



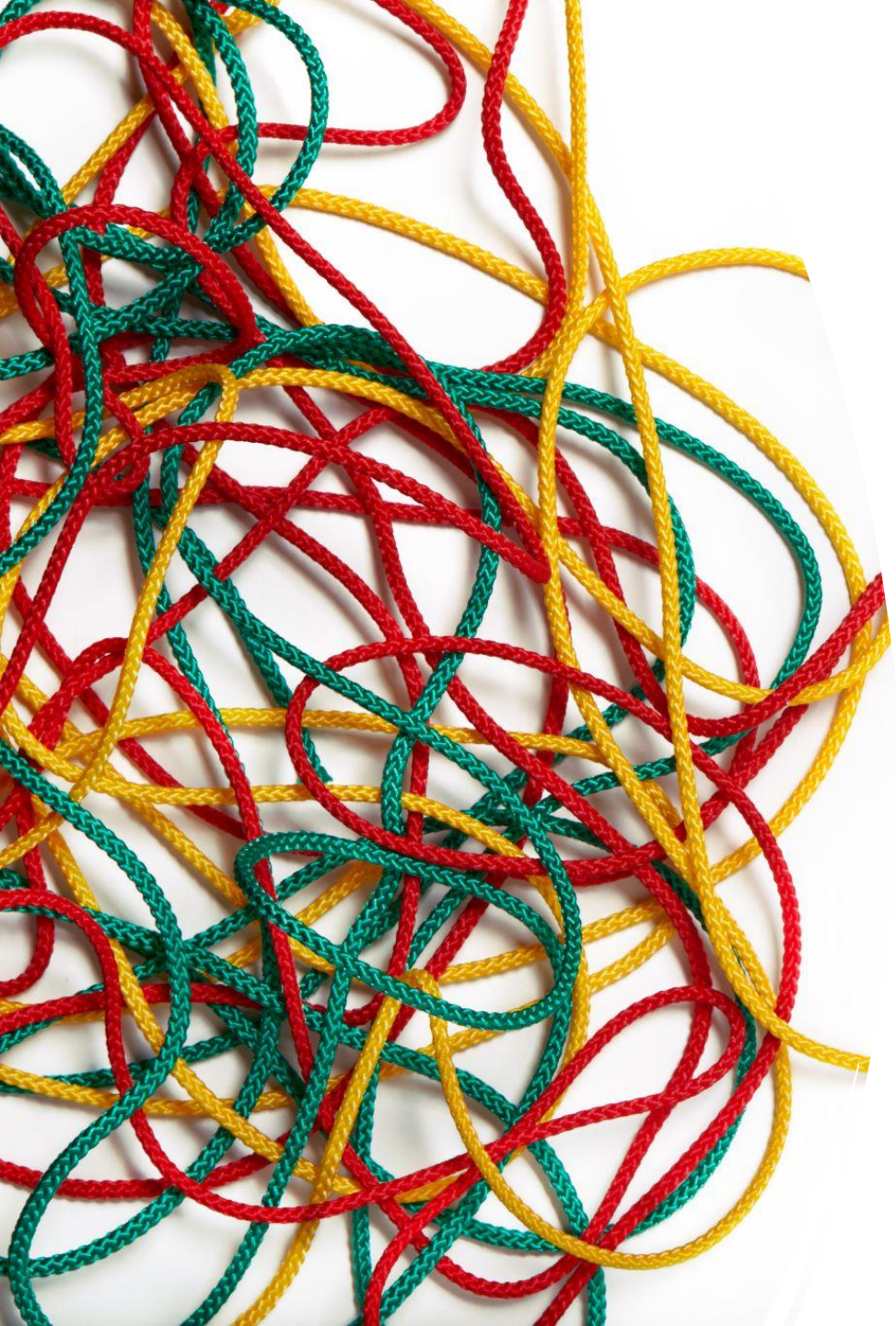
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# A Boltzmann equation approach to string thermodynamics

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# Why string thermodynamics: pheno aspects

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- Heat up a box of anything: stringy modes will eventually be excited.



# Why string thermodynamics: pheno aspects

Review: Cicoli *et al* '23

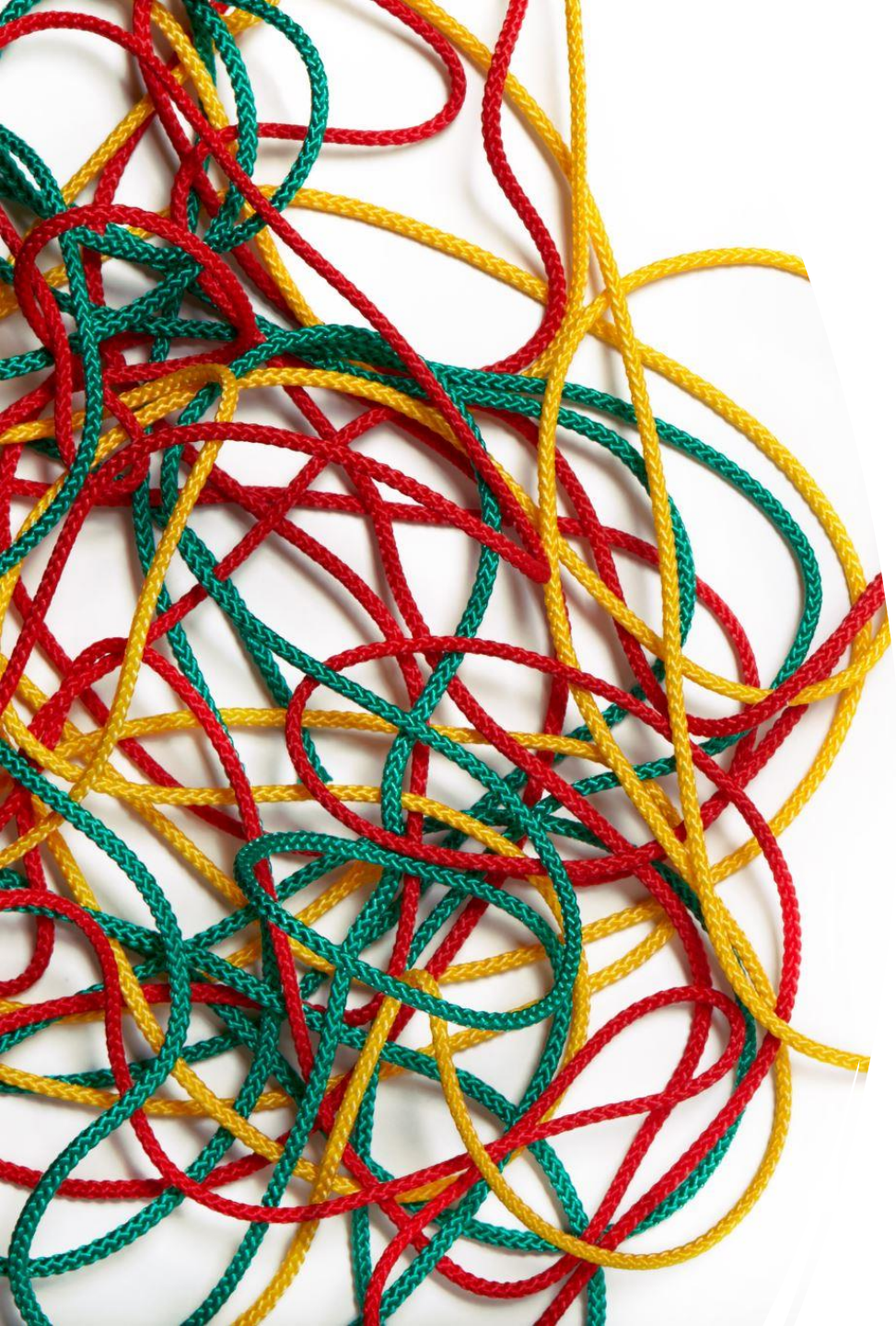
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  - It is common in the early Universe to find energy densities of order one in string units.
  - Influences in reheating, possible GW spectrum, moduli problem...



# Why string thermodynamics: pheno aspects

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- Heat up a box of anything: stringy modes will eventually be excited.
  - It is common in the early Universe to find energy densities of order one in string units.
  - Influences in reheating, possible GW spectrum, moduli problem...
  - Most likely out of equilibrium processes are important!

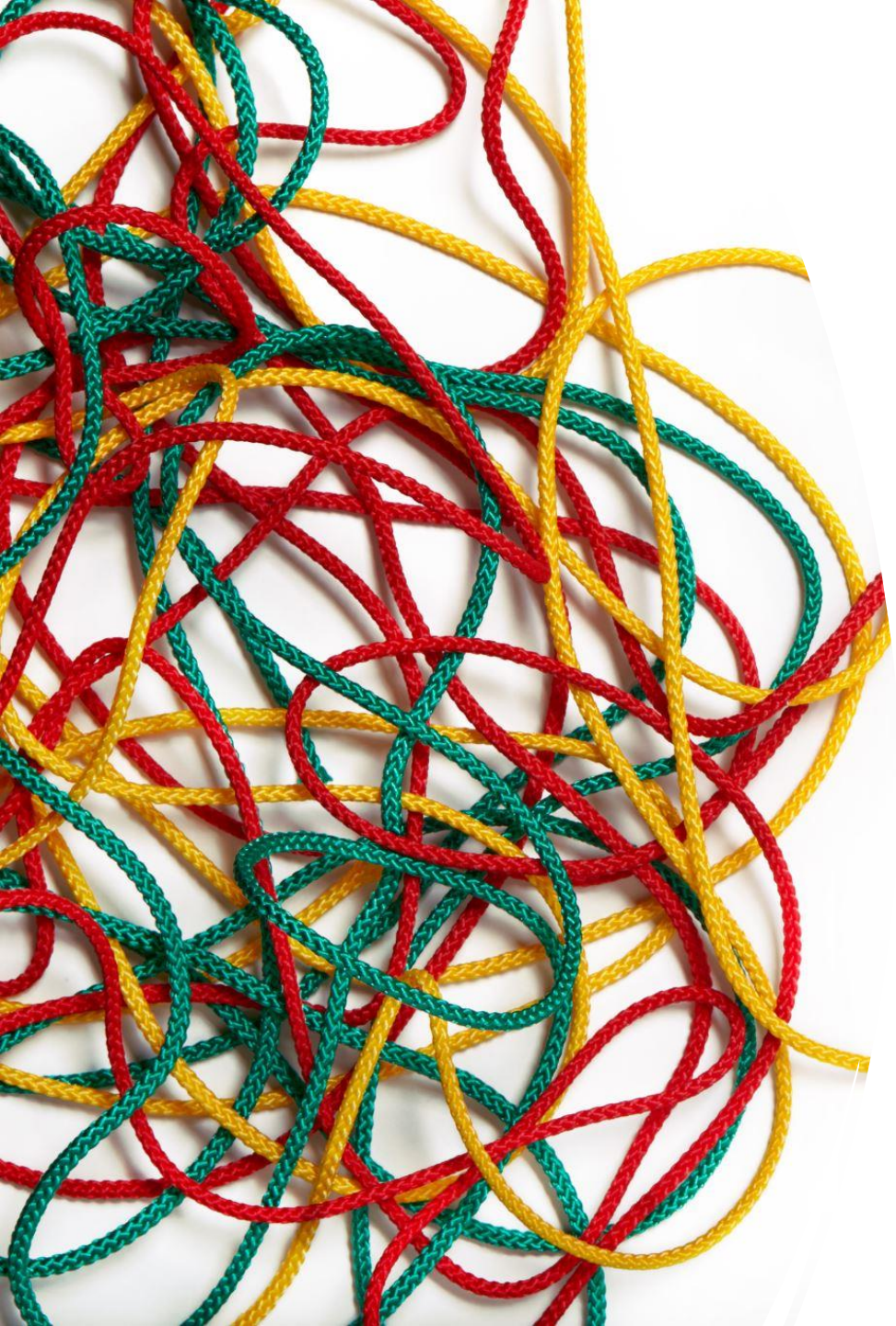


$d=0$  case: Lowe and Thorlacius'95  
 $d=0$  with D-branes: Lee and Thorlacius'97

# Why Boltzmann equations

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- Equilibrium is a subtle concept in presence of gravity:
  - Gravity is dynamical: Jeans instability.
  - Expanding Universe.



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# Why Boltzmann equations

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- An invaluable window to out-of-equilibrium physics.
- Equilibrium is a subtle concept in presence of gravity:
  - Gravity is dynamical: Jeans instability.
  - Expanding Universe.
- Knowledge of interaction rates allows to estimate when (and whether) equilibrium is a good approximation.

# A phase space for string theory

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- Consider a string in a D-dimensional spacetime, with d noncompact directions.
- The phase space is given by the position, noncompact momenta, oscillator level, winding and KK modes.

$$E_l^2 = k^2 + \underbrace{\frac{2}{\alpha'}(N + \tilde{N} - 2) + \sum_{i=1}^{d_c} \left(\frac{n_i}{R_i}\right)^2 + \left(\frac{\omega_i R_i}{\alpha'}\right)^2}_{M^2 = (\mu l)^2}$$

- Hence describe the thermodynamics with a distribution on a generalized phase space  $f(r, k, N, \sigma) \equiv f(E_l, l)$

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- In thermodynamics, we are interested in how the average string looks like.
- Consider the averaged semi-inclusive decay rate:

- $N \leftrightarrow V_1 + N_2$  
$$\frac{F(N, N_2)}{\mathcal{G}(N)} \equiv \frac{1}{\mathcal{G}(N)} \sum_{\Phi_N} \sum_{\Phi_{N_2}} |\langle \Phi_{N_2} | V_1(k) | \Phi_N \rangle|^2$$

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$$F(N, N_2) = \oint_C \frac{dz}{z} z^{-N} \oint_{C_2} \frac{dz_2}{z_2} z_2^{-N_2} \text{Tr} \left[ z^{\hat{N}} V_1^\dagger(k, 1) z_2^{\hat{N}} V_1(k, 1) \right]$$

# Understanding the typical string

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- The string prefers to decay into configurations with small external kinetic energy and with KK and winding mode energies proportional to the length.
- It follows that the typical string is described by its length (approximately, level).

Mañes'01

Chen *et al*'05

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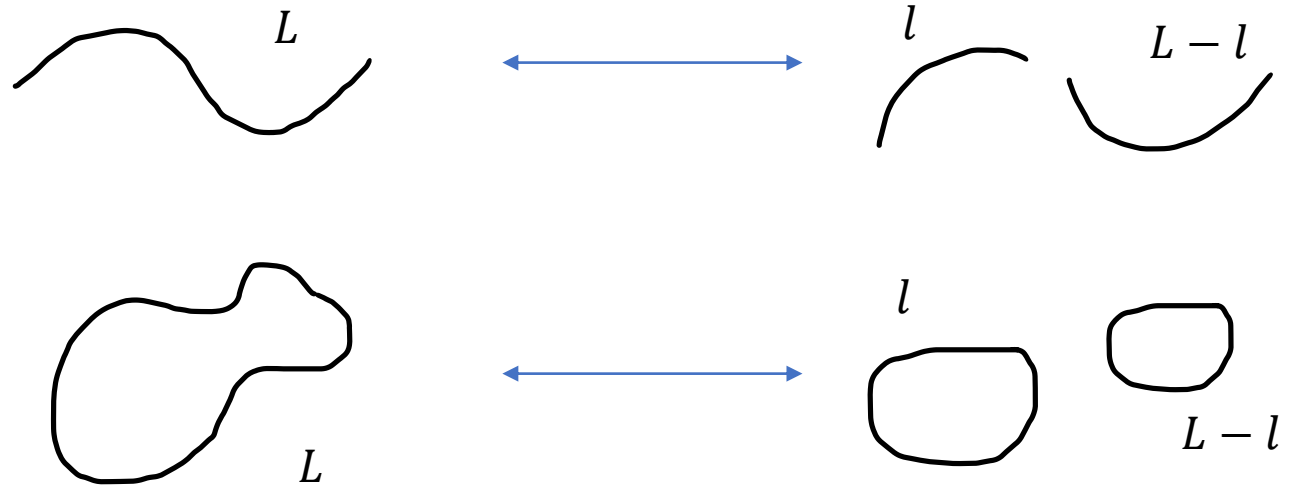
- The string prefers to decay into configurations with small external kinetic energy and with KK and winding mode energies proportional to the length.
- It follows that the typical string is described by its length (approximately, level).
- The decay rate looks like:

Mañes'01

Chen *et al*'05

$$\frac{d\Gamma_o}{dl} \sim g_s ,$$

$$\frac{d\Gamma_{cl}}{dl} \sim g_s^2 L \left( \frac{L}{l(L-l)} \right)^{d/2}$$



# The Boltzmann equation for the typical string

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- Recap: we want to describe the thermodynamics of string theory in terms of the average string, well described by the length and the number of non-compact dimensions.
- We have computed the interaction rates, so we can write:

$$\begin{aligned} \frac{\partial n_c(l)}{\partial t} = & \frac{\kappa}{2} \int_{l_c}^{l-l_c} dl' \left( \frac{n_c(l') l' n_c(l-l')(l-l')}{V} - l n_c(l) \left( \frac{l}{l'(l-l')} \right)^{d/2} \right) + \\ & + \kappa \int_{l+l_c}^{\infty} dl' \left( l' n_c(l') \left( \frac{l'}{l(l'-l)} \right)^{d/2} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V} \right). \end{aligned}$$

## Consistency check

- The equilibrium solution obtained from imposing detailed balance reads

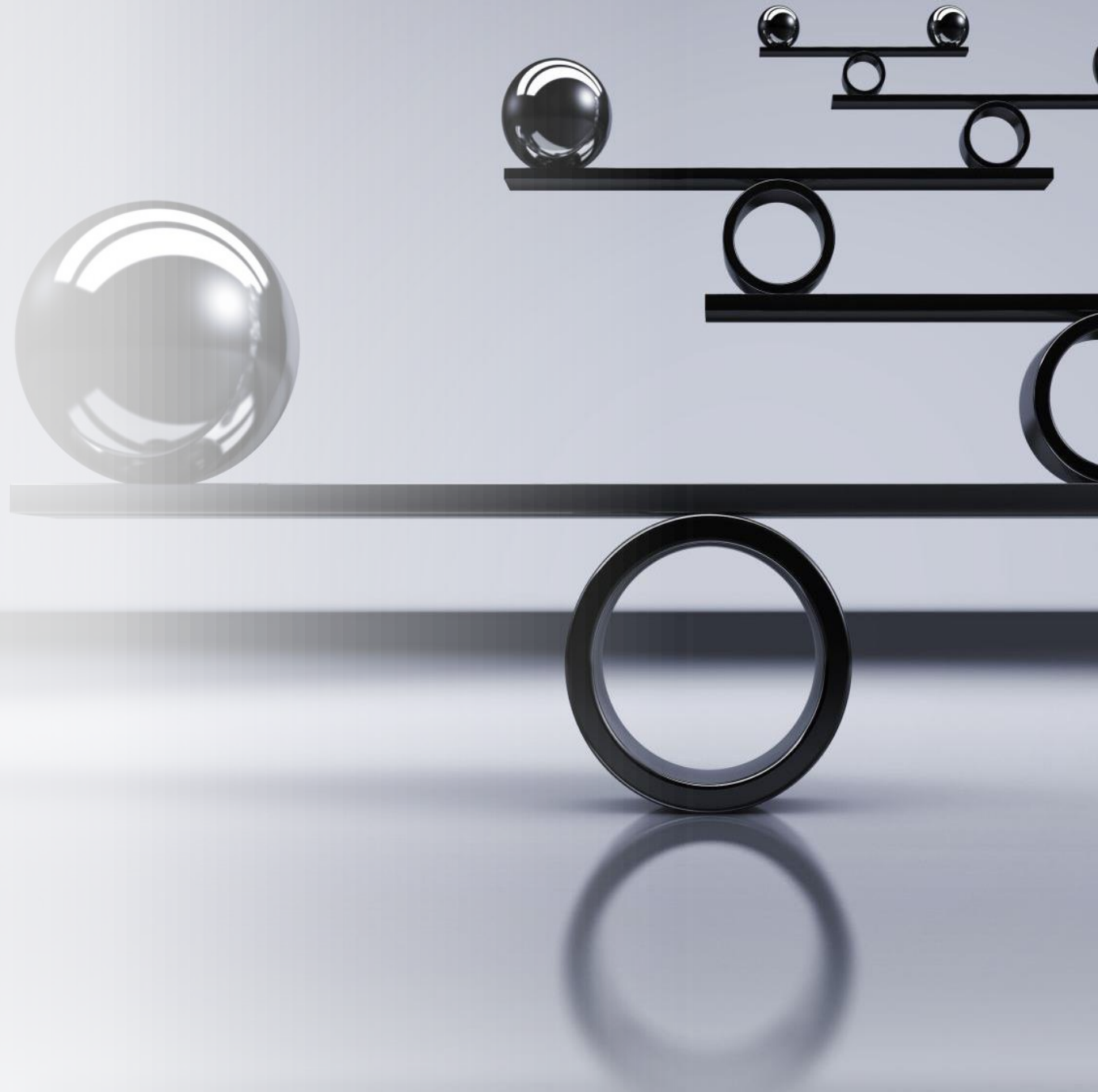
$$n_c(l) = V \frac{e^{-l/L}}{l^{1+d/2}},$$

$$1/L = \beta - \beta_H.$$

- These results agree with general considerations in equilibrium thermodynamics, provided winding and KK modes are taken into account.

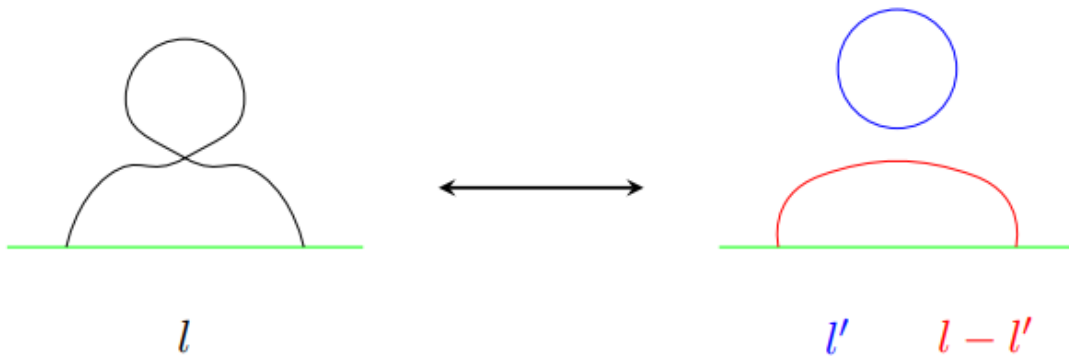
Deo *et al*'89

Deo *et al*'92



# Semiclassical strings: a random walk interpretation

The random walk interpretation allows us to conjecture the form of other interaction rates:



$$\frac{d\Gamma}{dl} \sim g_s^2 \frac{1}{l'^{d/2}} \int_{l_c}^{l-l_c} dl_x \left( \frac{l-l'}{l_x(l-l'-l_x)} \right)^{(d-p)/2}.$$

Non-trivial check: detailed balance must be satisfied *in every interaction* by an equilibrium solution agreeing with the general case at high energies.

# Boltzmann equations with open strings

$$\begin{aligned}\frac{\partial n_c(l)}{\partial t} = & + \frac{b}{2N} V_{\perp} \frac{n_o(l)}{l^{d/2}} - a \frac{N}{V_{\perp}} l n_c(l) + \\ & + \frac{\kappa}{2} \int_{l_c}^{l-l_c} dl' \left( \frac{n_c(l') l' n_c(l-l')(l-l')}{V} - l n_c(l) \left( \frac{l}{l'(l-l')} \right)^{d/2} \right) + \\ & + \kappa \int_{l+l_c}^{\infty} dl' \left( \frac{n_c(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V} \right) + \\ & + \kappa \int_{l+l_c}^{\infty} dl' \left( \frac{(l'-l) n_o(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_o(l'-l)}{V} \right).\end{aligned}$$

$$\begin{aligned}\frac{\partial n_o(l)}{\partial t} = & + a \frac{N}{V_{\perp}} l n_c(l) - \frac{b}{2N} V_{\perp} \frac{n_o(l)}{l^{d/2}} \\ & + \int_{l_c}^{l-l_c} dl' \left( \frac{b}{2N V_{\parallel}} n_o(l') n_o(l-l') - a \frac{N}{V_{\perp}} n_o(l) \right) \\ & + \int_{l+l_c}^{\infty} dl' \left( 2a \frac{N}{V_{\perp}} n_o(l') - \frac{b}{N V_{\parallel}} n_o(l) n_o(l'-l) \right) \\ & + \kappa \int_{l_c}^{l-l_c} dl' \left( \frac{l'(l-l') n_c(l') n_o(l-l')}{V} - n_o(l) \frac{l-l'}{l'^{d/2}} \right) \\ & + \kappa \int_{l+l_c}^{\infty} dl' \left( \frac{n_o(l') l}{(l'-l)^{d/2}} - \frac{l n_o(l)(l'-l) n_c(l'-l)}{V} \right)\end{aligned}$$

# Boltzmann equations with D- branes

$$\begin{aligned}\frac{\partial n_c(l)}{\partial t} = & \frac{b}{2N} \frac{n_o(l)}{l^{p/2}} - a \frac{N}{V_{\perp}} l n_c(l) \\ & + \frac{\kappa}{2} \int_{l_c}^{l-l_c} dl' \left( \frac{n_c(l') l' n_c(l-l')(l-l')}{V} - l n_c(l) \left( \frac{l}{l'(l-l')} \right)^{d/2} \right) + \\ & + \kappa \int_{l+l_c}^{\infty} dl' \left( \frac{n_c(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V} \right) + \\ & + \kappa \int_{l+l_c}^{\infty} dl' \int_{l_c}^{l'-l-l_c} \frac{dl_x}{(l_x(l'-l-l_x))^{(d-p)/2}} \left( \frac{n_o(l') l'^{(d-p)/2}}{l^{d/2}} - \frac{l n_c(l)}{V} n_o(l'-l)(l'-l)^{(d-p)/2} \right).\end{aligned}$$

$$\begin{aligned}\frac{\partial n_o(l)}{\partial t} = & + a \frac{N}{V_{\perp}} l n_c(l) - \frac{b}{2N} \frac{n_o(l)}{l^{p/2}} \\ & + \int_{l_c}^{l-l_c} dl' \left( \frac{b}{2N V_{\parallel}} n_o(l') n_o(l-l') - a N n_o(l) \left( \frac{l}{l'(l-l')} \right)^{(d-p)/2} \right) \\ & + \int_{l+l_c}^{\infty} dl' \left( 2a N n_o(l') \left( \frac{l'}{l(l'-l)} \right)^{(d-p)/2} - \frac{b}{N V_{\parallel}} n_o(l) n_o(l'-l) \right) \\ & + \kappa \int_{l_c}^{l-l_c} dl' \int_{l_c}^{l'-l-l_c} \frac{dl_x}{(l_x(l-l'-l_x))^{(d-p)/2}} \left( \frac{l' n_c(l') n_o(l-l')}{V} (l-l')^{(d-p)/2} - \frac{n_o(l)}{l^{d/2}} l^{(d-p)/2} \right) \\ & + \kappa \int_{l+l_c}^{\infty} dl' \int_{l_c}^{l-l_c} \frac{dl_x}{(l_x(l-l_x))^{(d-p)/2}} \left( \frac{n_o(l')}{(l'-l)^{d/2}} l'^{(d-p)/2} - \frac{n_o(l)(l'-l) n_c(l'-l)}{V} l^{(d-p)/2} \right)\end{aligned}$$

# Out of equilibrium: equilibration rates

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- Consider perturbations around the equilibrium solution in  $d=0$ :

$$\frac{\partial \delta n(l, t)}{\partial t} = - \left( \frac{l^2}{2} + lL \right) \delta n(l, t) + \int_0^l dl' l' \delta n(l', t) \left( e^{\frac{-(l-l')}{L}} - 1 \right) - E \left( e^{-l/L} - 1 \right) ,$$

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- We find zero-energy solutions of the form:

$$\delta n(l, t) = \frac{e^{-l/L}}{L} + \sqrt{\frac{\pi(c + tL^2)}{2}} \frac{e^{-\frac{l}{L} + A(t)^2}}{L} \text{Erf} \left( \sqrt{\frac{c + tL^2}{2}} \frac{l}{L} + A(t), A(t) \right) ,$$

- Qualitatively, find a length-dependent equilibration rate:  $\delta n(l, t) \sim \delta n(l, 0) e^{-\left(\frac{l^2}{2} + lL\right)t}$

# Conclusions and future directions

- We have described string thermodynamics in terms of the *typical* string, described by its length and the number of non-compact directions.
- The equilibrium conditions we find agree with general equilibrium considerations, a non-trivial consistency check of the interactions.
- We are able to probe the out-of-equilibrium regime, showing explicit behaviour of fluctuations and computing equilibration rates.
- We plan on applying these results to do phenomenology:
  - Warm inflation? String gas cosmology? Reheating? GWS?!?!?