Phases of scalar and fermionic field theories in thermal Anti-de Sitter Spaces

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The Outline

Introduction

- Effective action $\rightarrow$ Principal tool for studying phases of a QFT
- Perturbative methods, to the leading order, involve the computation of one-loop determinants.
- This talk has two parts
  1. We describe a method for computing one-loop partition function for scalar and fermionic fields in thermal AdS $\rightarrow$ Method of images and Eigenfunctions of Laplacian and Dirac operators on Euclidean AdS.
  2. We employ these to study phases of scalar and fermionic field theories in thermal AdS_{d+1}.
- Changes in infrared behaviour of theories in AdS space lead to deviations from flat space results. AdS space acts like a box that regulates infrared behaviour. These changes are captured by the Effective Potential. Ultraviolet behavior however remains the same.

Various Methods

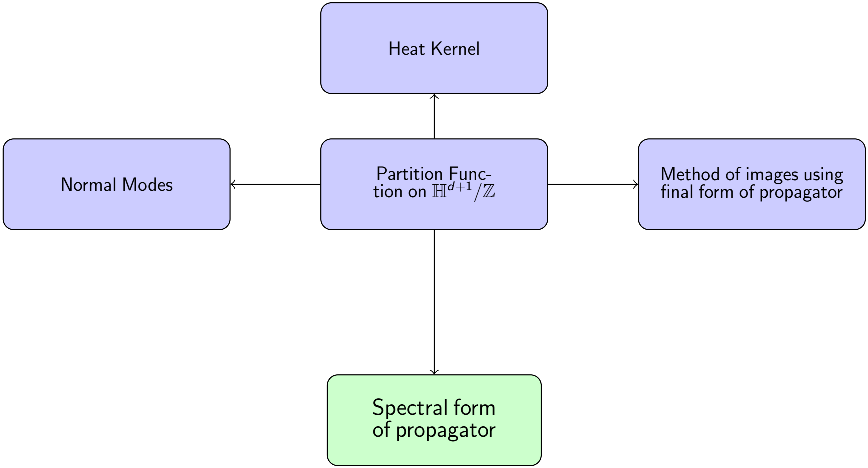


Figure: Partition Function on $\mathbb{H}^{d+1}/\mathbb{Z}$

Geometry of thermal AdS_{d+1}:

- Thermal AdS, defined in terms of global coordinates by compactifying the time circle, leads to $\mathbb{H}^{d+1}/\mathbb{Z}$ identification in Poincaré coordinates.
- The Poincaré metric on AdS_3 after an Euclidean continuation is

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 + dz d\bar{z}\right)$$

- Coordinate transformation

$$z = \tanh\rho\ e^{t+i\theta} \quad , \quad \bar{z} = \tanh\rho\ e^{t-i\theta} \quad \text{and} \quad y = \frac{e^t}{\cosh\rho}$$

- Metric obtained in global coordinates

$$ds^2 = \cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2, \qquad 0 < \rho < \infty, \quad 0 \le \phi \le 2\pi$$

- The identification $t \sim t + \beta$ and $\phi \sim \phi + \theta$ with $\tau = \frac{1}{2\pi}(\theta + i\beta)$ translates into the identification that leads to thermal AdS or $\mathbb{H}^3/\mathbb{Z}$ (can be generalized for all d).
- The action of $\gamma^n \in \mathbb{Z}$ on coordinates is

$$\gamma^n(y, z) = (e^{-n\beta} y, e^{2\pi i n\tau} z)$$

Methodology and Basic Setup

- For a given action

$$S = -\int d^{d+1}x\sqrt{g}\left[\bar{\psi}(D + m_f + g\phi)\psi + \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)\right]$$

- Effective potential

$$\begin{aligned} V_{eff}(\phi_{cl}) &= -\frac{1}{\mathcal{V}_{d+1}}\log Z_f^{(1)} - \frac{1}{\mathcal{V}_{d+1}}\log Z_b^{(1)} + V(\phi_{cl}) \\ &= -\frac{1}{\mathcal{V}_{d+1}}\mathrm{tr}\log[D + M_f(\phi_{cl})] + \frac{1}{2\mathcal{V}_{d+1}}\mathrm{tr}\log[-\Box_E + V''(\phi_{cl})] + V(\phi_{cl}) \end{aligned}$$

- log of the trace can be obtained by integrating the following:

$$\boxed{\frac{1}{2\mathcal{V}_{d+1}}\mathrm{tr}\left[\frac{1}{-\Box_E + V''(\phi_{cl})}\right] \quad \text{and} \quad \frac{1}{\mathcal{V}_{d+1}}\mathrm{tr}\left[\frac{1}{D + M_f(\phi_{cl})}\right]}$$

- Solutions of eigenvalue equations corresponding to respective differential operators:

$$\psi_{\vec{k},\lambda}(\vec{x}, y) = \phi_\lambda(y)e^{\pm i\vec{k}.\vec{x}} \ , \ \phi_\lambda(ky) = (ky)^{d/2}K_{i\lambda}(ky) \rightarrow \text{ Scalar wavefunction}$$

$$\psi_{\vec{k},\lambda}(\vec{x}, y) = \begin{pmatrix} \tilde{\psi}_+(ky) \\ \frac{i\Gamma_j k_j}{k}\tilde{\psi}_-(ky) \end{pmatrix} e^{i\vec{k}.\vec{x}} \ , \ \tilde{\psi}_\pm(ky) = (ky)^{\frac{d+1}{2}}K_{i\lambda\mp\frac{1}{2}}(ky) \rightarrow \text{ Spinor wavefunction}$$

- generalized eigenfunctions in thermal AdS obey respective periodicities under thermal identification

$$\boxed{\Psi_{\vec{k},\lambda}(x) = \frac{1}{\mathcal{N}} \sum_{n=-\infty}^{\infty} \mathcal{R}(\gamma^n)\ \psi_{\vec{k},\lambda}(\gamma^n x)}$$

- where $\mathcal{R}(\gamma^n)$ is a one dimensional representation of the group $\mathbb{Z}$ and can be written as $\mathcal{R}(\gamma^n) = e^{2\pi i n a}$ with $a = 1$ for bosons and $a = \frac{1}{2}$ for fermions.

- $\mathcal{N}$ is a normalization constant which regularizes the sum.

Partition Functions

Scalar Field:

- Trace at zero temperature:

$$\boxed{\frac{1}{L^2}\mathrm{tr}\left[\frac{1}{-\Box_E + V''(\phi_{cl})}\right] = \frac{\mathcal{V}_{d+1}}{L^{d+1}}\frac{\Gamma\left(d/2+\nu\right)\Gamma(1/2-d/2)}{\Gamma\left(1-d/2+\nu\right)(4\pi)^{(d+1)/2}}}$$

- One-loop partition function for even d at finite temperature:

$$\boxed{\log Z = \sum_{n=1}^{\infty}\frac{e^{-n\beta\nu}}{n}\prod_{i=1}^{d/2}\frac{e^{-n\beta}}{|1-e^{2\pi i n\tau_i}|^2}}$$

- One-loop partition function for odd d at finite temperature:

$$\boxed{\log Z = \sum_{n=1}^{\infty}\frac{e^{-n\beta(1/2+\nu)}}{n|1-e^{-n\beta}|}\prod_{i=1}^{(d-1)/2}\frac{e^{-n\beta}}{|1-e^{2\pi i n\tau_i}|^2}}$$

Fermion Field:

- Trace at zero temperature:

$$\boxed{\mathrm{tr}\left[\frac{1}{D+M_f}\right] = \mathrm{sgn}(M_f)\,\frac{\mathcal{V}_{d+1}2^{\frac{d+1}{2}}}{(4\pi)^{(d+1)/2}}\,\frac{\Gamma\left(\frac{d+1}{2}+|M_f|\right)\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{d}{2}+|M_f|\right)}}$$

- One-loop partition function for even d at finite temperature:

$$\boxed{\log Z_\tau^{(1)} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n\beta|M_f|}\prod_{i=1}^{d/2}\frac{2e^{-n\beta}}{|1-e^{2\pi i n\tau_i}|^2}}$$

- One-loop partition function for odd d at finite temperature:

$$\boxed{\log Z_\tau^{(1)} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{2e^{-n\beta(|M_f|+1/2)}}{|1-e^{-n\beta}|}\prod_{i=1}^{(d-1)/2}\frac{2e^{-n\beta}}{|1-e^{2\pi i n\tau_i}|^2}}$$

- Unlike finite temperature part, zero temperature contribution to one-loop correction is proportional to the divergent $\mathrm{Vol}(\mathbb{H}^d/\mathbb{Z})$.
- Regularized volume can be obtained using Euclidean metric in global coordinates with radial coordinate cutoff $\rho = \rho_0$ and thermal AdS period $\theta = \beta$.

$$\mathrm{Vol}(\mathbb{H}^2/\mathbb{Z}) = -\beta \quad , \quad \mathrm{Vol}(\mathbb{H}^3/\mathbb{Z}) = -\frac{\pi\beta}{2} \quad , \quad \mathrm{Vol}(\mathbb{H}^4/\mathbb{Z}) = \frac{2\pi\beta}{3}$$

- Or using dimensional regularization [C.R. Graham, 2000; D.E. Diaz and H. Dorn '07] regularized volume of $\mathbb{H}^{d+1}/\mathbb{Z}$ is $\mathcal{V}_{d+1} = \mathcal{V}(\mathbb{H}^{d+1})\beta/(2\pi)$ where

$$\begin{aligned} \mathcal{V}(\mathbb{H}^{d+1}) &= \frac{(-\pi)^{d/2}}{\Gamma((d+2)/2)}\left[\psi(1+d/2) - \log\pi\right] \qquad \text{for even } d \\ &= (-1)^{(d+1)/2}\frac{\pi^{(d+2)/2}}{\Gamma((d+2)/2)} \qquad \text{for odd } d \end{aligned}$$

- The phases depend on the sign of the renormalized volume: results using the two regularization schemes match for odd d but differ for even d.

Applications: Phases of single scalar model in AdS_3

- Lagrangian for the ϕ^4 theory

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4$$

- The effective potential for the ϕ^4 at finite temperature can thus be written as

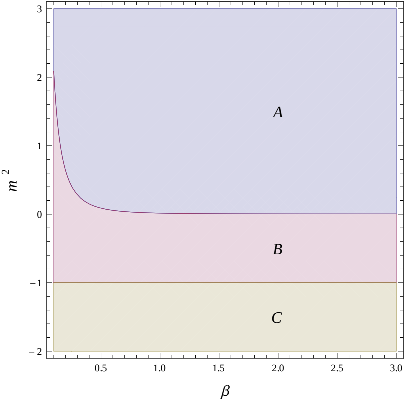
$$V_{eff}(\phi_{cl}) \quad = \quad V(\phi_{cl}) - \frac{1}{\mathcal{V}_{d+1}}[\log Z^{(1)} + \log Z_\tau^{(1)}]$$

- where

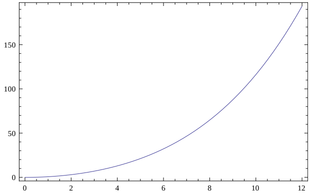
$$\nu = \sqrt{1+M^2} = \left[1 + \frac{\lambda}{2}\phi_{cl}^2 + m^2\right]^{1/2} .$$

- The complete expression for the one-loop corrected effective potential is

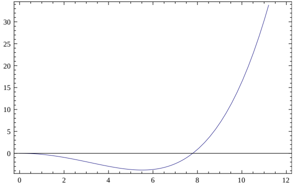
$$V_{eff}(\phi_{cl}) = \frac{1}{2} m^2 \phi_{cl}^2 + \frac{\lambda}{4!}\phi_{cl}^4 - \frac{\nu^3}{12\pi} + \frac{2}{\pi\beta}\sum_{n=1}^{\infty} \frac{e^{-\beta n(1+\nu)}}{n(1-e^{-\beta n})^2}$$



(a) Phases on the $\beta - m^2$ plane



(b) A: $m^2 = 1.5$, $\beta = 2.0$



(c) B: $m^2 = -0.5$, $\beta = 2.0$

Figure: Phases and potentials for $\lambda = 0.1$.

Applications: Phases of Large N $O(N)$ model in AdS_3

- Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}m^2(\phi^i)^2 + \frac{\lambda}{4}\left[(\phi^i)^2\right]^2 \qquad \text{where} \quad i = 1, \cdots, N$$

$$\xrightarrow{Large\ \ N} \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{m^2}{2}(\phi^i)^2 - \frac{1}{2\lambda}\sigma^2 + \frac{1}{\sqrt{N}}\sigma(\phi^i)^2$$

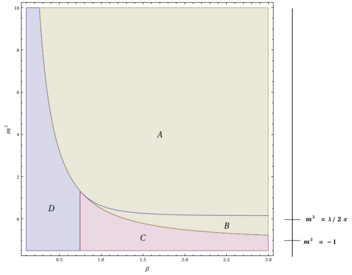
- Effective potential to the leading order in $1/N$

$$\boxed{V_{eff}(\phi^i_{cl}, \sigma_{cl}) = N\left[\frac{M^2}{2}(\phi^i_{cl})^2 - \frac{(M^2-m^2)^2}{8\lambda} - \frac{1}{\mathcal{V}_{d+1}}(\log Z^{(1)} + \log Z^{(1)}_\tau)\right]}$$

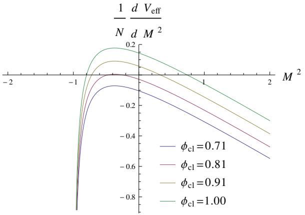
- for AdS_3

$$\boxed{\frac{V^0_{eff}(M^2, \phi^i_{cl})}{N} = -\frac{\left(M^2-m^2\right)^2}{8\lambda} + \frac{1}{2}(\phi^i_{cl})^2 M^2 - \frac{(1+M^2)^{\frac{3}{2}}}{12\pi} - \frac{1}{\mathcal{V}_3}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-n\beta\left(1+\sqrt{1+M^2}\right)}}{|1-e^{2\pi i n\tau}|^2}}$$

where $M^2 = m^2 + 2\sigma_{cl}$.



(a) Phases on the $\beta - m^2$ plane.



(b) Roots of the saddle point equation, $m^2 = -0.8$, $\beta = 1$, $n = 10$ and different values of ϕ_{cl}.

Figure: Phases and roots of saddle point equation for AdS_3 with negative renormalized volume.

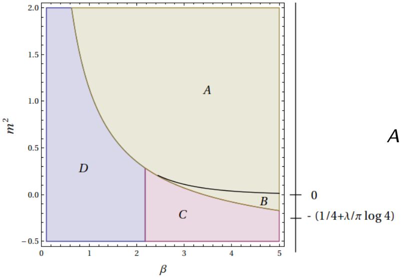
A: $m^2 = 4.0, \beta = 1.7$ B: $m^2 = -0.32, \beta = 2.5$ C: $m^2 = -0.9, \beta = 1.5$ D: $m^2 = 0.5, \beta = 0.4$

Figure: Representative plots of the potential corresponding to different regions for $\lambda = 1$.

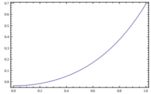
$\longrightarrow$ There exists a region in $\beta - m^2$ plane for the theory where both symmetry breaking and symmetry preserving phases coexist.

$\longrightarrow$ One gets a broken symmetry phase at high temperatures.

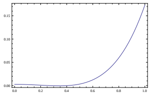
Applications: Phases of Large N $O(N)$ model in AdS_2



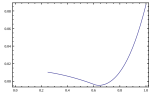
(a)



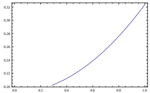
A: $m^2 = 1.1$, $\beta = 3.5$



B: $m^2 = -0.1$, $\beta = 4.7$



C: $m^2 = -0.3$, $\beta = 3.0$



D: $m^2 = -0.4$, $\beta = 1.2$

Figure: (a) Phases in AdS_2 on the $\beta - m^2$ plane. Representative potential plots (extreme right) corresponding to different regions for $\lambda = 0.5$.

$\longrightarrow$ In finite temperature theory in AdS_2 there occurs a symmetry breaking phase, unlike flat space where Coleman-Mermin-Wagner theorem [N.D. Mermin and H. Wagner, 1966; S.R. Coleman, 1973] prohibits continuous symmetry breaking (also noted in [T. Inami and H. Ooguri, 1985] and for large N $O(N)$ model in [Carmi et al. '19]).

Applications: Phases of Yukawa Model in AdS_3

- The general form of the effective potential is

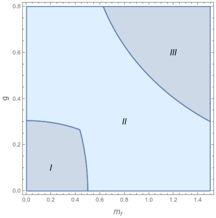
$$\boxed{\begin{aligned} V_{eff} = \frac{1}{2} m_s^2 \phi_{cl}^2 + \frac{\lambda_3}{3!}\phi_{cl}^3 + \frac{\lambda_4}{4!}\phi_{cl}^4 + \lambda_1 \phi_{cl} - \frac{1}{2\mathcal{V}_{d+1}} \int_{M_s^2}^{\infty} \mathrm{tr}\left[\frac{1}{-\Box_E + M_s^2}\right] dM_s^2 \\ -\frac{1}{\mathcal{V}_{d+1}} \int_0^{M_f} \mathrm{tr}\left[\frac{1}{D + M_f}\right] dM_f + \text{counterterms} \end{aligned}}$$

- For AdS_3 we get

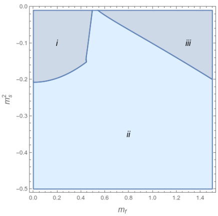
$$\boxed{V_{eff} = \frac{1}{2} m_s^2 \phi_{cl}^2 + \frac{\lambda_3}{3!}\phi_{cl}^3 + \frac{\lambda_4}{4!}\phi_{cl}^4 + \lambda_1 \phi_{cl} - \frac{\nu^3}{12\pi} + \frac{1}{2\pi}\left(\frac{|M_f|^3}{3} - \frac{|M_f|}{4}\right)}$$

where $M_s^2 = m_s^2 + \lambda_3 \phi_{cl} + \frac{\lambda_4}{2}\phi_{cl}^2$ and $M_f = m_f + g\phi_{cl}$.

- zero temperature results



(a)



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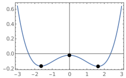
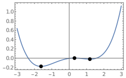
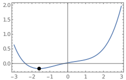
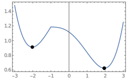




I: $m_f = 0.05$, $g = 0.05$ II: $m_f = 1$, $g = 0.4$





III: $m_f = 1.2$, $g = 0.6$ B: $\beta = 1$, $m_s^2 = -0.2$

Figure: Regions in m_f-g plane for $m_s^2 = -0.2$ (a), and m_f-m_s^2 plane for $g = 0.3$ (b). Representative potentials (extreme right) at zero temperature. B is potential plot at finite temperature for region B in the first finite temperature phase plot.

- At finite temperature

$$\boxed{V_{eff} = V^0_{eff} + \frac{2}{\pi\beta}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-n\beta\left(1+\sqrt{1+M_s^2}\right)}}{|1-e^{-n\beta}|^2} - \frac{4}{\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{e^{-n\beta(1+|M_f|)}}{|1-e^{-n\beta}|^2}}$$

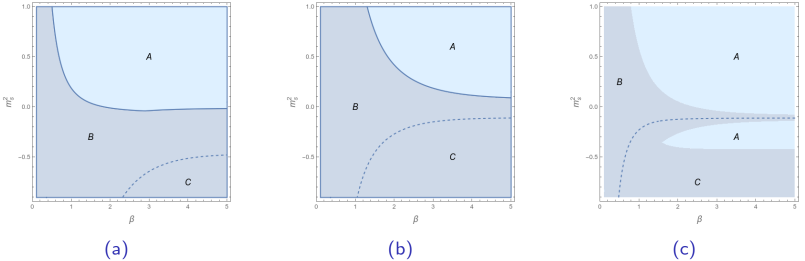


Figure: Phase plots in $m_s^2 - \beta$ plane for (a) $m_f = 0.2$, $g = 0.2$ (b) $m_f = 0.1$, $g = 0.4$ (c) $m_f = 1$, $g = 0.8$ which are points in region I, II and III of the zero temperature plot respectively.

Applications: Phases of Gross Neveu Model in $AdS_{3,2}$

- Effective potential as function of σ_{cl} at leading order in $1/N$

$$\boxed{\frac{V_{\it eff}}{N} = -\frac{\sigma_{cl}^2}{2g} - \mathrm{tr}\log\left(D + m_f + \sigma_{cl}\right) \xrightarrow{AdS_3} -\frac{\sigma_{cl}^2}{2g} + \frac{1}{2\pi}\left(\frac{1}{3}|m_f + \sigma_{cl}|^3 - \frac{1}{4}|m_f + \sigma_{cl}|\right)}$$

- At finite temperature

$$\boxed{\frac{V_{\it eff}}{N} = \frac{V^0_{\it eff}}{N} - \frac{4}{\pi\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{e^{-n\beta(1+|m_f+\sigma_{cl}|)}}{|1-e^{-n\beta}|^2}.}$$

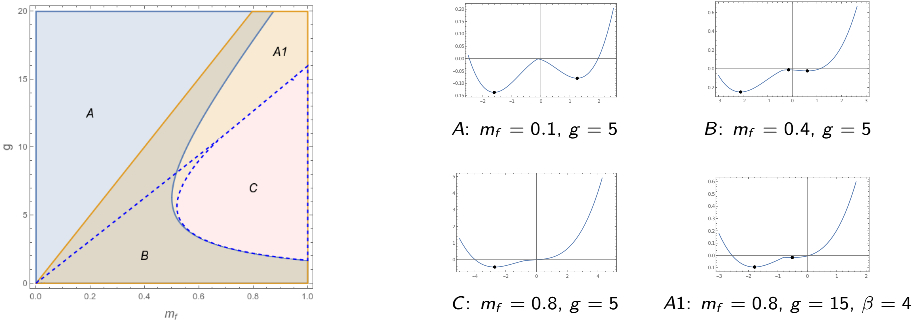
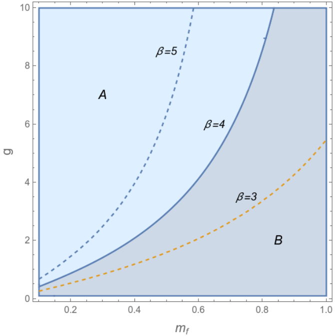
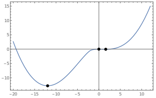


Figure: Regions (left) corresponding to various representative potentials (right). Solid lines correspond to boundaries at zero temperature and the dashed lines for $\beta = 4$ and $n = 10$.

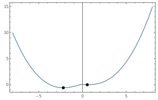
$\longrightarrow$ The discrete chiral symmetry, restored at high temperatures in flat space [K.G. Klimenko,1988; Rosenstein, 1989], remains broken at all temperatures in AdS_3. Also, no first order transition exists.



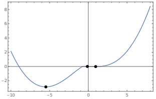
(a)



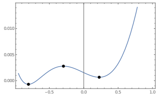
$T = 0,\ m_f = 2,\ g = 10$



$A\colon\ m_f = 0.2,\ g = 8$



$B\colon\ m_f = 0.8,\ g = 8$



$R\colon\ m_f = 0.95,\ g = 50,\ \beta = 4$

Figure: Regions in (m_f, g) space (left) and potential plots (right). The solid boundary separates the regions A and B for $\beta = 4$. The corresponding boundaries for $\beta = 3, 5$ appear as dashed lines.

Summary

- We gave a derivation for one-loop partition functions using eigenfunctions of Laplacian and Dirac operators in Euclidean AdS and method of images applied to Green's function.
- We studied phases of scalar and fermionic theories on thermal AdS_{d+1} and identified regions in corresponding parameter spaces for $d = 1, 2, 3$.
- We confirmed for a finite temperature theory in AdS for the $O(N)$ model there occurs a symmetry breaking phase in two dimensions, in contrast to flat space where the Coleman-Mermin-Wagner theorem prohibits continuous symmetry breaking.
- Scalars can have negative mass upto the Breitenlohner-Freedman (BF) bound $\rightarrow$ Unlike flat space, there exists a region in AdS where both symmetry breaking and symmetry preserving phases coexist.
- Symmetry breaking occurs at high temperature for cases with negative renormalized volume.
- For the Yukawa theories, for all cases at zero temperature we found a phase boundary where the two minima exchange dominance. At finite temperature this is observed in $\mathrm{AdS}_{2,3}$.
- The discrete chiral symmetry in the Gross Neveu model, restored at high temperatures in flat space, remains broken at all temperatures in $\mathrm{AdS}_{2,3}$.

Further Directions of Work

- Further research involves other theories of fermion and vector fields in thermal AdS spaces.
- An interesting exercise would be to consider asymptotically AdS black hole geometry.
- Another direction of research is the study of correlation functions in thermal AdS and to understand the implications of this study on dual boundary theory.

Acknowledgements:

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Thank You!

- For scalars the zero temperature trace can be computed as follows

$$
\begin{aligned}
& \frac{1}{L^2}\mathrm{tr}\left[\frac{1}{-\Box_E + V''(\phi_{cl})}\right] \\
= \quad & \frac{1}{L^2}\int d^{d+1}x\sqrt{g}\int d\lambda\ \mu(\lambda)\int \frac{d^d k}{(2\pi)^d}\frac{1}{(L^{d+1}k^d)}\langle\lambda, k|\left[\frac{1}{-\Box_E + V''(\phi_{cl})}\right]|y,\vec{x}\rangle\langle y,\vec{x}|\lambda, k\rangle \\
= \quad & \frac{1}{L^{d+1}}\int d^{d+1}x\sqrt{g}\int\frac{d\lambda\ \mu(\lambda)}{\lambda^2+\nu^2}\int\frac{d^d k}{(2\pi)^d}y^d K^2_{i\lambda}(ky) \\
= \quad & \frac{\mathcal{V}_{d+1}}{L^{d+1}}\frac{\Gamma\left(d/2+\nu\right)\Gamma(1/2-d/2)}{\Gamma\left(1-d/2+\nu\right)(4\pi)^{(d+1)/2}}
\end{aligned}
$$

- Where $\nu = \sqrt{\left(\frac{d}{2}\right)^2 + L^2 V''(\phi_{cl})}$ and $\mu(\lambda) = \frac{2\lambda}{\pi^2}\sinh(\pi\lambda)$

- This expression has been derived using various approaches before, for example in [C. P. Burgess and C. A. Lutken, 1985 ; R. Camporesi, 1990 etc].
- normalizations

$$
\begin{aligned}
\int d^{d+1}x\sqrt{g}|x\rangle\langle x| \quad &= \quad 1 \qquad ; \qquad |x\rangle = |\vec{x}\rangle\otimes|y\rangle \\
\langle y,\vec{x}|\lambda,\vec{k}\rangle \quad &= \quad e^{i\vec{k}.\vec{x}}(ky)^{d/2}K_{i\lambda}(ky)
\end{aligned}
$$

- Thus

$$
\int\frac{d^d k}{(2\pi)^d}\frac{1}{(L^{d+1}k^d)}\int d\lambda\ \mu(\lambda)\langle\lambda,\vec{k}|\lambda',\vec{k}'\rangle = 1
$$

- For fermions

$$\mu(\lambda) = \frac{1}{\pi\Gamma\left(\frac{1}{2}+i\lambda\right)\Gamma\left(\frac{1}{2}-i\lambda\right)} = \frac{1}{\pi^2}\cosh(\pi\lambda)$$

- We can thus compute the required zero temperature fermion trace as follows

$$\begin{aligned}
\mathrm{tr}\left[\frac{1}{D+M_f}\right] &= \int d^{d+1}x\sqrt{g}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^d}\int_{-\infty}^{\infty}\frac{d\lambda\ \mu(\lambda)}{i\lambda+M_f}\psi^\dagger_{\vec{k},\lambda}(\vec{x},y)\psi_{\vec{k},\lambda}(\vec{x},y) \\
&= 2^{\frac{d-1}{2}}M_f\int d^{d+1}x\sqrt{g}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^d}\int_{-\infty}^{\infty}\frac{d\lambda\ \mu(\lambda)}{\lambda^2+M_f^2}\times \\
&\times\quad (ky)^{d+1}\left[K_{i\lambda-\frac{1}{2}}(ky)K_{-i\lambda-\frac{1}{2}}(ky)+K_{i\lambda+\frac{1}{2}}(ky)K_{-i\lambda+\frac{1}{2}}(ky)\right] \\
&= \frac{\mathcal{V}_{d+1}2^{\frac{d+1}{2}}M_f}{(4\pi)^{(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2+M_f^2}\frac{\Gamma\left(\frac{d+1}{2}+i\lambda\right)\Gamma\left(\frac{d+1}{2}-i\lambda\right)}{\Gamma\left(\frac{1}{2}+i\lambda\right)\Gamma\left(\frac{1}{2}-i\lambda\right)} \\
&= \mathrm{sgn}(M_f)\frac{\mathcal{V}_{d+1}2^{\frac{d+1}{2}}}{(4\pi)^{(d+1)/2}}\frac{\Gamma\left(\frac{d+1}{2}+|M_f|\right)\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{d}{2}+|M_f|\right)}
\end{aligned}$$

To make the method of images manifest consider the two-point function,

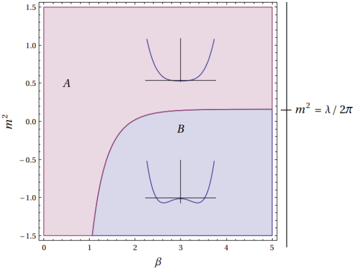
$$
\begin{aligned}
& \langle x| \left[\frac{1}{-\Box_E + V''(\phi_{cl})} \right] |x'\rangle \\
= \quad & \frac{1}{\mathcal{N}^2} \sum_{n,n'} \int \frac{d^2k}{(2\pi)^2} \int \frac{d\lambda\ \mu(\lambda)}{\lambda^2+\nu^2} (e^{-n\beta} y)(e^{-n'\beta} y') K_{i\lambda}(ke^{-n\beta} y) K_{i\lambda}(ke^{-n'\beta} y') e^{-i\vec{k}.(\gamma^n \vec{x})} e^{i\vec{k}.(\gamma^{n'} \vec{x}')} \\
= \quad & \frac{1}{\mathcal{N}} \sum_{n} \int \frac{d^2k}{(2\pi)^2} \int \frac{d\lambda\ \mu(\lambda)}{\lambda^2+\nu^2} (y)(e^{-n\beta} y') K_{i\lambda}(ky) K_{i\lambda}(ke^{-n\beta} y') e^{-i\vec{k}.\vec{x}} e^{i\vec{k}.(\gamma^n \vec{x}')} \\
= \quad & \frac{1}{\mathcal{N}} \sum_{n} G(x, \gamma^n x')
\end{aligned}
$$

Thus,

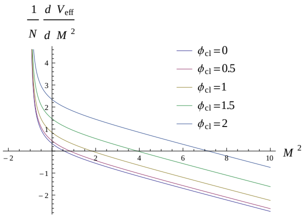
$$
\begin{aligned}
\mathrm{tr} \left[\frac{1}{-\Box_E + V''(\phi_{cl})} \right] &= \frac{1}{\mathcal{N}} \sum_n \int_{\mathbb{H}^3} d^3x \sqrt{g}\ G(x, \gamma^n x) \\
&= \sum_n \int_{\mathbb{H}^3/\mathbb{Z}} d^3x \sqrt{g}\ G(x, \gamma^n x)
\end{aligned}
$$

Each copy of the fundamental region gives the same answer. This cancels the normalization $\mathcal{N}$ in the denominator.

Applications: Phases of Large N $O(N)$ model in AdS_3 with positive volume



(a) Phases on the $\beta - m^2$ plane.



(b) Roots of the saddle point equation, $m^2 = -0.5$, $\beta = 1$, $n = 10$ and different values of ϕ_{cl}.

Figure: Phases and roots of saddle point equation for AdS_3 with positive renormalized volume.

- The saddle point equation has a solution for all values of ϕ_{cl}, m^2 and β.
- The corresponding C and D regions of the negative volume case are thus absent here.
- The phase boundary given by $M^2 = 0$ asymptotes to $m^2 = \lambda/(2\pi)$.

AdS_4

- Expanding trace at zero temperature and adding counter-terms

$$\frac{V_{eff}(\phi^i_{cl},\sigma_{cl})}{N} = -\frac{(M^2-m^2)^2}{8\lambda} + \frac{M^2}{2}(\phi^i_{cl})^2 - \frac{1}{\mathcal{V}_{d+1}}(\log Z^{(1)} + \log Z^{(1)}_\beta) + M^2\frac{\delta m^2}{4\lambda} - M^4\delta\left(\frac{1}{8\lambda}\right)$$

- renormalization conditions (at zero temperature)

$$\frac{1}{N}\frac{\partial}{\partial M^2}V^0_{eff}(\phi^i_{cl},\sigma_{cl})\Big|_{M^2=\phi^i_{cl}=0} = \frac{m^2}{4\lambda} \text{ and } \frac{1}{N}\frac{\partial^2}{\partial (M^2)^2}V^0_{eff}(\phi^i_{cl},\sigma_{cl})\Big|_{M^2=0} = -\frac{1}{4\lambda}$$

- renormalized effective potential at zero temperature

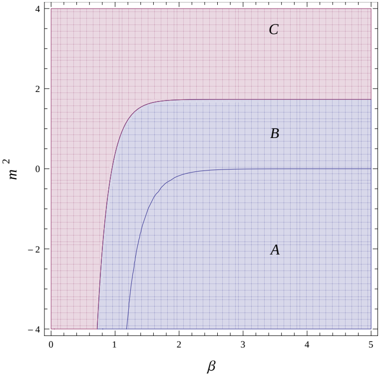
$$\frac{V(M^2,\phi^i_{cl})}{N} = -\frac{(M^2-m^2)^2}{8\lambda} + \frac{1}{2}M^2(\phi^i_{cl})^2 + \frac{1}{2}\int_0^{M^2} dM^2 \mathrm{tr}\left[\frac{1}{-\Box_E+M^2}\right]_{\mathrm{ren}} - \frac{M^4}{96\pi^2}[\psi^{(1)}(1)+\psi^{(1)}(3)]$$

- where,

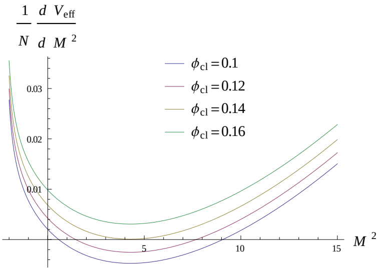
$$\mathrm{tr}\left[\frac{1}{-\Box_E+M^2}\right]_{\mathrm{ren}} = \frac{(2+M^2)}{16\pi^2}\left[\psi^{(0)}\left(\nu-\frac{1}{2}\right)+\psi^{(0)}\left(\nu+\frac{3}{2}\right)+2\gamma-\frac{3}{2}\right].$$

- The saddle point equation

$$\begin{aligned} 0 = \frac{1}{N}\frac{\partial V}{\partial M^2} &= \frac{m^2-M^2}{4\lambda} + \frac{(\phi^i_{cl})^2}{2} + \frac{(2+M^2)}{32\pi^2}\left[\psi^{(0)}\left(\nu-\frac{1}{2}\right)+\psi^{(0)}\left(\nu+\frac{3}{2}\right)+2\gamma-\frac{3}{2}\right] \\ &- \frac{M^2}{48\pi^2}[\psi^{(1)}(1)+\psi^{(1)}(3)] + \frac{3}{4\pi}\sum_{n=1}^{\infty}\frac{e^{-n\beta(\frac{3}{2}+\sqrt{\frac{9}{4}+M^2})}}{|1-e^{-n\beta}|^3\sqrt{\frac{9}{4}+M^2}} \end{aligned}$$



(a) Phases on AdS_4.



(b) Roots of the saddle point equation, $m^2 = -1$, $\beta = 2$, $n = 10$ and different values of ϕ_{cl}.

Figure: Phases and roots of saddle point equation for AdS_4.

- On boundary separating regions A and B, the m^2 value approaches zero for large β as in AdS_2. Corresponding value on boundary separating regions A and C at zero temperature is obtained from the condition that the two roots of saddle point equation coincide. For $\lambda = 70$ this gives $m^2 \sim 1.734$.

Applications: Gross Neveu model in AdS_4.

- Counterterms have the following form $\sigma_{cl}\ \delta\lambda_1 + \frac{1}{2}\sigma_{cl}^2\ \delta\left(\frac{1}{g}\right) + \frac{1}{3!}\sigma_{cl}^3\ \delta\lambda_3 + \frac{1}{4!}\sigma_{cl}^4\ \delta\lambda_4$.
- renormalization conditions at $\sigma_{cl} = 0$

$$\frac{1}{N}\frac{\partial V_{eff}^0}{\partial\sigma_{cl}} = 0 \qquad \frac{1}{N}\frac{\partial^2 V_{eff}^0}{\partial\sigma_{cl}^2} = -\frac{1}{g} \qquad \frac{1}{N}\frac{\partial^3 V_{eff}^0}{\partial\sigma_{cl}^3} = \lambda_3 \qquad \frac{1}{N}\frac{\partial^4 V_{eff}^0}{\partial\sigma_{cl}^4} = \lambda_4\ .$$

- zero temperature effective potential

$$\begin{aligned}
V_{eff}^0 &= -\frac{\sigma_{cl}^2}{2g} + \lambda_3\frac{\sigma_{cl}^3}{3!} + \lambda_4\frac{\sigma_{cl}^4}{4!} - \int_{m_f}^{M_f}\frac{M_f(M_f^2-1)}{4\pi^2}\left(\psi^{(0)}(|M_f|-1) + \psi^{(0)}(|M_f|+2)\right)dM_f \\
&+ \frac{\sigma_{cl}^4}{4!}\left[\frac{3}{2\pi^2}\left(\psi^{(0)}(m_f-1) + \psi^{(0)}(m_f+2)\right) + \frac{9m_f}{2\pi^2}\left(\psi^{(1)}(m_f-1) + \psi^{(1)}(m_f+2)\right)\right. \\
&+ \left.\frac{3}{4\pi^2}\left(3m_f^2-1\right)\left(\psi^{(2)}(m_f-1) + \psi^{(2)}(m_f+2)\right) + \frac{\left(m_f^3-m_f\right)}{4\pi^2}\left(\psi^{(3)}(m_f-1) + \psi^{(3)}(m_f+2)\right)\right] \\
&+ \frac{\sigma_{cl}^3}{3!}\left[\frac{3m_f}{2\pi^2}\left(\psi^{(0)}(m_f-1) + \psi^{(0)}(m_f+2)\right) + \frac{\left(3m_f^2-1\right)}{2\pi^2}\left(\psi^{(1)}(m_f-1) + \psi^{(1)}(m_f+2)\right)\right. \\
&+ \left.\frac{\left(m_f^3-m_f\right)}{4\pi^2}\left(\psi^{(2)}(m_f-1) + \psi^{(2)}(m_f+2)\right)\right] \\
&+ \frac{\sigma_{cl}^2}{2!}\left[\frac{\left(3m_f^2-1\right)}{4\pi^2}\left(\psi^{(0)}(m_f-1) + \psi^{(0)}(m_f+2)\right) + \frac{\left(m_f^3-m_f\right)}{4\pi^2}\left(\psi^{(1)}(m_f-1) + \psi^{(1)}(m_f+2)\right)\right] \\
&+ \sigma_{cl}\left[\frac{\left(m_f^3-m_f\right)}{4\pi^2}\left(\psi^{(0)}(m_f-1) + \psi^{(0)}(m_f+2)\right)\right]
\end{aligned}$$

- leading behavior for large values of σ_{cl}

$$-\int_{m_f}^{M_f} \frac{M_f(M_f^2-1)}{4\pi^2}\left(\psi^{(0)}\left(|M_f|-1\right)+\psi^{(0)}\left(|M_f|+2\right)\right) dM_f \sim -\sigma_{cl}^4 \log(\sigma_{cl})$$

- adding kinetic term for the σ field

$$\mathcal{L}' = \bar{\psi}^i \left(D + m_f + g\sigma\right)\psi^i + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} m_s^2 \sigma^2 + \lambda_3 \frac{\sigma^3}{3!} + \lambda_4 \frac{\sigma^4}{4!}$$

$\rightarrow$ **Gross-Neveu-Yukawa model** [Zinn-Justin, 1991].

- To study the large N behavior we re-scale $\sigma \to \sqrt{N}\sigma$, $g \to g/\sqrt{N}$, $\lambda_3 \to \lambda_3/\sqrt{N}$, $\lambda_4 \to \lambda_4/N$, write $\sigma = \sigma_{cl} + \delta\sigma$ and integrate over the fluctuations

$$\begin{aligned}\frac{V^0_{eff}}{N} &= \frac{1}{2} m_s^2 \sigma_{cl}^2 + \frac{\lambda_3}{3!}\sigma_{cl}^3 + \frac{\lambda_4}{4!}\sigma_{cl}^4 - \frac{1}{2\mathcal{V}_{d+1}} \int_{M_s^2}^{\infty} \mathrm{tr}\left[\frac{1}{-\Box_E + M_s^2}\right] dM_s^2 \\ &- \frac{1}{\mathcal{V}_{d+1}} \int_0^{M_f} \mathrm{tr}\left[\frac{1}{D + M_f}\right] dM_f + \text{counterterms}\end{aligned}$$

- with $M_s^2 = m_s^2 + \lambda_3 \sigma_{cl} + \lambda_4 \sigma_{cl}^2/2$ and $M_f = m_f + g\sigma_{cl}$, which is essentially same as the Yukawa model.