrams

(Expansion of the path integral

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, \mathcal{D}A\, e^{iS}$$
)

Perturbative QFT

Feynman, Dyson, Schwinger (1940s-1950s)

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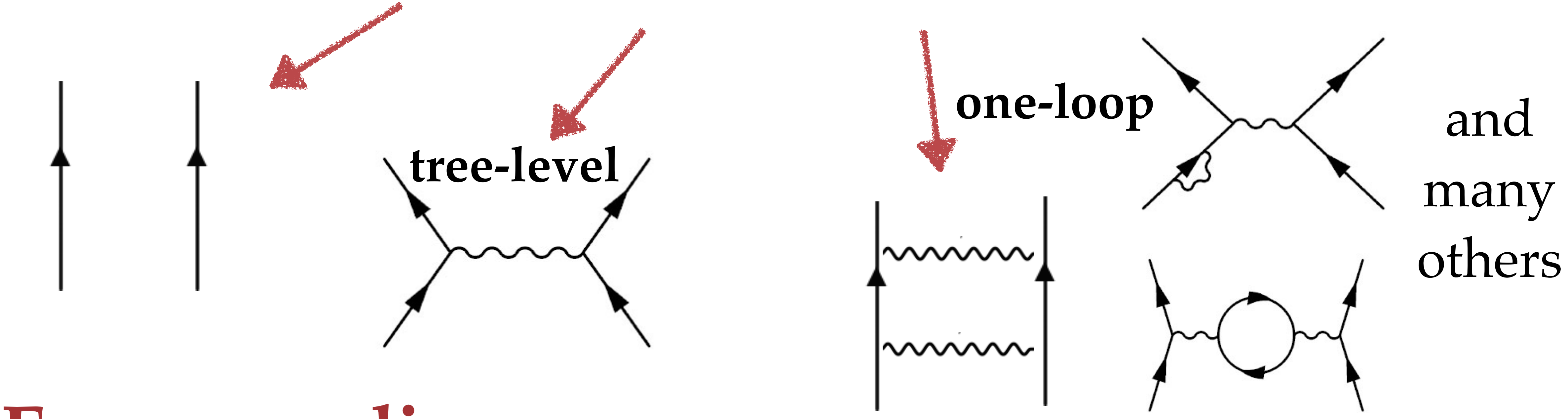
$$A(ee \to ee) = 1 + q^2 \cdot A_0 + q^4 \cdot A_1 + q^6 \cdot A_2 + \dots$$



Feynman diagrams

Loop expansion

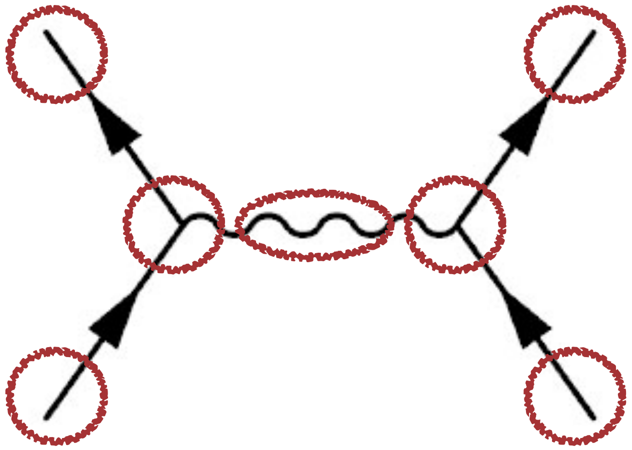
$$\left(\begin{array}{c} \text{Expansion of the path integral} \\ Z = \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, \mathcal{D}A \, e^{iS} \end{array} \right)$$

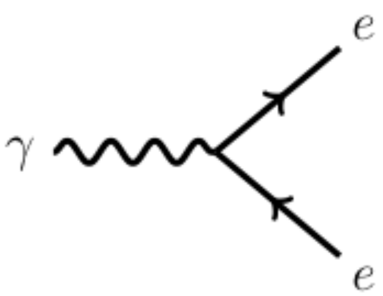
Quantum Field Theory

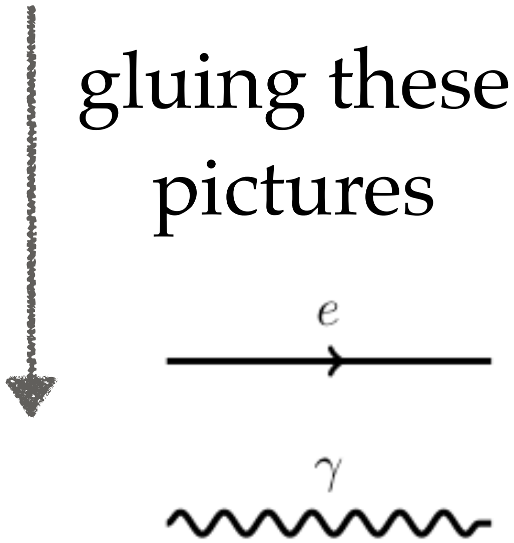
Feynman, Dyson, Schwinger (1940s-1950s)

- Simple diagrammatics: draw all Feynman diagrams
- Each diagram: contribution to amplitude

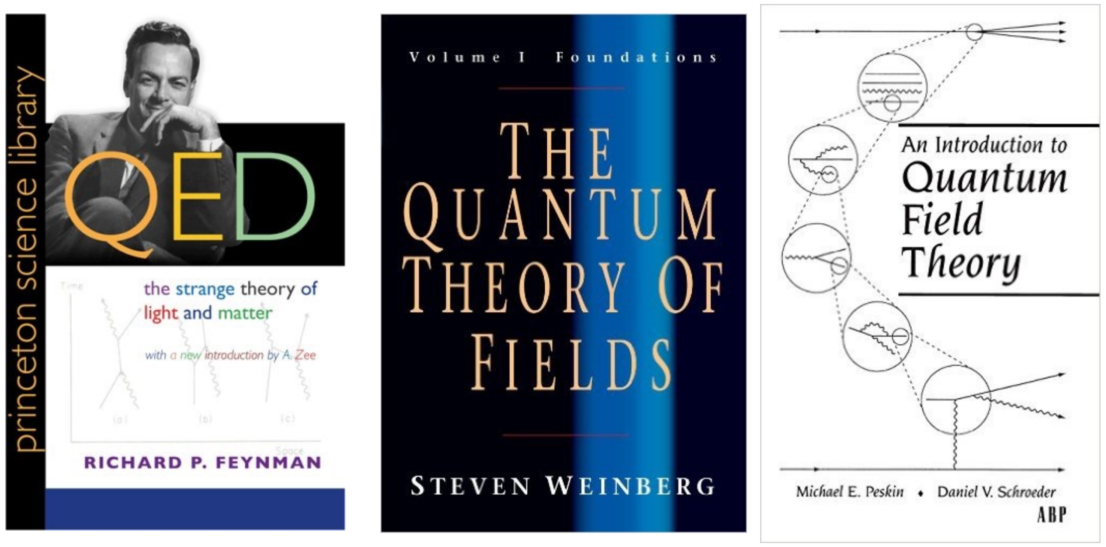
gluing these pictures





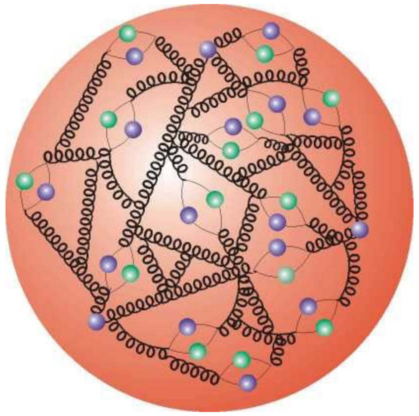


Feynman rules: prescription how to convert diagram into formula

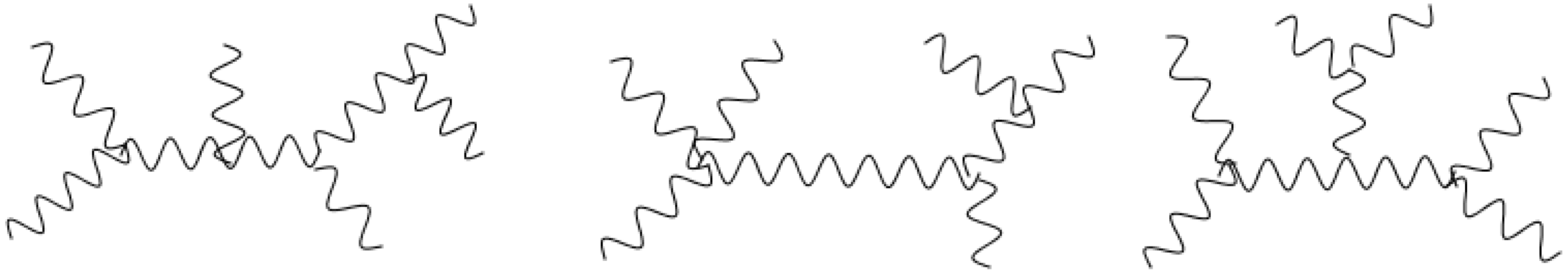


QCD background and new physics

- Distinguish new physics from Standard model
  - Accurate theoretical predictions of background needed
- Colliders: protons at high energies
  - Main component is scattering of gluons



- Standard procedure: Feynman diagrams



Status of the art: early 1980s

- Most complicated process: $gg \rightarrow ggg$ at leading order

Brute force calculation
24 pages of result

$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$

- Perhaps not every question has a simple answer…..

New collider

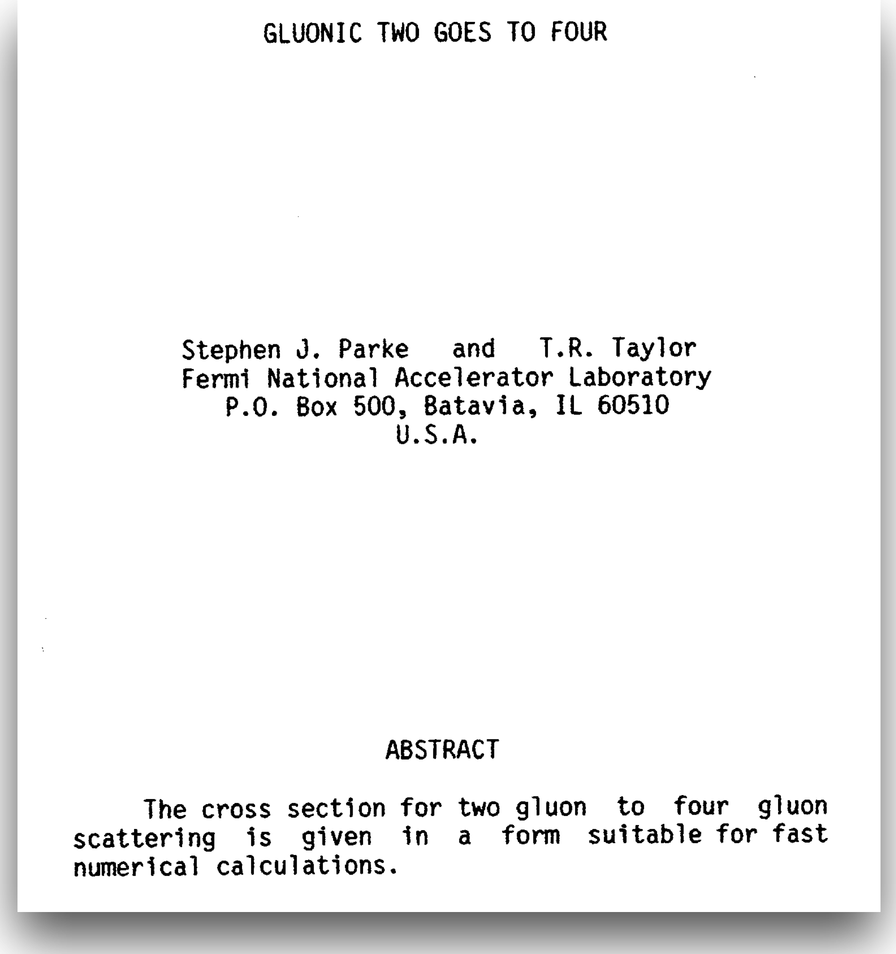
- 1983: Superconducting Super Collider approved
- Energy 40 TeV: many gluons!

- Demand for calculations, next on the list: $gg \to gggg$

Hidden simplicity in scattering amplitudes

Parke, Taylor (1985)

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations
- Paper with 14 pages of result



GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

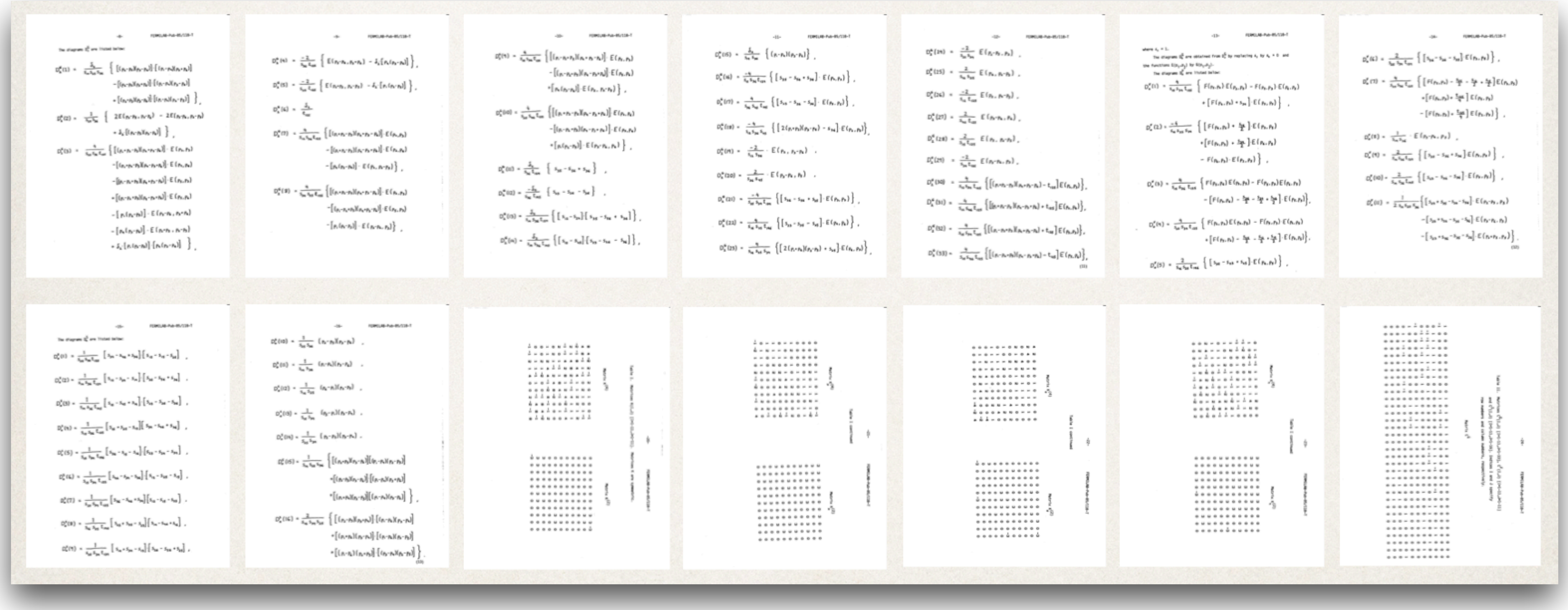
ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

Hidden simplicity in scattering amplitudes

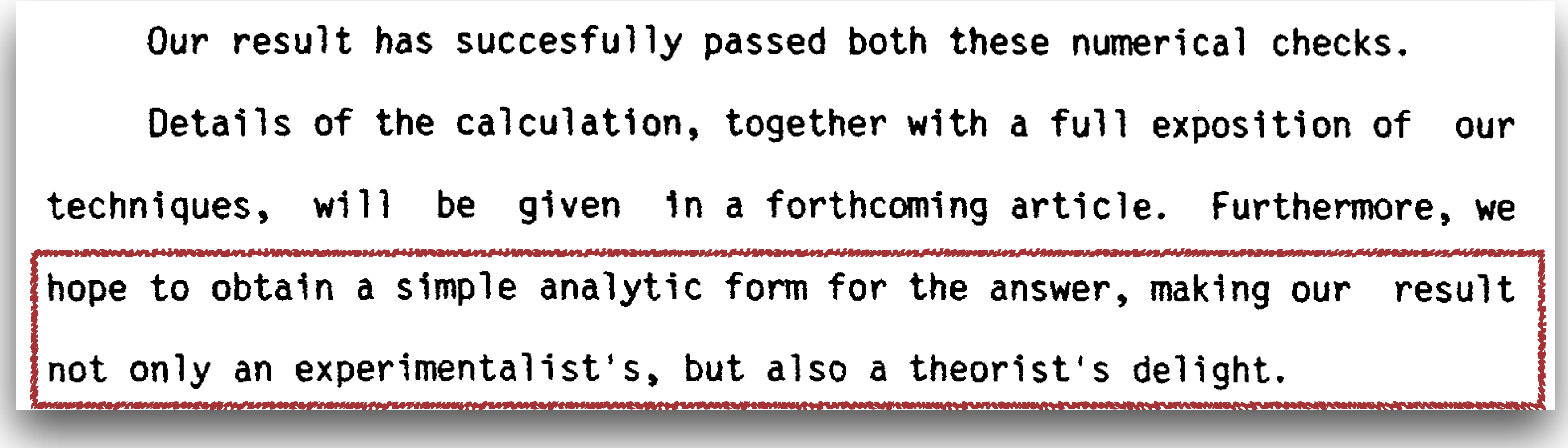
Parke, Taylor (1985)

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations



Hidden simplicity in scattering amplitudes

Parke, Taylor (1985)

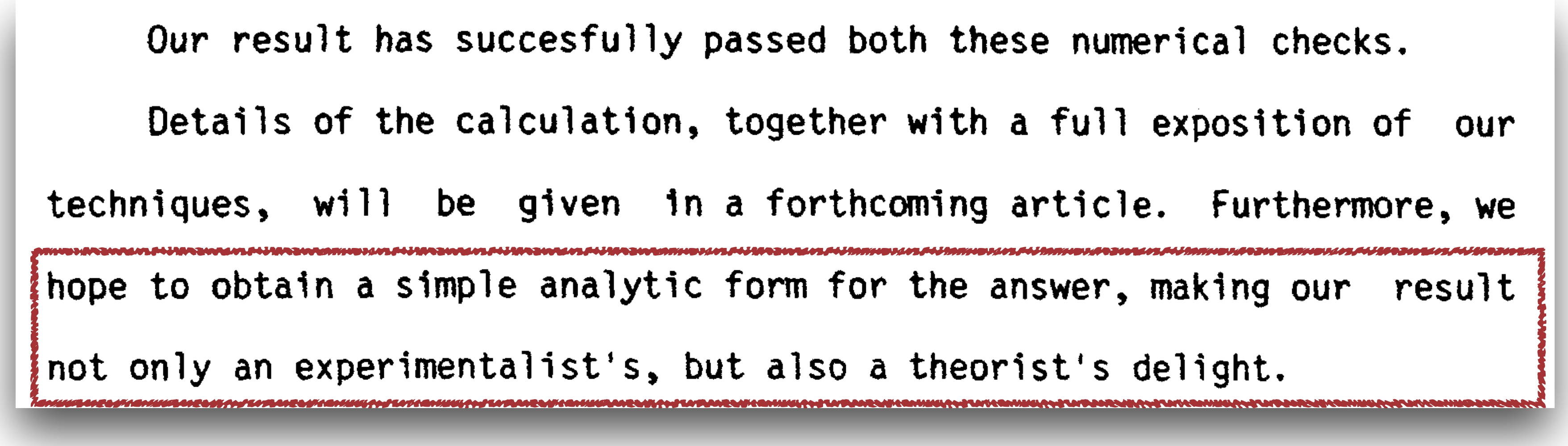


Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

Hidden simplicity in scattering amplitudes

Parke, Taylor (1985)



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- Within a year they realized

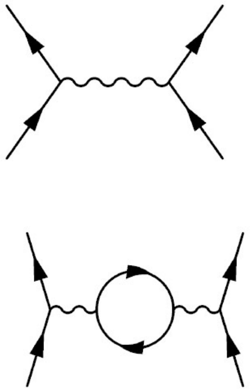
$$|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}$$

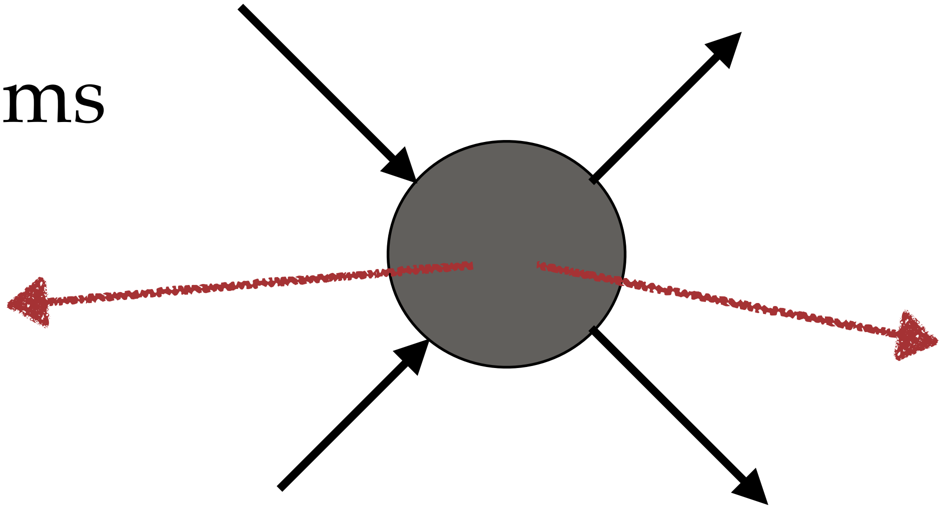
- Final result is much simpler than individual diagrams!

Change of strategy

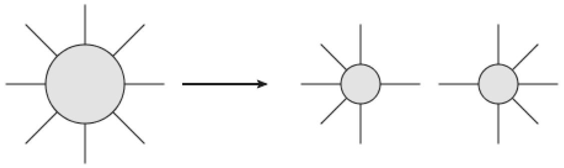
What is the scattering amplitude?

Feynman diagrams





Unique object fixed by physical properties



Was not successful

(1960s)

The Analytic S-Matrix

R.J. EDEN
P.V. LANDSHOFF
D.I. OLIVE
J.C. POLKINGHORNE

Cambridge University Press

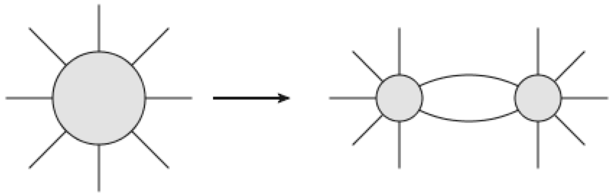
Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory

Life without Feynman diagrams

- New efficient methods of calculations
  - Unitarity methods

Bern, Dixon, Kosower (1990s)



BlackHat collaboration
QCD background for LHC

- Recursion relations

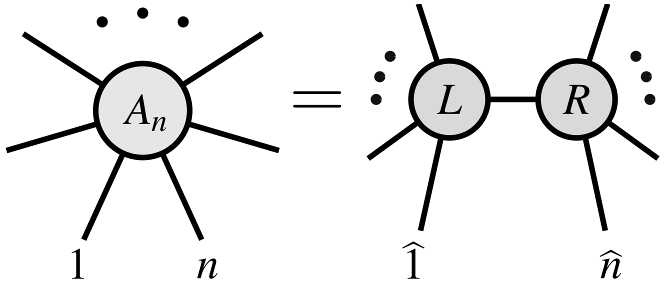
Britto, Cachazo, Feng, Witten (2005)

Cohen, Elvang, Kiermaier (2010)

Arkani-Hamed, Bourjaily, Cachazo,
Caron-Huot, Trnka (2010)

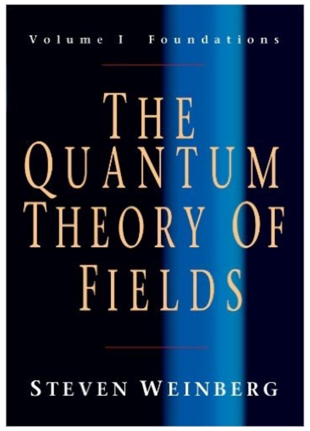
Cheung, Kampf, Novotny,
Shen, Trnka (2015)

Build amplitude
recursively from
simpler amplitudes



	$gg \to 4g$	$gg \to 5g$	$gg \to 6g$
Feynman diagrams	220	2485	34300
Terms in recursion	3	6	20

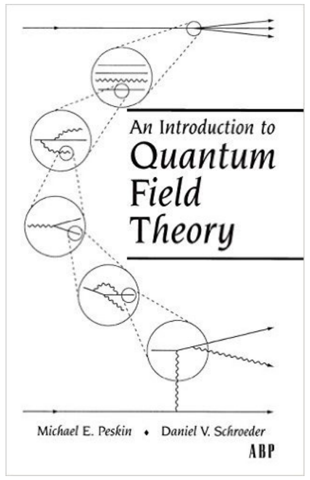
How to define / calculate
the perturbative S-matrix in QFT?



Volume I Foundations

THE QUANTUM THEORY OF FIELDS

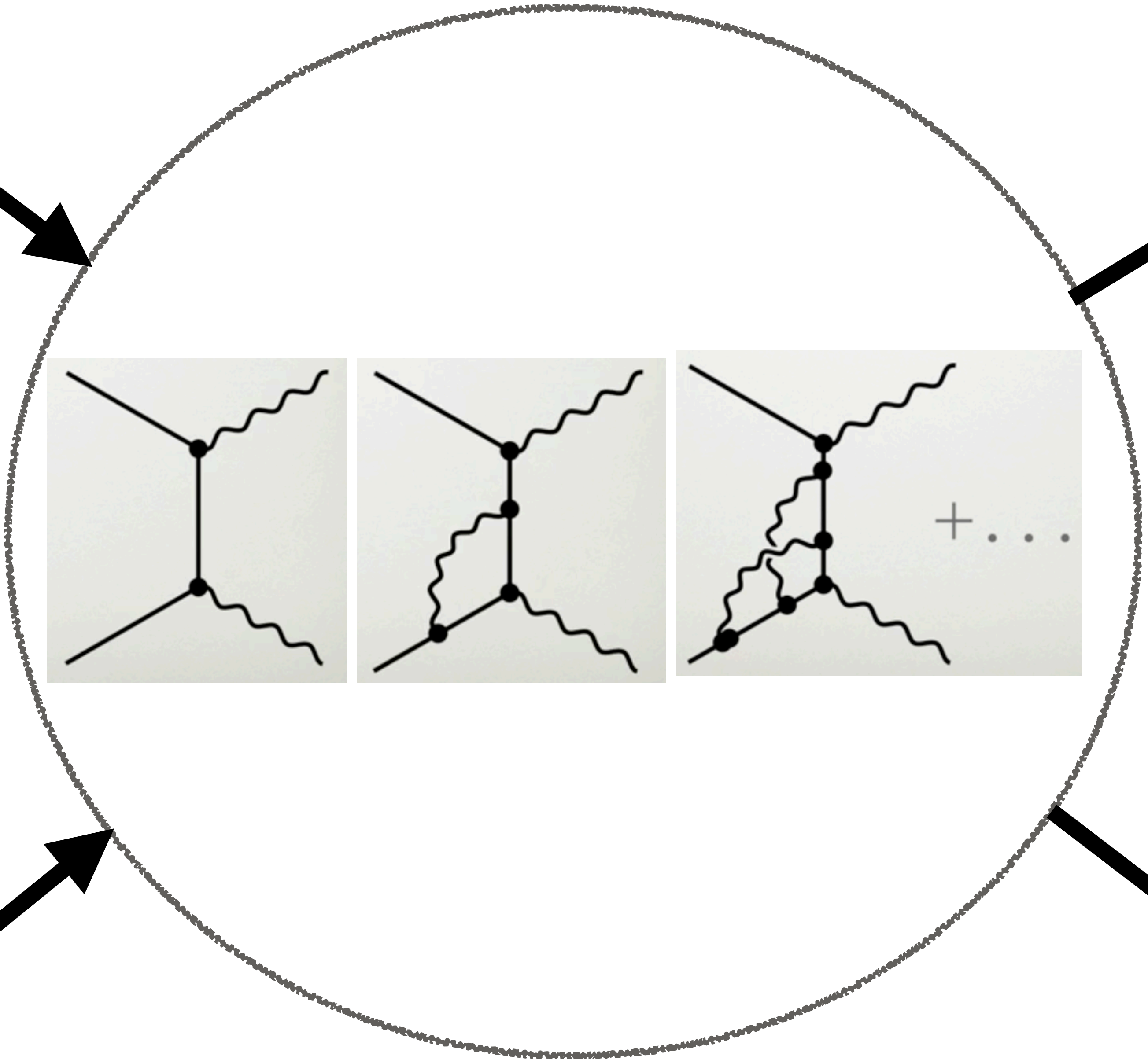
STEVEN WEINBERG



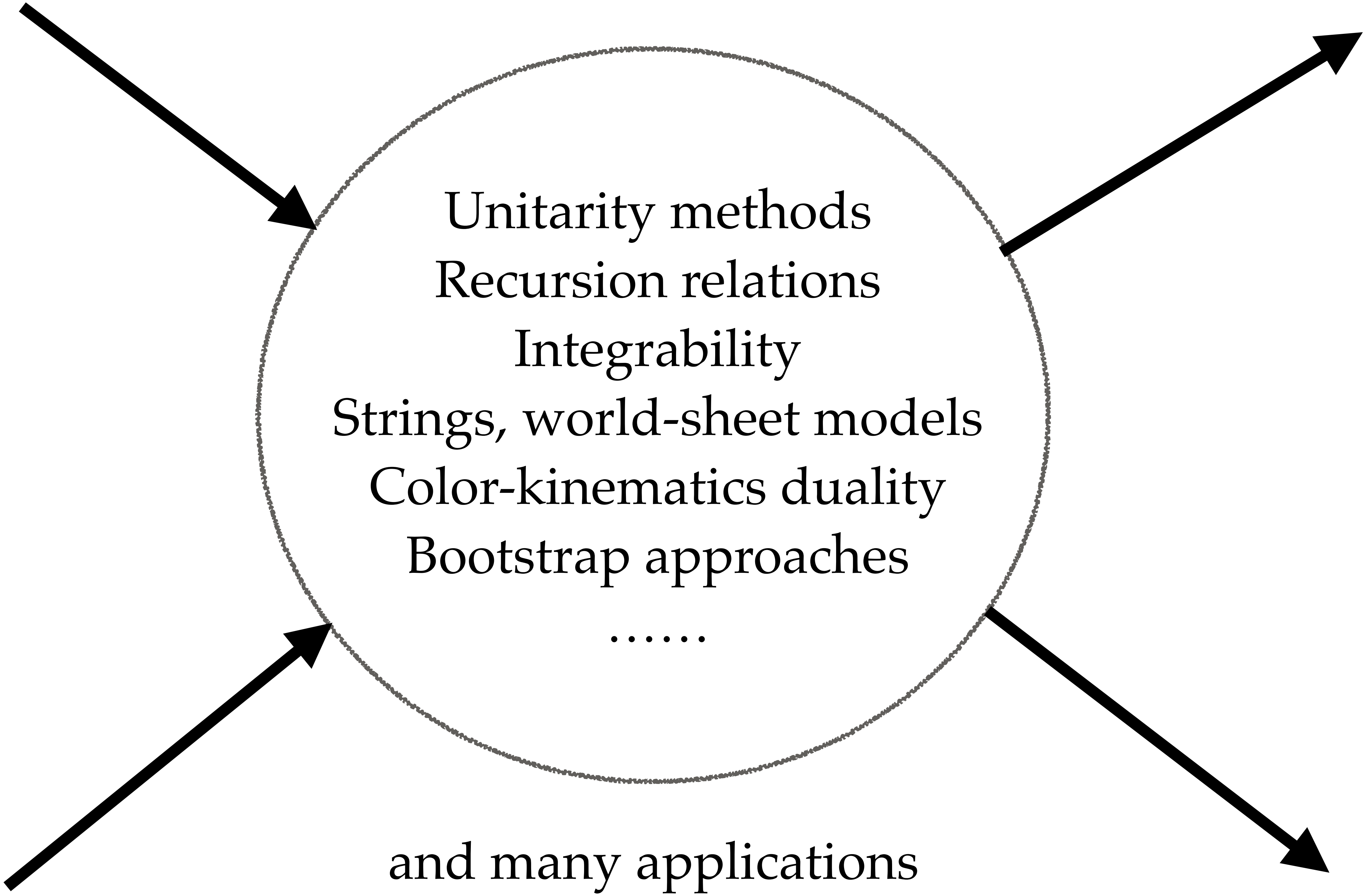
An Introduction to Quantum Field Theory

Michael E. Peskin • Daniel V. Schroeder

ABP



New picture?

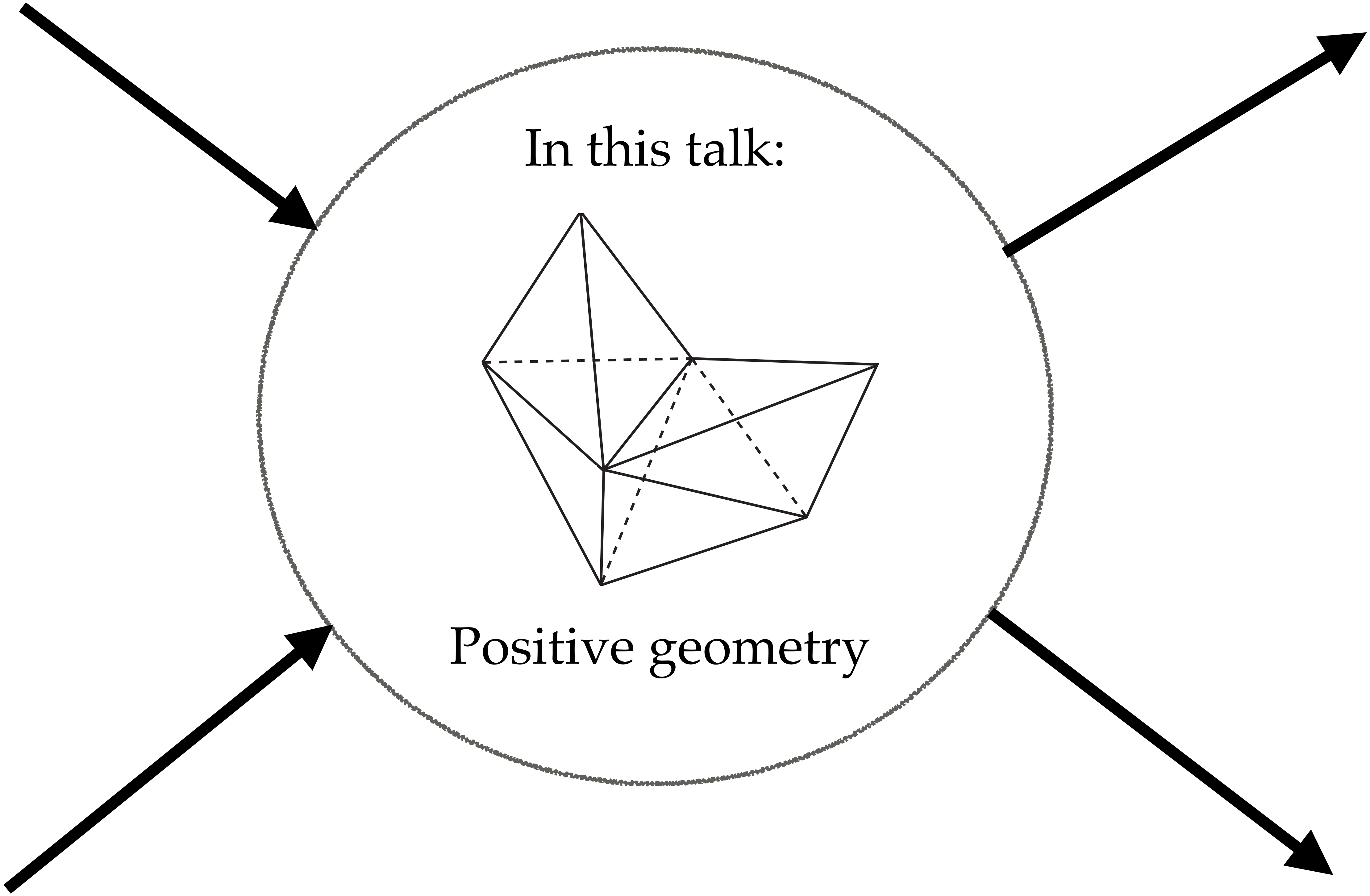


Unitarity methods
Recursion relations
Integrability
Strings, world-sheet models
Color-kinematics duality
Bootstrap approaches

……

and many applications

New picture?



In this talk:

Positive geometry

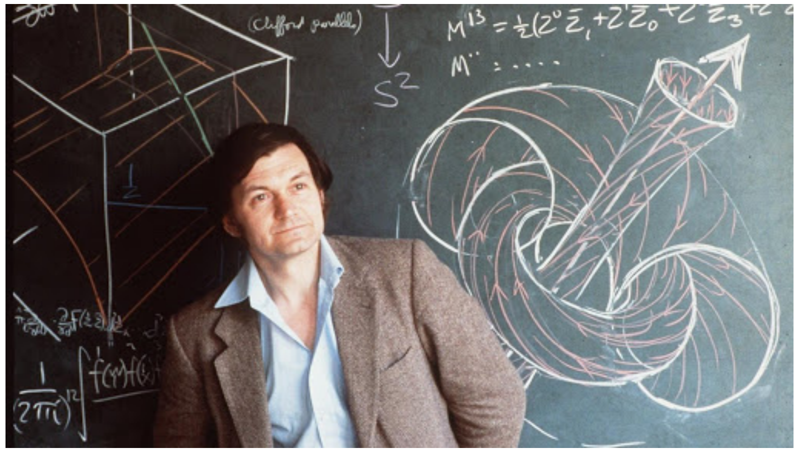
Short story of Amplituhedron

Amplitude as a volume

Hodges (2009)

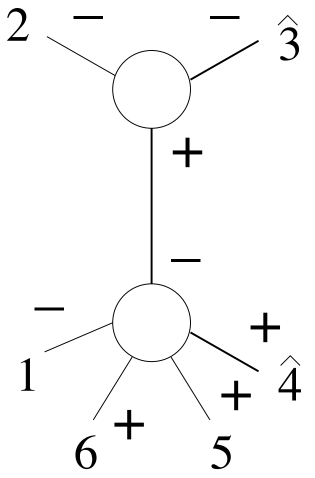
In 2009 Hodges studied recursion relations for gluon amplitudes

He wanted to use twistor variables introduced by Penrose in 1970s in his own attempt for quantum gravity

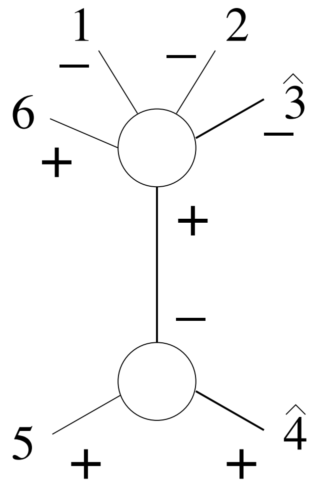


For particular six-gluon amplitude at tree-level he took the result

$A_6 =$



$+$



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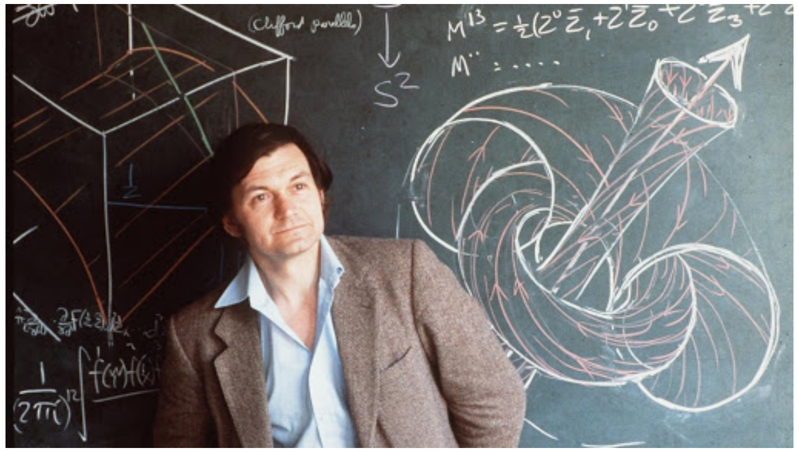


For particular six-gluon amplitude at tree-level he took the result and rewrote using momentum twistor variables

$$A_6 = \frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle} - \frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

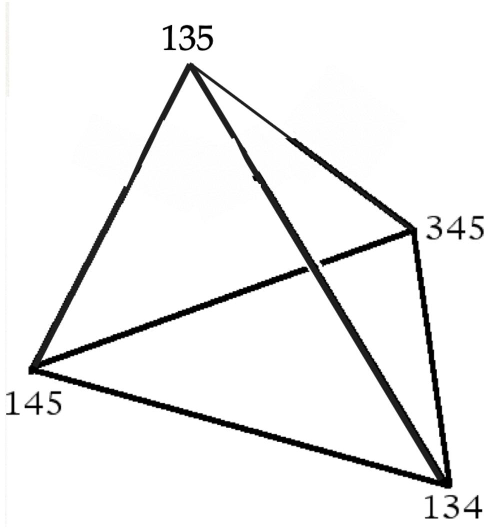
And the expressions look familiar to him

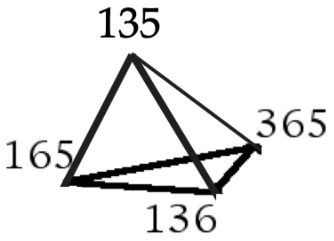
Amplitude as a volume

Hodges (2009)

They are **volumes** of tetrahedra in momentum twistor space!

$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle} \quad - \quad \frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$





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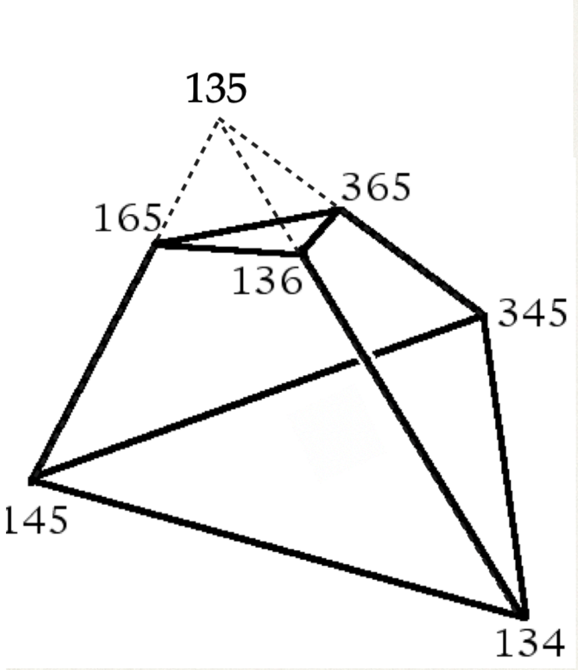

These two pieces subtract, we are triangulating polyhedron

Amplitude as a volume

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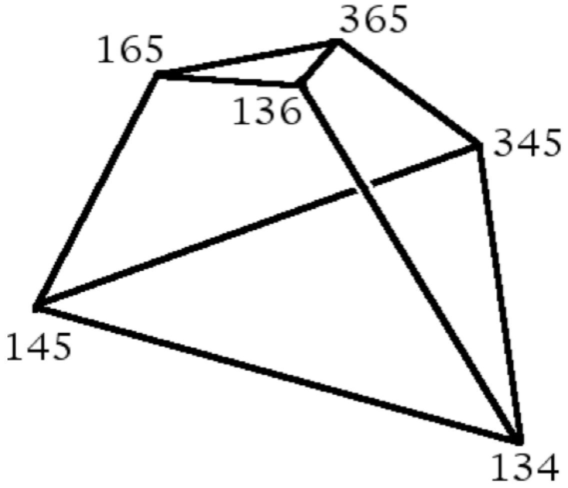
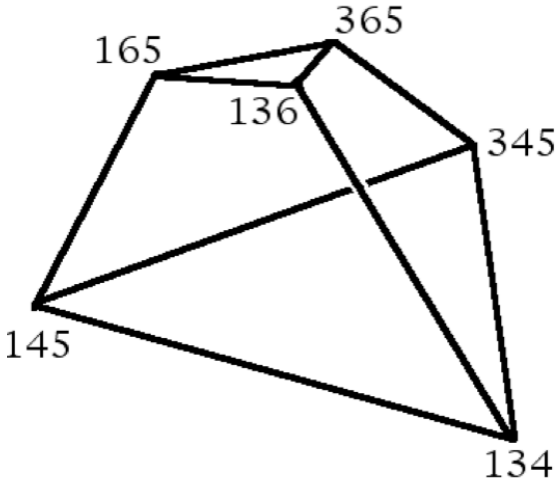


Amplitude is a volume of polyhedron!

Amplitude as a volume

Hodges (2009)



There is some triangulation in terms
220 pieces = Feynman diagrams

This was true for a simplest six-gluon amplitude, but did not seem to work for all tree-level amplitudes, neither loops.

We need “bigger space” to fit all amplitudes there.

The Amplituhedron

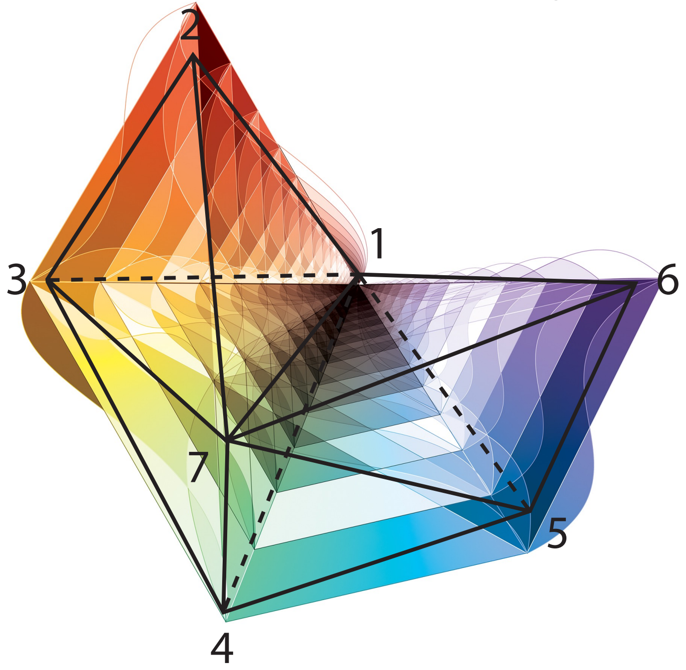
Arkani-Hamed, Trnka (2013), Arkani-Hamed, Thomas, Trnka (2017)

Hodges’ observation was not accidental

Special cases the amplitudes correspond to polyhedra, but the general space is the **Amplituhedron**

Higher-point amplitudes, loops

Multi-dimensional “curvy” spaces

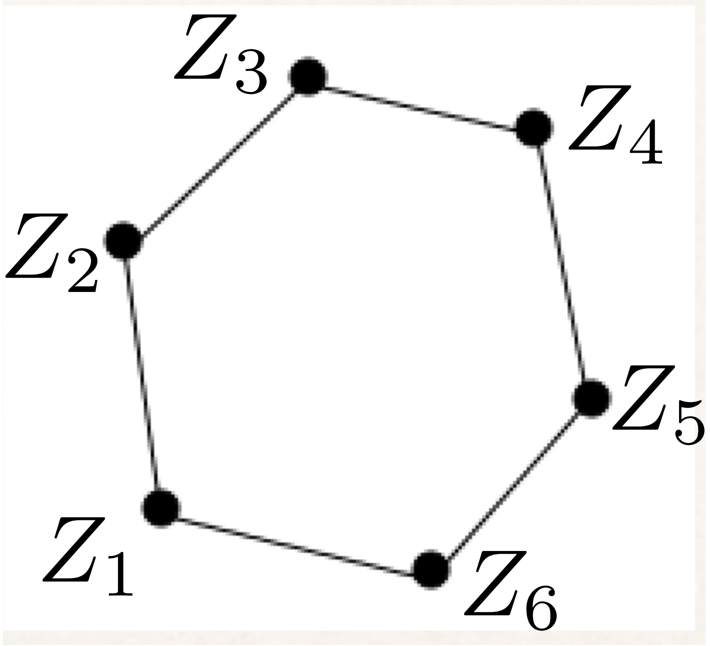


Gluon amplitudes (in planar N=4 SYM) are volumes of the Amplituhedron

The Amplituhedron

Arkani-Hamed, Trnka (2013), Arkani-Hamed, Thomas, Trnka (2017)

Toy example: polygon in the plane - points Z kinematical data



The polygon is a proxy for the Amplituhedron

The Amplituhedron

Arkani-Hamed, Trnka (2013), Arkani-Hamed, Thomas, Trnka (2017)

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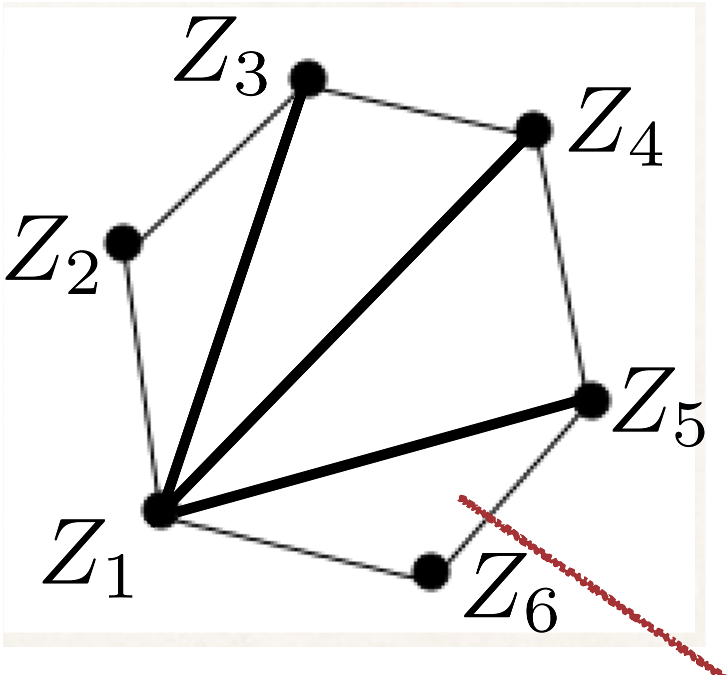
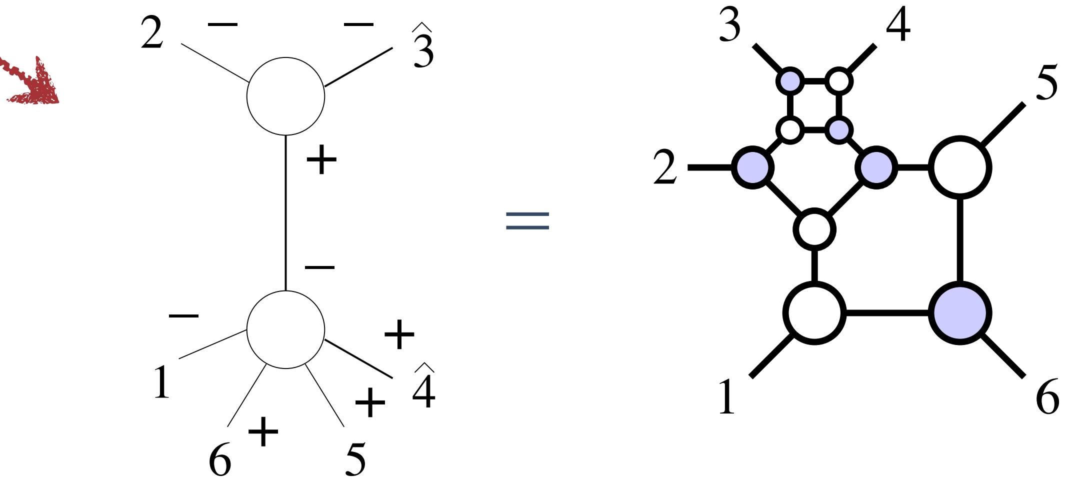


The polygon is a proxy for the Amplituhedron

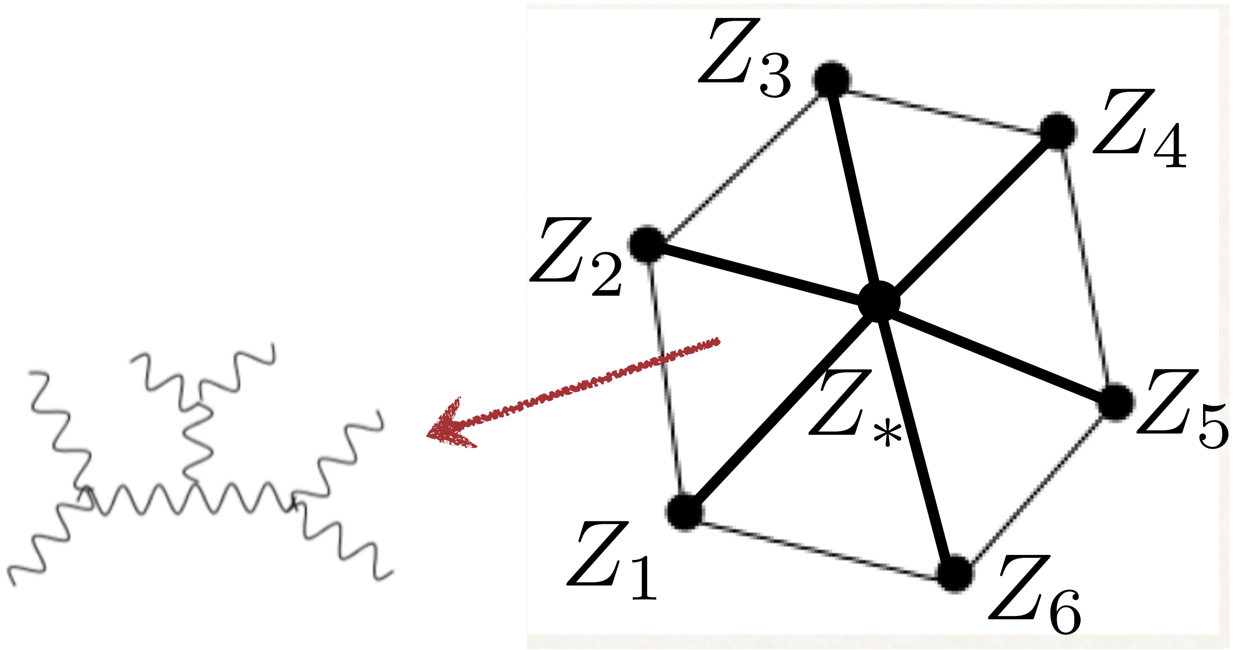
Certain triangulations correspond to the terms in the recursion relations and on-shell diagrams



The Amplituhedron

Arkani-Hamed, Trnka (2013), Arkani-Hamed, Thomas, Trnka (2017)

Toy example: polygon in the plane - points Z kinematical data



The polygon is a proxy for the Amplituhedron

Other triangulations correspond to **Feynman diagrams**

The reference point Z_* is related to the gauge choice

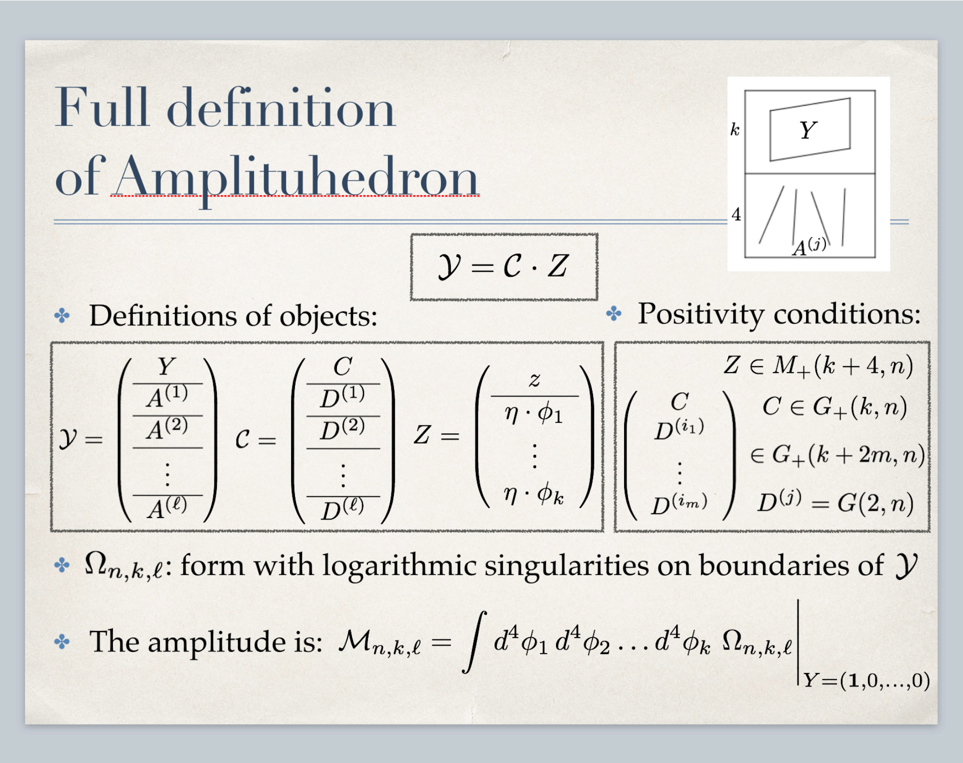
Invariant definition of the "amplitude": **area** of the polygon

The Amplituhedron

Arkani-Hamed, Trnka (2013), Arkani-Hamed, Thomas, Trnka (2017)

Full definition of the Amplituhedron:

geometric region
specified by a set of
inequalities
geometry labeled by n, k, ℓ



Full definition of Amplituhedron

$$\boxed{\mathcal{Y} = \mathcal{C} \cdot Z}$$

- Definitions of objects:

$$\mathcal{Y} = \begin{pmatrix} Y \\ \hline A^{(1)} \\ \hline A^{(2)} \\ \hline \vdots \\ \hline A^{(\ell)} \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} C \\ \hline D^{(1)} \\ \hline D^{(2)} \\ \hline \vdots \\ \hline D^{(\ell)} \end{pmatrix} \quad Z = \begin{pmatrix} z \\ \hline \eta \cdot \phi_1 \\ \vdots \\ \eta \cdot \phi_k \end{pmatrix}$$

- Positivity conditions:

$$Z \in M_+(k+4, n)$$
$$C \in G_+(k, n)$$
$$\begin{pmatrix} C \\ D^{(i_1)} \\ \vdots \\ D^{(i_m)} \end{pmatrix} \in G_+(k+2m, n)$$
$$D^{(j)} = G(2, n)$$

- $\Omega_{n,k,\ell}$: form with logarithmic singularities on boundaries of $\mathcal{Y}$
- The amplitude is: $\mathcal{M}_{n,k,\ell} = \int d^4\phi_1\, d^4\phi_2 \ldots d^4\phi_k\; \Omega_{n,k,\ell}\Big|_{Y=(\mathbf{1},0,\ldots,0)}$

↓

differential volume form on this space:
tree-level amplitudes and loop integrands
in the planar N=4 super Yang-Mills theory

tree-level = QCD
loops = simpler

n number of particles ℓ number of loops k helicity number

Planar integrand in N=4 SYM

N=4 SYM amplitudes

- $\mathcal{N}=4$ superfield

$$\Phi = G_+ + \tilde{\eta}_A \Gamma_A + \frac{1}{2}\tilde{\eta}^A\tilde{\eta}^B S_{AB} + \frac{1}{6}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\overline{\Gamma}^D + \frac{1}{24}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\tilde{\eta}^D G_-$$

- Superamplitudes: $\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}$ k is SU(4) R-charge

↓

Component amplitudes with power $\tilde{\eta}^{4k}$

Contain amplitudes with k negative and (n-k) positive helicity gluons but also many others

Loop amplitudes

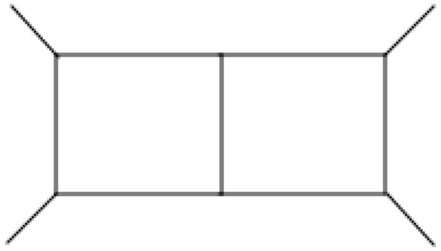
- Tree-level amplitude: rational function of kinematics

- Loop amplitude

$$\mathcal{A} = \sum_{FD} \int \underbrace{\mathcal{I}_j}_{\text{Obtain from Feynman rules}} \, d^4\ell_1 \ldots d^4\ell_L$$

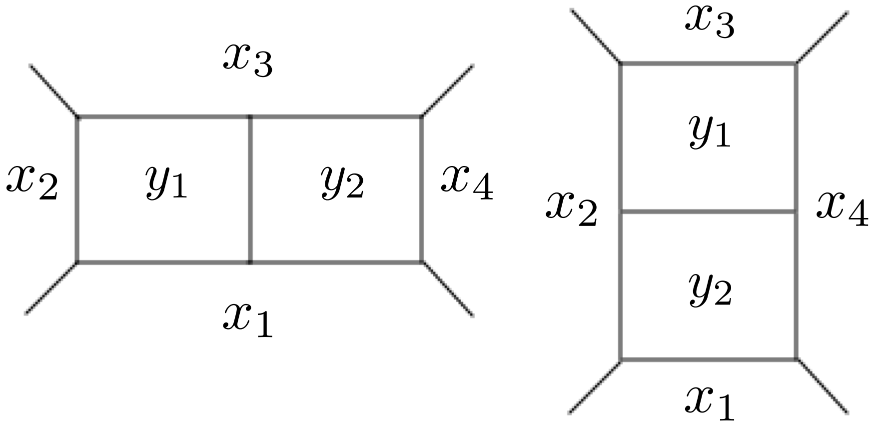
- We can rewrite it as:

$$\mathcal{A} = \sum_k \underbrace{c_k}_{\text{Kinematical coefficients}} \int \underbrace{\mathcal{I}_k}_{\text{Basis integrals}} \, d^4\ell_1 \ldots \ell_L$$



Planar integrand

- Planar (large N) limit: we can define global variables



$$k_1 = (x_1 - x_2)$$
$$\ell_1 = (x_3 - y_1)$$

etc

Dual variables

- Switch integral and the sum:

$$\mathcal{A} = \sum_k c_k \int \mathcal{I}_k \, d^4\ell_1 \ldots \ell_L = \int \mathcal{I} \, d^4\ell_1 \ldots d^4\ell_L$$

Loop integrand

Planar integrand

- Loop integrand is a rational function of momenta
    - Get the final amplitude: still want to integrate

$$A^{L-loop} = \int d^4\ell_1 \dots d^4\ell_L \, \mathcal{I}$$

$$A^{1-loop} \sim \mathrm{Li}_2, \log, \zeta_2$$

$$A^{L-loop} \sim \, ?$$

polylogs, elliptic polylogs,…

IR divergencies

Study the integrand instead

- simpler (rational) function
- many variables (loop momenta)
- properties of the amplitude non-trivially encoded in the integrand

Calculation of the integrand

- Laborious process
    - Feynman diagrams: draw all diagrams, evaluate using Feynman rules — huge number of terms even for low loop order
    - Unitarity methods: construct basis, fix coefficients using cuts — better, still scales badly with L
- New (surpising) definition: positive geometry
    - Provides all-loop order definition, turns physics problem into a mathematical problem

Positive geometry

Positive geometry

- Geometric space defined using a set of inequalities

$$F_k(x_i) \geq 0$$

polynomials   parametrize kinematics

- Define the differential form on this space $\Omega(x_i)$
  - Special form: logarithmic singularities on the boundaries

$$\Omega(x_i) \sim \frac{dx_i}{x_i} \quad \text{near boundary } x_i = 0$$

Simple examples

- Example: 1d interval

$$F(x) = x > 0$$

form: $\Omega = \frac{dx}{x} \equiv \mathrm{dlog}\, x$

$$F_1(x) = x - x_1 > 0$$
$$F_2(x) = x_2 - x > 0$$

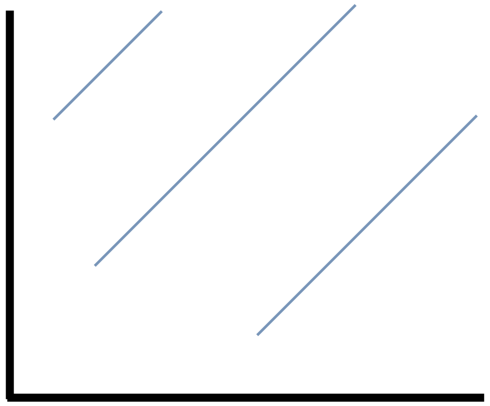
normalization: singularities are unit

$$\Omega = \frac{dx\,(x_1 - x_2)}{(x-x_1)(x-x_2)} = \mathrm{dlog}\left(\frac{x-x_1}{x-x_2}\right)$$

Simple examples

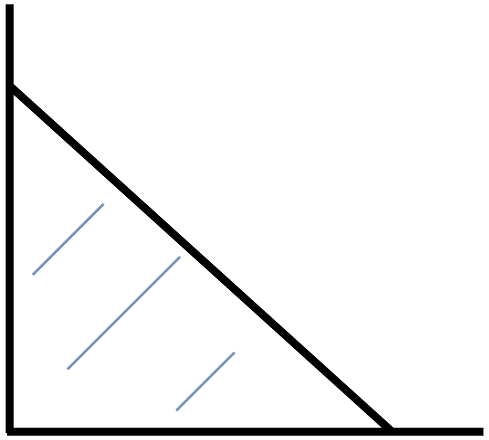
- Example: 2d region

$$x > 0$$
$$y > 0$$



$$x = (0, \infty) \qquad y = (0, \infty)$$

$$\Omega = \frac{dx}{x}\frac{dy}{y}$$



$$x > 0$$
$$y > 0$$
$$1 - x - y > 0$$

$$x = (0, 1 - y) \qquad y = (0, \infty)$$

$$\Omega = \frac{(y-1)\,dx}{x(x+y-1)} \wedge \frac{dy}{y} = \frac{(y-1)\,dx\,dy}{xy(x+y-1)}$$

- General positive geometry: more than just boundaries

Positive Grassmannian

- Consider space of (2x4) matrices modulo GL(2)

Real Grassmannian $G(2,4)$ $\quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$

- Positive Grassmannian $G_+(2,4)$

All (2x2) minors $(ij) > 0$ ⟶ not all of them are boundaries

Shouten identity $\quad (13)(24) = \underbrace{(12)(34) + (14)(23)}_{\text{all positive}}$

↓

product positive

$\boxed{(13),(24) > 0}$ ↙ ↘ $(13),(24) < 0$

only boundaries $(12),(23),(34),(14) = 0$

can not set (13)=0
unless set several
others to zero

Positive geometry for integrand

- Loop integrand in planar N=4 SYM is parameterized by a set of parameters x_i
- All singularities are logarithmic = think about the integrand as the differential form
- Is there a positive geometry that reproduces it?

Amplituhedron

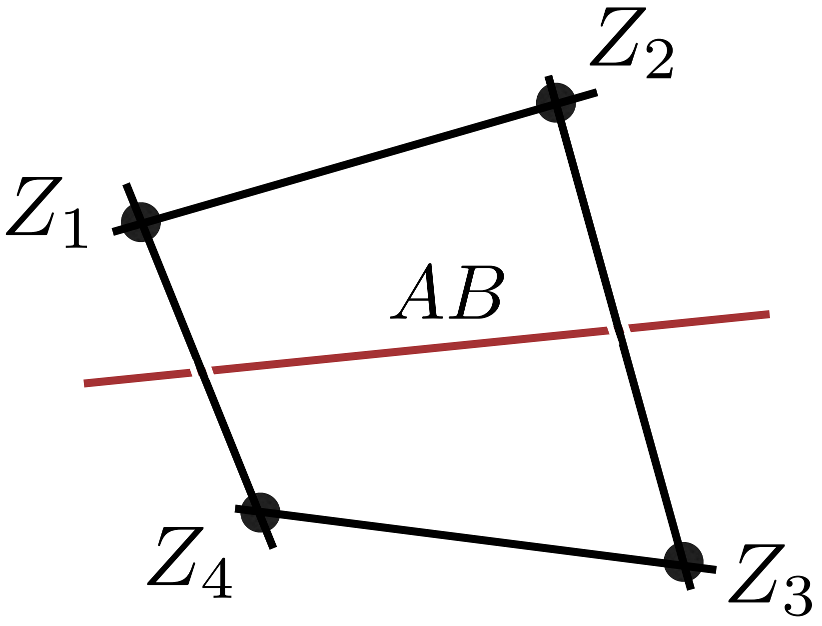
(Arkani-Hamed, JT 2013)

(Arkani-Hamed, Thomas, JT 2017)

Four-point Amplituhedron

- Convenient kinematical variables: momentum twistors

(Hodges, 2009)



external Z_1, Z_2, Z_3, Z_4

points in $\mathbb{P}^3$

loops $(AB)_j$ → lines

- Fixed convex external data: $\langle 1234 \rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d > 0$

- Amplituhedron: configuration space of all lines $(AB)_j$

One-loop Amplituhedron

- One-loop Amplituhedron: configuration of all lines (AB) which satisfy following conditions

$$\langle AB12\rangle, \langle AB23\rangle, \langle AB34\rangle, \langle AB14\rangle > 0, \qquad \langle AB13\rangle, \langle AB24\rangle < 0$$

where $\langle AB12\rangle = \epsilon_{abcd} Z_A^a Z_B^b Z_1^c Z_2^d$

- Convenient parametrization

$$Z_A = Z_1 + xZ_2 + yZ_4 \qquad Z_B = Z_3 - zZ_2 + wZ_4$$

the space reduces to

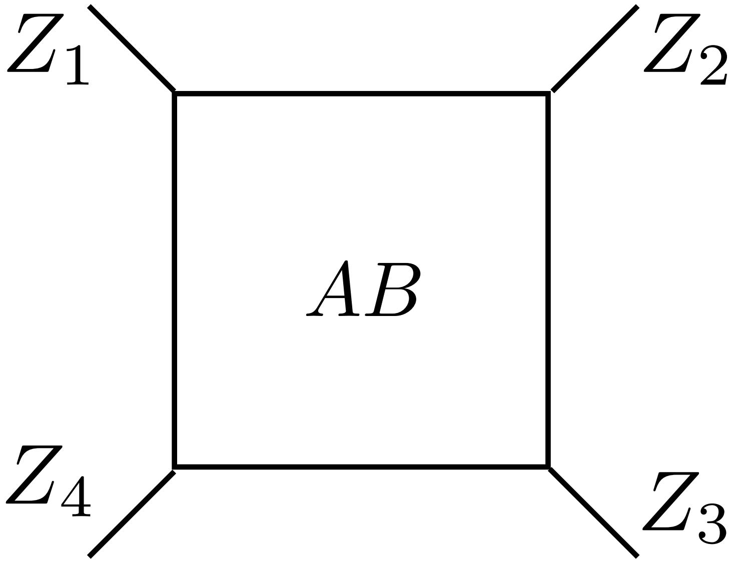
$$x, y, z, w > 0$$

One-loop Amplituhedron

- Logarithmic form on this space

$$\Omega = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{d\mu_{AB}\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

- This corresponds to the one-loop box integral



measure

$$d\mu_{AB} = \langle AB\, d^2A\rangle\langle AB\, d^2B\rangle$$

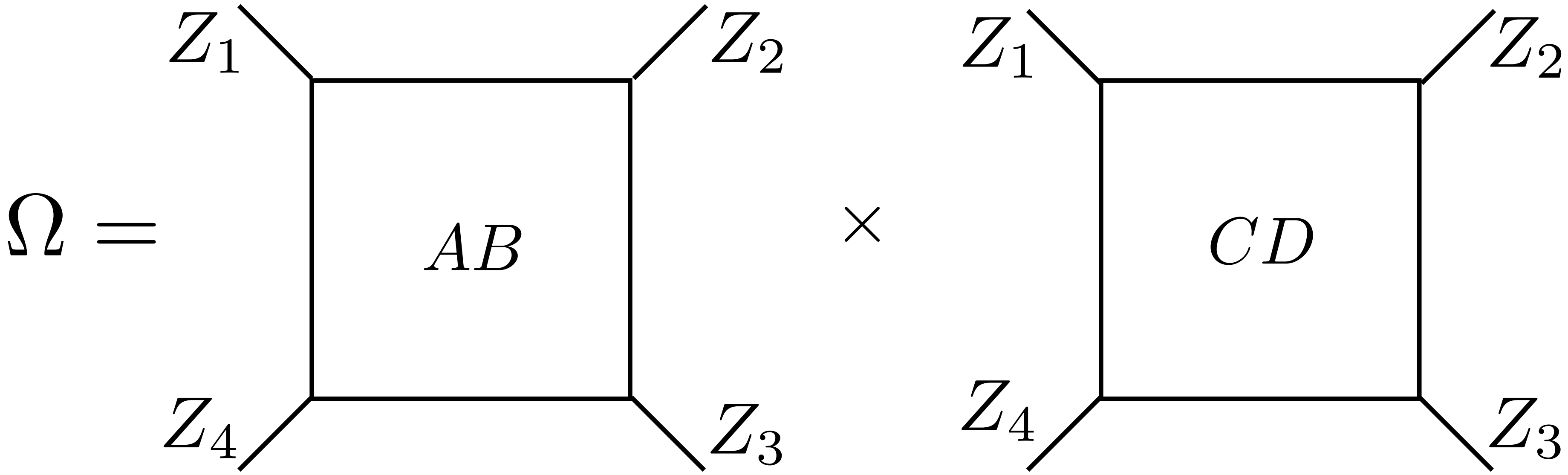
Two-loop Amplituhedron

- Configuration of two lines (AB) and (CD)

each line lives in the one-loop Amplituhedron

$$\langle AB12\rangle, \langle AB23\rangle, \langle AB34\rangle, \langle AB14\rangle > 0, \qquad \langle AB13\rangle, \langle AB24\rangle < 0$$
$$\langle CD12\rangle, \langle CD23\rangle, \langle CD34\rangle, \langle CD14\rangle > 0, \qquad \langle CD13\rangle, \langle CD24\rangle < 0$$

if nothing else is imposed: square of one-loop problem



Two-loop Amplituhedron

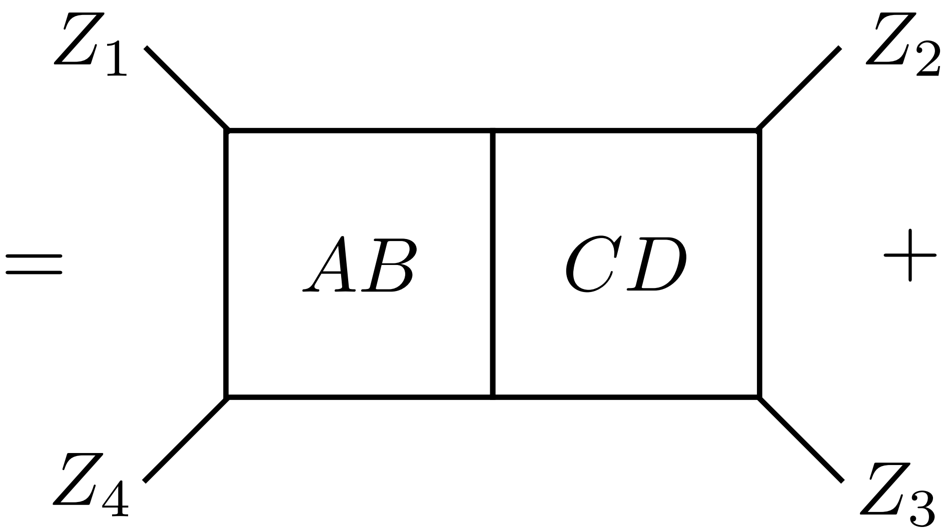
- Impose mutual positivity condition $\langle ABCD\rangle > 0$

$$x_1, y_1, z_1, w_1 > 0 \qquad x_2, y_2, z_2, w_2 > 0$$

$$D_{12} = -(x_1 - x_2)(w_1 - w_2) - (y_1 - y_2)(z_1 - z_2) > 0$$

- Logarithmic form: two-loop integrand

$$\Omega = \frac{x_1 w_2 + x_2 w_1 + y_1 z_2 + y_2 z_1}{x_1 y_1 z_1 w_1 x_2 y_2 z_2 w_2 D_{12}} =$$



$+\,\mathrm{cycl}$

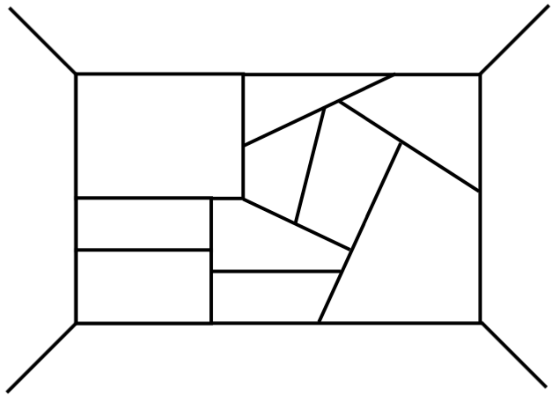
L-loop Amplituhedron

- At L-loops we have configuration of L lines $(AB)_i$
  - each line in the one-loop Amplituhedron
  - for any two lines we impose $\langle (AB)_i (AB)_j \rangle > 0$
- In our usual parametrization

$$x_i, y_i, z_i, w_i > 0 \qquad D_{ij} = -(x_i - x_j)(w_i - w_j) - (y_i - y_j)(z_i - z_j) > 0$$

This is a very complicated space and captures the complexity of L-loop integrand

$\Omega =$



$+$ many others

L-loop Amplituhedron

- Big open problem: triangulation

solving the problem of 4pt scattering to all loop orders
= exact amplitude

- We can calculate certain singularities = cuts to all loop orders, but not amplitudes yet

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Langer, Srikant, JT 2018) (Diaz, Heslop 2021)

- We focus on a simpler problem instead where we can solve the problem to all loops and even integrate! (using differential equation)

Logarithm of the amplitude and IR finiteness

BDS Ansatz

(Bern, Dixon, Smirnov, 2005)

- Dual conformal symmetry is extremely restrictive
- The kinematical structure of 4pt and 5pt amplitudes are **fixed to all loops** up to constant factors

$$M_n = \sum_{L=0}^{\infty} g^{2L} M_n^{(L)}(\epsilon) = \exp\left[\sum_{\ell=1}^{\infty} g^{2\ell}\left(f^{(\ell)}\underbrace{(\epsilon)M_n^{(1)}}(\ell\epsilon) + C^{(\ell)} + \mathcal{O}(\epsilon)\right)\right]$$

n>6: dual conformal invariant IR
finite remainder function $R_n^{(\ell)}$

leading $\frac{1}{\epsilon^2}$ behavior

mild divergence
for logarithm

$$\ln M = \frac{h(g)}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

’t Hooft coupling
$g^2 \equiv g_{\mathrm{YM}}^2 N/(16\pi^2)$

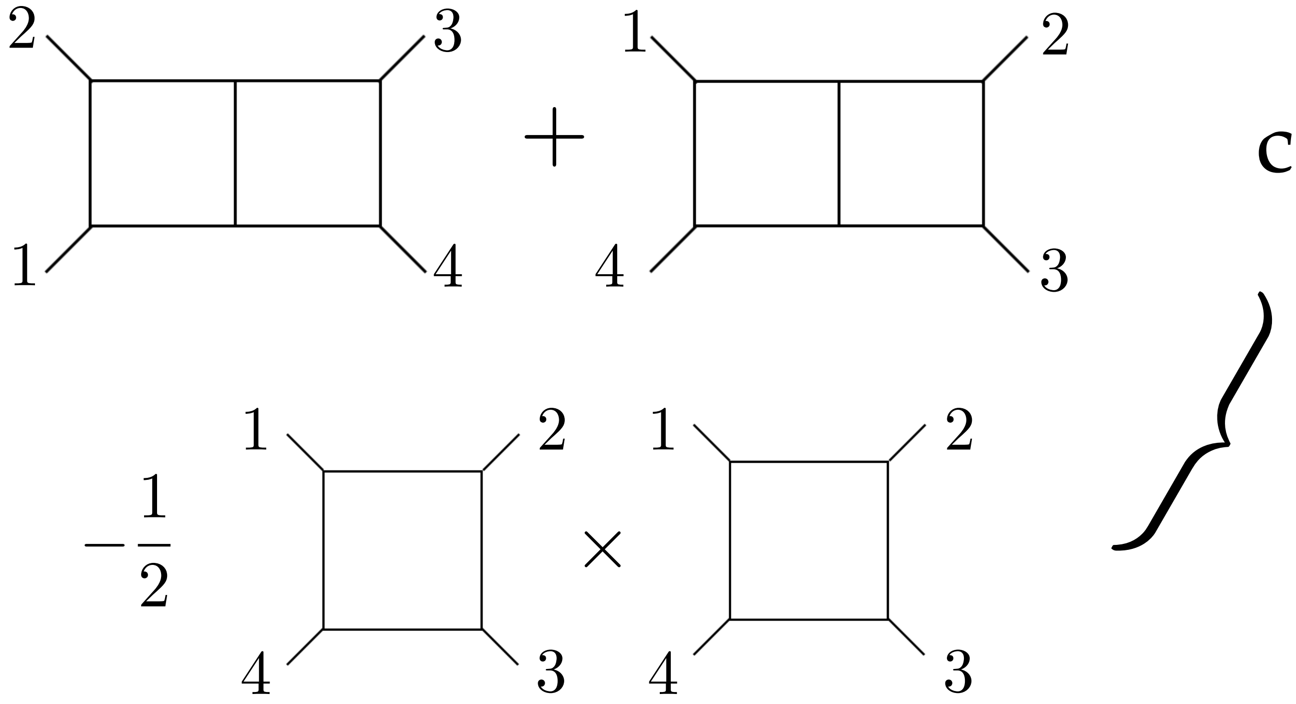
Logarithm of the amplitude

- Expand logarithm in terms of combinations of amplitudes

$$\ln M = \ln\Big(1 + g^2 M^{(1)} + g^4 M^{(2)} + g^6 M^{(2)} + \ldots\Big)$$

$$= g^2 M^{(1)} + g^4 \Big[\underbrace{M^{(2)} - \frac{1}{2}(M^{(1)})^2}_{\ln M_2}\Big] + g^6 \Big[M^{(3)} - \overbrace{M^{(2)} M^{(1)}}^{\ln M_3} + \frac{1}{3}(M^{(1)})^3\Big] + \ldots$$

each term $1/\epsilon^6$ divergent,
combination only $1/\epsilon^2$ divergent



$\boxed{\mathcal{I}_L : \text{integrand for } \ln M_L}$

cut structure encodes
mild IR divergencies

Integrating the logarithm

- Property of $\mathcal{I}_L$ in collinear regions: to generate IR divergence need to integrate **over all loop momenta**

$$\begin{tikzcd}
\mathcal{I}_L(\ell_1,\ell_2,\ldots,\ell_L) \arrow[d, "\text{integrate over } \ell_L"] & \text{rational function} \arrow[l] \\
\mathcal{I}_L(\ell_1,\ell_2,\ldots,\ell_{L-1}) \arrow[d, "\text{integrate over } \ell_{L-1}"] & \\
\vdots \quad\quad \vdots \arrow[d, "\text{integrate over } \ell_2"] & \\
\mathcal{I}_L(\ell_1) \arrow[d, "\text{integrate over } \ell_1"] & \\
\ln M_L & \text{IR divergent} \arrow[l]
\end{tikzcd}$$

all are IR finite

Closest we can get from the integrand to the amplitude without dealing with IR divergencies and regulators (we can stay in D=4)

Defining $\mathcal{F}(g,z)$

- The L-loop object is a function of a single cross-ratio

$$\mathcal{F}_L(z) \equiv \mathcal{I}_L(AB) \longleftarrow$$

interesting trascendental function,
depends on a single cross-ratio

dictated by the
dual conformal symmetry

$$z = \frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB23\rangle\langle AB14\rangle}$$

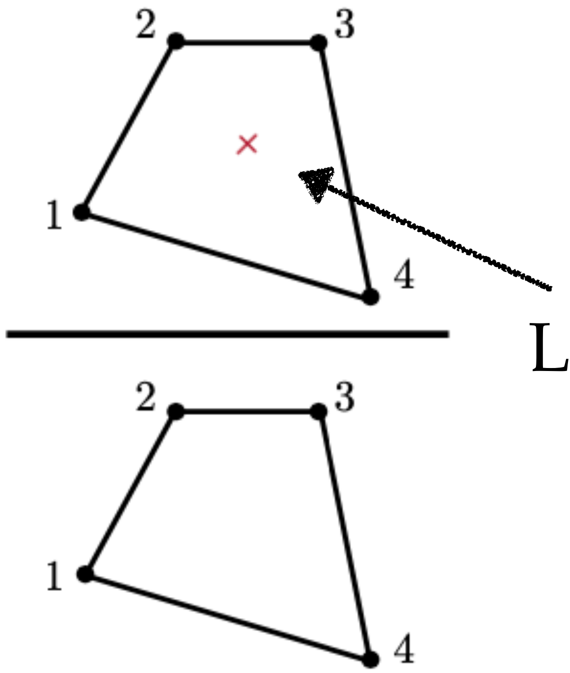
- Define a non-perturbative IR finite object

$$\boxed{\mathcal{F}(g,z) = \sum_{L=1}^{\infty} g^{2L}\mathcal{F}_L(z)}$$

Relation to Wilson loops

(Englund, Roiban, 2011) (Alday, Buchbinder, Tseytlin, 2011)

- Same object appeared earlier in the context of Wilson loops



Lagrangian insertion

$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{1}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} F(g;z)$$

$$F(g;z) = -g^2 \mathcal{F}(g,z) \qquad z = \frac{x_{20}^2 x_{40}^2 x_{13}^2}{x_{10}^2 x_{30}^2 x_{24}^2}$$

$$x_{ij} = (x_i - x_j)^2$$

- **weak coupling**: expansion in g^2

(Alday, Heslop, Sikorowski, 2012) (Alday, Henn, Sikorowski, 2013)

$$\mathcal{F}(g,z) = 1 + g^2(\log^2 z + \pi^2) + \ldots$$

- **strong coupling**: expansion in $1/g$ (string tension)

(Alday, Buchbinder, Tseytlin, 2011)

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z) + (z+1)\log z\right] + \cdots$$

exact result for $\mathcal{F}(g,z)$ too hard to obtain

Cusp anomalous dimension

- After integrating $\mathcal{F}(g,z)$ we recover $\ln M$

leading IR divergence: **cusp anomalous dimension**

$$\ln M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma_{\text{cusp}}^{(L)}}{(L\epsilon)^2} + \mathcal{O}\left(1/\epsilon\right) \longrightarrow \boxed{\Gamma_{\text{cusp}}(g) = \sum_{L\geq 1} g^{2L} \Gamma_{\text{cusp}}^{(L)}}$$

exact result from integrability

(Beisert, Eden, Staudacher, 2006)

We can extract $\Gamma_{\text{cusp}}(g)$ from $\mathcal{F}(g,z)$

(Alday, Henn, Sikorowski, 2013)

(Henn, Korchemsky, Mistlberger, 2019)

(Arkani-Hamed, Henn, JT, 2021)

$$g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}(g) = -\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi \, F(g, z = e^{i\phi})$$

Our goal

- Focus on $\mathcal{F}(g, z)$: calculate to all loops for a generic kinematics — exact in g, z : too hard….

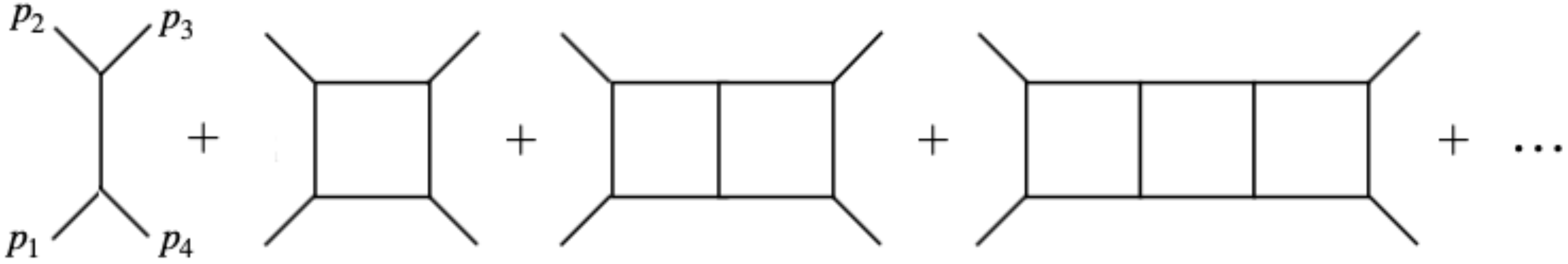
- Amplituhedron picture for loop integrand: suggests a new type of expansion, based on the "number of loops in the loop space"— an approximation exact in g, z

- Calculate the leading term, expand at strong coupling, so far completely inaccessible using our methods

Ladder resummation

- Our simplest case we will study later is somewhat analogous to the ladder resummation

(Broadhurst, Davydychev, 2010)



- Integral representation: strong coupling expansion exponentially suppressed

$$F(g,z) = \mathcal{O}(e^{-g})$$

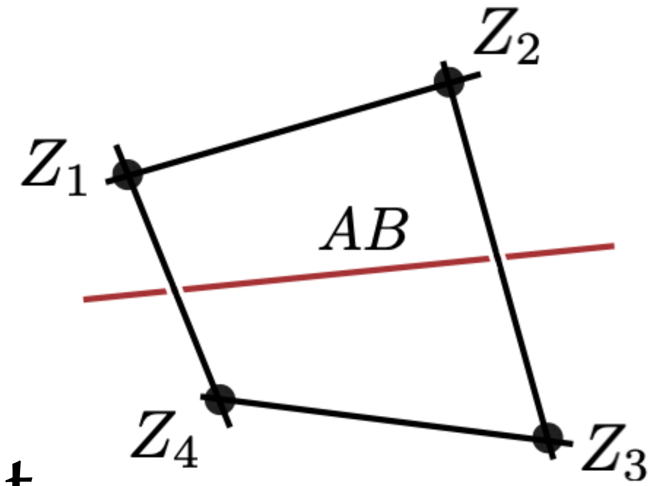
Negative Amplituhedron geometry

(Arkani-Hamed, Henn, JT 2021)

Four-point Amplituhedron

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

- Configuration of L lines in momentum twistor space



Each line: 4 positive coordinates

$$x_i, y_i, z_i, w_i > 0$$

(parametrize loop momentum)

For any pair we get a quadratic constraint

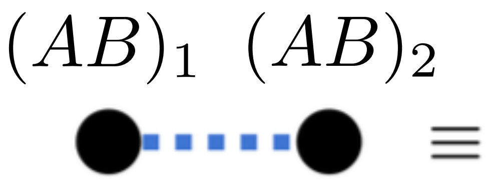
$$\langle (AB)_i (AB)_j \rangle = -(x_i - x_j)(w_i - w_j) - (y_i - y_j)(z_i - z_j) > 0$$

- The canonical dlog form on this space

  is a 4pt L-loop integrand

Graphical notation

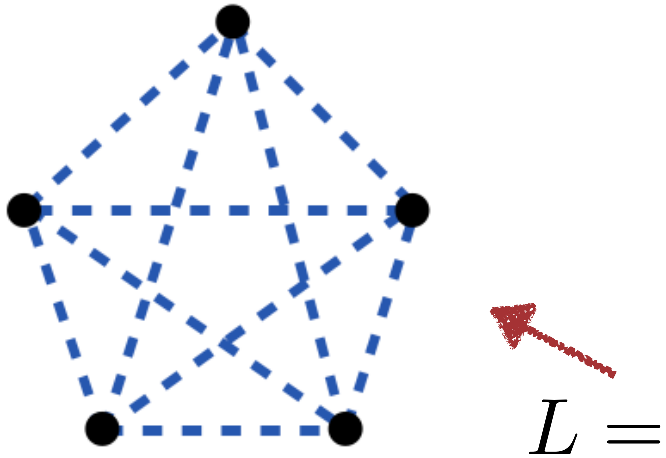
- We introduce a graphical notation:
  - vertex: loop line $(AB)_i$
  - blue dashed link: mutual positivity condition $\langle (AB)_i (AB)_j \rangle > 0$
- We denote the dlog form on the two-loop space



$$\equiv \Omega(\mathcal{A}_2)$$

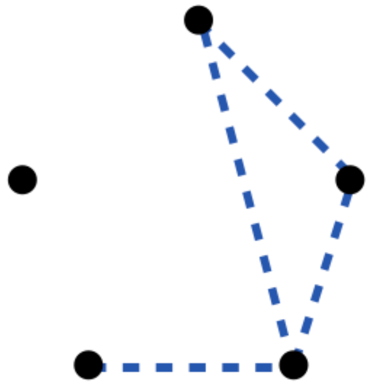
- The L-loop integrand dlog form: complete graph

$$\Omega(\mathcal{A}_L) =$$



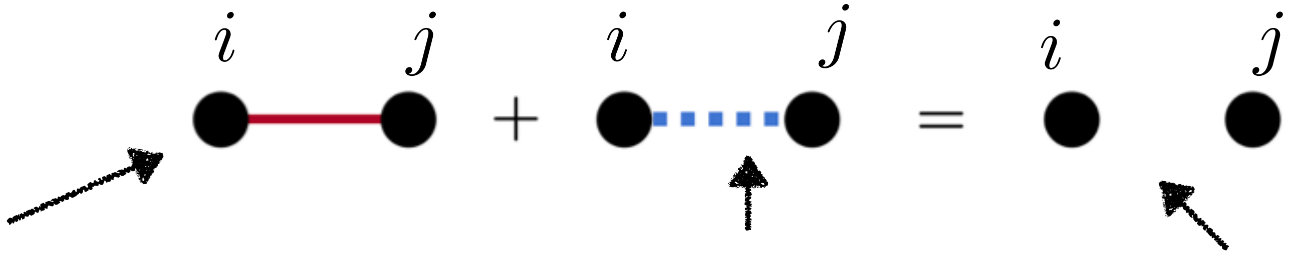
$L = 5$

we can have a simpler positive geometry with fewer links



Negative geometries

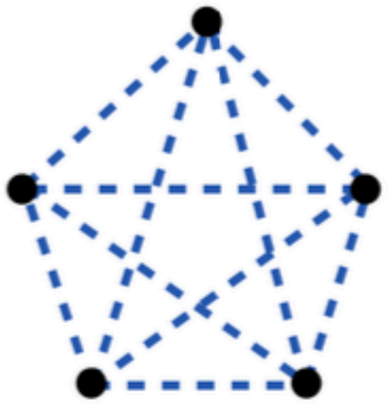
- Replace positive "links" by negative



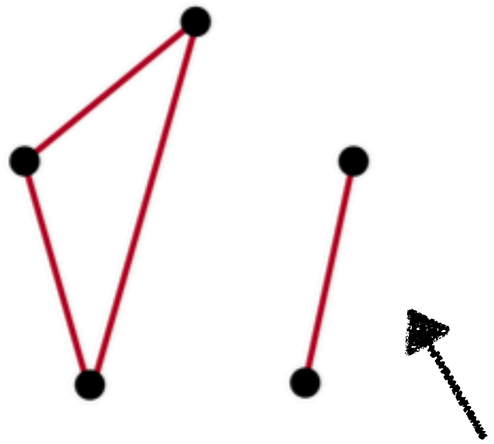
$\langle (AB)_i (AB)_j \rangle < 0$ $\langle (AB)_i (AB)_j \rangle > 0$ no mutual condition

- New formula for L-loop integrand dlog form:

$$\Omega(\mathcal{A}_L) =$$



$$= \sum_{\text{all}\, G} (-1)^{E(G)}$$



sum over all graphs

differential form for negative geometry

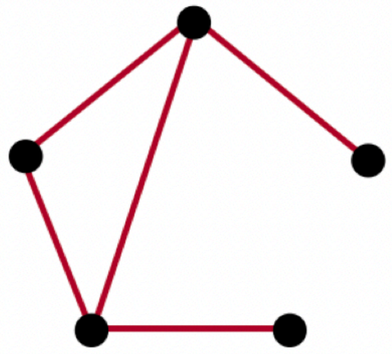
Exponentiation

- We can now write a formal sum over all loops

$$\Omega(g) = \sum_{L=0}^{\infty} (-g^2)^L \Omega(\mathcal{A}_L) \qquad \text{where } \Omega(\mathcal{A}_0) = 1$$

- The formula for $\Omega(g)$ exponentiates

$$\Omega(g) = \exp\left\{ \sum_{\substack{\text{all connected} \\ \text{graphs } G}} (-1)^{E(G)} \, (g^2)^L \qquad \right\}$$

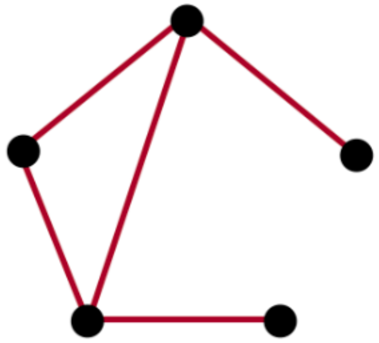


- We take the logarithm of both sides and expand in g

Geometry of $\mathcal{F}(g,z)$

- The L-loop logarithm: **connected graphs** with L vertices

$$\log \Omega(g)\Big|_{(-g^2)^L} = \widetilde{\Omega}_L = \sum_{\substack{\text{all connected}\\ \text{graphs } G\\ \text{with } L \text{ vertices}}} (-1)^{E(G)}$$

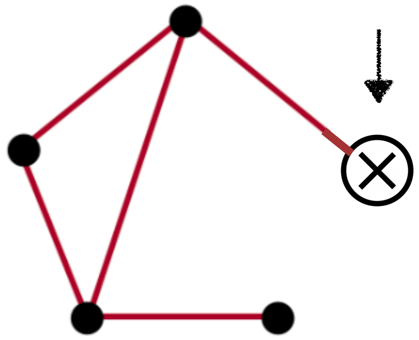


Manifest IR behavior: each term only $\dfrac{1}{\epsilon^2}$ divergent after integration

- Freezing one of the loops:

$$\mathcal{F}_L(g,z) = \sum_{\substack{\text{all connected}\\ \text{graphs } G\\ \text{with } L \text{ vertices}}} (-1)^{E(G)} \frac{1}{S}$$

symmetry factor



frozen

Integrate over all loops except one

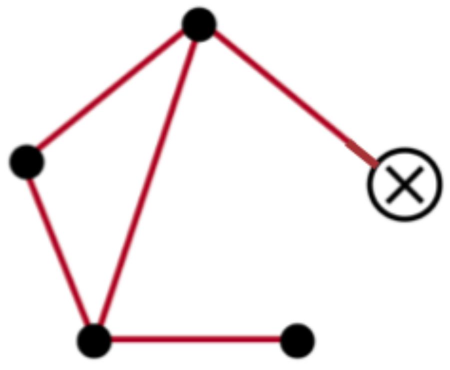
↓

Manifest IR finiteness

Geometry of $\mathcal{F}(g, z)$

new expansion

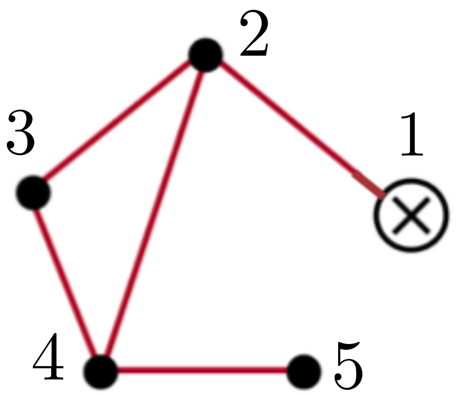
$$\mathcal{F}_L(g,z) = \sum_{\substack{\text{all connected}\\ \text{graphs } G\\ \text{with } L \text{ vertices}}} (-1)^{E(G)} \, \frac{1}{S}$$



L = number of vertices in the diagram

- each term represents a negative geometry

one of 5-loop contributions



$$(AB)_i \in \mathcal{A}^{(1)}$$

$$\langle (AB)_1 (AB)_2 \rangle < 0$$

$$\langle (AB)_2 (AB)_3 \rangle < 0$$

$$\langle (AB)_2 (AB)_4 \rangle < 0$$

$$\langle (AB)_3 (AB)_4 \rangle < 0$$

$$\langle (AB)_4 (AB)_5 \rangle < 0$$

not Amplituhedron, some other geometry

- then we integrate over $(AB)_2, (AB)_3, (AB)_4, (AB)_5$
- result is a IR finite function of the cross-ratio z
- no obvious physical interpretation

Low loop examples

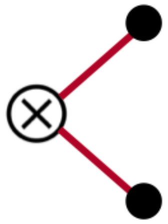
Tree-level

$$\otimes = 1$$

One-loop

$$= \left[\pi^2 + \log^2 z\right]$$

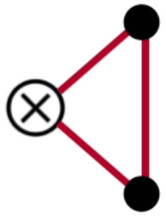
Two-loops



$$= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$$

$$= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$$

simple “tree” geometries



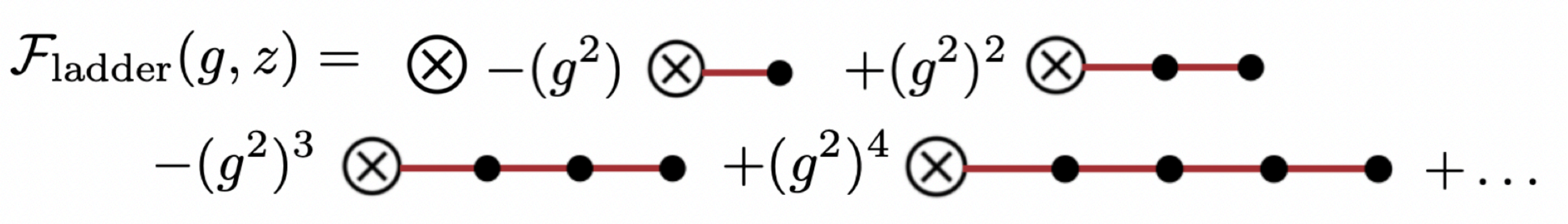
$$\begin{aligned} = &-\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right] \\ &+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2 \\ &- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18} \end{aligned}$$

complicated “one-loop” geometry

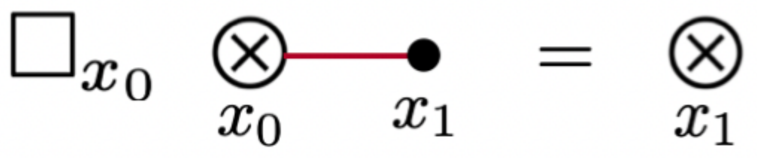
“Ladder” expansion

- To determine the integrand and integrate an arbitrary negative geometry form is complicated

- We approximate $\mathcal{F}(g,z)$ by the simplest geometries



- Property of the integrand: Laplace operator acting on subgraph



in dual variables

Closed formula for “ladders”

- Applying this operation on $\mathcal{F}_{\text{ladder}}(g,z)$ we get

$$\frac{1}{2}(z\partial z)^2\mathcal{F}_{\text{ladder}}(g,z)+g^2\mathcal{F}_{\text{ladder}}(g,z)=0$$

Liouville-type equation

which is solved by

$$\boxed{\mathcal{F}_{\text{ladder}}(g,z)=\frac{\cos(\sqrt{2}g\log z)}{\cosh(\sqrt{2}g\pi)}}$$

satisfying certain boundary conditions

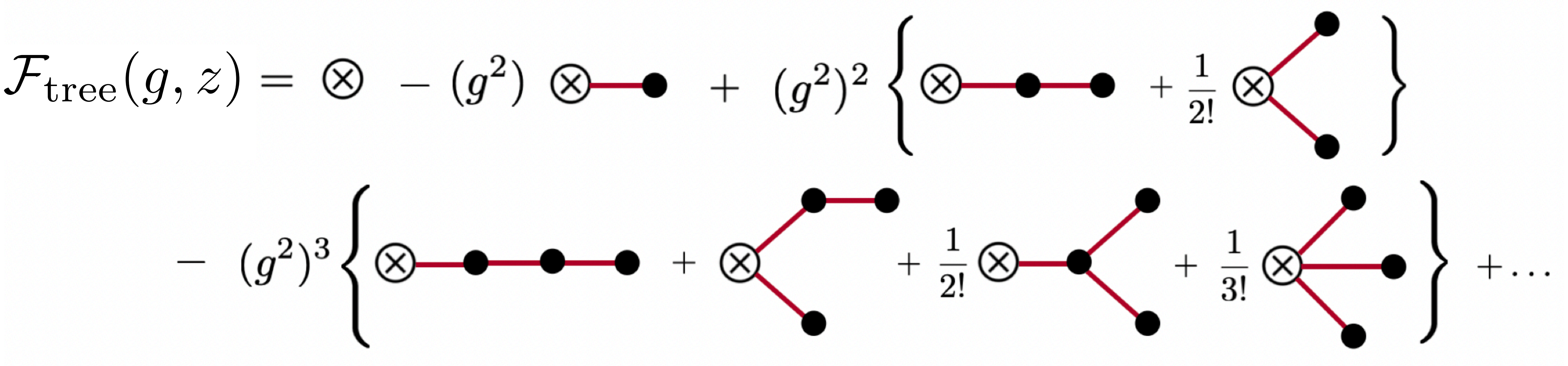
- Weak coupling: only consider a small subset of diagrams

- Strong coupling: $|\mathcal{F}_{\text{ladder}}(g;z)|\leq\frac{1}{\cosh(\sqrt{2}g\pi)}\leq 2e^{-\sqrt{2}g\pi}$

but the exact result behaves as $\mathcal{F}(g,z)\sim g+1+\mathcal{O}\left(\frac{1}{g}\right)$

Resummation of all "trees"

- Consider a collection of all "tree" negative geometries



As it turns out this is still easily computable

- Same differential operator: solve

$$\boxed{\mathcal{F}_{\text{tree}}(g, z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}\,, \quad \text{where} \quad \frac{A}{2g\cos\frac{\pi A}{2}} = 1}$$

Strong coupling expansion

- Expand the result at strong coupling

$$\boxed{\mathcal{F}_{\text{tree}}(g,z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)}$$

in comparison to the exact result

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z)+(z+1)\log z\right] + \cdots + \mathcal{O}\left(\frac{1}{g}\right)$$

Our "tree" approximation misses the leading term but does have $1/g$ expansion at strong coupling

Gamma cusp

- Gamma cusp controls the leading IR divergence

$$\log M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + \mathcal{O}(1/\epsilon) \longrightarrow \Gamma_{\text{cusp}}(g) = \sum_{L\geq 1} g^{2L}\Gamma^{(L)}_{\text{cusp}}$$

Exact result from integrability

(Beisert, Eden, Staudacher, 2006)

$$\Gamma_{\text{cusp}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ \boxed{2g - \frac{3\log 2}{2\pi} + \cdots \quad g \gg 1} \end{cases}$$

- Can be also extracted from $\mathcal{F}(g,z)$

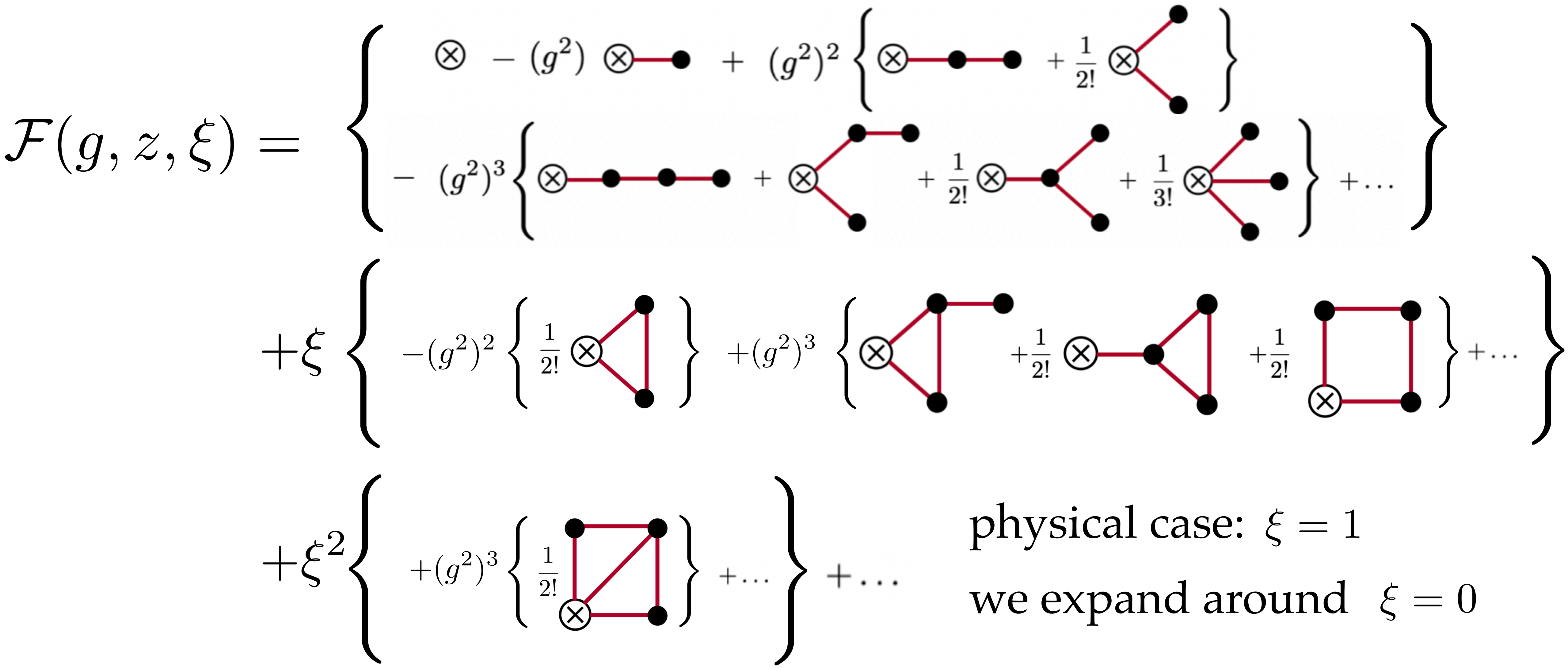
(Alday, Henn, Sikorowski, 2013)
(Henn, Korchemsky, Mistlberger 2019)
(Arkani-Hamed, Henn, JT, 2021)

qualitatively correct behavior at strong coupling ⟵

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \ldots & g \ll 1 \\ \boxed{\frac{8}{\pi}g - 1 + \ldots \quad g \gg 1} \end{cases}$$

Loops of loops expansion

- The geometry suggests to add one more parameter



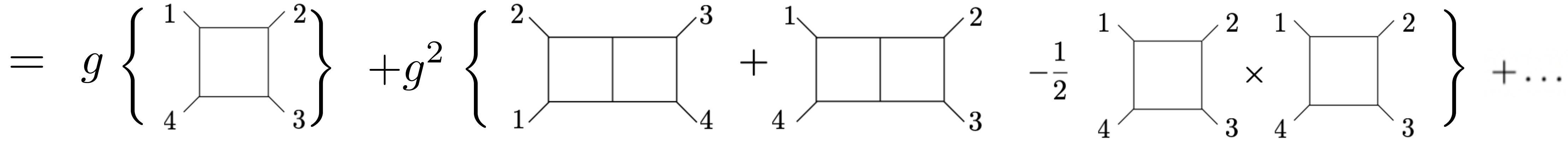
physical case: $\xi = 1$

we expand around $\xi = 0$

Comparison of expansions

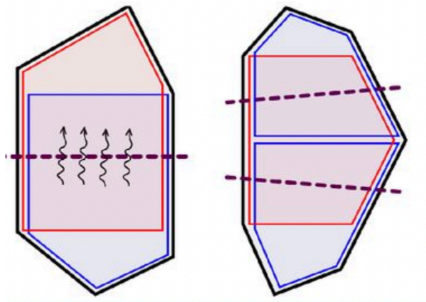
$$\mathcal{F}(g, z, \xi = 1)$$

- **Perturbative expansion**: for generic z and $\xi = 1$



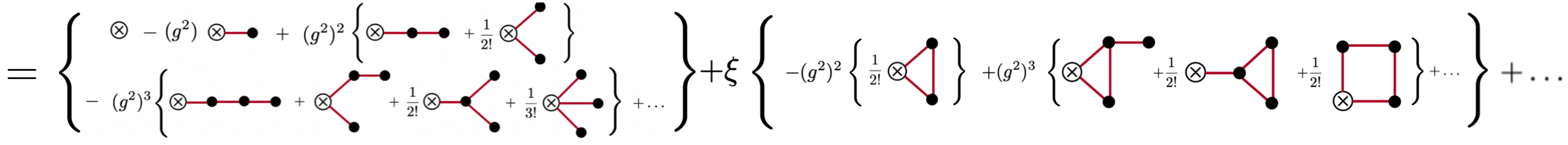
- **Near collinear expansion**: for generic g and $\xi = 1$

  (if the OPE can be run for our object… probably yes)



- **Negative geometry expansion**: for generic g and z

  (if we can resum a certain class of diagrams like we did for trees)



Summary & Outlook

Summary

- Our first attempt to use the Amplituhedron picture for planar N=4 SYM amplitudes for resummation and strong coupling.

- We considered an IR finite object $\mathcal{F}(g,z)$ derived from the logarithm of the amplitude (freezing one loop, integrate others)

- New expansion in terms of negative geometries and organization using "loop of loops". We solved for $\mathcal{F}(g,z)$ at for "trees"

- Extracted gamma cusp, expanded at strong coupling and compared to the exact result $\Gamma_{\rm cusp}(g)$: surprisingly good behavior

Outlook

- Can we restore $\mathcal{F}(g,z) \sim g$ behavior at strong coupling? Calculate subleading one-loop negative geometries: integrands known, need to integrate or find a differential equation…

  (Arkani-Hamed, Henn, JT, in progress)

- Higher points, richer analytic structure even at lower loops

  (Arkani-Hamed, Chicherin, Henn, JT, in progress)

- Deformed negative geometries: relation to integrability?

  (Arkani-Hamed, Henn, JT, in progress)

- General negative geometries, IR divergencies, applications

  (Brown, Oktem, Paranjape, JT, in progress)

- Strong coupling geometry? Emergence of strings?

Thank you!