Non-perturbative decoupling of massive fermions

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UNIVERSITY OF HELSINKI

CERN Lattice Coffee Talk, December 13, 2022

Outline

1. **Motivation**

   > flavor-number-dependency of running coupling (massless fermions)

   > fermion mass effects

2. **Lattice methods**

   > used data

   > gradient flow running coupling

   > finite volume effects

3. **Results I**

   > lattice gradient flow coupling data fitted

4. **Using different renormalization schemes**

   > massive GF scheme 1-loop beta function

   > relating the massive GF and massive BF-MOM schemes

5. **Results II**

   > Comparing lattice GF data with 2-loop BF-MOM

   > Potential effects from 3-loop non-universality

Beta function and running coupling in SU(N) gauge theories

- Gauge coupling beta function (massless fermions):

$$\mu \frac{dg^2}{d\mu} = \beta(g^2)$$

  > running gauge coupling g^2 at energy scale μ.

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$$\beta^{(n_l)}(g^2) = -2\,g^2 \sum_{n=0}^{n_l-1} \beta_n \left(\frac{g^2}{(4\pi)^2}\right)^{n+1}$$

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  [F. Herzog et al. JHEP02(2017)090]

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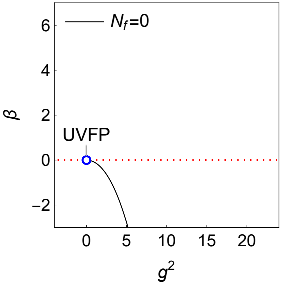

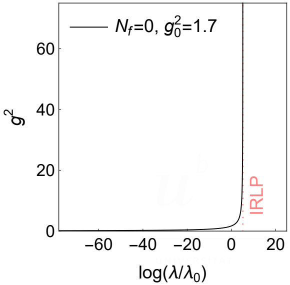

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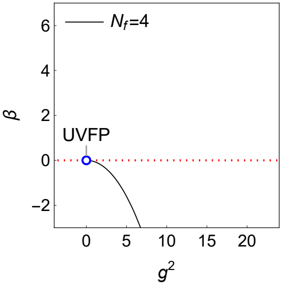

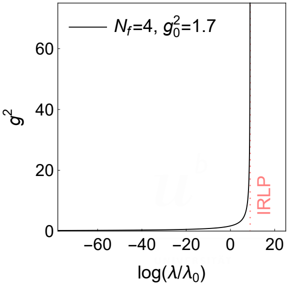

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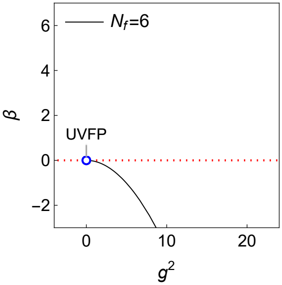

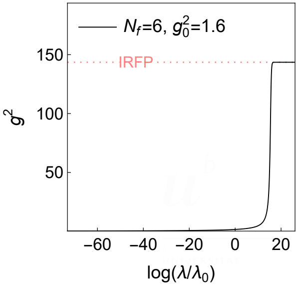

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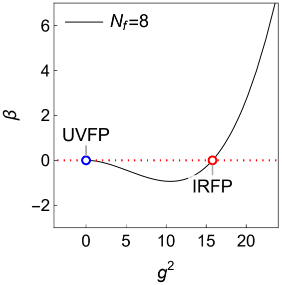

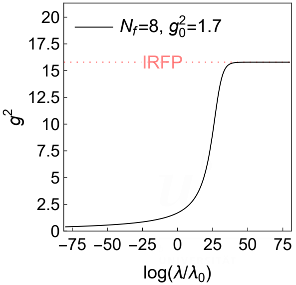

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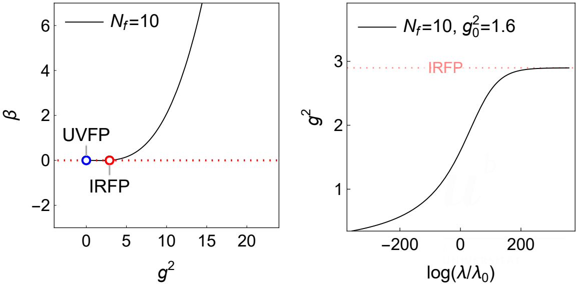

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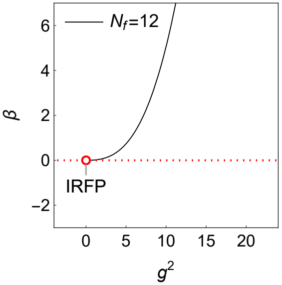

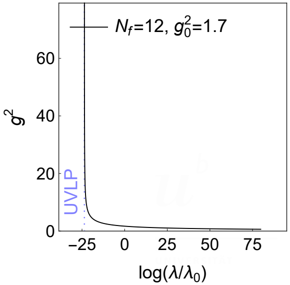

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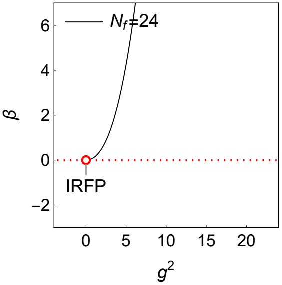

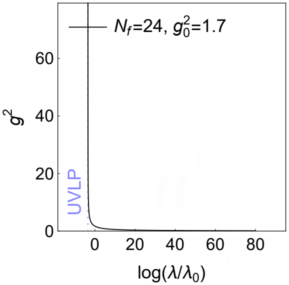

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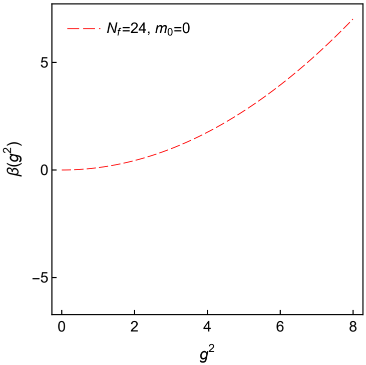

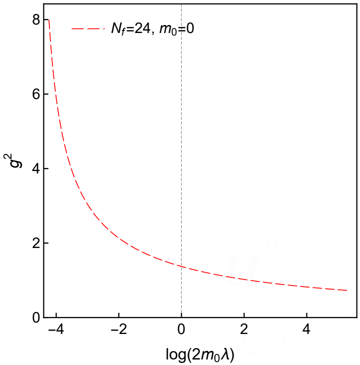

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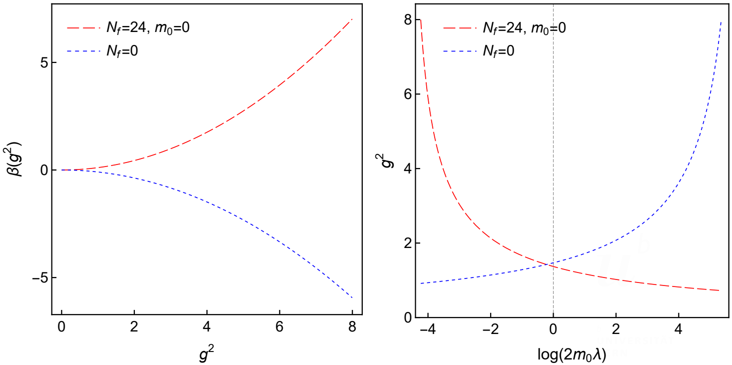

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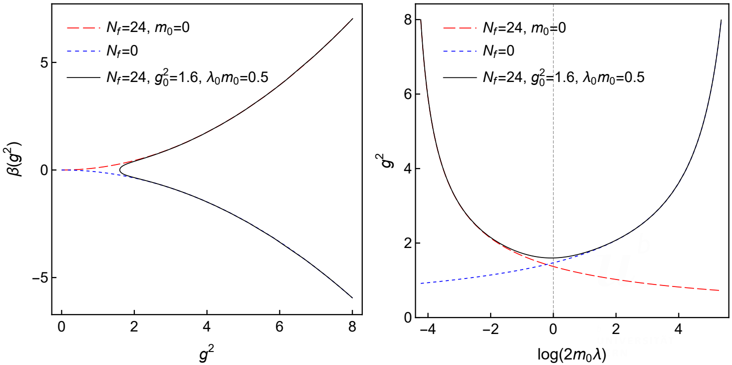

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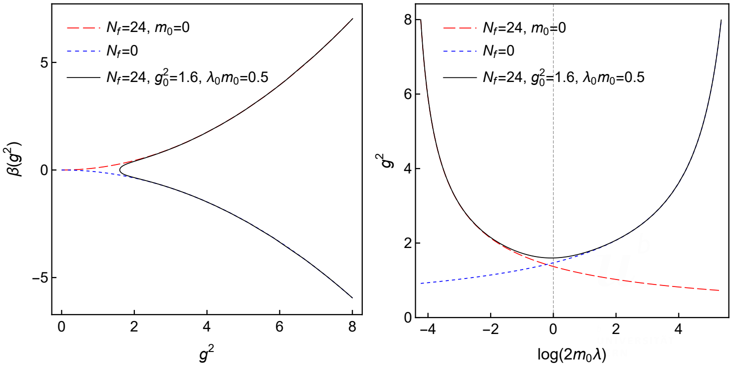

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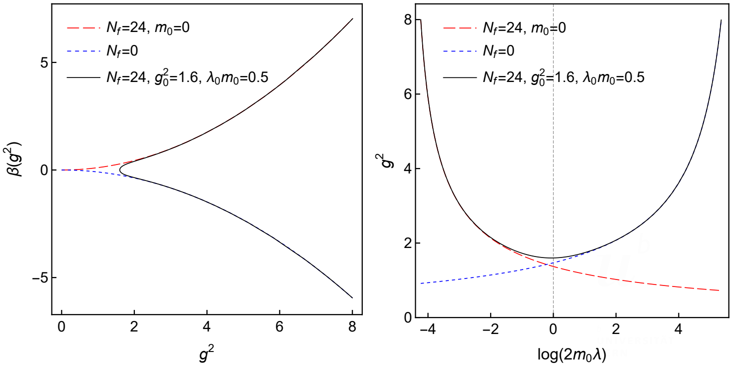

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- 2-loop BF-MOM scheme:

  [Jegerlehner & Tarasov NPB549(1999)]

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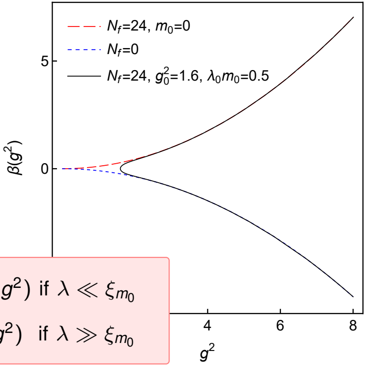

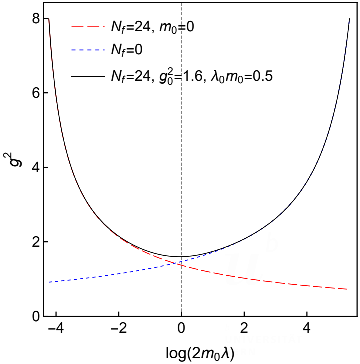

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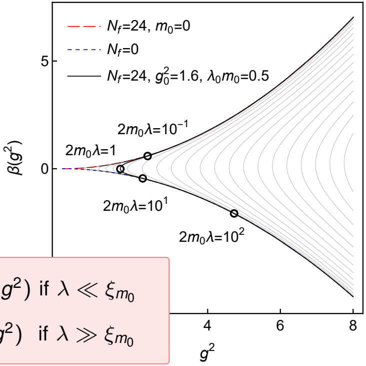
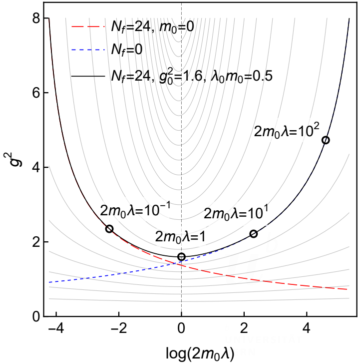
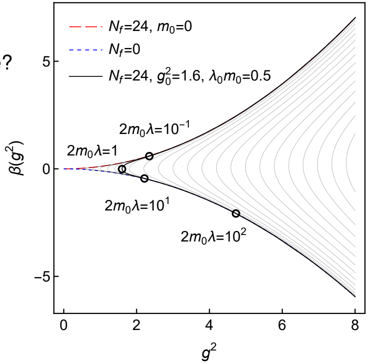
[PRL 129, 131601 (2022)], [arXiv:2110.13882 [hep-lat]]

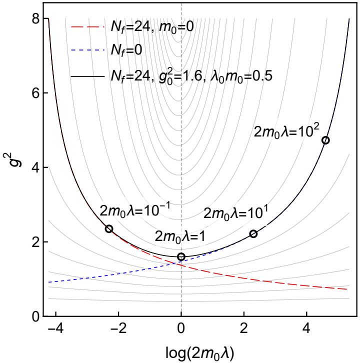




Running coupling on the lattice

- Can this behavior be observed on the lattice?





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Running coupling on the lattice

- Can this behavior be observed on the lattice?

➔ can use HMC data from previous study: [Rantaharju et al. PRD 104, 114504 (2021)]

> SU(2) lattice gauge theory with $N_f = 24$ massive Wilson fermions

> lattices of size $V = L^4$ ($L = 32a, 40a, 48a$) with periodic (gauge), resp. time-antiperiodic (fermions) BC

> lattice action: $S = S_G(U) + S_F(V) + c_{\mathrm{SW}}\, S_{\mathrm{SW}}(V)$

➔ U : fundamental, unsmeared gauge link matrices

➔ V : HEX-smeared gauge link matrices (corresponding to U) [Capitani et al. JHEP11(2006)028]

➔ S_G, S_F : Wilson gauge and Wilson fermion actions

➔ $c_{\mathrm{sw}}\, S_{\mathrm{sw}}$: clover term with Sheikholeslami-Wohlert coefficient $c_{\mathrm{SW}} = 1$

> 3 values of inverse gauge coupling, $\beta \in \{\, -0.25, 0.001, 0.25 \,\}$

> for each value of β 4-6 different values of fermion hopping parameter κ

> PCAC quark mass: $a\, m_q = \left. \dfrac{(\partial_4^* + \partial_4) f_A(x_4)}{4\, f_P(x_4)} \right|_{x_4 = L/2}$ (axial and pseudo-scalar current correlators f_A, f_P)

> gradient flow running coupling (clover-energy, Lüscher-Weisz action)

Running coupling on the lattice

- Gradient flow (GF) scheme in the continuum: $g^2_{\mathrm{GF}}(\lambda) = \dfrac{2\,\pi^2\,\lambda^4\,\langle E(\lambda)\rangle}{3(N^2-1)}$ [Lüscher JHEP08(2010)071]
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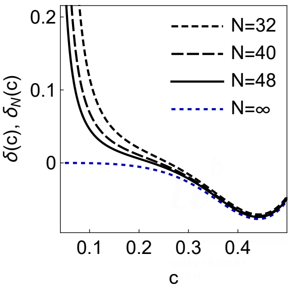

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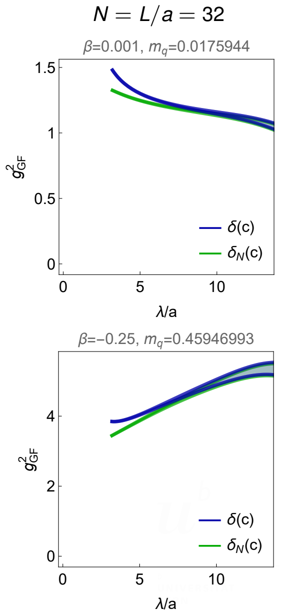

2. Lattice methods

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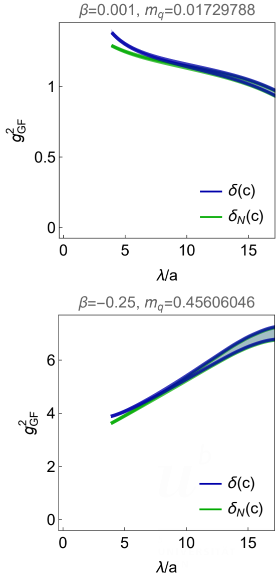
$N = L/a = 40$



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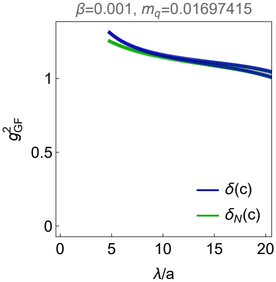

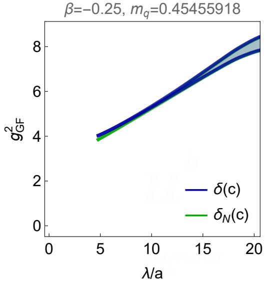

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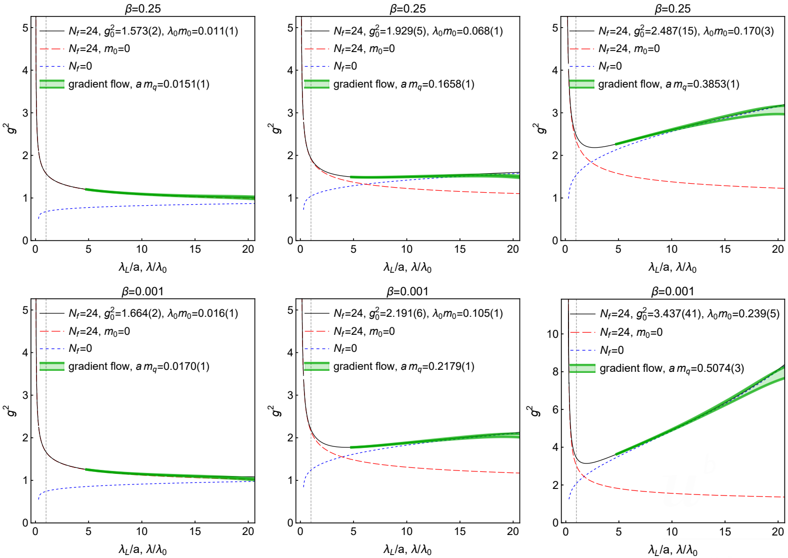
- Will consider $g^2_{\mathrm{GF}}(\lambda_L, L)$ as function of λ_L at fixed L (no step scaling).

Lattice GF-data

- $V = (48a)^4$ lattice:



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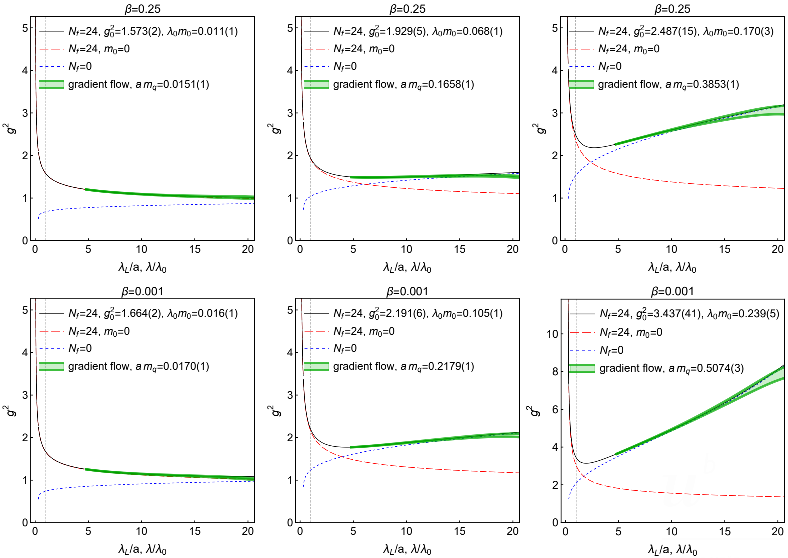

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3. Results I

Lattice GF-data

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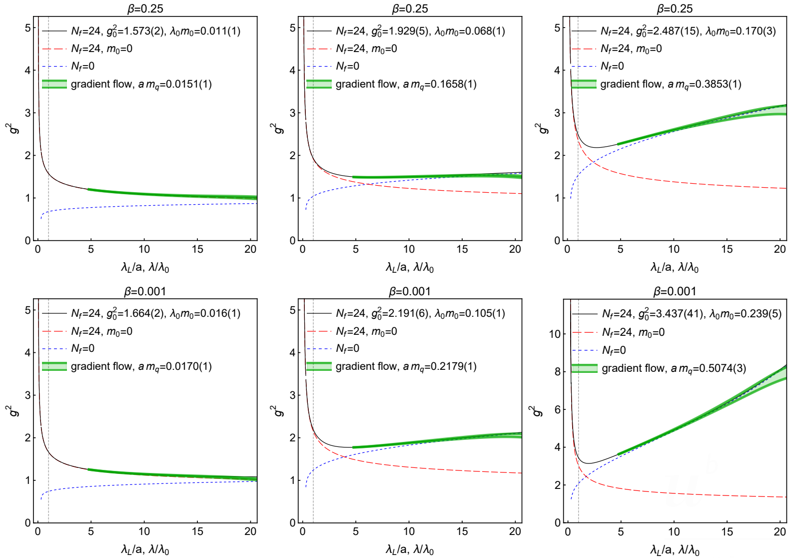
- Fit with 2-loop BF-MOM

  set: $\lambda_0 = a$, $\lambda = \lambda_L$, fit: g_0^2, m_0

  $\Rightarrow r_s = \frac{\lambda_0 m_0}{a\,m_q} \approx 0.4 - 0.5$



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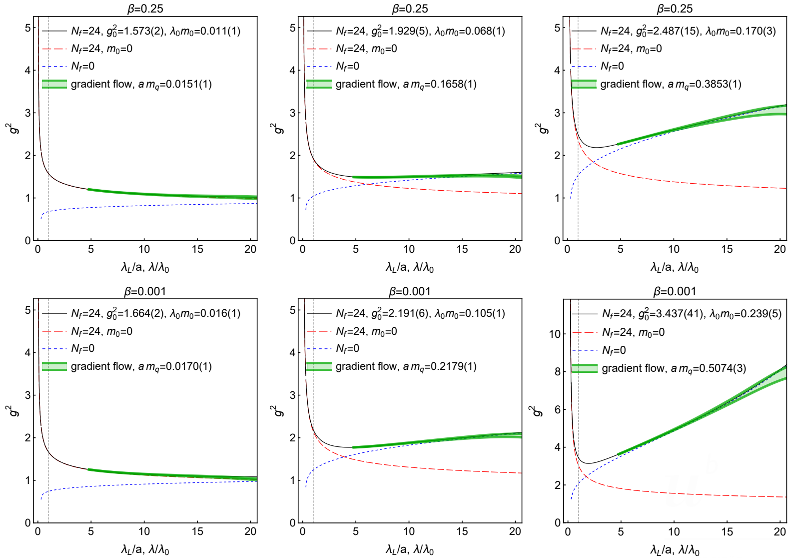

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Perturbative massive GF scheme beta function

- Continuum GF running coupling: $$u_{\mathrm{GF}}(\lambda) = g^2_{\mathrm{GF}}(\lambda) = \frac{2\,\pi^2\,\lambda^4\,\langle E(\lambda)\rangle}{3\,(N^2-1)} \quad \text{with} \quad \lambda = \sqrt{8\,t}$$

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- use 1-loop $\overline{\rm MS}$ result for leading m_q-contribution to $\langle E(t)\rangle$ [Harlander & Neumann JHEP06(2016)161]

➔ $u_{\rm GF}(\lambda) = u_{\overline{\rm MS}}(\lambda_0)\left(1 + k_1(\lambda/\lambda_0)\dfrac{u_{\overline{\rm MS}}(\lambda_0)}{(4\pi)^2} + l_1(\lambda\, m(\lambda_0))\dfrac{u_{\overline{\rm MS}}(\lambda_0)}{(4\pi)^2} + \mathcal{O}\big(u^2_{\overline{\rm MS}}(\lambda_0)\big)\right)$ (*)

> old: $k_1(\lambda/\lambda_0) = (2\,\log(\lambda/\lambda_0) + \gamma_E)\,\beta_0 + N\left(\dfrac{52}{9} - 3\,\log(3)\right) - N_f\left(\dfrac{4}{9} - \dfrac{4}{3}\log(2)\right)$ [Lüscher JHEP08(2010)071]

> new: $l_1(y) = -\dfrac{4}{3}\,T_R\,N_f\,\Omega_{1q}\big(-1/(2y)^2\big)$

4. Using different renormalization schemes

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- define GF beta function in terms of λ-derivative of (*): [Harlander PoS(LATTICE2021)489]

➔ $$\beta_{\rm GF} = -\lambda\frac{{\rm d}u_{\rm GF}}{{\rm d}\lambda} = -\left(\lambda\frac{{\rm d}k_1(\lambda/\lambda_0)}{{\rm d}\lambda} + \lambda\frac{{\rm d}l_1(\lambda\, m(\lambda_0))}{{\rm d}\lambda}\right)\frac{u^2_{\overline{\rm MS}}(\lambda_0)}{(4\pi)^2} + \mathcal{O}\big(u^3_{\overline{\rm MS}}\big) \text{ (**)}$$

> invert (*): $$u_{\overline{\rm MS}}(\lambda_0) = u_{\rm GF}(\lambda)\left(1 - k_1(\lambda/\lambda_0)\frac{u_{\rm GF}(\lambda)}{(4\pi)^2} - l_1(\lambda\, m(\lambda_0))\frac{u_{\rm GF}(\lambda)}{(4\pi)^2} + \mathcal{O}\big(u^2_{\rm GF}(\lambda)\big)\right)$$

> and use it in (**) $\Rightarrow$ $$\beta_{\rm GF}(u_{\rm GF},\lambda\, m) = -\underbrace{\left(\lambda\frac{{\rm d}k_1(\lambda/\lambda_0)}{{\rm d}\lambda} + \lambda\frac{{\rm d}l_1(\lambda\, m(\lambda_0))}{{\rm d}\lambda}\right)}_{2\,\beta_{0,\rm GF}}\frac{u^2_{\rm GF}}{(4\pi)^2} + \mathcal{O}\big(u^3_{\rm GF}\big)$$

Perturbative massive GF scheme beta function

- $\beta_{\mathrm{GF}}(u_{\mathrm{GF}}, \lambda\, m) = -2\,\beta_{0,\mathrm{GF}}(x)\,\dfrac{u_{\mathrm{GF}}^2}{(4\pi)^2} + \mathcal{O}(u_{\mathrm{GF}}^3)$

  with $\beta_{0,\mathrm{GF}}(x) = \beta_0 + \dfrac{4}{3}\,T_R\,N_f\,x\,\dfrac{\mathrm{d}\Omega_{1q}(x)}{\mathrm{d}x}$ and $x = -1/(2\,\lambda\,m)^2$

  > at 1-loop: $m = m_0$ (const.)

  > $\beta_{0,\mathrm{GF},N_f}(x) = \begin{cases} \beta_{0,\overline{\mathrm{MS}},N_f=24} & \text{if } \lambda \ll 1/m_0 \\ \beta_{0,\overline{\mathrm{MS}},N_f=0} & \text{if } \lambda \gg 1/m_0 \end{cases}$

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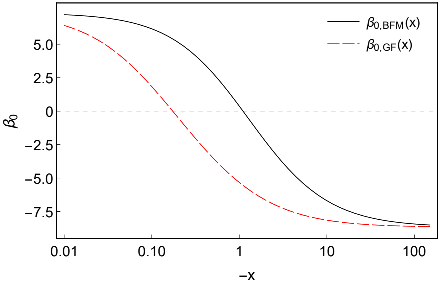
  > decoupling not near $2\,\lambda\, m_0 \approx 1 \Rightarrow$ mismatch between λ and m_0



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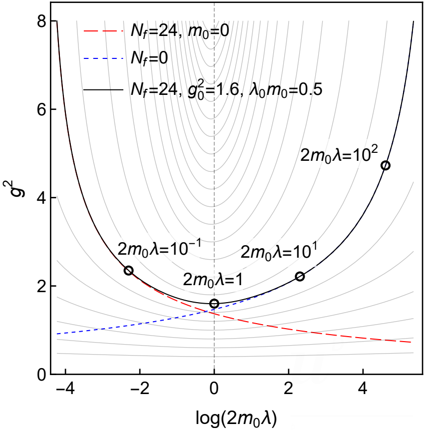

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➔ set:

> $\lambda^{(\mathrm{BFM})} = \lambda$, $\lambda^{(\mathrm{GF})} = \rho_s \lambda$
>
> $\lambda_0^{(\mathrm{BFM})} = \lambda_0$, $\lambda_0^{(\mathrm{GF})} = \rho_s \lambda_0$
>
> $u_{\mathrm{GF}}(\lambda_0^{(\mathrm{GF})}) = u_{\mathrm{BFM}}(\lambda_0^{(\mathrm{BFM})}) = u_0$
>
> $m_{\mathrm{GF}}(\lambda^{(\mathrm{GF})}) = m_{\mathrm{BFM}}(\lambda^{(\mathrm{BFM})}) = m_0$ (assuming pole masses)

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➔ require: $u_{\mathrm{GF}}(\lambda^{(\mathrm{GF})}) = u_{\mathrm{BFM}}(\lambda^{(\mathrm{BFM})})$ for $\log(\lambda/\lambda_0) \approx 0$, $\lambda_0 = 1/(2\,m_0)$.

Relating GF and BF-MOM (BFM) scheme

- can match GF and BFM by rescaling $\lambda^{(\mathrm{GF})} = \rho_s \, \lambda^{(\mathrm{BFM})}$ with some $\rho_s > 0$
- scheme matching usually done at trivial fixed point $u = g^2 = 0$ (universality) ➔ not available in our case!
- if decoupling of fermions physical $\Rightarrow$ should be described equivalently in all schemes

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➔ Expand both sides $u_{\mathrm{R}}\big(\lambda^{(\mathrm{R})}\big) = u_0 + \sum_{n=1}^{\infty} c_n^{(\mathrm{R})}\big(u_0, x_0^{(\mathrm{R})}\big) \dfrac{\log^n\big(\lambda^{(\mathrm{R})}/\lambda_0^{(\mathrm{R})}\big)}{n!}$

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➔ $c_n^{(\mathrm{GF})}(u_0, x_0/\rho_s^2) = c_n^{(\mathrm{BFM})}(u_0, x_0) \;\forall n \in \mathbb{N}$

4. Using different renormalization schemes

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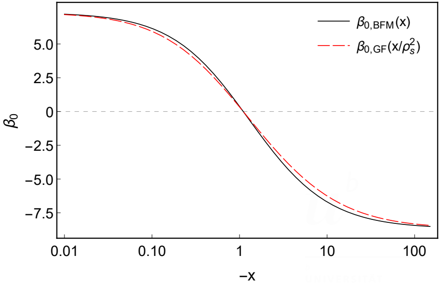
at 1-loop: $\beta_{0,\mathrm{GF}}(x_0/\rho_s^2) = \beta_{0,\mathrm{BFM}}(x_0) \quad \Rightarrow \quad \boxed{\rho_s = 2.535945\ldots}$



[PRL 129, 131601 (2022)], [arXiv:2110.13882 [hep-lat]]

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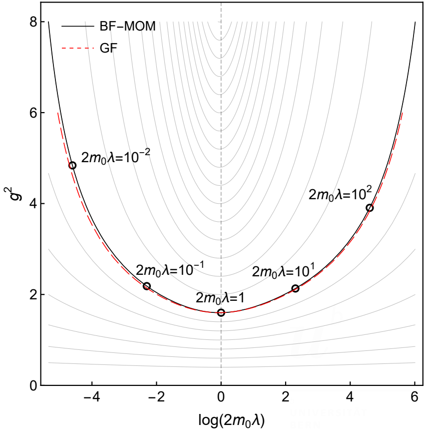

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Lattice GF-data compared to 2-loop BF-MOM (no fit)

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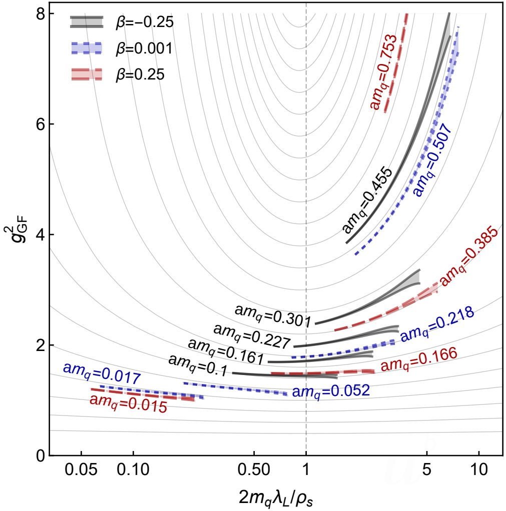

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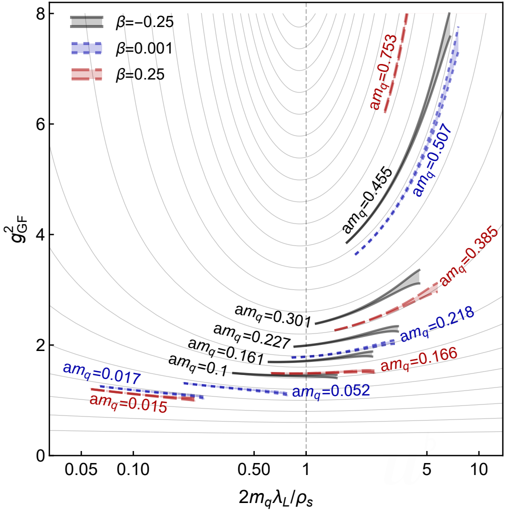
  [Dalla Brida & Ramos Eur.Phys.J.C 79,720(2019)]



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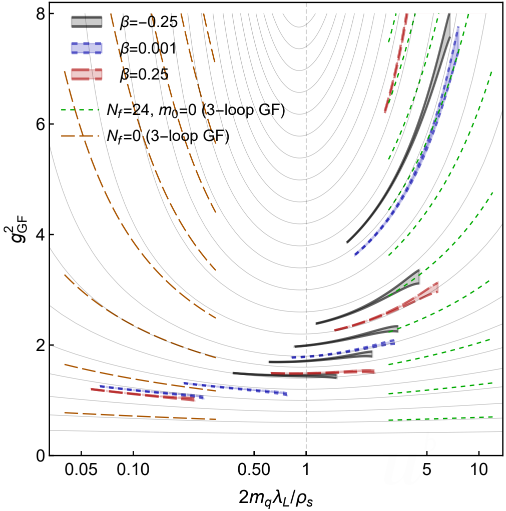

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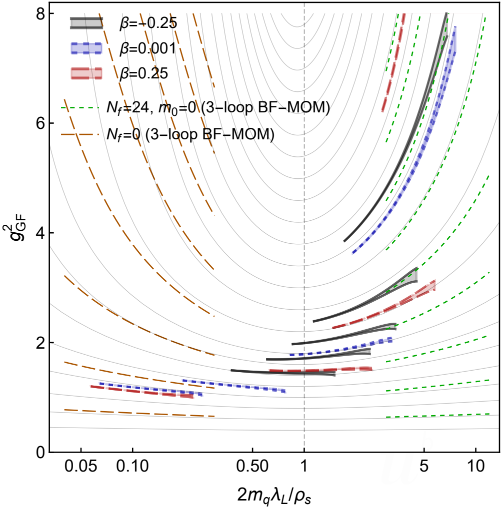

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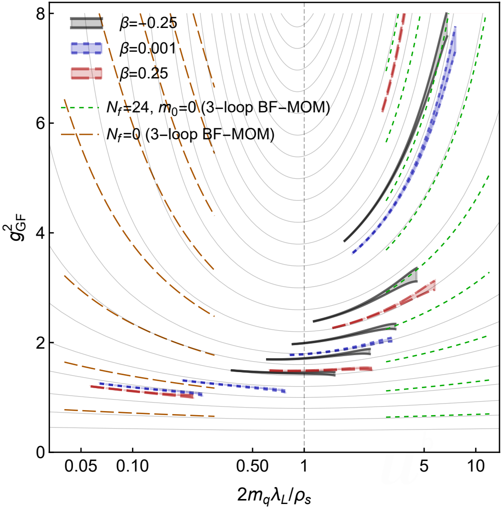

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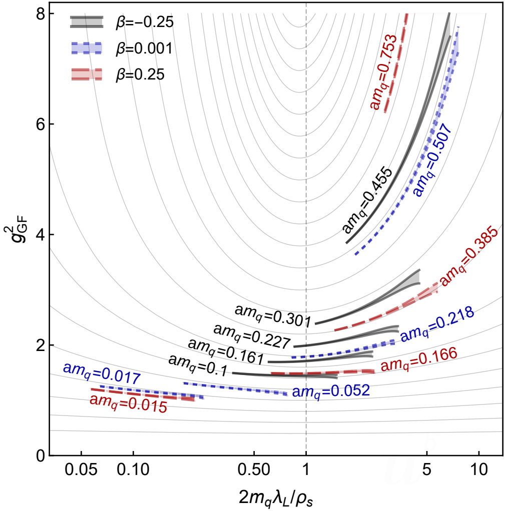
Summary

- Non-perturbative demonstration of decoupling of massive fermions at scale $\lambda \sim 1/m$



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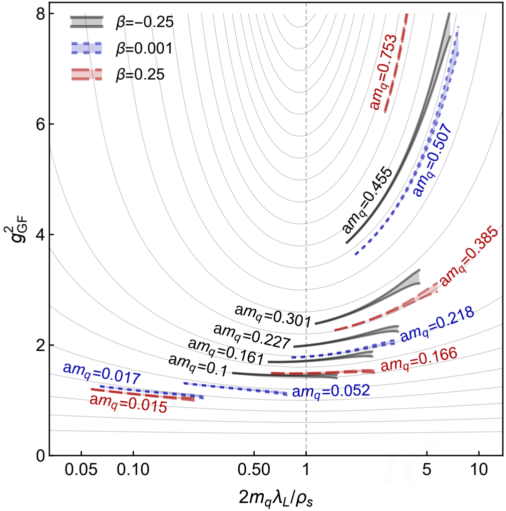

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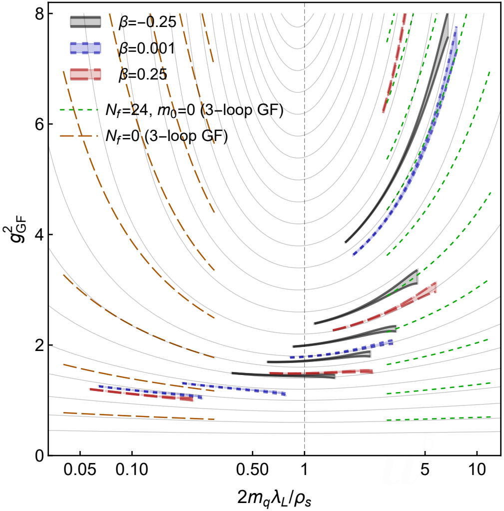

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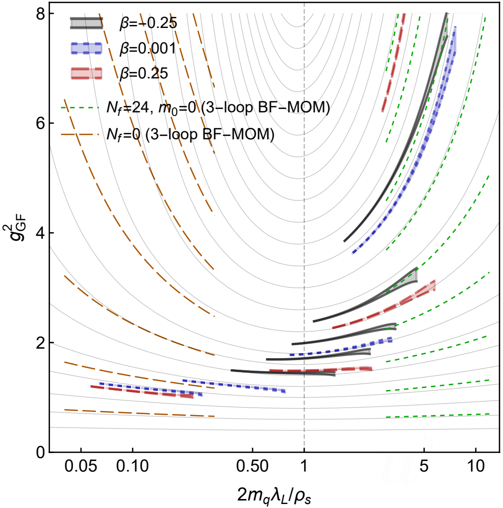

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Thank you for your attention!

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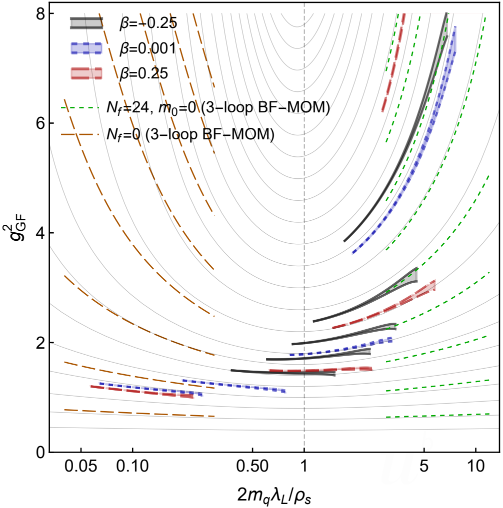

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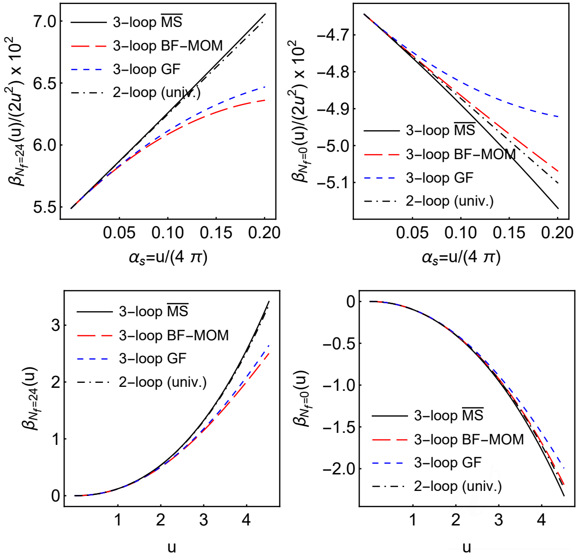
3-loop non-universal effects

- Comparisons of the 3-loop beta function for SU(2) gauge theory with $N_f = 24$ massless flavors (left) and $N_f = 0$ flavors (right) in the $\overline{\mathrm{MS}}$ (solid, black), BF-MOM (long dashes, red) and GF scheme (short dashes, blue). For comparison also the corresponding 2-loop beta function is shown (dot-dashed, black).



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