

# Primordial Black Holes and Stochastic Inflation beyond slow roll

## SUSY 2023 @ Southampton

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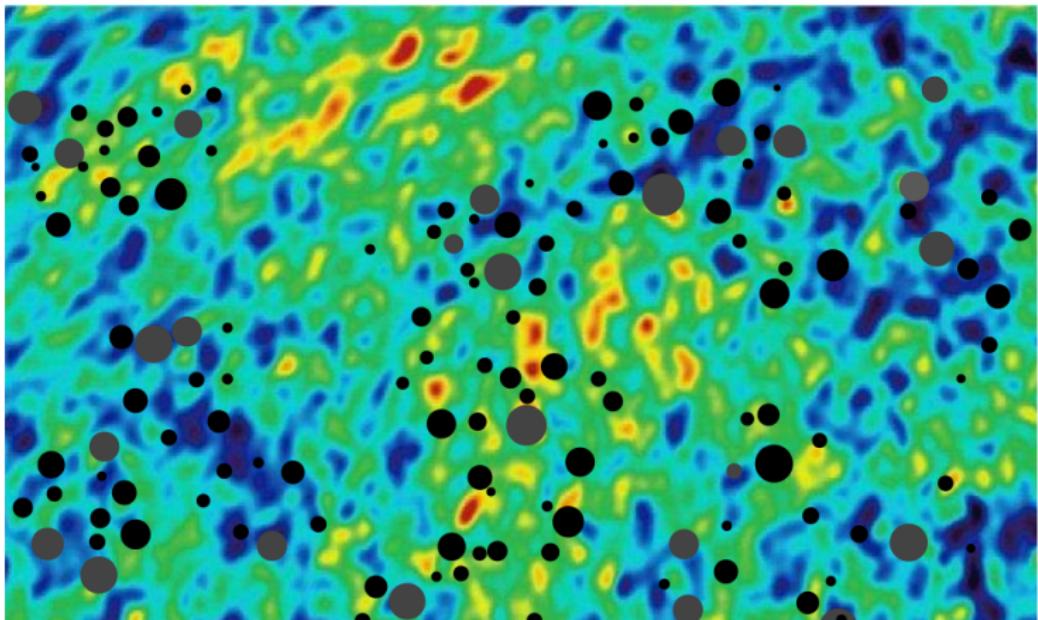
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18 July 2023

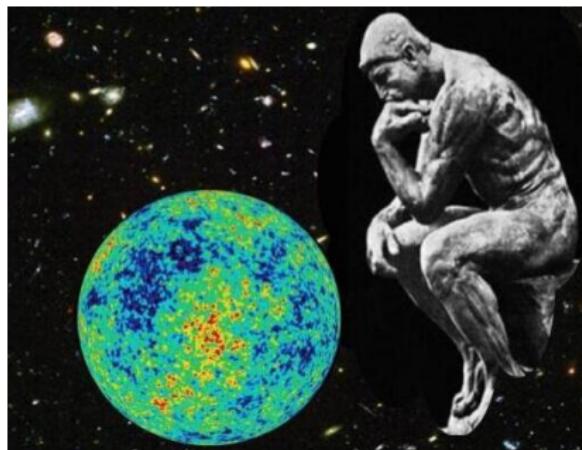
# Primordial Black Holes (PBHs)



Candidates for Dark Matter, Hawking Radiation,  
Baryogenesis, Reheating, seeds of SMBHs etc.  
**Extremely interesting rich phenomenology!**

# Inflation, Quantum fluctuations and PBHs

CMB → LSS



- Adiabatic  $\zeta(\vec{x})$
- Almost **scale-invariant**

$$\mathcal{P}_\zeta = A_S \left( \frac{k}{k_*} \right)^{n_S}$$

$$A_S \simeq 2 \times 10^{-9}, \ n_S \simeq -0.035$$

- Nearly **Gaussian**

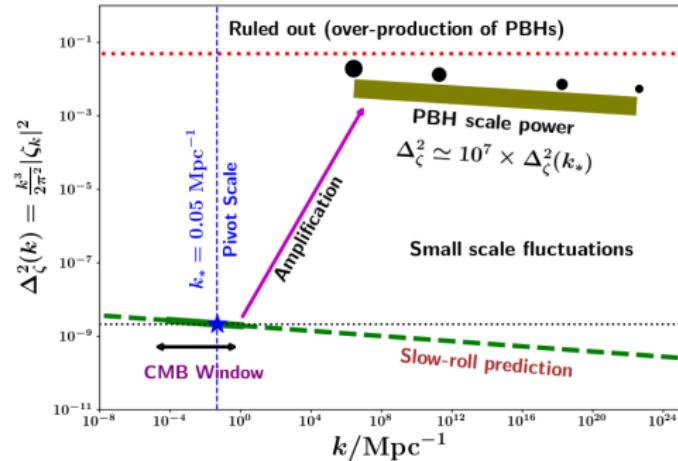
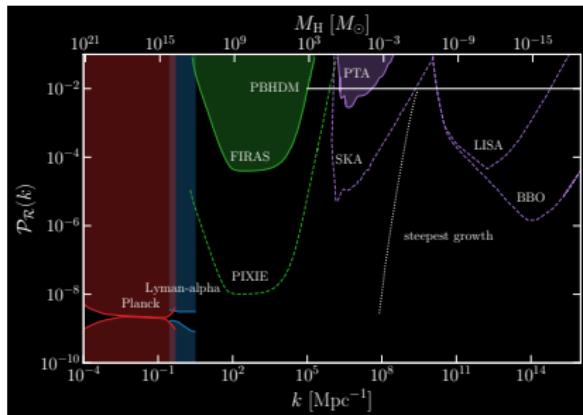
$$P[\zeta] = \mathcal{B} \exp \left[ \frac{-\zeta^2}{2\sigma^2} (1 + \textcolor{red}{f}_{\text{NL}} \zeta + \dots) \right]$$

- **LSS, CMB** ⇒ Large-scale tiny quantum fluctuations  
→ **PBHs, GW<sup>(2)</sup>s** ⇒ Small-scale larger fluctuations ?

# What we know from Observations

CMB probes scales  $\mathbf{k} \in [0.0005, 0.5] \text{ Mpc}^{-1} \Rightarrow \Delta N \simeq 7$

Small-scale power spectrum is not constrained!

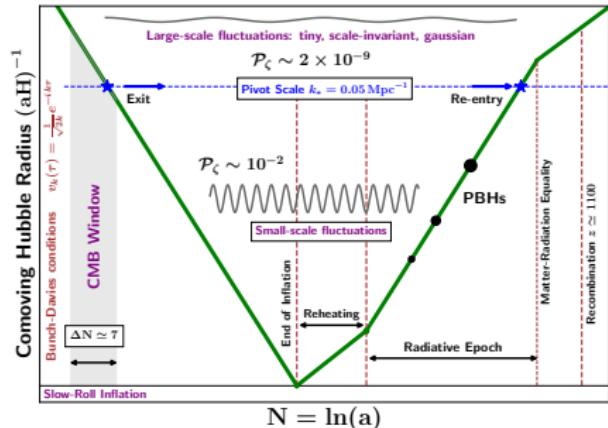
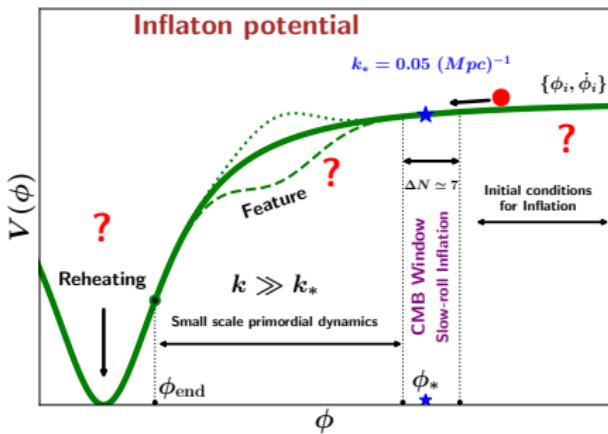


Possibility of enhancement of small-scale fluctuations!

\*\*Green and Kavanagh, J. Phys. G 48 (2021) 4, 043001

# Single-field Inflation beyond the CMB Window

⇒ Scope for non-trivial small-scale dynamics



CMB scales :  $P_\zeta \sim k^{-0.035}$  (Slightly red – tilted);  $\eta_H \simeq -0.018$

Small-scale growth :  $P_\zeta \sim k^{n_s (\leq 4)}$  (Blue – tilted);  $\eta_H \geq 3/2$

\*\*Byrnes et. al JCAP 06(2019) 028

# Large Quantum Fluctuations

- ① Breakdown of scale-invariance at small-scales

$$\boxed{\epsilon_H = -\frac{d\ln H}{dN}, \quad \eta_H = \epsilon_H - \frac{1}{2} \frac{d\ln \epsilon_H}{dN} \quad ; \quad N = \ln(a)}$$

- ② Breakdown of Gaussian nature of primordial fluctuations

For  $\zeta \gg 1$

$$\boxed{P[\zeta] \neq \mathcal{B} \exp \left[ \frac{-\zeta^2}{2 \int_{k_1}^{k_2} d\ln k \mathcal{P}_\zeta(k)} \left( 1 + \textcolor{red}{f_{NL}} \zeta + \textcolor{red}{g_{NL}} \zeta^2 + \dots \right) \right]}$$

\*\*Celoria et. al JCAP 06 (2021) 051

# Breakdown of Scale-invariance via feature

A feature: an inflection point or a local bump/dip at low scales slows down the inflaton

⇒ Breaking of scale invariance!!

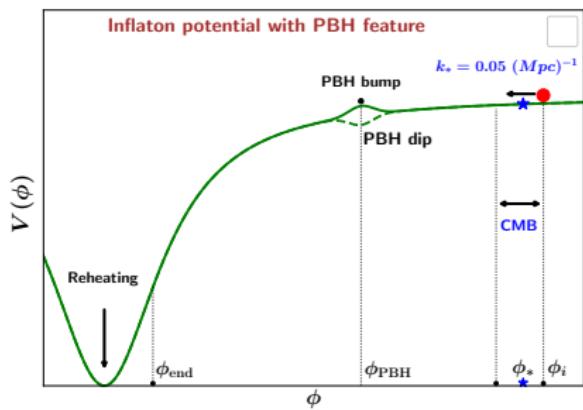
At small scales  $\epsilon_H \ll 1, \eta_H \gtrsim 3$

Violation of slow-roll

Criteria for PBH from single field

Inflation—

- ① Large scales satisfying with CMB constraints.
- ② Intermediate scale feature to enhance power for PBH formation.
- ③ Successful Reheating mechanism.



\*\*Motohashi, Hu PRD 96(2017) 6, Cole et. al arXiv:2304.01997

# Computing Power spectrum

$$\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta_k|^2 \Big|_{k \ll aH}$$

Mukhanov-Sasaki variable  $\boxed{\mathbf{v}_k = z \times \zeta_k}$ ;  $z = am_p \sqrt{2\epsilon_H}$

In spatially-flat gauge

$$\boxed{\frac{d^2 v_k}{dN^2} + (1 - \epsilon_H) \frac{dv_k}{dN} + \left[ \left( \frac{k}{aH} \right)^2 + \mathbf{M}_{\text{eff}}^2(N) \right] v_k = 0}$$

where the **effective mass term** is

$$\boxed{\mathbf{M}_{\text{eff}}^2 = -\frac{1}{(aH)^2} \left[ 2 + 2\epsilon_H + 2\epsilon_H^2 - 3\eta_H + \eta_H^2 - 3\epsilon_H \eta_H - \frac{d\eta_H}{dN} \right]}$$

**Background dynamics dependent and complicated!**

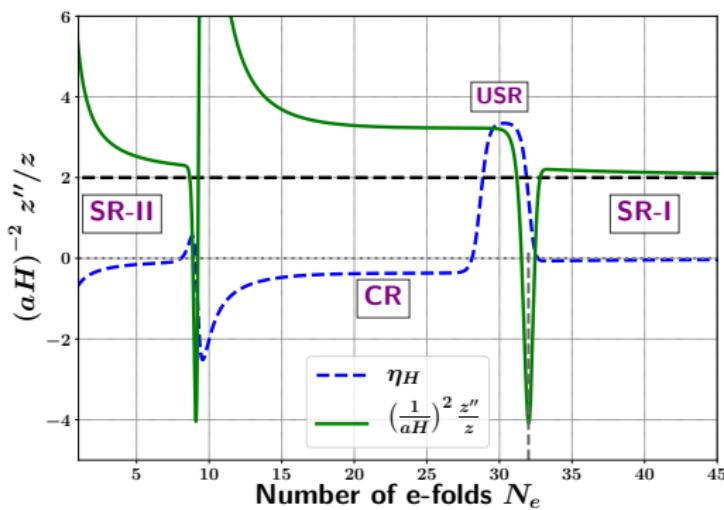
# Typical Inflationary Dynamics

**SR-I** (CMB scale)  $\rightarrow$  **USR**  $\rightarrow$  **CR**  $\rightarrow$  **SR-II**

$$\eta_H : \quad \eta_1 \longrightarrow \eta_2 \longrightarrow \eta_3 \longrightarrow \eta_4 \quad \text{Wands Duality}$$

Background

Reason for duality



For  $\epsilon_H \ll 1$ ,

$$\frac{\mathbf{M}_{\text{eff}}^2}{(aH)^2} \simeq 2 - 3\eta_H + \eta_H^2 - \frac{d\eta_H}{dN}$$

Assuming

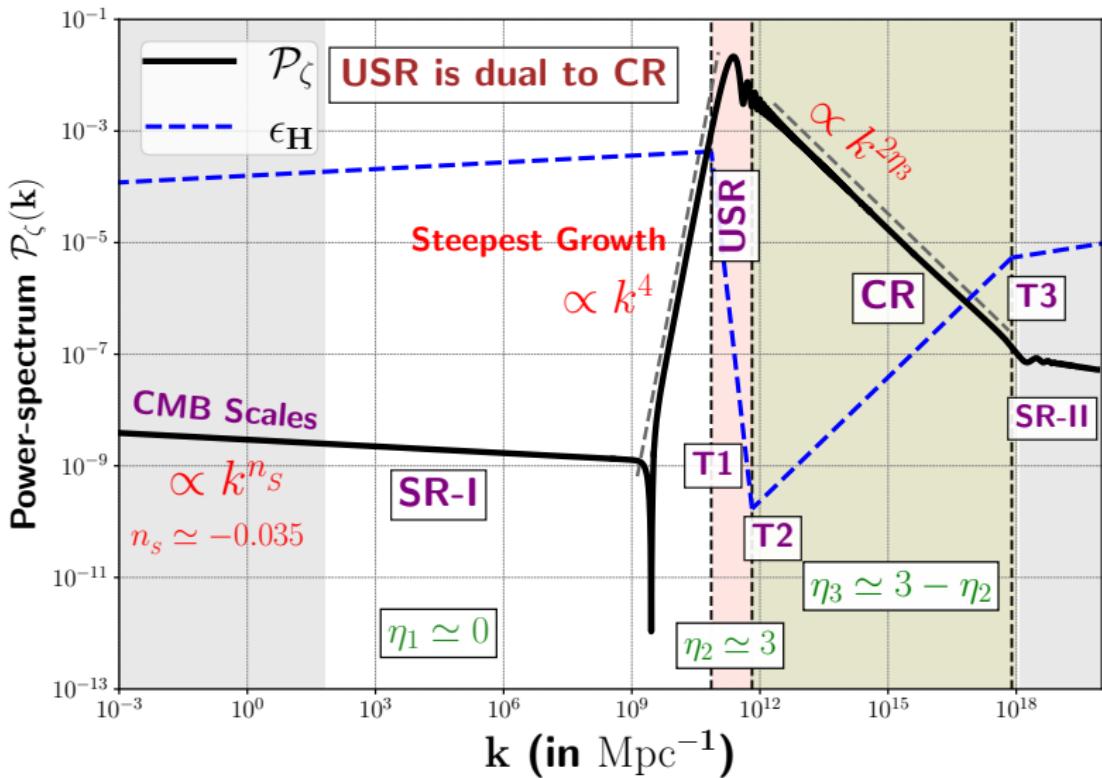
$$\eta_H = \frac{3}{2} + C \tanh \left[ C \left( N_e - \tilde{N}_e \right) \right]$$

$$\nu^2 \equiv \frac{\mathbf{M}_{\text{eff}}^2}{(aH)^2} + \frac{1}{4} \simeq \text{const.}$$

\*\*SSM, Sahni JCAP 04(2020) 007,

\*\*Karam et. al JCAP 03(2023) 013

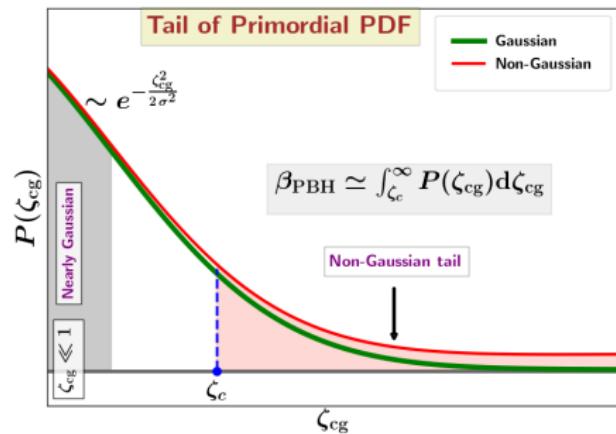
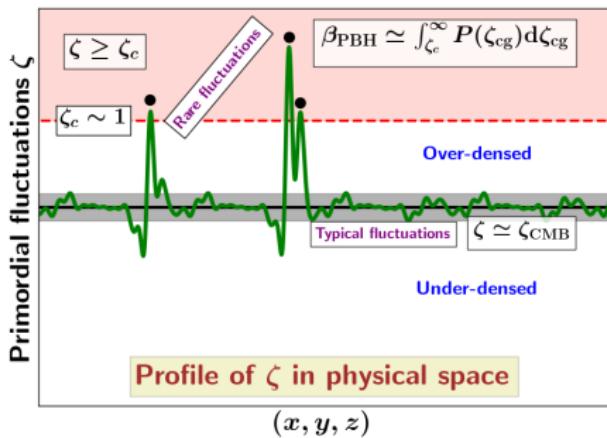
# Typical Power-spectrum



# Statistics of Primordial Fluctuations

Is the Primordial PDF  $P[\zeta]$  Gaussian or Non-Gaussian?

Non-Gaussian for  $\zeta \gg 1$  in general



PBHs from Rare Peaks: Sensitive to the tail of PDF

# Non-Perturbative Methods for full PDF

Approach - I

Classical Non-linear  $\delta N$  formalism

Approach - II

Semi-classical Approximation

Approach - III

Stochastic Inflation

# Stochastic Inflation: Effective IR description

Coarse-grained description

$$\phi = \Phi + \varphi \quad , \quad \pi_\phi = \Pi + \pi$$

## Langevin Equations (Non-linear)

$$\frac{d\Phi}{dN} = D_\Phi + \xi_\phi; \quad \frac{d\Pi}{dN} = D_\Pi + \xi_\pi$$

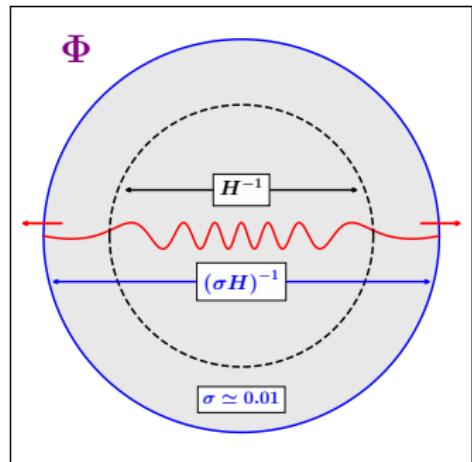
$$\frac{dF_{cg}}{dN} = \text{Drift}_{cl} + \text{Diffusion}_Q$$

## Gaussian White noise statistics

$$\langle \xi_i(N) \xi_j(N') \rangle = \Sigma_{ij}(N) \delta_D(N - N')$$

## Noise Matrix elements

$$\Sigma_{ij}(N) = (1 - \epsilon_H) \frac{k^3}{2\pi^2} \phi_{ik}(N) \phi_{jk}^*(N) \Big|_{k=\sigma aH}$$



Coarse-graining scale  
 $k = \sigma aH, \quad \sigma \ll 1$

\*\*A. A. Starobinsky (1986)

# PDF from first-passage time analysis

$$\frac{d\Phi}{dN} = D_\Phi + \xi_\phi; \quad \frac{d\Pi}{dN} = D_\Pi + \xi_\pi$$

First-passage no. of e-folds  $\mathcal{N}$

and PDF  $P(\mathcal{N})$

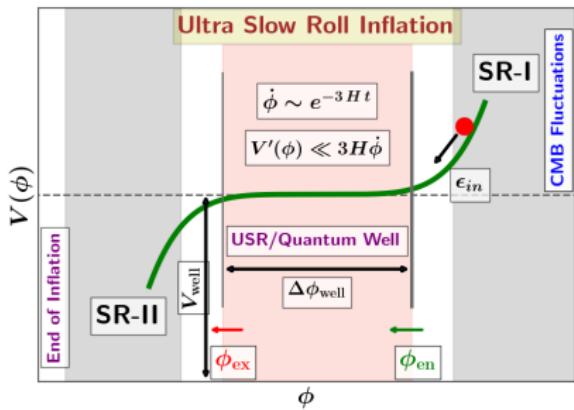
Subject to boundary conditions

- ① Reflecting boundary at  $\Phi = \phi_{\text{en}}$  :

$$\left. \frac{\partial}{\partial \Phi} P(\mathcal{N}) \right|_{\Phi=\phi_{\text{en}}} = 0$$

- ② Absorbing boundary at  $\Phi = \phi_{\text{ex}}$  :

$$P(\mathcal{N}) \Big|_{\Phi=\phi_{\text{ex}}} = \delta_D(\mathcal{N})$$



- Numerical Simulations
- **Fokker-Planck Equation**  
(for analytical treatment)

# Langevin $\longrightarrow$ Fokker-Planck Equation

PDF of first-passage number of e-foldings  $\mathcal{N}$ : **Adjoint FPE**

$$\frac{\partial P}{\partial \mathcal{N}} = \left[ D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} + \frac{1}{2} \Sigma_{\phi\phi} \frac{\partial^2}{\partial \Phi^2} + \Sigma_{\phi\Pi} \frac{\partial^2}{\partial \Phi \partial \Pi} + \frac{1}{2} \Sigma_{\pi\Pi} \frac{\partial^2}{\partial \Pi^2} \right] P(\mathcal{N})$$

$$P(\mathcal{N}) \equiv P_{\Phi, \Pi}(\mathcal{N})$$

## Stochastic $\delta\mathcal{N}$ Formalism

Statistics of  $\mathcal{N} \rightarrow$  Statistics of  $\zeta_{\text{cg}}$  :  $P[\mathcal{N}] \longrightarrow P[\zeta_{\text{cg}}]$

$$\boxed{\zeta_{\text{cg}} \equiv \zeta(\Phi) = \mathcal{N} - \langle \mathcal{N}(\Phi) \rangle}; \quad \langle \mathcal{N}(\Phi) \rangle = \int_0^\infty \mathcal{N} P(\mathcal{N}) d\mathcal{N}$$

## Abundance of PBHs

$$\boxed{\beta \sim \int_{\zeta_c}^\infty P(\zeta_{\text{cg}}) d\zeta_{\text{cg}}}$$

\*\*Pattison et. al JCAP 04 (2021) 080

# Quasi de Sitter approximation

Mode functions  $\{\phi_k, \pi_k\} \rightarrow \text{dS}$

$$\Sigma_{\phi\phi} \simeq \left(\frac{H}{2\pi}\right)^2, \quad \Sigma_{\phi\pi}, \Sigma_{\pi\pi} \ll \Sigma_{\phi\phi}$$

The Langevin equations become

$$\frac{d\Phi}{dN} = D_\Phi + \frac{H}{2\pi} \xi; \quad \frac{d\Pi}{dN} = D_\Pi$$

with single **Gaussian white noise**  $\xi$  satisfying

$$\langle \xi(N) \rangle = 0, \quad \text{and} \quad \langle \xi(N) \xi(N') \rangle = \delta_D(N - N')$$

**Adj. Fokker-Planck Equation becomes**

$$\frac{\partial P(\mathcal{N})}{\partial \mathcal{N}} = \left[ \frac{H^2}{8\pi^2} \frac{\partial^2}{\partial \Phi^2} + D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} \right] P(\mathcal{N})$$

# PDF for flat Quantum Well: Pure diffusion

$$V(\Phi) = V_0, \quad H^2 \simeq \frac{V_0}{3m_p^2}$$

Leading to

**PDF**

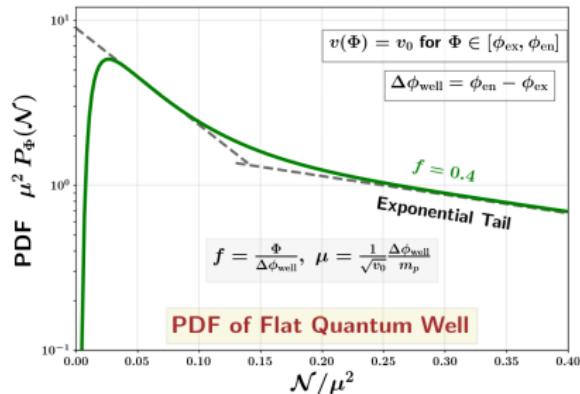
$$P(\mathcal{N}) = \sum_{n=0}^{\infty} A_n(\Phi) e^{-\Lambda_n \mathcal{N}}$$

with  $\Lambda_n = (2n+1)^2 \frac{\pi^2}{4} \frac{1}{\mu^2}$

$$A_n = (2n+1) \frac{\pi}{\mu^2} \sin \left[ (2n+1) \frac{\pi}{2} \left( \frac{\Phi}{\Delta\Phi} \right) \right]$$

**Control Parameter :**

$$\mu = 2\sqrt{2}\pi \frac{\Delta\phi_{\text{well}}}{H}$$



**Exponential Tail**  
**Highly Non-Gaussian!!**

\*\*Pattison et. al JCAP 10(2017) 046; Ezquiaga et. al. JCAP 03(2020) 029

# Additional Complications

- General form of the feature

$$V(\phi) = V_0 \pm \frac{1}{2} m^2 \phi^2 \pm \frac{\mu}{2} \phi^3 + \frac{\lambda}{4} \phi^4 \pm \dots$$

- When inflaton **drift** is included

$$\frac{\partial}{\partial \mathcal{N}} P(\mathcal{N}) = \left[ \frac{\Sigma_{\Phi\Phi}}{2} \frac{\partial^2}{\partial \Phi^2} + \left( D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} \right) \right] P(\mathcal{N})$$

- Beyond the de Sitter mode functions for noise**

$$\frac{\partial P}{\partial \mathcal{N}} = \left[ D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} + \frac{\Sigma_{\Phi\Phi}}{2} \frac{\partial^2}{\partial \Phi^2} + \Sigma_{\Phi\Pi} \frac{\partial^2}{\partial \Phi \partial \Pi} + \frac{\Sigma_{\Pi\Pi}}{2} \frac{\partial^2}{\partial \Pi^2} \right] P(\mathcal{N})$$

# Recently concluded work

**SSM, Edmund J. Copeland and Anne M. Green,**

*“Primordial black holes and stochastic inflation beyond slow roll: I - Noise Matrix Elements”*

[[arXiv:2303.17375](https://arxiv.org/abs/2303.17375)]

# Computing Noise Matrix Elements

$$\Sigma_{ij}(N) = (1 - \epsilon_H) \frac{k^3}{2\pi^2} \phi_{ik}^*(N) \phi_{jk}(N) \Big|_{k=\sigma aH}; \quad \phi_{ik} \equiv \{\phi_k, \pi_k\}$$

$$\phi_k(N) = \frac{v_k(N)}{a}, \quad \pi_k(N) = \frac{d\phi_k}{dN}$$

Mukhanov-Sasaki variable  $\textcolor{blue}{v_k}$  in spatially-flat gauge

$$\boxed{\frac{d^2 v_k}{dN^2} + (1 - \epsilon_H) \frac{dv_k}{dN} + \left[ \left( \frac{k}{aH} \right)^2 + \textcolor{green}{M_{\text{eff}}^2} \right] v_k = 0}$$

where the **effective mass term** is

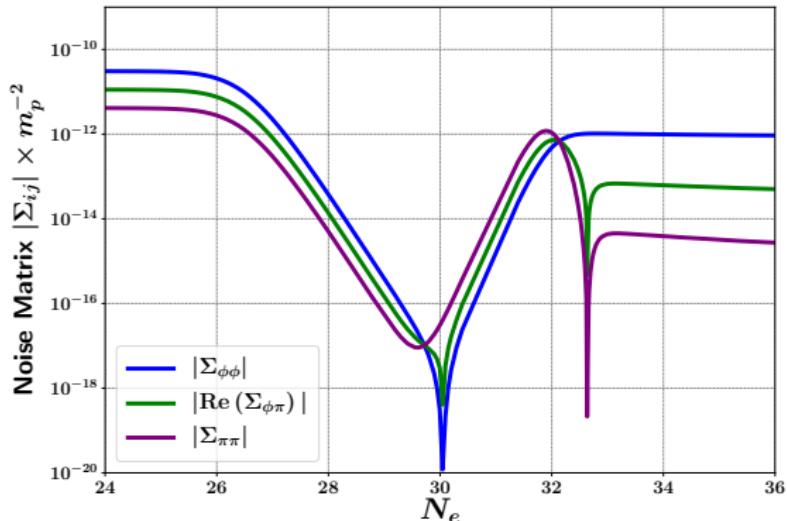
$$\boxed{\textcolor{green}{M_{\text{eff}}^2} (aH)^{-2} = 2 + 2\epsilon_H + 2\epsilon_H^2 - 3\eta_H + \eta_H^2 - 3\epsilon_H\eta_H - \frac{d\eta_H}{dN}}$$

Background dynamics dependent and complicated

# Numerical Noise Matrix Elements

Potential with a tiny Gaussian bump/dip feature

$$V(\phi) = V_0 \frac{\phi^2}{\phi^2 + M^2} \left[ 1 \pm A \exp \left( -\frac{1}{2} \left( \frac{\phi - \phi_0}{\Delta\phi} \right)^2 \right) \right]$$



$\Sigma_{ij}$  evolves and swaps hierarchy!

\*\*Mishra et. al JCAP 04(2020) 007

# Analytical approx: Sharp transitions

Assume  $|\epsilon_H| \ll |\eta_H|$  and  $\epsilon_H \ll 1$  (**qdS** approx.)

$$\Rightarrow -\mathbf{M}_{\text{eff}}^2(aH)^{-2} \simeq 2 - 3\eta_H + \eta_H - \frac{1}{aH} \eta_H$$

And  $\eta_H \rightarrow$  combination of **Step functions**

$$\eta_H(\tau) = \eta_1 + (\eta_2 - \eta_1) \Theta(\tau - \tau_1)$$

For which

$$-\mathbf{M}_{\text{eff}}^2(aH)^{-2} \simeq \mathcal{A} \tau \delta_D(\tau - \tau_1) + \left( \nu_1^2 - \frac{1}{4} \right) + (\nu_2^2 - \nu_1^2) \Theta(\tau - \tau_1)$$

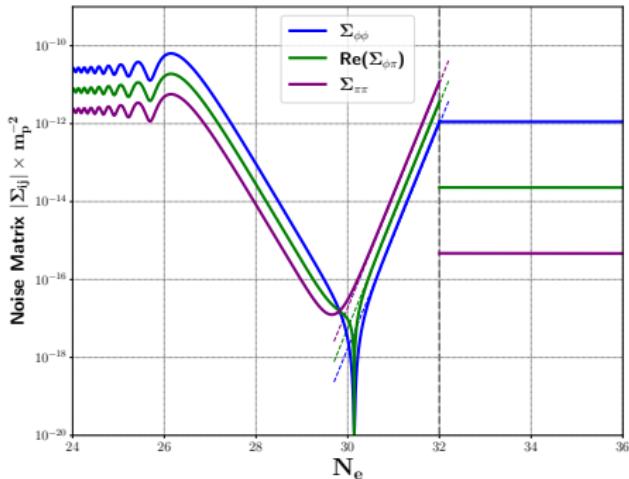
Where the **strength of transition** is  $\mathcal{A} = \eta_2 - \eta_1$  and

**Order of Hankel**  $\nu_{1,2}^2 = \left( \frac{3}{2} - \eta_{1,2} \right)^2$

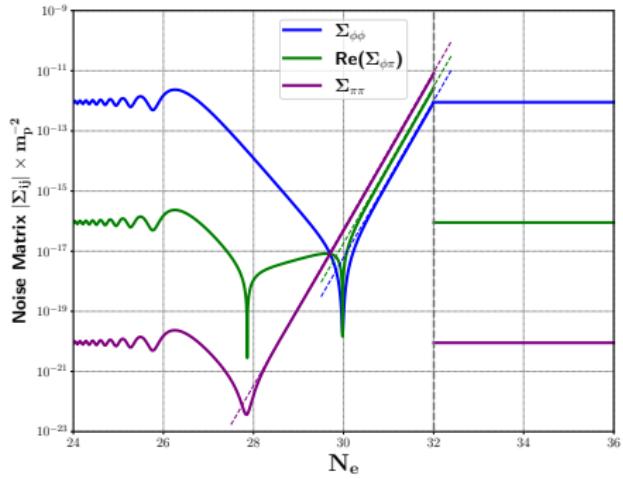
# Results from Analytical Techniques

$$\boxed{\eta_H(\tau) = \eta_1 + (\eta_2 - \eta_1) \Theta(\tau - \tau_1)}, \quad \text{Conformal time } \tau = \frac{-1}{aH}$$

$$\eta_1 \simeq -0.02; \quad \eta_2 \simeq 3.3$$



Reproduces numerical results



dS approximation

# Primary Conclusions

- ① During **SR-I** phase,  $\Sigma_{\phi\phi}^{\text{SR}} \simeq \left(\frac{H}{2\pi}\right)^2$

$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \left|\nu_1 - \frac{3}{2}\right| : \left(\nu_1 - \frac{3}{2}\right)^2}$$

- ② Immediately after the transition,  $\Sigma_{ij} \propto e^{-2\mathcal{A}N}$ , and

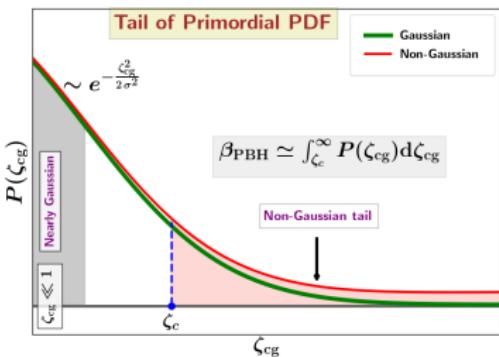
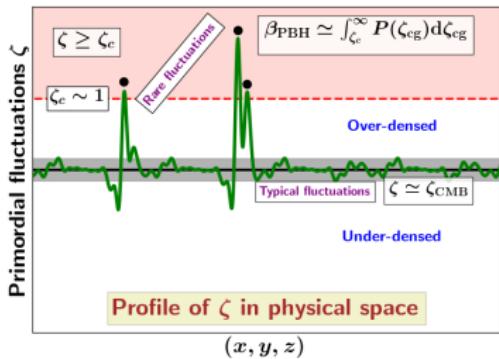
$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \mathcal{A} : \mathcal{A}^2}$$

- ③ During **CR** phase,  $\Sigma_{\phi\phi}^{\text{CR}} \simeq 2^{2(\nu_2 - \nu_1)} \left[ \frac{\Gamma(\nu_2)}{\Gamma(\nu_1)} \right]^2 \sigma^{2(\nu_1 - \nu_2)} \Sigma_{\phi\phi}^{\text{SR}}$

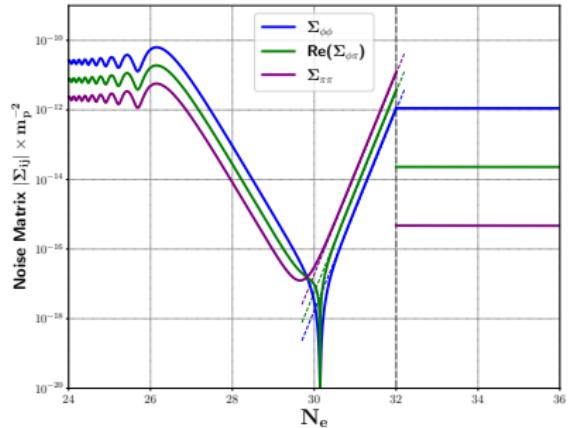
$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \left|\nu_2 - \frac{3}{2}\right| : \left(\nu_2 - \frac{3}{2}\right)^2}$$

⇒ Strongest diffusion during Constant-Roll epoch!

What is the nature of PDF  $P[\zeta]$ ? Work in Progress



PBHs  $\longrightarrow$  Large QFs  
 $\Rightarrow$  Sensitive to the tail



Stochastic Inflation formalism

(Fokker-Planck Equation)

Noise-Matrix Elements

$$\frac{\partial P}{\partial \mathcal{N}} = \left[ D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} + \frac{1}{2} \boldsymbol{\Sigma}_{\phi\phi} \frac{\partial^2}{\partial \Phi^2} + \boldsymbol{\Sigma}_{\phi\pi} \frac{\partial^2}{\partial \Phi \partial \Pi} + \frac{1}{2} \boldsymbol{\Sigma}_{\pi\pi} \frac{\partial^2}{\partial \Pi^2} \right] P(\mathcal{N})$$

# Caveats

- ➊ Mode functions evolved in a fixed (deterministic background). *\*\*Figueroa et. al 2021*
- ➋ Computed in spatially-flat gauge. *\*\*Pattison et. al 2019*
- ➌ Only a single transition was considered analytically (duality).
- ➍ Both  $\Phi$  and  $\Pi$  were treated stochastically. *\*\*Tomberg 2022*
- ➎  $\beta_{\text{PBH}}$  in terms of  $\zeta$  rather than  $\delta$ . *\*\*Tada, Vennin 2020*

Questions & Comments are most welcome.

Delighted to be here at SUSY 2023!

# EXTRA SLIDES

# Set-up for Analytical Computation

- $\eta_H \rightarrow$  is piece-wise constant  $\Rightarrow \eta_i \simeq \text{const.}$
- $\nu_i$  is piecewise (positive) constant.
- Introduce new time variable  $T = -k\tau = \frac{k}{aH}$

$$\boxed{\text{MS Eqn} \Rightarrow \frac{d^2v_k}{dT^2} + \left[ 1 - \frac{\nu^2 - 1/4}{T^2} \right] v_k = 0}$$

General solution is given by

- ➊ dS mode functions

$$v_k(T) = \frac{1}{\sqrt{2k}} \left[ \alpha_k \left( 1 + \frac{i}{T} \right) e^{iT} + \beta_k \left( 1 - \frac{i}{T} \right) e^{-iT} \right]$$

- ➋ Beyond dS approximation

$$v_k(T) = \sqrt{T} \left[ C_1 H_\nu^{(1)}(T) + C_2 H_\nu^{(2)}(T), \right]$$

# Determining Co-efficients

- Apply Bunch-Davies initial conditions for modes exiting before the transition  $T > T_1$

$$v_k(T) \Big|_{T \rightarrow \infty} \rightarrow \frac{1}{\sqrt{2k}} e^{iT}$$

- Apply **Israel Junction matching** conditions at transition

$$v_k^A(\tau_1) = v_k^B(\tau_1) \quad (\text{Continuity})$$

$$\frac{d}{d\tau} v_k^A \Big|_{\tau_1^+} - \frac{d}{d\tau} v_k^B \Big|_{\tau_1^-} = \int_{\tau_1^-}^{\tau_1^+} d\tau \frac{z''}{z} v_k^A(\tau) \quad (\text{Differentiability})$$

# Application of the technique

- A **single instantaneous transition** from SR  $\rightarrow$  USR using **dS** mode functions.
- A **single instantaneous transition** from SR  $\rightarrow$  USR using Hankel functions.

# Application of the technique

- A single instantaneous transition from SR  $\rightarrow$  USR using **dS** mode functions.
- A single instantaneous transition from SR  $\rightarrow$  USR using Hankel functions.

Why a single transition ?

Wands duality between USR and CR

# In the absence of transition

For  $\nu = \text{constant}$

$$\Sigma_{\phi\phi} = 2^{2(\nu - \frac{3}{2})} \left[ \frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 \left( \frac{H}{2\pi} \right)^2 T^{2(\frac{3}{2} - \nu)} \left[ 1 + \frac{1}{2(-1 + \nu)} T^2 + \dots \right]$$

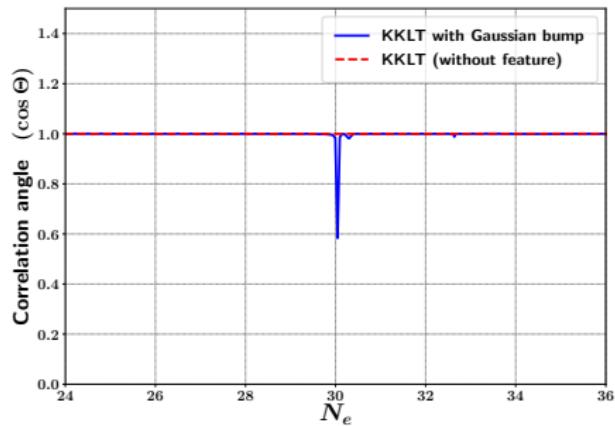
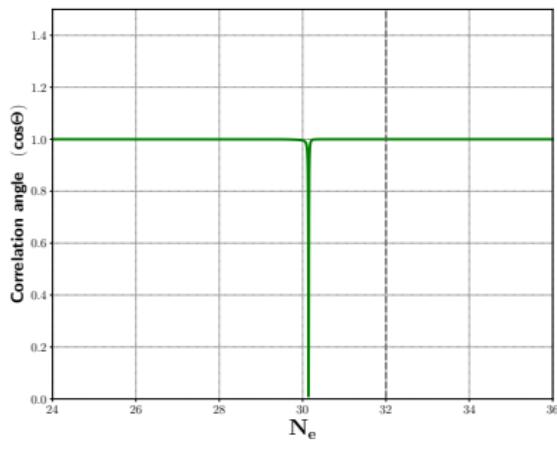
$$\Sigma_{\phi\pi} = 2^{2(\nu - \frac{3}{2})} \left[ \frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 \left( \frac{H}{2\pi} \right)^2 \left( \frac{3}{2} - \nu \right) T^{2(\frac{3}{2} - \nu)} \left[ 1 + \frac{2(5 - 2\nu)}{4(\nu - 1)(3 - 2\nu)} T^2 + \dots \right]$$

$$\Sigma_{\pi\pi} = 2^{2(\nu - \frac{3}{2})} \left[ \frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 \left( \frac{H}{2\pi} \right)^2 \left( \frac{3}{2} - \nu \right)^2 T^{2(\frac{3}{2} - \nu)} \left[ 1 + \frac{2(7 - 2\nu)}{4(\nu - 1)(3 - 2\nu)} T^2 + \dots \right]$$

# Correlation

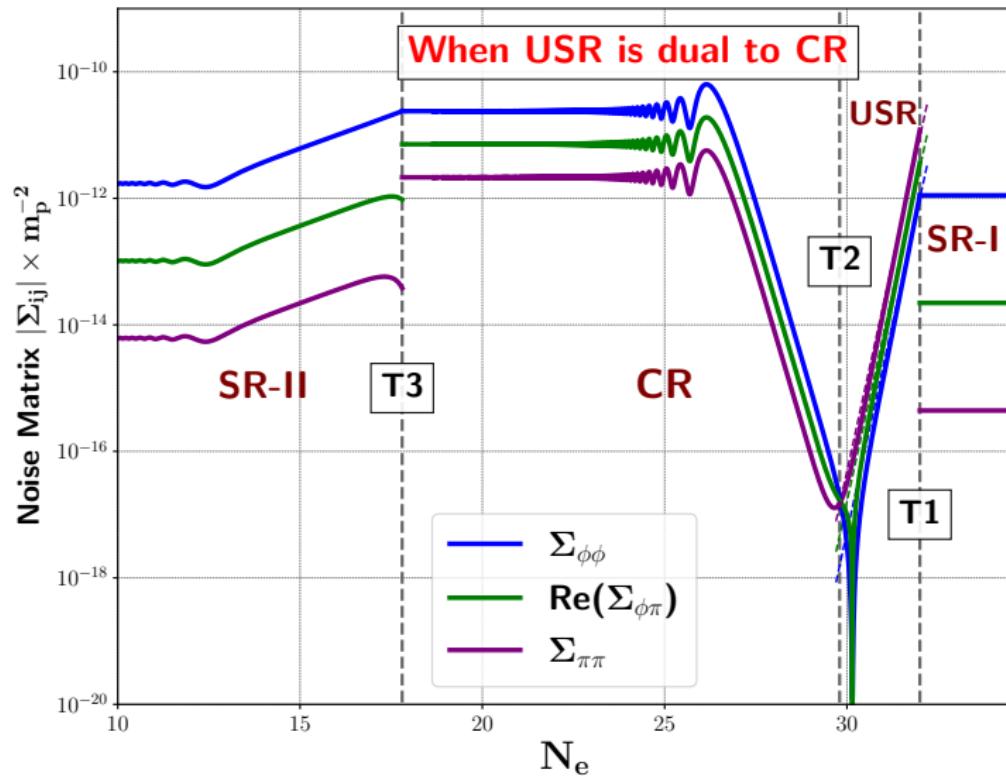
$$\gamma = \frac{|\text{Re}(\Sigma_{\phi\pi})|}{\sqrt{\Sigma_{\phi\phi}\Sigma_{\pi\pi}}}$$

With  $\gamma^2 = 1 - \det(\Sigma_{ij}) / (\Sigma_{\phi\phi}\Sigma_{\pi\pi})$

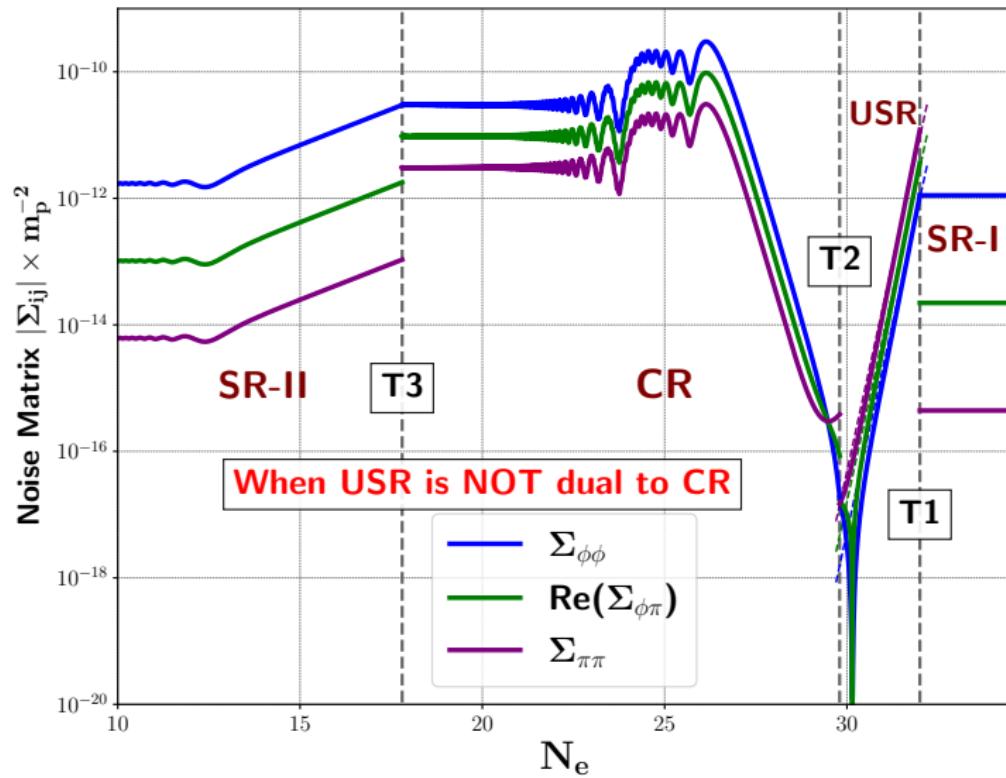


# Multiple Transitions

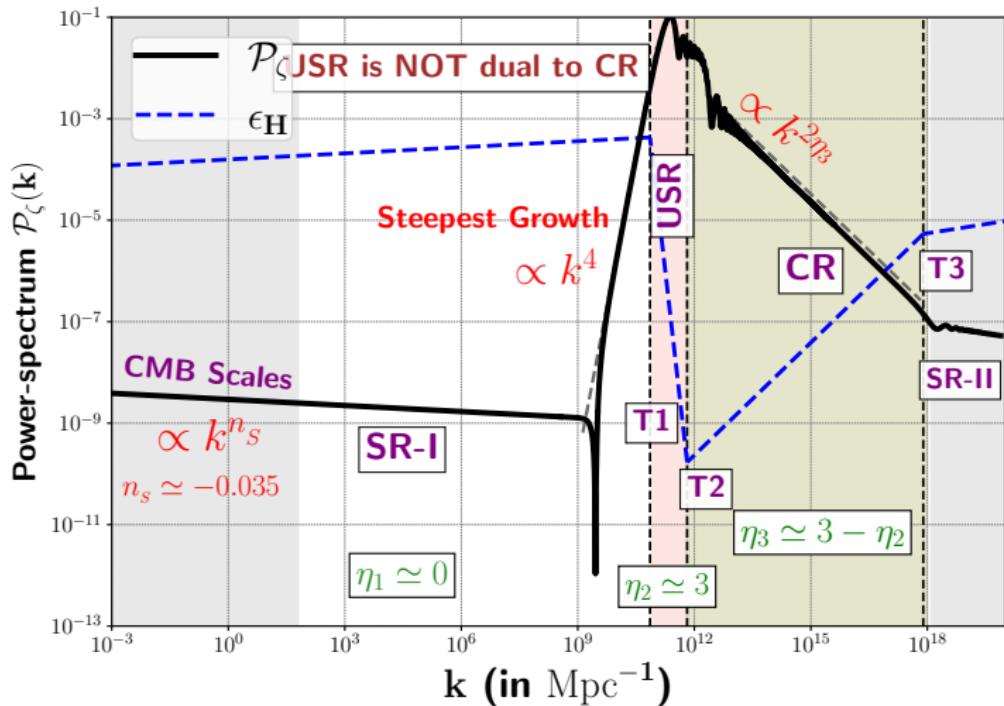
# Noise-Matrix: Three Transitions



# Noise-Matrix: No Duality



# Power-spectrum: No Duality



# From $\mathcal{N} \rightarrow \zeta_{\text{cg}}$ (Stochastic $\delta\mathcal{N}$ Formalism)

By **Stochastic  $\delta\mathcal{N}$  formalism**,

$$\boxed{\zeta_{\text{cg}} \equiv \zeta(\Phi) = \mathcal{N} - \langle \mathcal{N}(\Phi) \rangle}$$

Average no. of e-folds  $\langle \mathcal{N}(\Phi) \rangle = \int_0^\infty \mathcal{N} P_\Phi(\mathcal{N}) d\mathcal{N}$

$$\Rightarrow \boxed{\langle \mathcal{N}(\Phi) \rangle = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n^2}}$$

**Threshold**

$$\boxed{\mathcal{N}_c = \zeta_c + \langle \mathcal{N}(\Phi) \rangle = \zeta_c + \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n^2}}$$

# Relevance for PBH Mass Function

$$\beta(\Phi) \equiv \int_{\zeta_c}^{\infty} P(\zeta_{\text{cg}}) d\zeta_{\text{cg}} = \int_{N_c}^{\infty} P_{\Phi}(N) dN$$

With  $P_{\Phi}(N) = \sum_n A_n e^{-\Lambda_n N}$ ,  $N_c = \zeta_c + \langle N(\Phi) \rangle$

And  $\Rightarrow \langle N(\Phi) \rangle = \sum_m \frac{\mathcal{A}_m(\Phi)}{\Lambda_m^2}$

We get  $\beta(\Phi) = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n} e^{-\Lambda_n [\zeta_c + \langle N(\Phi) \rangle]}$

# PBH Mass Function: Gaussian vs Non-Gaussian

**Stochastic NG**

$$\beta^{\text{NG}}(\Phi) = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n} e^{-\Lambda_n [\zeta_c + \langle \mathcal{N}(\Phi) \rangle]}$$

**Classical Gaussian**

$$\beta^{\text{G}}(\Phi) = \frac{\sigma_{\text{cg}}}{\sqrt{2\pi}\zeta_c} e^{-\frac{\zeta_c^2}{2\sigma_{\text{cg}}^2}}$$

With

$$\sigma_{\text{cg}}^2(\Phi) = \int_{k(\Phi)}^{k_e} \frac{dk}{k} \mathcal{P}_\zeta(k)$$

**Gaussian approx. MIGHT under-estimate PBH abundance by several orders of magnitude**